Random field theory in classical dynamical systems

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Problem Statement. I

- **The problem:** Let $u_\alpha(\mathbf{r}, t)$ be a space-time vector field governed by the equation
$$\frac{\partial u_\alpha}{\partial t} = \Lambda_\alpha[u] + f_\alpha$$
If f_α is a random field, then what are the statistical properties of u_α?
- Note that the functional Λ_α is local in time.
- **Motivation:** Most statistical theories of turbulence, model turbulence via a randomly forced Navier-Stokes equation.

Problem Statement. II

- We are interested in the quadratic case:

$$\frac{\partial u_\alpha}{\partial t} = V_{\alpha\beta\gamma} u_\beta u_\gamma + L_{\alpha\beta} u_\beta + f_\alpha$$

- Repeated indices represent summation over components and space-time integration.
- For example, using $\mathbf{x}_k = (\mathbf{r}_k, t_k)$, we write

$$L_{\alpha\beta} u_\beta = \sum_\beta \int \mathrm{d}\mathbf{x}_2 \; L_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2) u_\beta(\mathbf{x}_2)$$

$$V_{\alpha\beta\gamma} u_\beta u_\gamma = \sum_{\beta\gamma} \int \mathrm{d}\mathbf{x}_1 \int \mathrm{d}\mathbf{x}_2 \; V_{\alpha\beta\gamma}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) u_\beta(\mathbf{x}_2) u_\gamma(\mathbf{x}_3)$$

Functional Calculus.

Variational derivative

- A functional $\mathcal{F} : u(\mathbf{x}) \to \ell$ maps a scalar or vector field $u(\mathbf{x})$ to a number ℓ. We write $\mathcal{F}[u] = \ell$.
- Let $\Delta_\alpha(x)$ be a gaussian peak with variance a. We define the *variational derivative* of $\mathcal{F}$ with respect to u via

$$\begin{aligned}\frac{\delta \mathcal{F}[u]}{\delta f(\xi)} &= \lim_{h \to 0} \lim_{a \to 0^+} \frac{\mathcal{F}[u(x) + h\Delta_\alpha(x - \xi)] - \mathcal{F}[u]}{h} \\ &= \lim_{h \to 0} \frac{\mathcal{F}[u(x) + h\delta_\alpha(x - \xi)] - \mathcal{F}[u]}{h}\end{aligned}$$

with $\delta(x)$ the Dirac delta function.

Functional Calculus.

Feynman path integral

- Assume that the functional $\mathcal{F}[u]$ can be discretized via a sequence of functions f_N:

$$f_N(u_1, u_2, \ldots, u_N) \to \mathcal{F}[u], \text{ as } N \to +\infty$$

- The Feynman path integral is defined as

$$\int \mathscr{D}u\mathcal{F}[u] = \lim_{N\to+\infty} \frac{1}{c} \int \frac{\mathrm{d}u_1}{c} \cdots \int \frac{\mathrm{d}u_N}{c} f_N(u_1, \ldots, u_N)$$

- Consider a linear *hyperfunctional* $\mathscr{A}\{\mathcal{F}\}$ that maps the functional $\mathcal{F}$ to a number such that

$$\forall\lambda_1, \lambda_2 \in \mathbb{R} : \mathscr{A}\{\lambda_1\mathcal{F}_1 + \lambda_2\mathcal{F}_2\} = \lambda_1\mathscr{A}\{\mathcal{F}_1\} + \lambda_2\mathscr{A}\{\mathcal{F}_2\}$$

- We can associate with $\mathscr{A}\{\mathcal{F}\}$ a *generalized functional* $\mathcal{A}[u]$ and write

$$\mathscr{A}\{\mathcal{F}\} = \int \mathscr{D}u\mathcal{A}[u]\mathcal{F}[u]$$

Random fields

Characteristic functional

- Let u be a random field, and let $F[u]$ be an analytical functional.
- $\langle F[u] \rangle$ denotes the ensemble average (or expected value) of $F[u]$, and can be represented via a probability hyperfunctional $\mathscr{P}[u]$ such that

$$\langle F[u] \rangle = \int \mathscr{D}u \; \mathscr{P}[u] F[u]$$

- We define the corresponding characteristic functional $C[p]$ of u as

$$C[p] = \langle \exp(i p_\alpha u_\alpha) \rangle = \int \mathscr{D}u \; \mathscr{P}[u] \exp(i p_\alpha u_\alpha)$$

- $C[p]$ contains all statistical information about the field u.
- For example, correlation functions of the field u_α can be obtained as variational derivatives of the characteristic functional

$$\langle u_\alpha \rangle = \int \mathscr{D}u \; \mathscr{P}[u] u_\alpha = \frac{1}{i} \left. \frac{\delta C[p]}{\delta p_\alpha} \right|_0$$

$$\langle u_\alpha u_\beta \rangle = \int \mathscr{D}u \; \mathscr{P}[u] u_\alpha u_\beta = \frac{1}{2! i^2} \left. \frac{\delta^2 C[p]}{\delta p_\alpha \delta p_\beta} \right|_0$$

Gaussian fields

- In general we have

$$\langle u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n} \rangle = \frac{1}{n! i^n} \left. \frac{\delta^n C[p]}{\delta p_{\alpha_1} \delta p_{\alpha_2} \cdots \delta p_{\alpha_n}} \right|_0$$

- A field u_α is *Gaussian* if and only if for any field c_α the random variable $x = c_\alpha (u_\alpha - \langle u_\alpha \rangle)$ is Gaussian.
- The characteristic functional $C[p]$ of a random Gaussian field with $\langle u_\alpha \rangle = 0$ is given by

$$C[p] = \exp(-(1/2) p_\alpha p_\beta F_{\alpha\beta})$$

with $F_{\alpha\beta} = \langle u_\alpha u_\beta \rangle$.
- It follows that

$$\langle u_\alpha u_\beta u_\gamma \rangle = 0$$
$$\langle u_\alpha u_\beta u_\gamma u_\delta \rangle = F_{\alpha\beta} F_{\gamma\delta} + F_{\alpha\gamma} F_{\beta\delta} + F_{\alpha\delta} F_{\beta\gamma}$$

Gaussian vs. non-Gaussian fields

- In general, for a Gaussian random field:
  - All odd-order correlators are zero

$$\left\langle u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_{2n+1}} \right\rangle = 0$$

  - Let Π_n be the set of all partitions $\ell \in \Pi_n$ of the set $\{1, 2, \ldots, 2n\}$ denoted as $\ell = \{\{\ell_1, \ell_2\}, \{\ell_3, \ell_4\}, \ldots, \{\ell_{2n-1}, \ell_{2n}\}\}$. Then, the general correlator is given by

$$\langle u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_{2n}} \rangle = \sum_{\ell \in \Pi_n} \prod_{m=1}^{n} \left\langle u_{\alpha_{\ell_{2m-1}}} u_{\alpha_{\ell_{2m}}} \right\rangle$$

- In a non-Gaussian field u_α, the correlator decomposes to a Gaussian and non-Gaussian (also called *connected*) contribution:

$$\langle u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n} \rangle = \langle u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n} \rangle_G + \langle u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n} \rangle_c$$

with the connected contribution given by

$$\langle u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n} \rangle_c = \frac{1}{n! i^n} \left. \frac{\delta^n \ln C[p]}{\delta p_{\alpha_1} \delta p_{\alpha_2} \cdots \delta p_{\alpha_n}} \right|_0$$

Relevance to hydrodynamic turbulence

The closure problem

- Turbulence is governed by the Navier-Stokes equations:

$$\frac{\partial u_\alpha}{\partial t} + \mathcal{P}_{\alpha\beta}\partial_\beta(u_\beta u_\gamma) = \nu\nabla^2 u_\alpha + \mathcal{P}_{\alpha\beta} f_\beta \tag{1}$$

  with $\mathcal{P}_{\alpha\beta} = \delta_{\alpha\beta} - \partial_\alpha\partial_\beta\nabla^{-2}$.
- As such, it corresponds to a general quadratic problem of the form:

$$\frac{\partial u_\alpha}{\partial t} = V_{\alpha\beta\gamma} u_\beta u_\gamma + L_{\alpha\beta} u_\beta + f_\alpha$$

- Kolmogorov energy cascade:
  - fluid stirred randomly at large length scales
  - energy is transfered to small scales where it is dissipated
  - at intermediate scales, the fluid forgets the details of forcing.
- The velocity field of turbulence is not random Gaussian. In a Gaussian velocity field, an energy cascade is not possible.
- This leads to the closure problem.

Relevance to hydrodynamic turbulence

Correlation and Response function

- Kraichnan proposed closure models based on a correlator $F_{\alpha\beta}$ and a Green's function $G_{\alpha\beta}$ such that

$$F_{\alpha\beta} = \langle u_\alpha u_\beta \rangle \qquad G_{\alpha\beta} = \left\langle \left. \frac{\delta u_\alpha}{\delta f_\beta} \right|_0 \right\rangle$$

- MSR theory: Martin-Siggia-Rose proposed a broad theoretical framework for building such theories
- MSR theory admits three equivalent formulations:
  - The path integral formulation proposed by Phythian
  - The variational differential equation formulation
  - The Dyson-Wyld formulation

MSR Theory

Characteristic functional

- Consider the general system $\partial u_\alpha/\partial t = \Lambda_\alpha[u] + f_\alpha$ with f_α random forcing with $\langle f_\alpha \rangle = 0$.
- Let $C[p]$ be the characteristic functional of f_α, and let $F[p] = \ln C[p]$
- MSR theory defines a characteritic functional $Z[\ell, m]$ for u_α given by

$$Z[\ell, m] = \int \mathscr{D}u \int \mathscr{D}p \ \exp(-iS[u,p])$$

  with $S[u,p]$ the *action* given by

$$S[u,p] = p_\alpha(\partial u_\alpha/\partial t - \Lambda_\alpha[u]) - F[p] + i\ell_\alpha u_\alpha - m_\alpha p_\alpha$$

- The correlator $F_{\alpha\beta}$ and response function $G_{\alpha\beta}$ are given by

$$F_{\alpha\beta} = \langle u_\alpha u_\beta \rangle = \left.\frac{\delta^2 Z[\ell, m]}{\delta \ell_\alpha \delta \ell_\beta}\right|_0$$

$$G_{\alpha\beta} = \left\langle \left.\frac{\delta u_\alpha}{\delta f_\beta}\right|_0 \right\rangle = \left.\frac{\delta^2 Z[\ell, m]}{\delta \ell_\alpha \delta m_\beta}\right|_0 = \langle u_\alpha (ip_\beta) \rangle$$

MSR Theory

Schwinger Equations

- In general, a mixed correlation-response function satisfies:

$$G^{(m,n)}_{\alpha_1\cdots\alpha_n\beta_1\cdots\beta_m} = \left\langle \prod_{k=1}^{n}\left(\frac{\delta}{\delta f_{\alpha_k}}\right)\prod_{l=1}^{m} u_{\beta_l}\right\rangle$$
$$= \prod_{k=1}^{n}\left(\frac{\delta}{\delta m_{\alpha_k}}\right)\prod_{l=1}^{m}\left(\frac{\delta}{\delta \ell_{\beta_l}}\right) Z[\ell, m]\Big|_0$$

- An equivalent formulation of MSR theory gives the characteristic functional in terms of the *Schwinger equations*:

$$\frac{\partial}{\partial t_\alpha}\frac{\delta Z}{\delta \ell_\alpha} = m_\alpha Z + \Lambda_\alpha\left(\frac{\delta}{\delta \ell_\alpha}\right) Z + G_\alpha\left[\frac{1}{i}\frac{\delta}{\delta m}\right] Z$$
$$\frac{\partial}{\partial t_\alpha}\frac{\delta Z}{\delta m_\alpha} = -\ell_\alpha Z - H_{\alpha\beta}\left(\frac{\delta}{\delta \ell}\right)\frac{\delta Z}{\delta m_\beta}$$

with $G_\alpha[p]$ and $H_{\alpha\beta}[u]$ given by

$$G_\alpha[p] = \frac{\delta F[p]}{\delta p_\alpha} \qquad H_{\alpha\beta}[u] = \frac{\delta \Lambda_\beta[u]}{\delta u_\alpha}$$

MSR Theory

The case of projected forcing

- To apply MSR to the Navier-Stokes equation, we consider the modified problem
$$\frac{\partial u_\alpha}{\partial t} = \Lambda_\alpha[u] + P_{\alpha\beta} f_\beta$$
where we can assume $P_{\alpha\beta} = P_{\beta\alpha}$.
- If $F[p]$ is the connected characteristic functional for f_α, then the corresponding characteristic functional for the modified force $g_\alpha = P_{\alpha\beta} f_\beta$ is denoted as $F_g[p]$.
- For Gaussian forcing with $\langle f_\alpha \rangle = 0$ and $\langle f_\alpha f_\beta \rangle = Q^0_{\alpha\beta}$, we can show that
$$iF_g[p] = -(1/2) Q_{\alpha\beta} p_\alpha p_\beta, \text{ with } Q_{\alpha\beta} = P_{\alpha\gamma} Q^0_{\gamma\delta} P_{\delta\beta}$$
- The action $S[u, p]$ must also be modified to take the form
$$S[u, p] = p_\alpha(\partial u_\alpha / \partial t - \Lambda_\alpha[u]) - F_g[p] + i\ell_\alpha u_\alpha - m_\alpha P_{\alpha\beta} p_\beta$$
- In response functions, written as ensemble averages, the ghost field p_α must be replaced with $q_\alpha = P_{\alpha\beta} p_\beta$.

MSR Theory

The quadratic problem. I.

- Now let us consider a problem of the form

$$\frac{\partial u_\alpha}{\partial t} = P_{\alpha\beta} V_{\beta\gamma\delta} u_\gamma u_\delta + L_{\alpha\beta} u_\beta + P_{\alpha\beta} f_\beta$$

  with $V_{\alpha\beta\gamma} = V_{\alpha\gamma\beta}$ and $P_{\alpha\beta} = P_{\beta\alpha}$ and u_α invariant under $P_{\alpha\beta}$ such that $P_{\alpha\beta} u_\beta = u_\alpha$.
- Forcing is assumed to be random Gaussian and satisfy $\langle f_\alpha \rangle = 0$ and $\langle f_\alpha f_\beta \rangle = Q^0_{\alpha\beta}$
- The characteristic functional for the linear problem (i.e. disregarding the $P_{\alpha\beta} V_{\beta\gamma\delta} u_\gamma u_\delta$ term) can be evaluated in closed form and it is given by

$$Z_0[\ell, m] = \exp((1/2)\ell_\alpha F^0_{\alpha\beta} \ell_\beta + \ell_\alpha G^0_{\alpha\beta} m_\beta)$$

  where $F^0_{\alpha\beta}$ and $G^0_{\alpha\beta}$ are the *bare correlator* and *bare response function* given by

$$\Gamma_{\alpha\gamma} G^0_{\gamma\beta} = P_{\alpha\beta} \text{ and } F^0_{\alpha\beta} = G^0_{\alpha\gamma} Q^0_{\gamma\delta} G^0_{\beta\delta}$$

- Here, $\Gamma_{\alpha\beta}$ is a generalized function kernel representing the operation $\Gamma_{\alpha\beta} u_\beta = \partial u_\alpha / \partial t - L_{\alpha\beta} u_\beta$

MSR Theory

The quadratic problem. II

- The characteristic functional $Z[\ell, m]$ for the quadratic problem is then obtained from $Z_0[\ell, m]$ by

$$Z[\ell, m] = \exp\left(\frac{\delta}{\delta m_\alpha} V_{\alpha\beta\gamma} \frac{\delta}{\delta \ell_\beta} \frac{\delta}{\delta \ell_\gamma}\right) Z_0[\ell, m]$$

- Expanding the exponential operator results in an infinite series of contributions.
- We use Feynman diagrams to keep track of the resulting terms and to introduce various simplifications

The Dyson-Wyld equations

- Various renormalizations:
  - Unlinked and weakly linked diagrams add up to zero and can be eliminated.
  - Dyson renormalization gives the equation

    $$G_{\alpha\beta} = G^0_{\alpha\beta} + G^0_{\alpha\gamma}\Sigma_{\gamma\delta}G_{\delta\beta}$$

    where $\Sigma_{\gamma\delta}$ is a sum of strongly-linked Feynman diagrams in terms of $F^0_{\alpha\beta}$ and $G^0_{\alpha\beta}$.
  - Wyld renormalization gives the equation

    $$F_{\alpha\beta} = G_{\alpha\gamma}(Q_{\gamma\delta} + \Phi_{\gamma\delta})G_{\beta\delta}$$

    where $\Phi_{\gamma\delta}$ is likewise a sum of strongly-linked Feynman diagrams in terms of $F^0_{\alpha\beta}$ and $G^0_{\alpha\beta}$.
  - Line renormalization: The expansions for $\Sigma_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ are further resummed in terms of *irreducible* Feynman diagrams in terms of $F_{\alpha\beta}$ and $G_{\alpha\beta}$.
- The resulting equations can be used to formulate closure models by truncating the expansions for $\Phi_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$

The 1-loop approximation

- For example the 1-loop approximation gives the following equations:

$$
\begin{aligned}
G_{\alpha\beta} &= G^0_{\alpha\beta} + G^0_{\alpha\gamma}\Sigma_{\gamma\delta}G_{\delta\beta} \\
F_{\alpha\beta} &= G_{\alpha\gamma}(Q_{\gamma\delta} + \Phi_{\gamma\delta})G_{\beta\delta} \\
\Sigma_{\alpha\beta} &\approx \Sigma^1_{\alpha\beta} = (V_{\alpha A\Gamma} + V_{\alpha\Gamma A})(V_{\beta B\Delta} + V_{\beta\Delta B})G_{AB}F_{\Gamma\Delta} \\
\Phi_{\alpha\beta} &\approx \Phi^1_{\alpha\beta} = V_{\alpha A\Gamma}(V_{\beta B\Delta} + V_{\beta\Delta B})F_{AB}F_{\Gamma\Delta}
\end{aligned}
$$

- In general, the operators $\Sigma_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ can be represented with a Feynman diagram expansion

$$\Sigma_{\alpha\beta} = \Sigma^1_{\alpha\beta} + \Sigma^2_{\alpha\beta} + \cdots \tag{2}$$

$$\Phi_{\alpha\beta} = \Phi^1_{\alpha\beta} + \Phi^2_{\alpha\beta} + \cdots \tag{3}$$

Conclusion

- MSR theory has been applied to the Navier-Stokes equations successfully using the *quasi-Lagrangian representation* of the velocity field
  - Establishes the perturbative locality of the downscale energy cascade
  - Explains the intermittency corrections to Kolmogorov theory
  - Can be used to derive the *fusion rules* governing generalized structure functions.
  - The fusion rules can in turn be used to explore, the non-perturbative locality, stability, dissipation scales, existence of anomalous sinks, etc.
- An open question: application of MSR theory to 2D Navier-Stokes turbulence and QG turbulence.
- Another open question: investigation of non-Gaussian forcing.

Thank you!