Energy and potential enstrophy flux constraints in quasi-geostrophic models

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Publications

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- E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Sys. B* **5**, 79-102
- E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Sys. B* **5**, 103-124.
- E. Gkioulekas and K.K. Tung (2007): *J. Fluid Mech.*, **576**, 173-189.
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- E. Gkioulekas (2014): *Physica D* **284**, 27-41

Outline

- Flux inequality for 2D turbulence
- Flux inequality for multi-layer symmetric QG model
  - Formulation
  - Dissipation rate spectra
  - symmetric streamfunction dissipation
- Flux inequality for two-layer QG model
  - asymmetric streamfunction dissipation
  - differential diffusion
  - extrapolated Ekman term

2D Navier-Stokes equations

- In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = d + f,$$

  where $\psi(x, y, t)$ is the streamfunction, and
  $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$, and

$$d = -[\nu(-\Delta)^{\kappa} + \nu_1(-\Delta)^{-m}]\zeta$$

- The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ, and is defined as

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}.$$

Energy and enstrophy spectrum. I

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2}\int \psi(x,y,t)\zeta(x,y,t)\,\mathrm{d}x\mathrm{d}y \quad G(t) = \frac{1}{2}\int \zeta^2(x,y,t)\,\mathrm{d}x\mathrm{d}y.$$

- Let $a^{<k}(\mathbf{x})$ be the field obtained from $a(\mathbf{x})$ by setting to zero, in Fourier space, the components corresponding to wavenumbers with norm greater than k:

$$\begin{aligned} a^{<k}(\mathbf{x}) &= \int \mathrm{d}\mathbf{y} P(k|\mathbf{x}-\mathbf{y})a(\mathbf{y}) \\ &= \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}_0 \int_{\mathbb{R}^2} \mathrm{d}\mathbf{k}_0 \, \frac{H(k-\|\mathbf{k}_0\|)}{4\pi^2} \exp(i\mathbf{k}_0\cdot(\mathbf{x}-\mathbf{x}_0))a(\mathbf{x}_0) \end{aligned}$$

- Filtered inner product:

$$\langle a,b\rangle_k = \frac{\mathrm{d}}{\mathrm{d}k}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\; a^{<k}(\mathbf{x})b^{<k}(\mathbf{x})$$

Energy and enstrophy spectrum. II

- Energy spectrum: $E(k) = -\langle \psi, \zeta \rangle_k$
- Enstrophy spectrum $G(k) = \langle \zeta, \zeta \rangle_k$
- Consider the conservation laws for $E(k)$ and $G(k)$:

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = -D_E(k) + F_E(k)$$
$$\frac{\partial G(k)}{\partial t} + \frac{\partial \Pi_G(k)}{\partial k} = -D_G(k) + F_G(k)$$

- In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2 \Pi_E(k) - \Pi_G(k) \leq 0$$

for all k not in the forcing range.

Energy and enstrophy spectrum. III

- Assuming a forced-dissipative configuration at steady state,

$$\Pi_E(k) = \int_k^{+\infty} D_E(q)\mathrm{d}q,$$
$$\Pi_G(k) = \int_k^{+\infty} D_G(q)\mathrm{d}q,$$

  and it follows that:

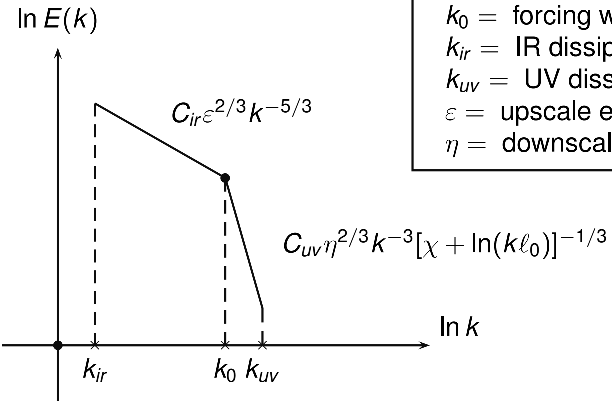
$$k^2\Pi_E(k) - \Pi_G(k) = \int_k^{+\infty} [k^2 D_E(q) - D_G(q)]\mathrm{d}q = \int_k^{+\infty} \Delta(k,q)\mathrm{d}q.$$

- For the case of two-dimensional Navier-Stokes turbulence, $D_G(k) = k^2 D_E(k)$, therefore

$$\Delta(k,q) = k^2 D_E(q) - D_G(q) = (k^2 - q^2) D_E(q) \leq 0$$

  so we get $k^2\Pi_E(k) - \Pi_G(k) < 0$.

KLB theory.



$k_0 =$ forcing wavenumber
$k_{ir} =$ IR dissipation wavenumber
$k_{uv} =$ UV dissipation wavenumber
$\varepsilon =$ upscale energy flux
$\eta =$ downscale enstrophy flux

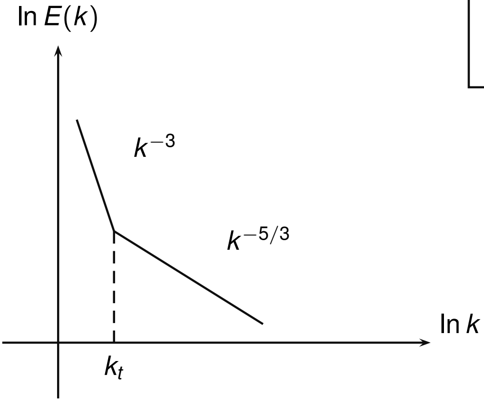
Cascade Directions

- Proofs that energy goes mostly upscale in 2D turbulence:
  - Fjørtøft (1953): Famous wrong argument.
  - Merilee and Warn (1975): Noticed error in Fjørtøft
  - Eyink (1996): Correct argument, but assumes inertial ranges.
  - Gkioulekas and Tung (2007): Flux inequality for 2D Navier-Stokes
- Linear Cascade Superposition hypothesis:
  - E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Sys. B* **5**, 79-102
  - E. Gkioulekas and K.K. Tung (2005), *Discr. Cont. Dyn. Sys. B* **5**, 103-124.
- Under coexisting downscale cascades of energy and enstrophy:

$$E(k) \approx C_1 \varepsilon_{uv}^{2/3} k^{-5/3} + C_2 \eta_{uv}^{2/3} k^{-3}$$

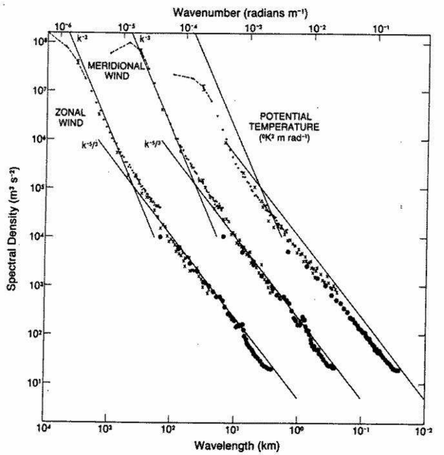
  with η_{uv} the downscale enstrophy flux and ε_{uv} the downscale energy flux.
- Transition wavenumber: $k_t = \sqrt{\eta/\varepsilon}$.

Nastrom-Gage spectrum schematic

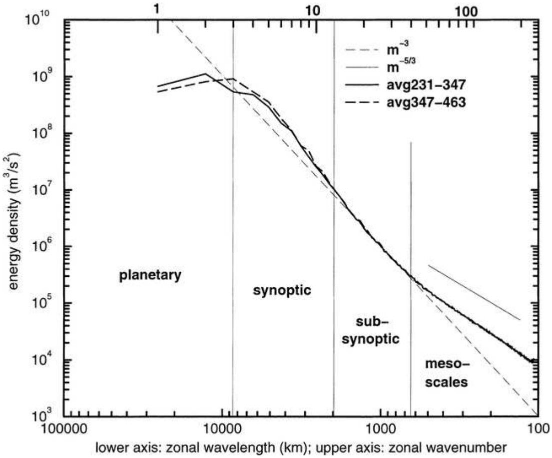


$k^{-3} \rightarrow$ 3000km – 800km
$k^{-5/3} \rightarrow$ 600km– $\ll$ 1km
$k_t \approx$ 700km $\approx k_R$

Nastrom-Gage spectrum



Tung and Orlando spectrum



Generalized multi-layer model. I.

- Consider the generalized form of an n-layer model:

$$\frac{\partial q_\alpha}{\partial t} + J(\psi_\alpha, q_\alpha) = d_\alpha + f_\alpha$$

$$d_\alpha = \sum_\beta \mathcal{D}_{\alpha\beta}\psi_\beta$$

$$\hat{q}_\alpha(\mathbf{k}, t) = \sum_\beta L_{\alpha\beta}(\|\mathbf{k}\|)\hat{\psi}_\beta(\mathbf{k}, t)$$

- We consider two types of forms for the dissipation term d_α:
  - Let $D_\alpha(k)$ be the spectrum of the operator $\mathscr{D}_\alpha$
  - **Streamfunction dissipation:**
    $d_\alpha = +\mathscr{D}_\alpha\psi_\alpha \implies D_{\alpha\beta}(k) = \delta_{\alpha\beta}D_\beta(k)$
  - **Symmetric streamfunction dissipation:** (all layers have the same operator)
    $d_\alpha = +\mathscr{D}\psi_\alpha \implies D_{\alpha\beta}(k) = \delta_{\alpha\beta}D(k)$

Generalized multi-layer model. II

- The energy spectrum $E(k)$ and the potential enstrophy spectrum $G(k)$ are given by:

$$E(k) = -\sum_{\alpha} \langle \psi_\alpha, q_\alpha \rangle_k = -\sum_{\alpha\beta} L_{\alpha\beta}(k) C_{\alpha\beta}(k)$$

$$G(k) = \sum_{\alpha} \langle q_\alpha, q_\alpha \rangle_k = \sum_{\alpha\beta\gamma} L_{\alpha\beta}(k) L_{\alpha\gamma}(k) C_{\beta\gamma}(k)$$

  with $C_{\alpha\beta}(k) = \langle \psi_\alpha, \psi_\beta \rangle_k$.
- The energy dissipation rate spectrum $D_E(k)$ and the layer-by-layer potential enstrophy dissipation rate spectra $D_{G_\alpha}(k)$ are given by

$$D_E(k) = 2 \sum_{\alpha\beta} D_{\alpha\beta}(k) C_{\alpha\beta}(k),$$

$$D_{G_\alpha}(k) = -2 \sum_{\beta\gamma} L_{\alpha\beta}(k) D_{\alpha\gamma}(k) C_{\beta\gamma}(k).$$

Multi-layer QG model. I.

- In a multi-layer quasigeostrophic model, the relation between q_α and ψ_α reads

$$
\begin{aligned}
q_1 &= \nabla^2\psi_1 + \mu_1 k_R^2(\psi_2 - \psi_1) \\
q_\alpha &= \nabla^2\psi_\alpha - \lambda_\alpha k_R^2(\psi_\alpha - \psi_{\alpha-1}) + \mu_\alpha k_R^2(\psi_{\alpha+1} - \psi_\alpha), \text{ for } 1 < \alpha < n \\
q_n &= \nabla^2\psi_n - \lambda_n k_R^2(\psi_n - \psi_{n-1})
\end{aligned}
$$

- Here $\lambda_\alpha, \mu_\alpha$ are the non-dimensional Froude numbers given by

$$
\begin{aligned}
\lambda_\alpha &= \frac{h_1}{h_\alpha}\frac{\rho_2 - \rho_1}{\rho_\alpha - \rho_{\alpha-1}}, \text{ for } 1 < \alpha \le n \\
\mu_\alpha &= \frac{h_1}{h_\alpha}\frac{\rho_2 - \rho_1}{\rho_{\alpha+1} - \rho_\alpha}, \text{ for } 1 \le \alpha < n
\end{aligned}
$$

  with $h_1, h_2, \ldots, h_n$, the thickness of layers from top to bottom, in pressure coordinates.
- For $h_1 = h_2 = \ldots = h_n$, we note that $\lambda_{\alpha+1} = \mu_\alpha$ for all $1 \le \alpha < n$.

Multi-layer QG model. II.

- The corresponding matrix $L_{\alpha\beta}(k)$ is given by:

$$L_{\alpha\alpha}(k) = \begin{cases} -k^2 - \mu_1 k_R^2, & \text{if } \alpha = 1 \\ -k^2 - (\lambda_\alpha + \mu_\alpha) k_R^2, & \text{if } 1 < \alpha < n \\ -k^2 - \lambda_n k_R^2, & \text{if } \alpha = n \end{cases}$$

$$L_{\alpha,\alpha+1}(k) = \mu_\alpha k_R^2, \text{ for } 1 \le \alpha < n$$

$$L_{\alpha-1,\alpha}(k) = \lambda_\alpha k_R^2, \text{ for } 1 < \alpha \le n$$

- We define:

$$\gamma_\alpha(k,q) = k^2 + \sum_\beta L_{\alpha\beta}(q) = k^2 - q^2 < 0, \text{ for } k < q$$

- We consider the case where $h_\alpha = h$ for all layers. Then, $L_{\alpha\beta}(k)$ is symmetric, and our theoretical framework becomes applicable.

Multi-layer QG model. III.

- Let $U_\alpha(k) = \langle \psi_\alpha, \psi_\alpha \rangle_k \geq 0$ for $1 \leq \alpha \leq n$. (streamfunction spectrum)
- Recall that:

$$k^2 \Pi_E(k) - \Pi_G(k) = \int_k^{+\infty} [k^2 D_E(q) - D_G(q)] \mathrm{d}q = \int_k^{+\infty} \Delta(k,q) \mathrm{d}q.$$

- PROPOSITION 1: In a generalized n-layer model, under symmetric streamfunction dissipation $d_\alpha = +\mathscr{D}\psi_\alpha$ with spectrum $D(k)$, we assume that $L_{\alpha\beta}(q) \geq 0$ when $\alpha \neq \beta$, and $L_{\alpha\beta}(q) = L_{\beta\alpha}(q)$, and $\gamma_\alpha(k,q) \leq 0$ when $k < q$ for all α. It follows that:

$$\Delta(k,q) \leq D(q) \sum_\alpha \gamma_\alpha(k,q) U_\alpha(q) \leq 0$$

- It follows that under symmetric streamfunction, the flux inequality is satisfied.
- The case $d_\alpha = \mathfrak{D} q_\alpha$ presents unexpected challenges, and may violate the flux inequality in models with more than 2 layers.

The two-layer model. I

- The governing equations for the two-layer quasi-geostrophic model are

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1$$
$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2$$
$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0$$

  where $\zeta_1 = \nabla^2\psi_1$; $\zeta_2 = \nabla^2\psi_2$; $T = (2/h)(\psi_1 - \psi_2)$. f is the Coriolis term; N the Brunt-Väisälä frequency; Q_0 is the thermal forcing on the temperature equation; d_1, d_2 the dissipation terms.
- The three equations are situated in three layers:
  - ψ_1: At 0.25Atm, top streamfunction layer
  - T: At 0.5Atm, temperature layer.
  - ψ_2: At 0.75Atm, bottom streamfunction layer

The two-layer model. II

- The potential vorticity is defined as

$$q_1 = \nabla^2\psi_1 + f + \frac{k_R^2}{2}(\psi_2 - \psi_1)$$
$$q_2 = \nabla^2\psi_2 + f - \frac{k_R^2}{2}(\psi_2 - \psi_1)$$

with $k_R \equiv 2\sqrt{2}f/(hN)$ and it satisfies

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1$$
$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2 + e_2$$

with $f_1 = (1/4)k_R^2 hQ_0$ and $f_2 = -(1/4)k_R^2 hQ_0$.

The two-layer model. III

- We use the following asymmetric dissipation configuration:

$$d_1 = \nu(-1)^{p+1}\nabla^{2p+2}\psi_1, \tag{1}$$

$$d_2 = (\nu + \Delta\nu)(-1)^{p+1}\nabla^{2p+2}\psi_2 - \nu_E\nabla^2\psi_s. \tag{2}$$

- Differential hyperdiffusion: $\Delta\nu > 0$.
- The Ekman term is given in terms of the streamfunction ψ_s at the surface layer ($p_s = 1$Atm) which is linearly extrapolated from ψ_1 and ψ_2 and it is given by $\psi_s = \lambda\psi_2 + \mu\lambda\psi_1$, with λ and μ given by

$$\lambda = \frac{p_s - p_1}{p_2 - p_1} \text{ and } \mu = \frac{p_2 - p_s}{p_s - p_1}. \tag{3}$$

- $0 < p_1 < p_2 < p_s \Longrightarrow -1 < \mu < 0$

The two-layer model. IV

- The dissipation term configuration corresponds to setting the generalized dissipation operator spectrum $D_{\alpha\beta}(k)$ equal to

$$D(k) = \begin{bmatrix} D_1(k) & 0 \\ \mu d(k) & D_2(k) + d(k) \end{bmatrix}, \tag{4}$$

with $D_1(q)$, $D_2(q)$, and $d(q)$ given by

$$D_1(k) = \nu k^{2p+2} \text{ and } D_2(k) = (\nu + \Delta\nu) k^{2p+2} \text{ and } d(k) = \lambda \nu_E k^2. \tag{5}$$

- The nonlinearity corresponds to an operator $\mathcal{L}_{\alpha\beta}$ with spectrum $L_{\alpha\beta}(k)$ given by

$$L(k) = -\begin{bmatrix} a(k) & b(k) \\ b(k) & a(k) \end{bmatrix}, \tag{6}$$

with $a(k)$ and $b(k)$ given by $a(k) = k^2 + k_R^2/2$ and $b(k) = -k_R^2$.

The two-layer model. V

- PROPOSITION 2:Assume streamfunction dissipation with both differential small-scale dissipation and extrapolated Ekman dissipation with $-1 < \mu < 0$. Assume also that $k^2 - a(q) - b(q) < 0$, and $b(q) < 0$, and $\Delta D(q) \equiv D_2(q) - D_1(q) \geq 0$, and also that $D_1(q)$, $\Delta D(q)$, and $d(q)$ satisfy

$$\frac{2D_1(q) + \mu d(q)}{\Delta D(q) + (\mu + 1)d(q)} > \frac{b(q)}{k^2 - a(q) - b(q)}. \tag{7}$$

  Then it follows that $\Delta(k, q) \leq 0$.
- For dissipation term configurations, choose one of
  - For $\mu = 0$ and $\lambda = 1$: standard Ekman at $p_s = p_2$.
  - For $\mu = -1/3$ and $\lambda = 3/2$: extrapolated Ekman at $p_s = 1$Atm.
- and also one of
  - For $\Delta\nu > 0$: differential small-scale diffusion
  - For $\Delta\nu = 0$: no differential small-scale diffusion

The two-layer model. VI

- For $\mu = 0$ and $\lambda = 1$: **standard** Ekman at $p_s = p_2$; and $\Delta\nu = 0$: **no** differential diffusion

$$\frac{\nu_E}{4\nu q^{2p}} \leq \frac{q^2 - k^2}{k_R^2} \Longrightarrow \Delta(k, q) \leq 0.$$

- For $\mu = -1/3$ and $\lambda = 3/2$: **extrapolated** Ekman at $p_s = 1$Atm; and $\Delta\nu > 0$: with differential diffusion

$$0 < \frac{\Delta\nu q^{2p} + \nu_E}{4\nu q^{2p} - \nu_E} < \frac{q^2 - k^2}{k_R^2} \Longrightarrow \Delta(k, q) \leq 0. \tag{8}$$

  - Note that the hypothesis requires that $\nu_E < 4\nu q^{2p}$ (thank extrapolated Ekman)
  - Increasing either ν_E or $\Delta\nu$ indicates a tendency towards violating the flux inequality.
  - For $\Delta\nu > 0$, the LHS of hypothesis will approach $\Delta\nu/(4\nu)$ and remain bounded for large wavenumbers q (thank differential diffusion)

The two-layer model. VII

- For $\mu = 0$ and $\lambda = 1$: **standard** Ekman at $p_s = p_2$; and $\Delta\nu > 0$: **with** differential diffusion

$$\frac{\Delta\nu q^{2p} + \nu_E}{4\nu q^{2p}} < \frac{q^2 - k^2}{k_R^2} \Longrightarrow \Delta(k, q) \leq 0, \qquad (9)$$

  - Hyperbolic blow-up is no longer possible.
  - For $\Delta\nu > 0$, the LHS of hypothesis will still approach $\Delta\nu/(4\nu)$ and remain bounded for large wavenumbers q
- For $\mu = -1/3$ and $\lambda = 3/2$: **extrapolated** Ekman at $p_s = 1$Atm; and $\Delta\nu = 0$: **no** differential diffusion

$$\frac{\nu_E}{4\nu q^{2p}} < \frac{q^2 - k^2}{k_R^2 + (q^2 - k^2)} \Longrightarrow \Delta(k, q) \leq 0. \qquad (10)$$

  - LHS vanishes with increasing q but RHS remains bounded.
  - Condition is still tighter.

The two-layer model. VIII

- Sufficient conditions can be derived in terms of the streamfunction spectra $U_1(q) = \langle \psi_1, \psi_1 \rangle_k$, $U_2(q) = \langle \psi_2, \psi_2 \rangle_k$, and $C_{12}(q) = \langle \psi_1, \psi_2 \rangle_k$
- Arithmetic-geometric mean inequality: $2|C_{12}(q)| \leq U_1(q) + U_2(q)$
- PROPOSITION 3: Assume streamfunction dissipation with $\Delta\nu = 0$, and standard Ekman (i.e. $\mu = 0$ and $\lambda = 1$) with $d(k) > 0$ and $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$ and $C_{12}(q) \leq U_2(q)$. Then, it follows that $\Delta(k, q) \leq 0$.
- PROPOSITION 4: Assume that $b(q) < 0$ and $k^2 - a(q) - b(q) < 0$. Assume also streamfunction dissipation with both differential small-scale dissipation and extrapolated Ekman dissipation with $-1 < \mu < 0$.
    1. If $C_{12}(q) \leq 0$, then $\Delta(k, q) \leq 0$.
    2. If $C_{12}(q) \leq \min\{U_1(q), U_2(q)\}$ and $U_1(q) + \mu U_2(q) \geq 0$, then $\Delta(k, q) \leq 0$.
- PROPOSITION 5: Assume that $k^2 - a(q) - b(q) < 0$ and $b(q) < 0$. We also assume streamfunction dissipation with extrapolated Ekman dissipation with $-1 < \mu < 0$ and symmetric small-scale dissipation with $D_1(q) = D_2(q)$. It follows that if $C_{12}(q) \leq \min\{U_1(q), U_2(q)\}$ then $\Delta(k, q) \leq 0$.

Conclusions

- Under symmetric dissipation, the flux inequality is satisfied unconditionally for multi-layer QG models
- Under asymmetric dissipation, the flux inequality is satisfied only when the asymmetry satisfies restrictions.
- The restrictions given are sufficient but not necessary.

Thank you!