On the denesting of nested square roots

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Introduction to denesting radicals. I. What are they?

- Using basic arithmetic, we can expand $(2+\sqrt{2})^2 = 2^2 + 2\cdot 2\sqrt{2} + (\sqrt{2})^2 = 6 + 4\sqrt{2}$ and then turn that around to write $\sqrt{6+4\sqrt{2}} = 2+\sqrt{2}$.
- The left-hand-side is an example of a *nested radical*, which is defined as an expression involving rational numbers, the basic four operations of arithmetic (addition, subtraction, multiplication, division), and roots, such that some root appears under another root.
- *Denesting* means rewritting the expression so that only rational numbers appear inside roots.
- Here, we limit ourselves to enesting expressions of the form $A = \sqrt{a \pm b\sqrt{p}}$ and $\sqrt{a\sqrt{p}+b\sqrt{q}}$ with a, b rational numbers and p, q rational positive numbers
- Such expressions may occur in solutions of quadratic or biquadratic equations, trigonometry problems, integrals of rational functions, and so on.

Introduction to denesting radicals. II. A broader problem

- Nested radicals involving a square root inside a cubic root routinely emerge when solving cubic equations
- More complicated examples of nested radicals were given by Ramanujan such as:

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}, \tag{1}$$

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} = (1/3)(\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}), \tag{2}$$

- Open question: Is there a systematic algorithm that can be used to denest radical expressions of arbitrary complexity?
  - Blomer 1982: Algorithm that can handle nested radicals with depth 2 (roots inside roots) but cannot handle depths greater than 2
  - Zippel 1992: Method that is able to denest some radicals with mixed roots
  - Landau 1992: Most general algorithm with shortcomings: the denested expression will use complex roots of unity; exponential time with respect to the depth of the expression;

Introduction to denesting radicals. III. Nested square roots

- For the simple case $A = \sqrt{a \pm b\sqrt{p}}$, Borodin (1985) showed that it can denest in only two ways, or not at all:

$$\sqrt{a \pm b\sqrt{p}} = \sqrt{x} \pm \sqrt{y}; \qquad \text{or} \qquad \sqrt{a \pm b\sqrt{p}} = \sqrt[4]{p}(\sqrt{x} \pm \sqrt{y}), \qquad (3)$$

  with x, y, p also positive rational numbers.
- In this talk, we will
  - Derive the necessary and sufficient conditions for the existence of denestings in accordance to Eq. (3), without however going as far as to show that these are the only possible denestings.
  - Show related examples.
- The proofs can serve as excellent examples for introducing concepts of proof in lower-level coursework, such as proof by contradiction, proof by cases, and quantified statements.

Definition: Direct vs indirect denesting

- $\mathbb{Q}$ represents the set of all rational numbers
- We define also $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ and $\mathbb{Q}_+^* = \{x \in \mathbb{Q} | x > 0\}$.
- We pose the problem of denesting the expression $A = \sqrt{a \pm b\sqrt{p}}$ via the following definition:

Definition 1

Let $A = \sqrt{a \pm b\sqrt{p}}$ with $a, b, p \in \mathbb{Q}_+^*$ and $\sqrt{p} \notin \mathbb{Q}$. We say that:

$$A \text{ denests directly } \iff \exists x, y \in \mathbb{Q}_+^* : A = \sqrt{x} \pm \sqrt{y},$$
$$A \text{ denests indirectly } \iff \exists x, y \in \mathbb{Q}_+^* : A = \sqrt[4]{p}(\sqrt{x} \pm \sqrt{y}).$$

- The goal is to derive necessary and sufficient conditions for the statements "A denests directly" and "A denests indirectly" and to calculate the corresponding rational numbers x, y.
- To show these theorems, we begin with showing the following lemmas

Preliminaries. I. Irrationality of root difference

Lemma 2

Let $a, b \in \mathbb{Q}_+^*$ be given. Then

$$(\sqrt{a} \notin \mathbb{Q} \wedge a \neq b) \implies \sqrt{a} - \sqrt{b} \notin \mathbb{Q}$$

Proof.

Assume that $\sqrt{a} \notin \mathbb{Q}$ and $a \neq b$. To show that $\sqrt{a} - \sqrt{b} \notin \mathbb{Q}$, we assume that $\sqrt{a} - \sqrt{b} \in \mathbb{Q}$ in order to derive a contradiction. Since, $a \neq b \implies \sqrt{a} - \sqrt{b} \neq 0$, we may write

$$\begin{aligned}
\sqrt{a} &= (1/2)[(\sqrt{a} + \sqrt{b}) + (\sqrt{a} - \sqrt{b})] \\
&= \frac{1}{2}\left[\frac{(\sqrt{a})^2 - (\sqrt{b})^2}{\sqrt{a} - \sqrt{b}} + (\sqrt{a} - \sqrt{b})\right] \\
&= \frac{1}{2}\left[\frac{a - b}{\sqrt{a} - \sqrt{b}} + (\sqrt{a} - \sqrt{b})\right],
\end{aligned}$$

and it follows that $\sqrt{a} - \sqrt{b} \in \mathbb{Q} \implies \sqrt{a} \in \mathbb{Q}$ which is a contradiction, since by hypothesis we have $\sqrt{a} \notin \mathbb{Q}$. We conclude that $\sqrt{a} - \sqrt{b} \notin \mathbb{Q}$. □

Preliminaries. II. Equality condition for mixed expressions

Lemma 3

Let $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ with $b_1 > 0$ and $b_2 > 0$ and $\sqrt{b_1} \notin \mathbb{Q}$. Then

$$a_1 \pm \sqrt{b_1} = a_2 \pm \sqrt{b_2} \Longleftrightarrow (a_1 = a_2 \wedge b_1 = b_2).$$

Proof.

($\Longrightarrow$): Assume that $a_1 \pm \sqrt{b_1} = a_2 \pm \sqrt{b_2}$. We distinguish between the following cases:
Case 1: Assume that $b_1 = b_2$. Then

$$\begin{aligned} a_1 \pm \sqrt{b_1} = a_2 \pm \sqrt{b_2} &\Longrightarrow a_1 \pm \sqrt{b_1} = a_2 \pm \sqrt{b_1} && [\text{ via } b_1 = b_2] \\ &\Longrightarrow a_1 = a_2, \end{aligned}$$

and we conclude that $a_1 = a_2 \wedge b_1 = b_2$.
Case 2: Assume that $b_1 \neq b_2$. Then

$$\begin{aligned} a_1 \pm \sqrt{b_1} = a_2 \pm \sqrt{b_2} &\Longrightarrow \pm(\sqrt{b_1} - \sqrt{b_2}) = a_2 - a_1 \\ &\Longrightarrow \sqrt{b_1} - \sqrt{b_2} \in \mathbb{Q}. && [\text{via } a_1, a_2 \in \mathbb{Q}] \end{aligned}$$

This is a contradiction because, using Lemma 2, we have
$(\sqrt{b_1} \notin \mathbb{Q} \wedge b_1 \neq b_2) \Longrightarrow \sqrt{b_1} - \sqrt{b_2} \notin \mathbb{Q}$. This means that this case does not materialize.
($\Longleftarrow$): Assume that $a_1 = a_2 \wedge b_1 = b_2$. Then, it trivially follows that $a_1 \pm \sqrt{b_1} = a_2 \pm \sqrt{b_2}$. □

Preliminaries. III. The fundamental linear-quadratic system

Lemma 4

Let $f(z) = z^2 - az + b$ with zeroes $z_1, z_2 \in \mathbb{R}$. It follows that

$$\begin{cases} x + y = a \\ xy = b \end{cases} \iff \begin{cases} x = z_1 \\ y = z_2 \end{cases} \vee \begin{cases} x = z_2 \\ y = z_1. \end{cases}$$

Proof.

Since $z_1, z_2 \in \mathbb{R}$ are zeroes of $f(z) = z^2 - az + b$, from the fundamental theorem of algebra, we write $z^2 - az + b = (z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1 z_2$, $\forall z \in \mathbb{R}$, and therefore $z_1 + z_2 = a$. It follows that $z_1 = a - z_2$ and $z_2 = a - z_1$. Then, we argue that:

$$\begin{cases} x + y = a \\ xy = b \end{cases} \iff \begin{cases} y = a - x \\ x(a - x) = b \end{cases} \iff \begin{cases} y = a - x \\ ax - x^2 = b \end{cases} \iff \begin{cases} y = a - x \\ x^2 - ax + b = 0 \end{cases}$$

$$\iff \begin{cases} y = a - x \\ x = z_1 \vee x = z_2 \end{cases} \iff \begin{cases} x = z_1 \\ y = a - z_1 \end{cases} \vee \begin{cases} x = z_2 \\ y = a - z_2 \end{cases}$$

$$\iff \begin{cases} x = z_1 \\ y = z_2 \end{cases} \vee \begin{cases} x = z_2 \\ y = z_1. \end{cases}$$

□

Preliminaries. IV. Positive zeroes Lemma

Lemma 5

$\forall x, y \in \mathbb{R} : \left(\begin{cases} x + y > 0 \\ xy > 0 \end{cases} \Longleftrightarrow \begin{cases} x > 0 \\ y > 0 \end{cases} \right).$

Proof.

($\Longrightarrow$): Let $x, y \in \mathbb{R}$ be given such that $x + y > 0$ and $xy > 0$. To show that $x > 0 \wedge y > 0$, we assume the negation of that statement, which reads $x \leq 0 \vee y \leq 0$, in order to derive a contradiction. Since the lemma remains invariant with respect to the exchange $x \leftrightarrow y$, we may assume, with no loss of generality, that $x \leq 0$ and then distinguish between the following cases:

Case 1: Assume that $y \leq 0$. Then $x \leq 0 \wedge y \leq 0 \Longrightarrow x + y \leq 0$, which is a contradiction, so this case does not materialize.

Case 2: Assume that $y > 0$. Then

$$\begin{cases} x \leq 0 \\ y > 0 \end{cases} \Longrightarrow \begin{cases} -x \geq 0 \\ y > 0 \end{cases} \Longrightarrow -xy \geq 0 \Longrightarrow xy \leq 0,$$

which is also a contradiction, since $xy > 0$, so this case also does not materialize. Since neither case materializes, we have an overall contradiction and we conclude that $x > 0 \wedge y > 0$

($\Longleftarrow$): Assume that $x > 0 \wedge y > 0$. It follows immediately that $x + y > 0 \wedge xy > 0$. □

Direct denesting theorem. I

Theorem 6

Let $A = \sqrt{a \pm \sqrt{b}}$ *with* $a, b \in \mathbb{Q}^*$ *and* $b > 0$ *and* $\sqrt{b} \notin \mathbb{Q}$ *and* $a \pm \sqrt{b} > 0$. *Then, it follows that*

$$A \text{ denests directly} \iff \begin{cases} \exists \delta \in \mathbb{Q}_+^* : a^2 - b = \delta^2 \\ a > 0 \end{cases},$$

$$\begin{cases} \delta = \sqrt{a^2 - b} \\ a > 0 \end{cases} \implies \sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\delta}{2}} \pm \sqrt{\frac{a-\delta}{2}}. \tag{4}$$

Proof.

Under the assumption $x > y > 0$, we establish the following equivalence:

$$\begin{aligned}
\sqrt{a \pm \sqrt{b}} = \sqrt{x} \pm \sqrt{y} &\iff a \pm \sqrt{b} = (\sqrt{x} \pm \sqrt{y})^2 && \text{[require } x > y\text{]} \\
&\iff a \pm \sqrt{b} = (\sqrt{x})^2 \pm 2\sqrt{x}\sqrt{y} + (\sqrt{y})^2 \\
&\iff a \pm \sqrt{b} = (x + y) \pm \sqrt{4xy} \\
&\iff \begin{cases} x + y = a \\ 4xy = b \end{cases} && \text{[via Lemma 4]} \\
&\iff \begin{cases} x + y = a \\ xy = b/4. \end{cases} && (5)
\end{aligned}$$

Direct denesting theorem. II. Proof continued

Define the quadratic $f(z) = z^2 - az + b/4$ and calculate its discriminant $\Delta = (-a)^2 - 4 \cdot 1 \cdot (b/4) = a^2 - b$. The corresponding zeroes are given by $z_1 = [a + \sqrt{a^2 - b}]/2$ and $z_2 = [a - \sqrt{a^2 - b}]/2$. Furthermore, they satisfy $z_1 + z_2 = a$ and $z_1 z_2 = b/4$. The main argument reads:

$$
\begin{aligned}
A \text{ denests directly} &\iff \exists x, y \in \mathbb{Q}_+^* : \sqrt{a \pm \sqrt{b}} = \sqrt{x} \pm \sqrt{y} && [\text{via Definition 1}] \\
&\iff \exists x, y \in \mathbb{Q}_+^* : \begin{cases} x + y = a \\ xy = b/4 \end{cases} && [\text{via Eq. (5) and } x, y \in \mathbb{Q}_+^*] \\
&\iff z_1 \in \mathbb{Q}_+^* \wedge z_2 \in \mathbb{Q}_+^* \\
&\iff \begin{cases} \sqrt{a^2 - b} \in \mathbb{Q}^* \\ z_1 > 0 \wedge z_2 > 0 \end{cases} && [\text{via } \sqrt{b} \notin \mathbb{Q}] \\
&\iff \begin{cases} \exists \delta \in \mathbb{Q}_+^* : a^2 - b = \delta^2 \\ z_1 z_2 > 0 \wedge z_1 + z_2 > 0 \end{cases} && [\text{via Lemma 5}] \\
&\iff \begin{cases} \exists \delta \in \mathbb{Q}_+^* : a^2 - b = \delta^2 \\ a > 0 \wedge b > 0 \end{cases} && \left[\text{via } \begin{cases} z_1 + z_2 = a \\ z_1 z_2 = b/4 \end{cases}\right] \\
&\iff \begin{cases} \exists \delta \in \mathbb{Q}_+^* : a^2 - b = \delta^2 \\ a > 0. \end{cases} && [\text{via } b > 0]
\end{aligned}
$$

The possibility $a^2 - b = 0$ is ruled out by the assumption $\sqrt{b} \notin \mathbb{Q}$ which is why we write $\sqrt{a^2 - b} \in \mathbb{Q}^*$ on the third to last statement above. Furthermore, the requirement $x > y$ is easy to satisfy with the choice $x = z_1$ and $y = z_2$.

Direct denesting theorem. III. Proof continued

For the second statement, using $\delta = \sqrt{a^2 - b}$ we note that since $z_1 = (a + \delta)/2$ and $z_2 = (a - \delta)/2$, it follows that

$$\sqrt{a \pm \sqrt{b}} = \sqrt{x} \pm \sqrt{y} \Longleftrightarrow \begin{cases} x + y = a \\ xy = b/4 \end{cases} \quad \text{[via Eq. (5)]}$$

$$\Longleftrightarrow \begin{cases} x = (a+\delta)/2 \\ y = (a-\delta)/2 \end{cases} \vee \begin{cases} x = (a-\delta)/2 \\ y = (a+\delta)/2 \end{cases} \quad \text{[via Lemma 4]}$$

$$\Longleftrightarrow \begin{cases} x = (a+\delta)/2 \\ y = (a-\delta)/2, \end{cases} \quad \text{[via } x > y\text{]}$$

and therefore, we obtain the denesting equation:

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\delta}{2}} \pm \sqrt{\frac{a-\delta}{2}}. \tag{6}$$

□

- The main result is the denesting equation
- The theorem also shows that when the formula fails to result in a successful direct denesting, that means that no such denesting is possible.

Indirect denesting theorem. I

Theorem 7

Let $A = \sqrt{a + b\sqrt{q}}$ *with* $a, b \in \mathbb{Q}^*$ *and* $q \in \mathbb{Q}_+^*$ *and* $\sqrt{q} \notin \mathbb{Q}_+^*$ *such that* $a + b\sqrt{q} > 0$*. Then the following statements hold:*

$$A \text{ denests indirectly } \iff \begin{cases} \exists \delta \in \mathbb{Q}_+^* : q(b^2 q - a^2) = \delta^2 \\ b > 0 \end{cases},$$

$$\begin{cases} \delta = \sqrt{q(b^2 q - a^2)} \\ a > 0 \wedge b > 0 \end{cases} \implies \sqrt{a + b\sqrt{q}} = \frac{1}{\sqrt[4]{q}} \left[\sqrt{\frac{bq + \delta}{2}} + \sqrt{\frac{bq - \delta}{2}} \right],$$

$$\begin{cases} \delta = \sqrt{q(b^2 q - a^2)} \\ a < 0 \wedge b > 0 \end{cases} \implies \sqrt{a + b\sqrt{q}} = \frac{1}{\sqrt[4]{q}} \left[\sqrt{\frac{bq + \delta}{2}} - \sqrt{\frac{bq - \delta}{2}} \right].$$

Proof.

To show the first statement, we begin with the observation that since $a + b\sqrt{q} > 0$, multiplying both sides with $\sqrt{q} > 0$ gives $a\sqrt{q} + bq > 0$. This enables us to write

$$A = \sqrt{a + b\sqrt{q}} = \sqrt{\frac{a\sqrt{q} + b(\sqrt{q})^2}{\sqrt{q}}} = \frac{1}{\sqrt[4]{q}} \sqrt{a\sqrt{q} + bq}$$

$$= \begin{cases} \dfrac{1}{\sqrt[4]{q}} \sqrt{bq + |a|\sqrt{q}}, & \text{if } a > 0 \\ \dfrac{1}{\sqrt[4]{q}} \sqrt{bq - |a|\sqrt{q}}, & \text{if } a < 0. \end{cases}$$

Indirect denesting theorem. II. Proof continued

Noting that $|a|\sqrt{q} = \sqrt{a^2}\sqrt{q} = \sqrt{a^2 q}$, it follows that

$$A = \begin{cases} \frac{1}{\sqrt[4]{q}} \sqrt{bq + \sqrt{a^2 q}}, & \text{if } a > 0 \\ \frac{1}{\sqrt[4]{q}} \sqrt{bq - \sqrt{a^2 q}}, & \text{if } a < 0. \end{cases} \tag{7}$$

The main argument proving the first statement reads:

$$\begin{aligned} A \text{ denests indirectly} &\iff \sqrt{bq \pm \sqrt{a^2 q}} \text{ denests directly} && \text{[via Eq. (7) and } q > 0] \\ &\iff \begin{cases} \exists \delta \in \mathbb{Q}_+^* : (bq)^2 - a^2 q = \delta^2 \\ bq > 0 \end{cases} && \text{[via Theorem 6]} \\ &\iff \begin{cases} \exists \delta \in \mathbb{Q}_+^* : q(b^2 q - a^2) = \delta^2 \\ b > 0. \end{cases} && \text{[via } q > 0] \end{aligned}$$

To show the next two statements, we combine Eq. (7) with the direct denesting equation given by Eq. (4), and distinguish between the following cases:
Case 1: For $a > 0$, we have:

$$\begin{aligned} A = \sqrt{a + b\sqrt{q}} &= \frac{1}{\sqrt[4]{q}} \sqrt{bq + \sqrt{a^2 q}} && \text{[via Eq. (7)]} \\ &= \frac{1}{\sqrt[4]{q}} \left[\sqrt{\frac{bq + \delta}{2}} + \sqrt{\frac{bq - \delta}{2}} \right]. && \text{[via Eq. (4)]} \end{aligned}$$

Indirect denesting theorem. III. Proof continued

Case 2: For $a < 0$, we have:

$$A = \sqrt{a + b\sqrt{q}} = \frac{1}{\sqrt[4]{q}}\sqrt{bq - \sqrt{a^2 q}} \qquad \text{[via Eq. (7)]}$$

$$= \frac{1}{\sqrt[4]{q}}\left[\sqrt{\frac{bq+\delta}{2}} - \sqrt{\frac{bq-\delta}{2}}\right], \qquad \text{[via Eq. (4)]}$$

and this conclude the proof.
□

- The theorem for indirect denesting corresponds to the following denesting equation:

$$\sqrt{a + b\sqrt{q}} = \begin{cases} \dfrac{1}{\sqrt[4]{q}}\left[\sqrt{\dfrac{bq+\delta}{2}} + \sqrt{\dfrac{bq-\delta}{2}}\right], & \text{if } a > 0 \\ \dfrac{1}{\sqrt[4]{q}}\left[\sqrt{\dfrac{bq+\delta}{2}} - \sqrt{\dfrac{bq-\delta}{2}}\right], & \text{if } a < 0, \end{cases}$$

  where $\delta = \sqrt{q(b^2 q - a^2)}$.
- As long as δ is a rational number, we have a successful indirect denesting.
- From the theorems we learn, in general, that expressions of the form $\sqrt{a \pm \sqrt{b}}$ do not have direct denesting when $a < 0$, however it is possible that they may have an indirect denesting.

Example of direct denesting

Example 8

Denest the expression $\sqrt{37+20\sqrt{3}}$.

Solution.

Using $a = 37$ and $b = (20\sqrt{3})^2$, we have

$$\begin{aligned}\delta^2 &= a^2 - b = 37^2 - (20\sqrt{3})^2 = 1369 - 400 \cdot 3 = 1369 - 1200 = 169 \\ &= 13^2 \Longrightarrow \delta = 13,\end{aligned}$$

and therefore

$$\begin{aligned}\sqrt{37+20\sqrt{3}} &= \sqrt{(a+\delta)/2} + \sqrt{(a-\delta)/2} = \sqrt{(37+13)/2} + \sqrt{(37-13)/2} \\ &= \sqrt{50/2} + \sqrt{24/2} = \sqrt{25} + \sqrt{12} = 5 + 4\sqrt{3}.\end{aligned}$$

□

Example of indirect denesting

Example 9

Denest the expression $\sqrt{3\sqrt{2}-4}$.

Solution.

Since $-4 < 0$, this expression does not have a direct denesting. Factoring out $\sqrt{2}$ gives

$$\begin{aligned}\sqrt{3\sqrt{2}-4} &= \sqrt{\sqrt{2}\left(3-\frac{4}{\sqrt{2}}\right)} = \sqrt[4]{2}\sqrt{3-\frac{4\sqrt{2}}{2}} = \sqrt[4]{2}\sqrt{3-2\sqrt{2}} \\ &= \sqrt[4]{2}\sqrt{1-2\sqrt{2}+(\sqrt{2})^2} = \sqrt[4]{2}\sqrt{(1-\sqrt{2})^2} = \sqrt[4]{2}|1-\sqrt{2}| \\ &= \sqrt[4]{2}(\sqrt{2}-1).\end{aligned}$$

□

- Indirect denesting is equivalent to factoring out the corresponding radical and then look for a direct denesting.
- In the above example, the denesting was obvious enough with term splitting.
- **Caution:** Use the identity $\sqrt{x^2} = |x|$ in order to remove the root

Another example of indirect denesting

Example 10

Denest the expression $\sqrt{-84+67\sqrt{7}}$.

Solution.

Since $-84 < 0$, there is no direct denesting, so we look for an indirect denesting. Using $a = -84$ and $b = 67$ and $q = 7$, we have

$$\begin{aligned}\delta^2 &= q(b^2 q - a^2) = 7(67^2 \cdot 7 - (-84)^2) = 7(4489 \cdot 7 - 7056) = 7(31423 - 7056) \\ &= 7 \cdot 24367 = 170569 = 413^2 \Longrightarrow \delta = 413,\end{aligned}$$

and from the indirect denesting identity, we have

$$\begin{aligned}\sqrt{-84+67\sqrt{7}} &= \frac{1}{\sqrt[4]{q}}\left[\sqrt{\frac{bq+\delta}{2}} - \sqrt{\frac{bq-\delta}{2}}\right] \\ &= \frac{1}{\sqrt[4]{7}}\left[\sqrt{\frac{67 \cdot 7 + 413}{2}} - \sqrt{\frac{67 \cdot 7 - 413}{2}}\right] \\ &= \frac{1}{\sqrt[4]{7}}\left[\sqrt{\frac{469+413}{2}} - \sqrt{\frac{469-413}{2}}\right] = \frac{1}{\sqrt[4]{7}}\left[\sqrt{\frac{882}{2}} - \sqrt{\frac{56}{2}}\right] \\ &= \frac{\sqrt{441}-\sqrt{28}}{\sqrt[4]{7}} = \frac{21-2\sqrt{7}}{\sqrt[4]{7}} = \frac{\sqrt{7}}{\sqrt[4]{7}}\left[\frac{21}{\sqrt{7}} - 2\right] = \sqrt[4]{7}\left[\frac{21\sqrt{7}}{7} - 2\right] \\ &= \sqrt[4]{7}(3\sqrt{7}-2).\end{aligned}$$

A related denesting problem

- Radicals of the form $\sqrt{a\sqrt{p}+b\sqrt{q}}$ with $a, b \in \mathbb{Q}$ and $p, q \in \mathbb{Q}_+^*$ can be also denested using direct denesting (Theorem 6) or indirect denesting (Theorem 7) after factoring out $\sqrt{p}$ or $\sqrt{q}$.
- It is is easy to show that $\sqrt{p}$ factorization is equivalent to $\sqrt{q}$ factorization, meaning that the radical can be denested by $\sqrt{p}$ factorization if and only if it can be denested with $\sqrt{q}$ factorization.

Example 11

Denest the expression $A = \sqrt{5\sqrt{2}+4\sqrt{3}}$.

Solution.

We note that

$$A = \sqrt{5\sqrt{2}+4\sqrt{3}} = \sqrt{\sqrt{2}\left(5+\frac{4\sqrt{3}}{\sqrt{2}}\right)} = \sqrt[4]{2}\sqrt{5+\frac{4\sqrt{2}\sqrt{3}}{2}} = \sqrt[4]{2}\sqrt{5+2\sqrt{6}}.$$

We attempt a direct denesting using $a = 5$ and
$\delta^2 = 5^2 - (2\sqrt{6})^2 = 25 - 4 \cdot 6 = 25 - 24 = 1 \Longrightarrow \delta = 1$, therefore

$$\sqrt{5+2\sqrt{6}} = \sqrt{\frac{a+\delta}{2}} + \sqrt{\frac{a-\delta}{2}} = \sqrt{\frac{5+1}{2}+\frac{5-1}{2}} = \sqrt{3}+\sqrt{2}$$

$$\Longrightarrow A = \sqrt{5\sqrt{2}+4\sqrt{3}} = \sqrt[4]{2}(\sqrt{3}+\sqrt{2}).$$

Conclusion

- In its most general form, the problem of denesting roots remains an open question for current research.
- Simple denesting techniques can be introduced in College Algebra or Precalculus courses, in the context of solving quadratic or biquadratic equations, or evaluating trigonometric numbers for unusual angles.
- They can also be introduced in Calculus coursework in the context of evaluating definite integrals of rational functions.
- The proofs of the lemmas and theorems are simple and make for excellent examples for introducing basic concepts of proof techniques, such as proof by contradiction, proof by cases, quantifiers. and so on.
- E. Gkioulekas: "On the denesting of nested square roots", *International Journal of Mathematical Education in Science and Technology*, (2017), published online first

Thank you!