Locality and stability of the cascades of two-dimensional turbulence.

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Publications

- This presentation is based on
  1. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 79-102
  2. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 103-124.
  3. E. Gkioulekas (2007), *Physica D*, **226**, 151-172
  4. E. Gkioulekas (2007), submitted to Phys. Rev. E, arXiv:0801.3006v1 [nlin.CD].
- Other relevant papers include:
  1. U. Frisch, *Proc. R. Soc. Lond. A* **434** (1991), 89–99.
  2. U. Frisch, *Turbulence: The legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge, 1995.
  3. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **52** (1995), 3840–3857.
  4. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **54** (1996), 6268–6284.

Outline

- Why study turbulence?
- Brief overview of K41 theory (3D turbulence)
- Frisch reformulation of K41 theory.
- KLB theory (2D turbulence).
- My reformulation of Frisch to address 2D turbulence
- Locality and stability of the cascades of 2D turbulence.
- Future directions.

Turbulence is everywhere

- Turbulent fluid flows are encountered everywhere in nature. The usual suspects:
  - Hydrodynanic turbulence: water flowing through a pipe. (Navier-Stokes)
  - Superfluid turbulence: liquid helium (Gross-Pitaevskii)
  - Geostrophic turbulence: atmosphere and ocean. (Phillips layers; QG)
  - Magnetohydrodynamic turbulence: plasmas in fusion reactors, the Sun.
- The **un**usual suspects:
  - Astronomical turbulence: interstellar medium, galaxy as a turbulent vortex.
  - Cosmological turbulence: Evolution of the universe itself as a Kolmogorov cascade?

Why study turbulence?

- **Engineering interest:** control large-scale aspects of turbulence to
  - Design airplanes.
  - Drag reduction in oil pipelines
  - Accelarate chemical combustion
  - Stabilize plasma in a nuclear fusion reactor
  - Propagation of laser through turbulence (SDI)
  - etc.
- **Scientific interest:** understand small-scale aspects of turbulence because
  - they are there.
  - seem governed by universal physical principles.
  - pose irresistably delightful paradoxes to the human mind.
  - Can we use a set of fundamental ideas to unify our understanding of radically different phenomena?

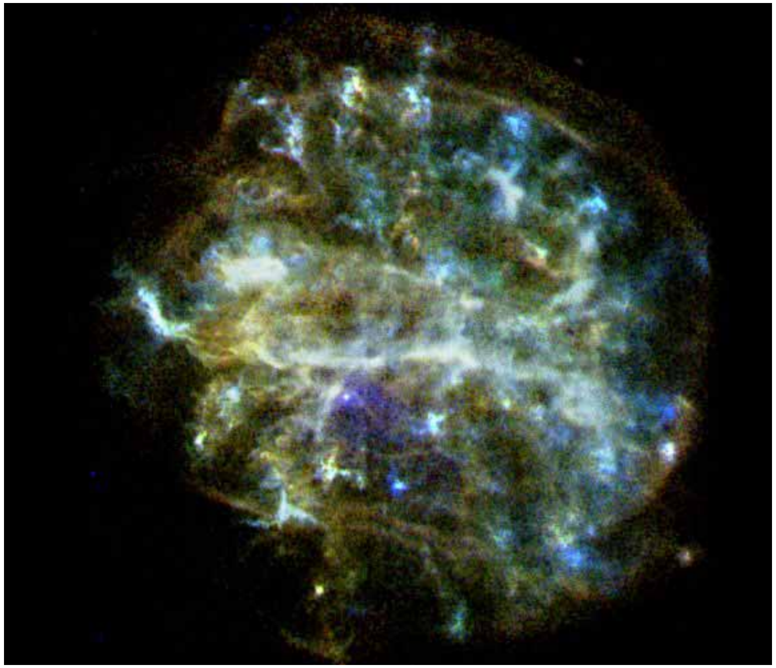
Turbulence is universal. I. An energy cascade

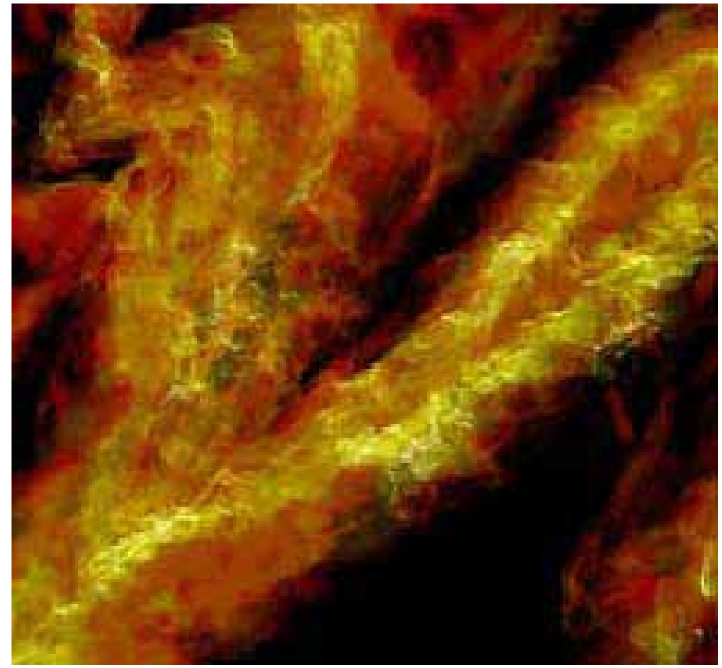
Richardson (1922) inspired by a Jonathan Swift poem:

So naturalists observe; a flea
hath smaller fleas that on him prey;
and these have smaller yet to bite'em
and so proceed ad infinitum.
Thus every poet, in his kind,
is bit by him that comes behind.

1. Replace: **flea** with **vortex**
2. **Bites** bite **energy** $\Longrightarrow$ energy cascade.
3. Formalized by Kolmogorov in 1941

Turbulence is universal. II. Astrophysical turbulence?



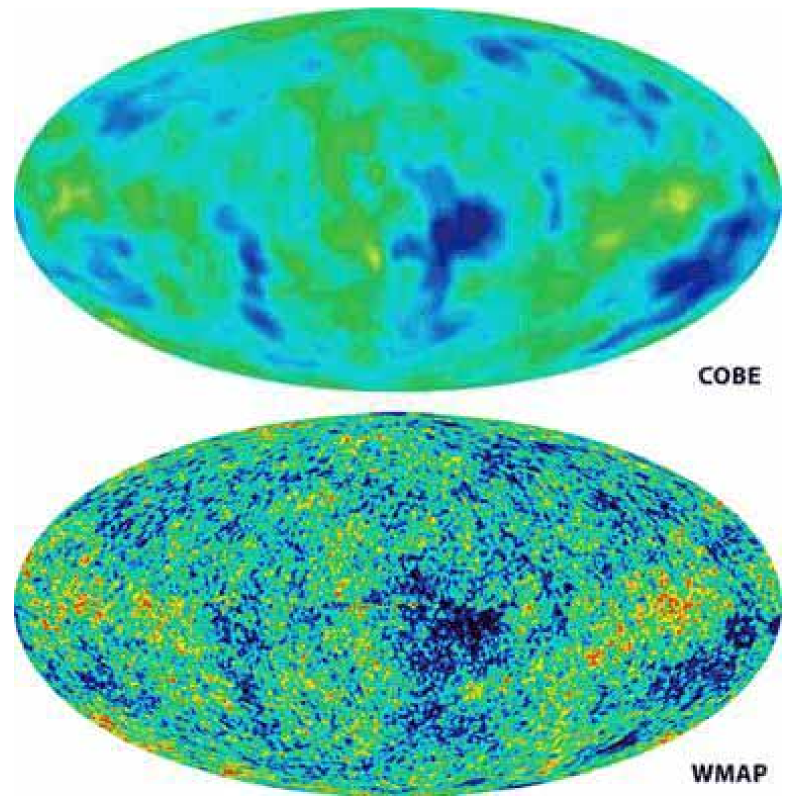


The obvious next question is whether the Kolmogorov cascade can be found elsewhere:

- in the debris of a supernova explosion? (left)
- in the interstellar medium? (right)

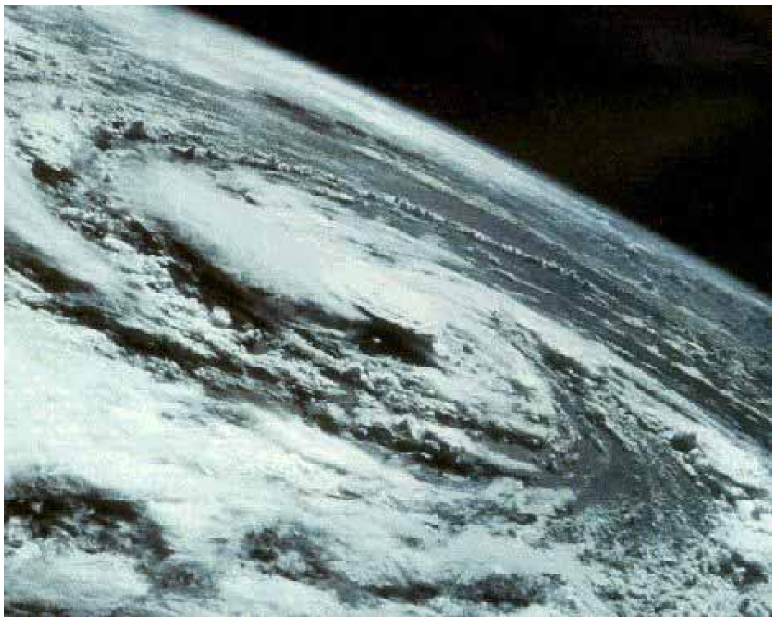
Turbulence is universal. III. Cosmological turbulence?

- This is the oldest picture of the Universe, obtained by analysing the cosmic fossil radiation emitted when it was 380,000 years old.
- The pattern in the picture represents the early clumps of matter that eventually evolved into today's galaxies.
- Can we understand this picture in terms of turbulence theory? (large-scale vortices at universe scale cascading into small-scale vortices at galactic scale).



Turbulence is universal. IV. Coherent vortices

- When turbulence is constrained into 2 dimensions, it develops an appetite for **coherent vortices** that look like this:




- Left: 1,000 km coherent vortex
- Right: 100,000 ly coherent vortex

K41 theory. I.

- In three-dimensional turbulence there is an energy cascade from large scales to small scales driven by the nonlinear term of the Navier-Stokes equations
- Kolmogorov (1941) predicts that the structure functions $S_n(\mathbf{x}, r\mathbf{e})$ of longitudinal velocity differences, defined as

$$S_n(\mathbf{x}, r\mathbf{e}) = \langle \{[\mathbf{u}(\mathbf{x} + r\mathbf{e}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \mathbf{e}\}^n \rangle \tag{1}$$

  are governed by self-similar scaling $S_n(\mathbf{x}, \lambda r\mathbf{e}) = \lambda^{\zeta_n} \lambda S_n(\mathbf{x}, r\mathbf{e})$ for scales r in the inertial range $\eta \ll r \ll \ell_0$ (intermediate asymptotics) with
  - $\ell_0 =$ forcing length scale
  - $\eta = (\nu^3/\varepsilon)^{1/4} =$ dissipation scale. (Kolmogorov microscale)
  - $\varepsilon =$ rate of energy injection
- Kolmogorov (1941) predicts that $\zeta_n = n/3$ and thus $S_n(\mathbf{x}, r\mathbf{e}) \sim C_n(\varepsilon r)^{n/3}$ in the inertial range.

K41 theory. II.

- Oboukhov (1941) argued that the energy spectrum $E(k)$ will scale as $E(k) \sim k^{-1-\zeta_2}$, and will thus be given by

$$E(k) \sim C\varepsilon^{2/3}k^{-5/3} \tag{2}$$

- 1962: First experimental confirmation of the Kolmogorov-Oboukhov prediction by measurement of oceanic currents.
- 1962: Kolmogorov predicts **intermittency corrections** to ζ_n:

$$\zeta_n = \frac{n}{3} - \frac{\mu n(n-3)}{18} \tag{3}$$

- Not self-consistent statistically, because ζ_n should not decrease.
- The existence of intermittency corrections confirmed by experimental measurements
- The problem of calculating ζ_n rigorously is still open.

Frisch reformulation of K41. I

- Define the Eulerian velocity differences w_α:

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) = u_\alpha(\mathbf{x}, t) - u_\alpha(\mathbf{x}', t). \tag{4}$$

- H1: Local homogeneity/isotropy/stationarity

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}+\mathbf{y}, \mathbf{x}'+\mathbf{y}, t) \ , \forall \mathbf{y} \in \mathbb{R}^d. \tag{5}$$

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}_0 + A(\mathbf{x}-\mathbf{x}_0), \mathbf{x}_0 + A(\mathbf{x}'-\mathbf{x}_0), t) \ , \forall A \in SO(d). \tag{6}$$

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t+\Delta t) \ , \forall \Delta t \in \mathbb{R}. \tag{7}$$

- H2: Self-similarity

$$w_\alpha(\lambda\mathbf{x}, \lambda\mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} \lambda^h w_\alpha(\mathbf{x}, \mathbf{x}', t) \tag{8}$$

- H3: Anomalous energy sink: energy will still be dissipated when $\nu \to 0^+$.

Frisch reformulation of K41. II

- The argument
  - H1 and H3 $\Longrightarrow$ 4/5 law $\Longrightarrow \zeta_3 = 1$
  - H2 $\Longrightarrow \zeta_n = nh$
  - Therefore: $\zeta_n = n/3 \Longrightarrow k^{-5/3}$ scaling
- 2005: Frisch questions self-consistency of local homogeneity
- Proof of 4/5 law
- 2007: These issues discussed further by Gkioulekas in
  - E. Gkioulekas (2007), *Physica D*, **226**, 151-172
- The above theory rules out intermittency corrections.
- To allow intermittency corrections we need a better theory which at the very least
  - Weakens H2
  - Tolerates H1 and H3
  - Leads to a calculation of the correct ζ_n exponents.

A happy breakthrough

- 2000: L'vov-Procaccia show that $K62$ is indeed a first-order correction. Next order correction gives:

$$\zeta_n = \frac{n}{3} - \frac{n(n-3)}{2}\delta_2[1 + 2\delta_2 b_2(n-2)] + O(\delta_2^3) \tag{9}$$

- What we don't know:
  - How to calculate the numerical coefficients in general
  - How to show that ζ_n are universal (if they are)
  - Lack of rigor in derivation of ζ_n. There are underlying unproven hypotheses.
  - Why the same ζ_n can be obtained from shell models of the energy cascade (ODE models)
  - How ζ_n behave in the limit $n \to +\infty$.
- Fun part: 2D turbulence does not have these intermittency corrections

Governing equations for 2D

- In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = -[\nu(-\Delta)^{\kappa} + \nu_1(-\Delta)^{-m}]\zeta + F, \tag{10}$$

  where $\psi(x, y, t)$ is the streamfunction and $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$.
- The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ, and is defined as

$$J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}. \tag{11}$$

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2}\int \psi(x, y, t)\zeta(x, y, t)\; dxdy \quad G(t) = \frac{1}{2}\int \zeta^2(x, y, t)\; dxdy. \tag{12}$$

Flux directions

- Assume that 2D turbulence is forced in a narrow band $[k_1, k_2]$ of wavenumbers.
- Let $\Pi_E(k)$ and $\Pi_G(k)$ be the rate with which energy and enstrophy are transfered by the nonlinearity $J(\psi, \zeta)$ from $[0, k]$ to $[k, +\infty)$.
- Then, under stationarity the fluxes $\Pi_E(k)$ and $\Pi_G(k)$ will satisfy the inequalities

$$\int_0^k q\Pi_E(q)\, dq < 0,\ \forall k > k_2 \quad \text{and} \quad \int_k^{+\infty} q^{-3}\Pi_G(q) > 0,\ \forall k < k_1. \tag{13}$$

- Thus in 2D turbulence **energy goes upscale** and **enstrophy goes downscale**.
- Further discussion in
  - R. Fjørtøft (1953), *Tellus*, **5**, 225-230.
  - P.E. Merilees and T. Warn (1975), *J. Fluid. Mech.*, **69**, 625–630.
  - E. Gkioulekas and K.K. Tung (2007), *J. Fluid Mech.*, **576**, 173-189.
- There is no known proof that **energy goes downscale** in 3D turbulence!

KLB theory I

- Kraichnan, Leith, and Batchelor (KLB) proposed that in two-dimensional turbulence there is an upscale energy cascade and a downscale enstrophy cascade. (1967)
- The energy spectrum in the upscale energy range is

$$E(k) = C_{ir}\varepsilon^{2/3}k^{-5/3}, \tag{14}$$

and in the downscale enstrophy range is

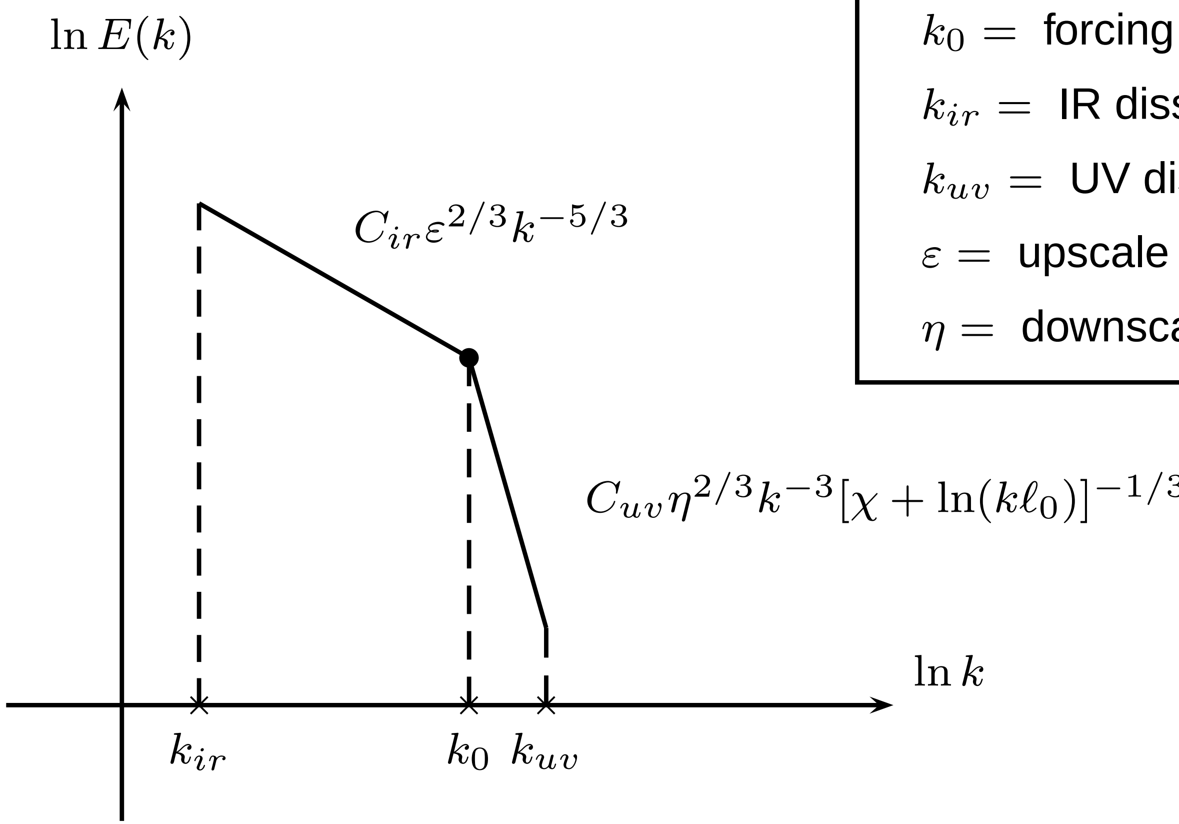
$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{15}$$

- Falkovich and Lebedev (1994) predict that the vorticity ζ structure functions have logarithmic scaling given by

$$\langle[\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]^n\rangle \sim [\eta \ln(\ell_0/r_{12})]^{2n/3}. \tag{16}$$

- Confirmed using spectral reduction by Bowman, Shadwick and Morrison (1999).

KLB theory II



$k_0 =$ forcing wavenumber

$k_{ir} =$ IR dissipation wavenumber

$k_{uv} =$ UV dissipation wavenumber

$\varepsilon =$ upscale energy flux

$\eta =$ downscale enstrophy flux

Open Questions

- Enstrophy cascade is difficult to reproduce numerically. It requires:
  - A large-scale sink (Ekman or hypodiffusion)
  - High numerical resolution
  - All published simulations so far have used hyperdiffusion.
- The inverse energy cascade is often disrupted by coherent structures.
  - Coherent structures give a dominant k^{-3} contribution to $E(k)$ even though they occupy a small percentage of the physical domain
  - Removing coherent structures artificially recovers the $k^{-5/3}$ spectrum.
- Eyink (2001): We know why the enstrophy cascade has no intermittency corrections.
- Why does the inverse energy cascade not have observable intermittency corrections?
- The underlying fundamental question is to explain why the cascades of 3D turbulence are robust and the cascades of 2D turbulence are not.

Revisions to the Frisch framework

- The KLB theory can be reformulated similarly.
- Such a theory implicitly assumes locality and universality of the two cascades.
- The conditions needed for the existence of universal cascades is **the** question!
- A deeper theory of 2D turbulence can be formulated as follows
  1. Begin with the Frisch reformulation of Kolmogorov theory in 3D turbulence.
  2. Replace anomalous sink assumption with the axiom of universality. (non-perturbative theory of L'vov and Procaccia)
  3. Weaken the multifractal self-similarity hypothesis.
  4. Adapt the non-perturbative theory of L'vov and Procaccia to 2D turbulence.
- Then, it is possible to:
  1. Deduce conditions for locality and stability of both cascades.
  2. Deduce existence of anomalous sinks from our axioms.

The new framework of hypotheses. I

- Define the Eulerian velocity differences w_α:

$$w_\alpha(\mathbf{x},\mathbf{x}',t) = u_\alpha(\mathbf{x},t) - u_\alpha(\mathbf{x}',t). \tag{17}$$

- The Eulerian generalized structure function is defined as

$$F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t) = \left\langle \left[\prod_{k=1}^{n} w_{\alpha_k}(\mathbf{x}_k,\mathbf{x'}_k,t) \right] \right\rangle, \tag{18}$$

where $\{\mathbf{X}\}_n = \{\mathbf{x},\mathbf{x}'\}_n$ is shorthand for a list of $2n$ position vectors.

- We also define the conditional correlations

$$\Phi_n(\{\mathbf{X}\}_n,\{\mathbf{Y}\}_m,\{\mathbf{w}_k\}_{k=1}^m,t) = \left\langle \left[\prod_{k=1}^{n} w_{\alpha_k}(\mathbf{X}_k,t) \right] \middle| \mathbf{w}(\mathbf{y}_k,\mathbf{y'}_k,t) = \mathbf{w}_k \right\rangle. \tag{19}$$

The new framework of hypotheses. II

- **Hypothesis 1:** The velocity field is locally stationary, locally homogeneous, and locally isotropic, defined as

$$\frac{\partial F_n(\{\mathbf{X}\}_n, t)}{\partial t} = 0, \forall t \in \mathbb{R} \tag{20}$$

$$\sum_{k=1}^{n} (\partial_{\alpha_k, \mathbf{x}_k} + \partial_{\alpha_k, \mathbf{x}'_k}) F_n(\{\mathbf{X}\}_n, t) = 0 \tag{21}$$

$$F_n(\{\mathbf{X}\}_n, t) = F_n(\mathbf{r}_0 + \mathcal{A}(\{\mathbf{X}\}_n - \mathbf{r}_0), t), \ \forall \mathcal{A} \in SO(2) \tag{22}$$

as long as the evaluations $\{\mathbf{X}\}_n$, $\{\mathbf{X}\}_n + \Delta\mathbf{r}$, $\mathbf{r}_0 + \mathcal{A}(\{\mathbf{X}\}_n - \mathbf{r}_0)$, lie within an inertial range.

- **Hypothesis 2:** The velocity field is self-similar in the sense that for every evaluation $\{\mathbf{X}\}_n$ within an inertial range

$$\exists \varepsilon > 0 : F_n(\lambda\{\mathbf{X}\}_n, t) = \lambda^{\zeta_n} F_n(\{\mathbf{X}\}_n, t), \ \forall \lambda \in (1 - \varepsilon, 1 + \varepsilon) \tag{23}$$

The new framework of hypotheses. III

- **Hypothesis 3:** Let $\{\mathbf{X}\}_n$ and $\{\mathbf{Y}\}_m$ represent the geometries of velocity differences and let $\mathcal{W} = \{\mathbf{w}(\mathbf{y}_k, \mathbf{y'}_k, t) = \mathbf{w}_k\}$. Then, if in the direct cascade they satisfy $\|\{\mathbf{X}\}_n\| \ll \|\{\mathbf{Y}\}_m\| \ll \ell_0$, or alternatively if in the inverse cascade they satisfy $\|\{\mathbf{X}\}_n\| \gg \|\{\mathbf{Y}\}_m\| \gg \ell_0$, then the conditional correlations Φ_n preserve local homogeneity, and local isotropy, with respect to $\{\mathbf{X}\}_n$, defined as

$$\begin{aligned} \Phi_n(\{\mathbf{X}\}_n, \mathcal{W}, t) &= \Phi_n(\{\mathbf{X}\}_n + \Delta\mathbf{r}, \mathcal{W}, t) \\ \Phi_n(\{\mathbf{X}\}_n, \mathcal{W}, t) &= \Phi_n(\mathbf{r}_0 + \mathcal{A}(\{\mathbf{X}\}_n - \mathbf{r}_0), \mathcal{W}, t), \ \forall \mathcal{A} \in SO(2) \end{aligned} \tag{24}$$

and also self similarity, with the same scaling exponents ζ_n, defined as

$$\exists \varepsilon > 0 : \Phi_n(\lambda\{\mathbf{X}\}_n, \mathcal{W}, t) = \lambda^{\zeta_n} \Phi_n(\{\mathbf{X}\}_n, \mathcal{W}, t), \ \forall \lambda \in (1-\varepsilon, 1+\varepsilon) \tag{25}$$

The new framework of hypotheses. IV

- Hypothesis 1 is incremental stationarity, homogeneity, and isotropy
- Hypothesis 2 is the L'vov–Procaccia *self-similarity* principle.
- Hypothesis 3 is the *universality principle.*
    - The events $\mathcal{W} = \{\mathbf{w}(\mathbf{y}_k, \mathbf{y'}_k, t) = \mathbf{w}_k\}$ partition the ensemble of all possible forcing histories into **subensembles** defined by the parameters $\{\mathbf{w}_k\}_{k=1}^m$.
    - Each choice of $\{\mathbf{Y}\}_m$ represents a distinct partition.
    - We assume that Hypothesis 1 and 2 hold for each subensemble $\{\mathbf{w}_k\}_{k=1}^m$ and for all possible partitions $\{\mathbf{Y}\}_m$. (with $\|\{\mathbf{X}\}_n\| \ll \|\{\mathbf{Y}\}_m\| \ll \ell_0$ if it is a downscale cascade or $\|\{\mathbf{X}\}_n\| \gg \|\{\mathbf{Y}\}_m\| \gg \ell_0$ if it is an upscale cascade)
- These hypotheses are an efficient *definition* of the concept of an "inertial range".
- The hypotheses are valid only a multidimensional domain of velocity differences geometries $\{\mathbf{X}\}_n \in \mathcal{I}_n$.
- The extent of this domain $\mathcal{I}_n$ *is* the extent of the inertial range itself.
- A different set of exponents ζ_n and region $\mathcal{J}_n$ is associated with each range.

The fusion rules hypothesis

- **Step 1:** Hypothesis 3 $\Longrightarrow$ the fusion rules hypothesis.
- Consider a geometry of velocity differences $\{\mathbf{X}\}_n$ such that $\|\{\mathbf{X}\}_n\| = 1$ and define

$$F_n^{(p)}(r, R) = F_n(r\{\mathbf{X}_k\}_{k=1}^{p}, R\{\mathbf{X}_k\}_{k=p+1}^{n}). \quad (26)$$

- The fusion rules give the scaling properties of $F_n^{(p)}$ in terms of the following general form:

$$F_n^{(p)}(\lambda_1 r, \lambda_2 R) = \lambda_1^{\xi_{np}} \lambda_2^{\zeta_n - \xi_{np}} F_n^{(p)}(r, R) \quad (27)$$

- A concise statement of the fusion rules hypothesis is that for the direct enstrophy cascade $\xi_{np} = \zeta_p$, and for the inverse energy cascade $\xi_{np} = \zeta_n - \zeta_{n-p}$ for $1 < p < n - 1$.
- We will also consider the case of "regular" violations to the fusion rules where the scaling exponents ξ_{np} satisfy $0 < \xi_{np} < \zeta_n$

Balance Equations and Locality. I

- **Step 2:** The fusion rules hypothesis $\Longrightarrow$ Locality
- We employ the balance equations introduced by L'vov and Procaccia (1996).
- The Navier-Stokes equations, where the pressure term has been eliminated, read

$$\frac{\partial u_\alpha}{\partial t} + \mathcal{P}_{\alpha\beta}\partial_\gamma(u_\beta u_\gamma) = \mathcal{D}u_\alpha + \mathcal{P}_{\alpha\beta}f_\beta, \tag{28}$$

where $\mathcal{P}_{\alpha\beta} = \delta_{\alpha\beta} - \partial_\alpha\partial_\beta\nabla^{-2}$ is the projection operator and $\mathcal{D}$ is the dissipation operator given by

$$\mathcal{D} \equiv (-1)^{\kappa+1}\nu_\kappa\nabla^{2\kappa} + (-1)^{m+1}\beta\nabla^{-2m} \tag{29}$$

- The balance equations are obtained by differentiating the definition of F_n with respect to time t and substituting the Navier-Stokes equations

Balance Equations and Locality. II

- Thus one obtains the equation:

$$\frac{\partial F_n}{\partial t} + \mathcal{O}_n F_{n+1} + I_n = \nu J_n + \beta H_n + Q_n \tag{30}$$

  where:
  - $\mathcal{O}_n F_{n+1}$ represents the local nonlinear interactions
  - I_n represents the sweeping interactions
  - Q_n represents the forcing term
  - νJ_n and βH_n represent the dissipation terms
- We propose that the locality of the interaction integral in $\mathcal{O}_n F_{n+1}$ is the mathematical definition that corresponds most closely with our physical conception of locality in a local eddy cascade.
- We assume, without proof, that the sweeping term I_n can be disregarded in the inertial range

Locality conditions

- For either a downscale or an upscale cascade the locality conditions are
  - UV locality: $\xi_{n+1,2} > 0, \ \forall n \in \mathbb{N} : n > 1$
  - IR locality: $\zeta_{n+1} \leq \xi_{n+1,2} + \xi_{n+1,n-1} \ \forall n \in \mathbb{N} : n > 1$
- **The fusion rules hypothesis implies the conditions above**
- Consider a regular violation of the fusion rules hypothesis with:

$$\xi_{np} = \zeta_p + \Delta\xi_{np} \text{ (downscale)} \quad (31)$$

$$\xi_{np} = \zeta_n - \zeta_{n-p} + \Delta\xi_{np} \text{ (upscale)} \quad (32)$$

- UV localiy is still maintained, because $0 < \xi_{np} < \zeta_n$
- For IR locality, the sufficient condition becomes

$$\Delta\xi_{n+1,2} + \Delta\xi_{n+1,n-1} \geq 0 \ \forall n \in \mathbb{N} : n > 1 \text{ (downscale)} \quad (33)$$

$$\Delta\xi_{n+1,2} + \Delta\xi_{n+1,n-1} \leq 0 \ \forall n \in \mathbb{N} : n > 1 \text{ (upscale)} \quad (34)$$

Stability of cascades. I

- Conclusion: Given the fusion rules hypothesis, both the inverse energy cascade and the enstrophy cascade are local.
- **Step 3:** Locality $\Longrightarrow$ stability
- **Locality** implies that the contributions D_{kn} to $\mathcal{O}F_{n+1}$ are also self-similar with scaling exponent δ_n and satisfy

$$D_{kn}(\lambda\{\mathbf{X}\}_n, t) = \lambda^{\zeta_{n+1}-1} D_{kn}(\{\mathbf{X}\}_n, t) \tag{35}$$

- **Statistical stability:** there should be a region $\mathcal{J}_n$ such that $Q_n(\{\mathbf{X}\}_n)$ is negligible relative to the contributions to $D_{kn}(\{\mathbf{X}\}_n)$ for all $\{\mathbf{X}\}_n \in \mathcal{J}_n$
- The forcing term Q_n is also self-similar with scaling exponent q_n and satisfies

$$Q_n(\lambda\{\mathbf{X}\}_n, t) = \lambda^{q_n} Q_n(\{\mathbf{X}\}_n, t) \tag{36}$$

Stability of cascades. II

- Assume that f_α is a delta-correlated stationary gaussian field with $\langle f_\alpha(\mathbf{x})\rangle = 0$, and

$$\left\langle f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_2, t_2)\right\rangle = 2\varepsilon C_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1 - t_2), \tag{37}$$

  where ε is constant, and $C_{\alpha\beta}$ is normalized such that $C_{\alpha\alpha}(\mathbf{x}, \mathbf{x}) = 1$.
- It can be shown that:

$$Q_{kn}^{\alpha_1 \cdots \alpha_{n-1}\beta}(\{\mathbf{X}\}_{n-1}, \mathbf{Y}, t) = \sum_{l=1}^{n-1} F_{n-2}^{\alpha_1 \cdots \alpha_{l-1}\alpha_{l+1}\cdots\alpha_{n-1}}(\{\mathbf{X}\}_{n-1}^{l}) Q_{\alpha_l \beta}(\mathbf{X}_l, \mathbf{Y}),$$

$$Q_{\alpha\beta}(\mathbf{X}, \mathbf{Y}) = 2\varepsilon[C_{\alpha\beta}(\mathbf{y}, \mathbf{x}) - C_{\alpha\beta}(\mathbf{y}', \mathbf{x}) - C_{\alpha\beta}(\mathbf{y}, \mathbf{x}') + C_{\alpha\beta}(\mathbf{y}', \mathbf{x}')].$$

- For Gaussian delta-correlated in time forcing $q_n = \zeta_{n-2} + q_2$
- We see that F_{n-2} provides feedback to Q_n, when the forcing is gaussian.
- For statistical stability we need this feedback to be negligible in the inertial range.

Stability of cascades. III

- It follows that the ratio Q_n/D_{kn} scales as

$$\frac{Q_n(R)}{D_{kn}(R)} \sim \left(\frac{R}{\ell_0}\right)^{\Delta q_n}. \tag{38}$$

  with $\Delta q_n = (\zeta_{n-2} + q_2) - (\zeta_{n+1} - 1)$.
- Stability conditions for downscale cascades
  - In a downscale cascade $q_2 = 2$
  - Downscale cascades: this ratio must vanish when $\ell_0 \to +\infty$
  - $\Longleftrightarrow \Delta q_n > 0,\ \forall n \in \mathbb{N}, n > 1$
  - $\Longleftrightarrow h < 1$ for monofractal scaling $\zeta_n = nh$
- The stability condition is neither satisfied nor broken, because $h = 1$!
- When the downscale energy flux is small enough, then $q_2 \geq 3$, and the stability condition is satisfied.

Stability of cascades. IV

- Recall that the ratio Q_n/D_{kn} scales as

$$\frac{Q_n(R)}{D_{kn}(R)} \sim \left(\frac{R}{\ell_0}\right)^{\Delta q_n}. \tag{39}$$

  with $\Delta q_n = (\zeta_{n-2} + q_2) - (\zeta_{n+1} - 1)$.
- Stability conditions for upscale cascades
  - In an upscale cascade $q_2 < 0$
  - Upscale cascades: this ratio must vanish when $\ell_0 \to 0$
  - $\iff \Delta q_n < 0, \ \forall n \in \mathbb{N}, n > 1$
  - $\iff h > (1 + q_2)/3$ for monofractal scaling $\zeta_n = nh$
- The stability condition is satisfied, because $h = 1/3$.
- However the inverse energy cascade can be disrupted by the sweeping interactions term I_n.

Concluding remarks

- **Paradox:** The constraint $0 < \zeta_2 < 2$ does not appear anywhere in our locality proof!
- The inequality $0 < \zeta_2 < 2$ can come in as a necessary condition
  - for the survival of locality under the Fourier integral
  - for **perturbative locality** for each Feynman diagram
- The enstrophy cascade is **non-perturbatively local** and borderline non-local **only** in the perturbative sense.
- Closure models unwittingly exchange **non-perturbative locality** with **perturbative locality**!
- Stability of cascades (required for universality) imposes constraints on ζ_3:
  - for downscale enstrophy cascade: $0 < \zeta_3 < 3$
  - for upscale inverse energy cascade: $\zeta_3 \geq 1$
- The stability of the downscale enstrophy cascade requires considerable separation between forcing and small-scale separation

Future directions

- Immediate concerns (2D turbulence):
  - Dissipation terms and anomalous sinks.
  - Non-gaussian forcing and stability.
  - The fusion rules hypothesis (numerical and theoretical investigation).
- Apply what we have learned to SQG turbulence and α-turbulence.
- Understand Phillips layer modes and QG turbulence.
- Superfluid turbulence.
- Compressible turbulence. (Moiseev-Shivamoggi theory)