Multilocality and fusion rules on the generalized structure functions in two-dimensional and three-dimensional Navier-Stokes turbulence

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Outline

- Brief review of turbulence phenomenology
- Development of analytical theories of turbulence
- Review of my extensions of the Lvov-Procaccia theory
- Locality and multilocality

Publications

- This presentation is based on
  1. E. Gkioulekas and K.K. Tung: *Discrete and Continuous Dynamical Systems B* **5**, (2005), 79-102
  2. E. Gkioulekas and K.K. Tung: *Discrete and Continuous Dynamical Systems B* **5**, (2005), 103-124.
  3. E. Gkioulekas: *Physica D* **226** (2007), 151-172
  4. E. Gkioulekas: *Phys. Rev. E* **78** (2008), 066302
  5. E. Gkioulekas: *Phys. Rev. E* **82** (2010), 046304
  6. E. Gkioulekas: *Phys. Rev. E* **94** (2016), 033105
- Other relevant papers include:
  1. U. Frisch, *Proc. R. Soc. Lond. A* **434** (1991), 89–99.
  2. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **52** (1995), 3840–3857.
  3. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **54** (1996), 6268–6284.

K41 theory. I.

- In three-dimensional turbulence there is an energy cascade from large scales to small scales driven by the nonlinear term of the Navier-Stokes equations
- Kolmogorov (1941) predicts that the structure functions $S_n(\mathbf{x}, r\mathbf{e})$ of longitudinal velocity differences, defined as

$$S_n(\mathbf{x}, r\mathbf{e}) = \langle \{[\mathbf{u}(\mathbf{x} + r\mathbf{e}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \mathbf{e}\}^n \rangle \tag{1}$$

  are governed by self-similar scaling $S_n(\mathbf{x}, \lambda r\mathbf{e}) = \lambda^{\zeta_n} S_n(\mathbf{x}, r\mathbf{e})$ for scales r in the inertial range $\eta \ll r \ll \ell_0$ (intermediate asymptotics) with
  - $\ell_0 =$ forcing length scale
  - $\eta = (\nu^3/\varepsilon)^{1/4} =$ dissipation scale. (Kolmogorov microscale)
  - $\varepsilon =$ rate of energy injection
- Kolmogorov (1941) predicts that $\zeta_n = n/3$ and thus $S_n(\mathbf{x}, r\mathbf{e}) \sim C_n(\varepsilon r)^{n/3}$ in the inertial range.

K41 theory. II.

- Oboukhov (1941) argued that the energy spectrum $E(k)$ will scale as $E(k) \sim k^{-1-\zeta_2}$, and will thus be given by

$$E(k) \sim C\varepsilon^{2/3}k^{-5/3} \tag{2}$$

- 1962: First experimental confirmation of the Kolmogorov-Oboukhov prediction by measurement of oceanic currents.
- 1962: Kolmogorov predicts **intermittency corrections** to ζ_n:

$$\zeta_n = \frac{n}{3} - \frac{\mu n(n-3)}{18} \tag{3}$$

- Not self-consistent statistically, because ζ_n should not decrease.
- The existence of intermittency corrections confirmed by experimental measurements
- The problem of calculating ζ_n rigorously is still open.

Governing equations for 2D turbulence

- In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = -[\nu(-\Delta)^p + \beta(-\Delta)^{-h}]\zeta + F, \tag{4}$$

  where $\psi(x, y, t)$ is the streamfunction and $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$.
- The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ, and is defined as

$$J(\psi, \zeta) = \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial y}. \tag{5}$$

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2}\int \psi(x, y, t)\zeta(x, y, t)\, dxdy \quad G(t) = \frac{1}{2}\int \zeta^2(x, y, t)\, dxdy. \tag{6}$$

KLB theory. I

Kraichnan, Leith, and Batchelor (KLB) proposed that in two-dimensional turbulence there is an upscale energy cascade and a downscale enstrophy cascade. The energy spectrum in the upscale energy range is

$$E(k) = C_{ir}\varepsilon^{2/3}k^{-5/3}, \tag{7}$$

and in the downscale enstrophy range is

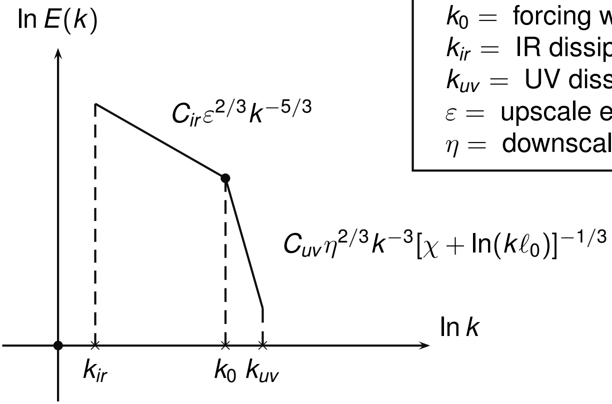
$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{8}$$

Falkovich and Lebedev (1994) predict that the vorticity ζ structure functions have logarithmic scaling given by

$$\langle[\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]^n\rangle \sim [\eta \ln(\ell_0/r_{12})]^{2n/3}. \tag{9}$$

Confirmed using spectral reduction by Bowman, Shadwick and Morrison (1999).

KLB theory. II



$k_0 =$ forcing wavenumber
$k_{ir} =$ IR dissipation wavenumber
$k_{uv} =$ UV dissipation wavenumber
$\varepsilon =$ upscale energy flux
$\eta =$ downscale enstrophy flux

Open questions with 2D turbulence

- Both cascades have been reproduced successfully in numerical simulations, however the cascades of 2D turbulence are not as robust as those of 3D turbulence
- The enstrophy cascade is hard to reproduce without using hyperdiffusion – first observed by Lindborg in 1999
- The inverse energy cascade readily manifests but tends to be disrupted by coherent vortices as it approaches steady state
- Lack of intermittency corrections to the downscale enstrophy cascade and inverse energy cascade.

Analytical theories.

- In the beginning: Quasinormal closure models.
- 1957: Kraichnan showed that they give negative $E(k)$.
- 1958: Kraichnan DIA theory $\Longrightarrow k^{-3/2}$ scaling.
- 1961: Wyld shows that DIA is 1-loop line-renormalized diagrammatic theory
- 1962: Experiments confirm $k^{-5/3}$ scaling.
- 1964: Kraichnan notes the need to eliminate the sweeping interactions via a Lagrangian transformation.
- 1965: LHDIA theory $\Longrightarrow$ Locality $\Longrightarrow k^{-5/3}$ scaling.
- 1973: Martin-Siggia-Rose theory (MSR theory)
- 1977: Phythian reformulates MSR theory in terms of path integrals.
- 1987: Belinicher-Lvov: quasi-Langrangian representation
- 1995-2000: Lvov-Procaccia theory – going beyond LHDIA

Structure of the Lvov-Procaccia theory

- MSR formalism applied on Navier-Stokes equations using the quasi-Lagrangian representation
- Perturbative theory
  - Uses the Schwinger-Dyson equations of MSR theory with line renormalization
  - Generalizes LHDIA $\Longrightarrow$ reproduces K41 scaling to finite order
  - Cause of intermittency: Ladder diagram divergences
  - Derivation of the fusion rules
  - Perturbative calculation of ζ_n
- Non-perturbative theory
  - Navier-Stokes equations $\Longrightarrow$ balance equations for generalized structure functions
  - Balance equations $+$ fusion rules $\Longrightarrow$ non-perturbative locality
  - Stability of cascades with respect to forcing
  - Transition to the dissipation range

Extensions of Lvov-Procaccia theory to 2D turbulence by Gkioulekas

- Coexisting cascades – linear superposition principle
- Generalized fusion rules to upscale cascades
- Generalized non-perturbative theory to 2D turbulence
  - Inverse energy cascade is stable with respect to forcing but can be disrupted by sweeping.
  - Downscale enstrophy cascade is marginally stable. Stability depends on downscale energy flux
  - Fusion rules imply anomalous sinks at small scales and large scales.
  - Logarithmic correction is essential in separating the dissipation range from the inertial range of downscale enstrophy cascade dissipated under regular diffusion. Not needed under hyperdiffusion.

Framework for non-perturbative theory

- Define the n^{th}-order generalized structure functions:

$$F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t) = \left\langle \prod_{k=1}^{n} w_{\alpha_k}(\mathbf{x}_k,\mathbf{x}'_k,t) \right\rangle$$

- Differentiating with respect to time t yields an equation of the form

$$\frac{\partial F_n}{\partial t} + \mathcal{O}_n F_{n+1} = I_n + \mathscr{D}_n F_n + Q_n$$

with:
  - $\mathcal{O}_n F_{n+1}$ representing the nonlinear local interactions that govern the cascades
  - I_n representing the sweeping interactions
  - $\mathscr{D}_n F_n$ representing the dissipation terms
  - Q_n representing the forcing terms

What is locality and multilocality. I

- The mathematical structure of $\mathcal{O}_n F_{n+1}$ takes the form

$$\mathcal{O}_n F_{n+1}(\{\mathbf{X}\}_n, t) = \sum_{k=1}^{n} \iint \mathrm{d}\mathbf{Y}_1 \mathrm{d}\mathbf{Y}_2 \mathcal{O}(\mathbf{X}_k, \mathbf{Y}_1, \mathbf{Y}_2) F_{n+1}(\{\mathbf{X}\}_n^k, \mathbf{Y}_1, \mathbf{Y}_2).$$

- Locality means that the integrals do not diverge in the IR and UV limits and the main contributions originates at the scale $R \sim \{\mathbf{X}\}_n$
- Locality implies that if $F_n \sim R^{\zeta_n} \Longrightarrow \mathcal{O}_n F_{n+1} \sim R^{\zeta_{n+1}-1}$.
- Multilocality: Show that the integrals of the terms that comprise $\mathcal{O}_n \mathcal{O}_{n+1} \cdots \mathcal{O}_{n+p-1} F_{n+p}$ are also local.
- Multilocality implies that $F_n \sim R^{\zeta_n} \Longrightarrow \mathcal{O}_n \mathcal{O}_{n+1} \cdots \mathcal{O}_{n+p-1} F_{n+p} \sim R^{\zeta_{n+p}-p}$
- Multilocality is needed by: previous argument to establish bridge relations; future study of dissipation scales.
- Special challenge by crossterms.

What is locality and multilocality. II

- For example:

$$\mathcal{O}_n \mathcal{O}_{n+1} F_{n+2}(\{\mathbf{X}\}_n, t) = \sum_{l=1}^{n} \iint d\mathbf{Z}_1 d\mathbf{Z}_2 \; \mathcal{O}(\mathbf{X}_l, \mathbf{Z}_1, \mathbf{Z}_2) \mathcal{O}_{n+1} F_{n+2}(\{\mathbf{X}\}_n^l, \mathbf{Z}_1, \mathbf{Z}_2)$$

$$= \sum_{l=1}^{n} \sum_{\substack{k=1 \\ k \neq l}}^{n} \iint d\mathbf{Z}_1 d\mathbf{Z}_2 \iint d\mathbf{Y}_1 d\mathbf{Y}_2 \, \mathcal{O}(\mathbf{X}_l, \mathbf{Z}_1, \mathbf{Z}_2) \mathcal{O}(\mathbf{X}_k, \mathbf{Y}_1, \mathbf{Y}_2) F_{n+2}(\{\mathbf{X}\}_n^{kl}, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Z}_1, \mathbf{Z}_2)$$

$$+ \sum_{l=1}^{n} \iint d\mathbf{Z}_1 d\mathbf{Z}_2 \iint d\mathbf{Y}_1 d\mathbf{Y}_2 \, \mathcal{O}(\mathbf{X}_l, \mathbf{Z}_1, \mathbf{Z}_2) \mathcal{O}(\mathbf{Z}_1, \mathbf{Y}_1, \mathbf{Y}_2) F_{n+2}(\{\mathbf{X}\}_n^l, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Z}_2)$$

$$+ \sum_{l=1}^{n} \iint d\mathbf{Z}_1 d\mathbf{Z}_2 \iint d\mathbf{Y}_1 d\mathbf{Y}_2 \, \mathcal{O}(\mathbf{X}_l, \mathbf{Z}_1, \mathbf{Z}_2) \mathcal{O}(\mathbf{Z}_2, \mathbf{Y}_1, \mathbf{Y}_2) F_{n+2}(\{\mathbf{X}\}_n^l, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Z}_1)$$

- The second and third terms are cross-terms and require a separate locality argument.
- *The main claim is that the fusion rules hypothesis implies both locality and multilocality in both the IR and UV limits for the downscale energy cascade of three-dimensional Navier-Stokes turbulence and the downscale enstrophy cascade and inverse energy cascade of two-dimensional Navier-Stokes turbulence.*

What are the fusion rules. I

- The fusion rules encapsulate mathematically the notion that deep inside the inertial range and far away from the forcing range, the statistical details of random forcing are forgotten. The same dynamic plays out between the small scales r and the large scales R
- Define the conditional generalized structure function

$$\Phi_{nm}^{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m}(\{\mathbf{X}\}_n, \{\mathbf{Y}\}_m, \{w\}_m, t) = \\ \left\langle \left[\prod_{\kappa=1}^{n} w_{\alpha_k}(\mathbf{X}_k, t) \middle| \, w_{\beta_k}(\mathbf{Y}_k, t) = w_k, \ \forall k \in \{1, \dots, m\} \right] \right\rangle . \quad (10)$$

- We postulate that for a downscale cascade, with $\{\mathbf{X}\}_n \sim r$ and $\{\mathbf{Y}\}_m \sim R$ with $r \ll R$ and r, R both in the inertial range

$$\Phi_{nm}(\{\mathbf{X}\}_n, \{\mathbf{Y}\}_m, \{w\}_m, t) = \tilde{F}_n(\{\mathbf{X}\}_n, t)\tilde{\Phi}_{nm}(\{\mathbf{Y}\}_m, \{\mathbf{w}\}_m, t) \quad (11)$$

- For the case of the inverse energy cascade we assume that the above equation holds when $\{\mathbf{X}\}_n \sim R$ and $\{\mathbf{Y}\}_m \sim r$, with $r \ll R$ and r, R both in the inertial range.

What are the fusion rules. II

- Via the Bayes theorem we can show that for $\{\mathbf{X}\}_{k=1}^{p} \sim r$ and $\{\mathbf{X}\}_{k=p+1}^{n} \sim R$ with $r \ll R$ and both r, R in the inertial range, the generalized structure function $F_n(\{\mathbf{X}\}_n, t)$ will give

$$F_n(\{\mathbf{X}\}_n, t) = \tilde{F}_p(\{\mathbf{X}\}_{k=1}^{p}, t)\Psi_{n,p}(\{\mathbf{X}\}_{k=p+1}^{n}, t)$$

  for a downscale cascade and

$$F_n(\{\mathbf{X}\}_n, t) = \tilde{F}_{n-p}(\{\mathbf{X}\}_{k=p+1}^{n}, t)\Psi_{n,n-p}(\{\mathbf{X}\}_{k=1}^{p}, t)$$

  for an upscale cascade

- Equivalently, for $\{\mathbf{X}\}_p \sim r$ and $\{\mathbf{Y}\}_{n-p} \sim R$ with $r \ll R$, with both r, R in the inertial range, we find that

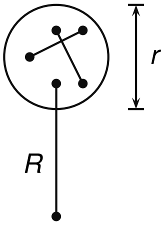
$$F_n(\lambda\{\mathbf{X}\}_p, \mu\{\mathbf{Y}\}_{n-p}) = \lambda^{\xi_{np}}\mu^{\zeta_n - \xi_{np}} F_n(\{\mathbf{X}\}_p, \{\mathbf{Y}\}_{n-p})$$

  with $\xi_{np} = \zeta_p$ for downscale cascades and $\xi_{np} = \zeta_n - \zeta_{n-p}$ for upscale cascades.

Fusion rule for $p = 1$

- Let $F_n^{(p)}(r, R)$ denote a generalized structure function with p velocity differences reduced to length scale r and the remaining $n - p$ velocity differences at length scale R.
- For $p = 1$ the leading order contribution to the fusion rule vanishes, both for upscale and downscale cascades.
- Define $R_{\min}$: the minimum distance between the small velocity difference from the other velocity differences
  - $r \ll R_{\min} \Longrightarrow F_n^{(1)}(r, R) \sim (r/R_{\min})R^{\zeta_n}$
  - $r \gg R_{\min} \Longrightarrow F_n^{(1)}(r, R) \sim r^{\xi_{n,1}} R^{\zeta_n - \xi_{n,1}}$
  - $\xi_{n,1} = \zeta_2$ for downscale cascades and $\xi_{n,1} = \zeta_n - \zeta_{n-2}$ for upscale cascades, for all $n > 3$
  - For $n = 3$: an additional cancellation will give $\xi_{3,1} = \zeta_3$ (both upscale and downscale cascades)
  - For $n = 2$: we get $\xi_{2,1} = \zeta_2$ (both upscale and downscale cascades)
- Main result: $\xi_{n+p,1} > 0 \Longrightarrow$ UV multilocality for $\mathcal{O}_n \mathcal{O}_{n+1} \cdots \mathcal{O}_{n+p-1} F_{n+p}$.

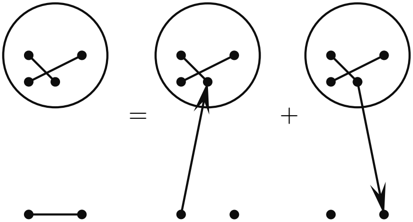
Fusion rule for $p = n - 1$



- The $p = n - 1$ fusion rule velocity difference geometry where one endpoint of the unfused velocity difference is within the r-blob where the other $n - 1$ velocity differences are gathered
- The leading order contribution to the fusion rule vanishes, both for upscale and downscale cascades.
- The next order contribution gives $F_n^{(n-1)}(r, R) \sim r^{\zeta_n} R^0$, thus $\xi_{n,n-1} = \zeta_n$, both for upscale and downscale cascades.
- Responsible for $\xi_{3,1} = \zeta_3$ and $\xi_{2,1} = \zeta_2$, both for upscale and downscale cascades.

The α scaling exponent

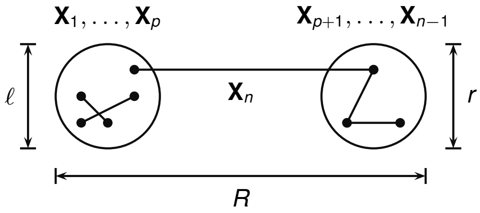
- Separating one velocity difference away from a group of velocity differences congregated inside a small-scale blob. The arrows indicate the direction of the velocity differences involved in the major cancellation of the leading $r^{\zeta_n} R^0$ contribution.



- Leading-order term cancellation in the limit $R \to +\infty$
- Next order term scaling as $r^{\zeta_n} R^0 (r/R)^{\alpha} \sim r^{\zeta_n + \alpha} R^{-\alpha}$, with $\alpha > 0$

two-blob geometry and IR multilocality

- The two-blob velocity difference geometry with $\ell \ll R$ and $r \ll R$ and ℓ, r, R all within the inertial range.



- For downscale cascades, the scaling exponent Δ_{np} of R is:
  - For downscale cascades: $\Delta_{np} = \zeta_n - \zeta_{p+1} - \zeta_{n-p} < 0$
  - For upscale cascades: $\Delta_{np} = -\alpha < 0$ (with α defined in the previous slide)
- IR multilocality of $\mathcal{O}_n \mathcal{O}_{n+1} \cdots \mathcal{O}_{n+p-1} F_{n+p}$ follows from $\Delta_{n+p,m} < 0$ for all m with $1 \leq m < p$.

Conclusion – Main Results

- Generalized fusion rules to upscale cascades
- The fusion rule for the new two-blob velocity difference geometry, both for upscale and downscale cascades
- Proof that the fusion rules imply locality and multilocality.
- IR multilocality for downscale cascades requires fusion rules with $p > 2$.

Thank you!