The effect of the asymmetric Ekman term on the phenomenology of the two-layer quasigeostrophic model

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February 22, 2020

Publications

- K.K. Tung and W.W. Orlando: *J. Atmos. Sci.* **60** (2003), 824-835.
- K.K. Tung and W.W. Orlando: *Discrete Contin. Dyn. Syst. Ser. B*, **3** (2003b), 145-162.
- K.K. Tung: *J. Atmos. Sci.*, **61** (2004), 943-948.
- E. Gkioulekas and K.K. Tung: *Discr. Cont. Dyn. Sys. B* **5** (2005), 79-102
- E. Gkioulekas and K.K. Tung: *Discr. Cont. Dyn. Sys. B* **5** (2005), 103-124.
- E. Gkioulekas and K.K. Tung: *J. Fluid Mech.*, **576** (2007), 173-189.
- E. Gkioulekas and K.K. Tung: *Discrete Contin. Dyn. Syst. Ser. B*, **7** (2007), 293-314
- E. Gkioulekas: *J. Fluid Mech.* **694** (2012), 493-523
- E. Gkioulekas: *Physica D* **284** (2014), 27-41
- E. Gkioulekas: *Physica D*, **403** (2020), 132372, 17 pp.

The two-layer model. I.

- The governing equations for the two-layer quasi-geostrophic model are

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1$$
$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2$$
$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0$$

  where $\zeta_1 = \nabla^2\psi_1; \zeta_2 = \nabla^2\psi_2; T = (2/h)(\psi_1 - \psi_2)$. f is the Coriolis term; N the Brunt-Väisälä frequency; Q_0 is the thermal forcing on the temperature equation; d_1, d_2 the dissipation terms.
- The Jacobian term $J(a, b)$ is defined as

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}.$$

The two-layer model. II.

- The three equations are situated in three layers:
  - ψ_1: At 0.25Atm, upper streamfunction layer
  - T: At 0.50Atm, temperature layer.
  - ψ_2: At 0.75Atm, lower streamfunction layer
- The potential vorticity is defined as

$$q_1 = \nabla^2\psi_1 + f + \frac{k_R^2}{2}(\psi_2 - \psi_1)$$
$$q_2 = \nabla^2\psi_2 + f - \frac{k_R^2}{2}(\psi_2 - \psi_1)$$

with $k_R \equiv 2\sqrt{2}f/(hN)$ and it satisfies

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1$$
$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2$$

with $f_1 = (1/4)k_R^2 hQ_0$ and $f_2 = -(1/4)k_R^2 hQ_0$.

The two-layer model. III.

- We use the following asymmetric dissipation configuration:

$$d_1 = \nu(-1)^{p+1}\nabla^{2p+2}\psi_1, \quad (1)$$

$$d_2 = (\nu + \Delta\nu)(-1)^{p+1}\nabla^{2p+2}\psi_2 - \nu_E \nabla^2 \psi_s. \quad (2)$$

- Differential hyperdiffusion: $\Delta\nu > 0$. [see Gkioulekas (2004)]
- $\psi_s = \lambda\psi_2 + \mu\lambda\psi_1$: Streamfunction at the surface boundary layer ($p_2 \leq p_s \leq 1\text{atm}$), with

$$\mu = \frac{p_2 - p_s}{p_s - p_1} \text{ and } \lambda = \frac{p_s - p_1}{p_2 - p_1} = \frac{1}{\mu + 1}. \quad (3)$$

- $\mu = 0$: Standard Ekman term with $\psi_s = \psi_2$.
- $\mu = -1/3$: Phillips Extrapolated Ekman term with $p_s = 1\text{atm}$.
- $\mu \in (-1/3, 0)$: Extrapolated Ekman term with $p_2 < p_s < 1\text{atm}$.

Energy and potential enstrophy spectrum. I.

- Two conserved quadratic invariants: energy E and potential enstrophy G_1, G_2 per layer:

$$E(t) = -\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\ [\psi_1(\mathbf{x},t)q_1(\mathbf{x},t) + \psi_2(\mathbf{x},t)q_2(\mathbf{x},t)], \tag{4}$$

$$G_1(t) = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\ q_1^2(\mathbf{x},t), \quad G_2(t) = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\ q_2^2(\mathbf{x},t), \tag{5}$$

- Filtered inner product:

$$\langle a, b\rangle_k = \frac{\mathrm{d}}{\mathrm{d}k} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\ a^{<k}(\mathbf{x}) b^{<k}(\mathbf{x})$$

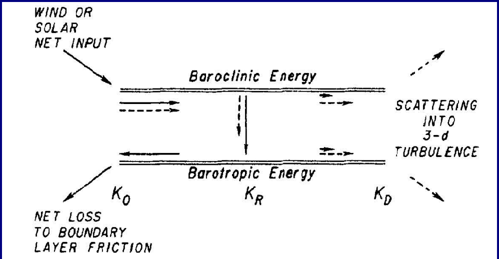
- The distribution of energy and potential enstrophy for each layer in Fourier space is described by:

$$E(k) = -\langle \psi_1, q_1\rangle_k - \langle \psi_2, q_2\rangle_k\,, \tag{6}$$

$$G_1(k) = \langle q_1, q_1\rangle_k\,, \quad G_2(k) = \langle q_2, q_2\rangle_k\,. \tag{7}$$

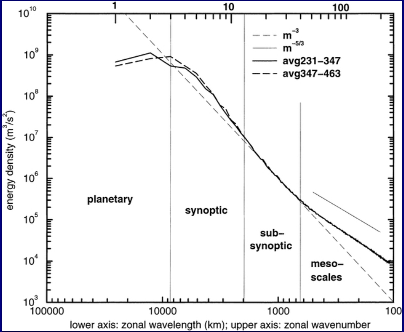
Salmon's phenomenology

- Barotropic vs Baroclinic energy spectrum:
  - Let $\psi = (\psi_1 + \psi_2)/2$ and $\tau = (\psi_1 - \psi_2)/2$
  - Barotropic energy spectrum: $E_K(k) = 2k^2 \langle \psi, \psi \rangle_k$
  - Baroclinic energy spectrum: $E_P(k) = 2(k^2 + k_R^2) \langle \tau, \tau \rangle_k$
- Salmon's phenomenology:
  - Energy injected as baroclinic at $k \ll k_R$.
  - Converted from baroclinic to barotropic at $k \sim k_R$.
  - Energy mostly barotropic at $k \ll k_R$.
  - Energy about half barotropic half baroclinic at $k \gg k_R$.



Tung and Orlando spectrum

- Tung-Orlando simulation: Coexisting downscale potential enstrophy cascade and downscale energy cascade
- Gkioulekas-Tung linear superposition principle:
  - Energy spectrum: $E(k) \approx C_1 \varepsilon_{uv}^{2/3} k^{-5/3} + C_2 \eta_{uv}^{2/3} k^{-3}$
  - Scaling transition at $k_t = \sqrt{\eta/\varepsilon}$



Previous work on two-layer QG model

- Conditions for observable transition to $k^{-5/3}$ scaling:
  - Violation of flux inequality $k^2\Pi_E(k) - \Pi_G(k) \leq 0$ for $k > k_t$
  - Energy and enstrophy injection rates should place $k_t \sim k_R$.
- Previous work on flux inequality on two-layer QG model:
  - E. Gkioulekas and K.K. Tung (2007): *Discrete Contin. Dyn. Syst. Ser. B*, **7**, 293-314
  - E. Gkioulekas (2014): *Physica D* **284**, 27-41
  - No transition to $k^{-5/3}$ scaling, with symmetric dissipation
  - Asymmetric dissipation needed $\Longrightarrow$ asymmetric Ekman term.
  - Concentration of potential enstrophy at top layer
- Previous work on forcing spectra on two-layer QG model:
  - E. Gkioulekas (2012): *J. Fluid Mech.* **694**, 493-523
  - Forcing spectra: $F_G(k) = (k^2 + k_R^2)F_E(k)$
  - Ratio of potential enstrophy to energy injection rates place k_t near k_R.
  - Unknown: **[effect of Ekman term]** on injection rates and potential enstrophy cascade.

Energy and potential enstrophy distributions. I.

- **New idea:** Define $\Gamma(k)$ and $P(k)$ such that:
  - $E_K(k) = [1 - P(k)]E(k)$ and $E_P(k) = P(k)E(k)$
  - $G_1(k) = \Gamma(k)G(k)$ and $G_2(k) = [1 - \Gamma(k)]G(k)$
  - $P(k)$: Controls distribution of energy between barotropic and baroclinic
  - $\Gamma(k)$: Controls distribution of potential enstrophy between upper and lower layers.
- **Major result:** $\Gamma(k)$ and $P(k)$ are rigorously restricted via:

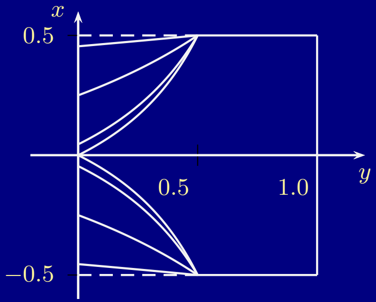
$$|2\Gamma(k) - 1| \leq \frac{k^2 + [1 - P(k)]k_R^2}{k^2 + k_R^2 P(k)}. \tag{8}$$

- In the limit $k \ll k_R$ (note: $k < k_R/10$ is close enough), Eq. (8) reduces to

$$|2\Gamma(k) - 1| \leq \min\left\{1, \frac{1 - P(k)}{P(k)}\right\}, \quad \text{for } k \ll k_R. \tag{9}$$

Energy and potential enstrophy distributions. II.

- Region constraining $\Gamma(k)$ and $P(k)$ using the representation $\Gamma(k) = 1/2 + x$ and $P(k) = 1 - y$.
- The smallest "pointy box" corresponds to the limit $k \ll k_R$.
- The larger boxes correspond to the wavenumber ratios $k/k_R = 10^{-1/2}, 1, 10^{1/2}$, with the box area increasing with the ratio k/k_R.



Ekman term dissipation rate spectra

- **Another major result:**

$$D_E(k) = \frac{[B_E^{(1)}(k)k^2 + B_E^{(2)}(k)k_R^2]d(k)E(k)}{k^2(k^2+k_R^2)},$$

$$D_{G_1}(k) = 0$$

$$D_{G_2}(k) = \frac{[B_G^{(1)}(k)k^4 + B_G^{(2)}(k)k^2k_R^2 + B_G^{(3)}(k)k_R^4]d(k)E(k)}{2k^2(k^2+k_R^2)},$$

with

$$B_E^{(1)}(k) = 2[1-\Gamma(k)] + \mu[1-2P(k)],$$

$$B_E^{(2)}(k) = [1-2\Gamma(k)P(k)] + \mu[1-P(k)],$$

$$B_G^{(1)}(k) = 4[1-\Gamma(k)] + 2\mu[1-2P(k)],$$

$$B_G^{(2)}(k) = [-4\Gamma(k)P(k) + 2P(k) - 2\Gamma(k) + 3] + \mu[3-2\Gamma(k)-4P(k)],$$

$$B_G^{(3)}(k) = (\mu+1)[1-2\Gamma(k)]P(k).$$

$D_{G_2}(k)$ in the limit $k \ll k_R$. I.

- In the limit $k \ll k_R$, the dominant contribution to $D_G(k)$ is given by
$$D_{G_2}(k) \sim \frac{\mu+1}{2}\left(\frac{k_R}{k}\right)^2 [1-2\Gamma(k)]P(k)d(k)E(k), \quad \text{with } k \ll k_R.$$
- All factors unconditionally positive or zero except $[1-2\Gamma(k)]$
- Stable fixed point dynamic:
  - $\Gamma(k) > 1/2 \Longrightarrow D_{G_2}(k) < 0 \Longrightarrow$ potential enstrophy injected at bottom layer $\Longrightarrow \Gamma(k)$ decreases.
  - $\Gamma(k) < 1/2 \Longrightarrow D_{G_2}(k) > 0 \Longrightarrow$ potential enstrophy removed at bottom layer $\Longrightarrow \Gamma(k)$ increases.
  - Recall: Energy is injected as baroclinic $\Longrightarrow P(k) \approx 1 \Longrightarrow \Gamma(k)$ is **initially** constrained in a narrow interval around $1/2$.
  - Ekman potential enstrophy dissipation rate spectrum $\Longrightarrow$ **[stable fixed point]** $\Gamma(k) \approx 1/2$
  - Diminished potential enstrophy dissipation $\Longrightarrow$ **[happy enstrophy cascade]**
- Energy mostly barotropic at $k \ll k_R \Longrightarrow P(k)$ close to 0 at $k \ll k_R \Longrightarrow$ Leading term suppressed (expect $P(k) \sim 0.1$)

$D_{G_2}(k)$ in the limit $k \ll k_R$. II.

- When $\Gamma(k)$ is near $1/2$ or $P(k)$ near 0, the subleading contribution to the potential enstrophy dissipation rate spectrum $D_{G_2}(k)$ becomes dominant.
- The sign of the subleading contribution is controlled by the numerical coefficient $B_G^{(2)}(k)$.
- We have shown that $\mu = -1/3 \implies B_G^{(2)}(k) > 0$
- Otherwise, under the assumption $0 \leq \Gamma(k) < 1$, we have:

$$\begin{cases} -1/3 < \mu < 0 \\ |2\Gamma(k) - 1| \leq \min \left\{ 1, \dfrac{1 - P(k)}{P(k)} \right\} \end{cases} \implies B_G^{(2)}(k) > 0, \tag{10}$$

$$\begin{cases} \mu = 0 \\ |2\Gamma(k) - 1| < \min \left\{ 1, \dfrac{1 - P(k)}{P(k)} \right\} \end{cases} \implies B_G^{(2)}(k) > 0. \tag{11}$$

- For $\Gamma(k) = 1/2$: Leading term zero, subleading term positive $\implies$ potential enstrophy removed from bottom layer $\implies \Gamma(k)$ increases.
- Stable fixed point shifts to $\Gamma(k) = 1/2 + \gamma_0(k)$ with $\gamma_0(k) > 0$

$D_{G_2}(k)$ in the limit $k \gg k_R$

- In the limit $k \gg k_R$, the dominant contribution to $D_{G_2}(k)$ is given by

$$D_{G_2}(k) \sim (1/2) B_G^{(1)}(k) d(k) E(k),$$
$$B_G^{(1)}(k) = 2[1 - \Gamma(k)] + \mu[1 - 2P(k)].$$

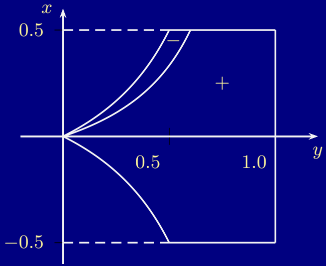
- For standard Ekman:
  - $\mu = 0 \Longrightarrow D_{G_2}(k) > 0 \Longrightarrow$ potential enstrophy will be dissipated from the bottom layer $\Longrightarrow$ potential enstrophy becomes increasingly concentrated in the top layer $\Longrightarrow$ helps violate flux inequality.
  - Expect $P(k)$ near $1/2$. When $P(k) < 1/2 \Longrightarrow B_G^{(2)}(k) > 0$, subleading contribution is dissipative. Otherwise fixed-point $\Gamma(k)$ for no potential enstrophy dissipation shifted slightly below 1
- For extrapolated Ekman: We showed that $\Gamma(k) < 5/6 \Longrightarrow B_G^{(1)}(k) > 0$. Same dynamic with stable fixed point at $5/6 \leq \Gamma(k) \leq 1$.

The energy dissipation rate spectrum $D_E(k)$. I.

- For standard Ekman dissipation $\Longrightarrow D_E(k) \geq 0 \Longrightarrow$ asymmetric Ekman term will always dissipate energy
- For extrapolated Ekman dissipation, in the limit $k \ll k_R$

$$D_E(k) \sim \frac{B_E^{(2)}(k)d(k)E(k)}{k^2} \quad \text{with } k \ll k_R, \tag{12}$$

- Expected barotropization $\Longrightarrow$ not in negative region
- In negative region, tend to increase $y \Longrightarrow$ **[leave]**

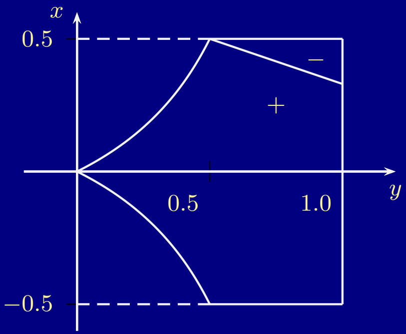


The energy dissipation rate spectrum $D_E(k)$. II.

- For extrapolated Ekman dissipation, in the limit $k \gg k_R$

$$D_E(k) \sim \frac{B_E^{(1)}(k)d(k)E(k)}{k^2} \quad \text{with } k \gg k_R. \tag{13}$$

- Barotropization necessary to be in the negative region.
- In negative region, tend to increase $y \Longrightarrow$ **[stay]**



Conclusion

- Constaint between $\Gamma(k)$ (potential enstrophy distribution between layers) and $P(k)$ (energy distribution between barotropic and baroclinic).
- For $k \ll k_R$, the tendency of the Ekman term is to stabilize the equipartition of potential enstrophy between the two layers towards a stable fixed point distribution in which the Ekman term does not dissipate potential enstrophy, which prevents distortion of potential enstrophy cascade.
- For $k \gg k_R$ the Ekman term is expected to dissipate potential enstrophy from the bottom layer.
- Standard Ekman term unconditionally dissipates energy over all wavenumbers k
- Extrapolated Ekman term has negative regions, both in the limit $k \ll k_R$ and $k \gg k_R$ where the Ekman term may be injecting energy

Thank you!