

On the elimination of the sweeping interactions from theories of hydrodynamic turbulence

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Some Publications.

- This presentation is based on
 1. E. Gkioulekas (2006): Ph.D. thesis, University of Washington (Advisor: Ka-Kit Tung)
 2. E. Gkioulekas: *Physica D*, under review. [nlin.CD/0506064]
- Other relevant papers include:
 1. U. Frisch, *Proc. R. Soc. Lond. A* **434** (1991), 89–99.
 2. U. Frisch, *Turbulence: The legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge, 1995.
 3. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **52** (1995), 3840–3857.
 4. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **54** (1996), 6268–6284.
 5. R.J. Hill, *J. Fluid. Mech.* **353** (1997), 67–81.
 6. U. Frisch, J. Bec, and E. Aurell, “Locally homogeneous turbulence: Is it a consistent framework?”, [nlin.CD/0502046], 2005.

K41 prediction

- In three-dimensional turbulence there is an energy cascade from large scales to small scales is driven by the nonlinear term of the Navier-Stokes equations
- Using dimensional analysis we get K41 prediction

$$S_n(\mathbf{x}, r\mathbf{e}) = \langle \{[\mathbf{u}(\mathbf{x} + r\mathbf{e}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \mathbf{e}\}^n \rangle \quad (1)$$

$$= C_n(\varepsilon r)^{n/3}, \text{ for } \eta \ll r \ll \ell_0 \quad (2)$$

$$E(k) = C\varepsilon^{2/3}k^{-5/3}, \text{ for } \ell_0^{-1} \ll k \ll \eta^{-1} \quad (3)$$

- Including intermittency corrections, the real behaviour in the inertial range is:

$$S_n(\mathbf{x}, r\mathbf{e}) = C_n(\varepsilon r)^{n/3}(r/\ell_0)^{\zeta_n - n/3} \quad (4)$$

$$E(k) \sim C\varepsilon^{2/3}k^{-5/3}(k\ell_0)^{5/3 - \zeta_2} \quad (5)$$

- How do we understand: dimensional analysis, intermittency, and universality?

Outline of presentation

- Review of the following ideas:
 - Similarity analysis
 - Frisch reformulation of K41
 - Analytical theories: MSR theory and L'vov-Procaccia theory
- Hierarchical definition of local homogeneity
- Sufficient condition to eliminate sweeping
- Same condition needed to prove 4/5 law
- Stronger condition needed to use the Belinicher-L'vov quasi-Lagrangian transformation
- Open question: More rigorous elimination of sweeping

Similarity analysis I



Similarity analysis is a generalization of dimensional analysis.

1. E. Hopf, *Statistical hydromechanics and functionals calculus*, J. Ratl. Mech. Anal. **1** (1952) 87–123.
2. S. Moiseev, A. Tur, V. Yanovskii, *Spectra and expectation methods of turbulence in a compressible fluid*, Sov. Phys. JETP **44** (1976) 556–561.
3. A.G. Sazontov, *The similarity relation and turbulence spectra in a stratified medium*, Izv. Atmos. Ocean. Phys. **15** (1979), 566–570.
4. S.S. Moiseev and O.G. Chkhetiani, *Helical scaling in turbulence*, JETP **83** (1996), 192–198.
5. H. Branover, A. Eidelman, E. Golbraikh, and S. Moiseev, *Turbulence and structures: chaos, fluctuations, and helical self organization in nature and the laboratory*, Academic Press, San Diego, 1999.

Similarity analysis II

- Assume gaussian delta-correlated forcing with forcing spectrum $F(k)$ parameterized as

$$F(k) = \varepsilon F_0(k\ell_0) \quad (6)$$

- Using the Hopf formalism, it can be shown rigorously that the energy spectrum satisfies

$$E(k, t|\nu, \varepsilon, \ell_0) = \lambda^{-(2\beta+1)} E(k, \lambda^{1-\beta}t|\lambda^{1+\beta}\nu, \lambda^{3\beta-1}\varepsilon, \lambda\ell_0) \quad (7)$$

- From the conditions $\partial E/\partial t = 0$ and $\partial E/\partial \beta = 0$ we find that

$$E(k, t|\nu, \varepsilon, \ell_0) = \varepsilon^{2/3} k^{-5/3} E_0(k\ell_0, k\eta) \quad (8)$$

with $\eta \equiv (\nu^3/\varepsilon)^{1/4}$. If F_0 fixed, then E_0 is fixed.

- Assume the limits $\ell_0 \rightarrow +\infty$ and $\eta \rightarrow 0$ converge (similarity assumption). Likewise for structure functions.

Frisch reformulation of K41. I

- Define the Eulerian velocity differences w_α :

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) = u_\alpha(\mathbf{x}, t) - u_\alpha(\mathbf{x}', t). \quad (9)$$

- H1: Local homogeneity/isotropy/stationarity

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \stackrel{\mathbf{x}, \mathbf{x}'}{\sim} w_\alpha(\mathbf{x} + \mathbf{y}, \mathbf{x}' + \mathbf{y}, t), \forall \mathbf{y} \in \mathbb{R}^d. \quad (10)$$

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \stackrel{\mathbf{x}, \mathbf{x}'}{\sim} w_\alpha(\mathbf{x}_0 + A(\mathbf{x} - \mathbf{x}_0), \mathbf{x}_0 + A(\mathbf{x}' - \mathbf{x}_0), t), \forall A \in SO(d). \quad (11)$$

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \stackrel{\mathbf{x}, \mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t + \Delta t), \forall \Delta t \in \mathbb{R}. \quad (12)$$

- H2: Self-similarity

$$w_\alpha(\lambda \mathbf{x}, \lambda \mathbf{x}', t) \stackrel{\mathbf{x}, \mathbf{x}'}{\sim} \lambda^h w_\alpha(\mathbf{x}, \mathbf{x}', t) \quad (13)$$

- H3: Anomalous energy sink

Frisch reformulation of K41. II

• The argument

• H1 and H3 \implies 4/5 law $\implies \zeta_3 = 1$

• H2 $\implies \zeta_n = nh$

• Therefore: $\zeta_n = n/3 \implies k^{-5/3}$ scaling

• 2005: Frisch questions self-consistency of local homogeneity

• Proof of 4/5 law

• 1959: Proof by Monin using local homogeneity (in Russian)

• 1975: Reprinted by Monin and Yaglom book

• 1995: Frisch proof uses global homogeneity

• 1996: Lindborg notes that the pressure gradient/velocity field correlations cannot be eliminated by local isotropy

• 1997: Problem corrected by Hill

• 1999: Rasmussen proof uses global homogeneity

• 2006: Gkioulekas notes that local homogeneity not sufficient.

Analytical theories.

- In the beginning: Quasnormal closure models.
- 1957: Kraichnan showed that they give negative $E(k)$.
- 1958: Kraichnan DIA theory $\implies k^{-3/2}$ scaling.
- 1961: Wyld shows that DIA is 1-loop line-renormalized diagrammatic theory
- 1962: Experiments confirm $k^{-5/3}$ scaling.
- 1964: Kraichnan notes the need to eliminate the sweeping interactions via a Lagrangian transformation.
- 1965: LHDIA theory \implies Locality $\implies k^{-5/3}$ scaling.
- 1973: Martin-Siggia-Rose theory (MSR theory)
- 1977: Phythian reformulates MSR theory in terms of path integrals.

MSR theory. I. Formulation

- In MSR formalism we have a quadratic problem of the form

$$\frac{\partial u_\alpha}{\partial t} = P_{\alpha\delta} V_{\beta\gamma\delta} u_\gamma u_\delta + \mathcal{D}_{\alpha\beta} u_\alpha + P_{\alpha\beta} f_\beta \quad (14)$$

with gaussian forcing: $Q_{\alpha\beta} = \langle f_\alpha f_\beta \rangle$.

- Define the correlators

$$F_{\alpha\beta} = \langle u_\alpha u_\beta \rangle, \quad G_{\alpha\beta} = \langle \delta u_\alpha / \delta f_\beta \rangle \quad (15)$$

- The Dyson-Wyld equations are

$$\frac{\partial G_{\alpha\beta}(t)}{\partial t} = \mathcal{D}_{\alpha\gamma} G_{\gamma\beta}(t) + P_{\alpha\beta} \delta(t) + \int_0^t dt_1 P_{\alpha\gamma} \Sigma_{\gamma\delta}(t_1) G_{\delta\beta}(t - t_1) \quad (16)$$

$$F_{\alpha\beta}(t) = \int dt_1 \int dt_2 G_{\alpha\gamma} [Q_{\gamma\delta}(t - t_1 + t_2) + \Phi_{\gamma\delta}(t - t_1 + t_2)] G_{\delta\beta}(t_2) \quad (17)$$

MSR theory. II. Diagram expansion

- The operators $\Sigma_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ can be represented with a Feynman diagram expansion

$$\Sigma_{\alpha\beta} = \Sigma_{\alpha\beta}^1 + \Sigma_{\alpha\beta}^2 + \dots \quad (18)$$

$$\Phi_{\alpha\beta} = \Phi_{\alpha\beta}^1 + \Phi_{\alpha\beta}^2 + \dots \quad (19)$$

- In Eulerian formulation, the 1-loop approximation gives DIA:

$$\Sigma_{\alpha\beta} \approx \Sigma_{\alpha\beta}^1 = (V_{\alpha A\Gamma} + V_{\alpha\Gamma A})(V_{\beta B\Delta} + V_{\beta\Delta B})G_{AB}F_{\Gamma\Delta} \quad (20)$$

$$\Phi_{\alpha\beta} \approx \Phi_{\alpha\beta}^1 = V_{\alpha A\Gamma}(V_{\beta B\Delta} + V_{\beta\Delta B})F_{AB}F_{\Gamma\Delta} \quad (21)$$

- Problem: In Eulerian formulation, IR divergences arise from sweeping interactions
 - 1987: Belinicher-L'vov quasi-Lagrangian transformation
 - 1995-2001: L'vov and Procaccia go beyond the LHDIA theory

Quasi-Lagrangian transformation. I. Definition

- Let $u_\alpha(\mathbf{x}, t)$ be the Eulerian velocity field, and let $\rho_\alpha(\mathbf{x}_0, t_0|t)$ be the position of the unique fluid particle initiated at (\mathbf{x}_0, t_0) at time t relative to its initial position at time t_0 .
- First, we introduce $v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t)$ as

$$\begin{aligned}\rho_\alpha(\mathbf{x}_0, t_0|t) &= \int_{t_0}^t d\tau u_\alpha(\mathbf{x}_0 + \rho(\mathbf{x}_0, t_0|\tau), \tau) \\ v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) &= u_\alpha(\mathbf{x} + \rho(\mathbf{x}_0, t_0|t), t).\end{aligned}\tag{22}$$

- Then, we subtract the velocity of the fluid particle uniformly:

$$\begin{aligned}w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) &= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - \frac{\partial}{\partial t} \rho_\alpha(\mathbf{x}_0, t_0|t) \\ &= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}_0, t) \\ &= u_\alpha(\mathbf{x} + \rho(\mathbf{x}_0, t_0|t), t) - u_\alpha(\mathbf{x}_0 + \rho(\mathbf{x}_0, t_0|t), t).\end{aligned}\tag{23}$$

Quasi-Lagrangian transformation. II. Navier-Stokes equations

Let $w_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, t)$ be defined as

$$W_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, \mathbf{x}', t) \equiv w_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, t) - w_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}', t) \quad (24)$$

$$= v_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, t) - v_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}', t). \quad (25)$$

Differentiating with respect to time gives an equation of the form

$$\frac{\partial W_\alpha}{\partial t} + \mathcal{V}_{\alpha\beta\gamma} W_\beta W_\gamma = \nu(\nabla_{\mathbf{x}}^2 + \nabla_{\mathbf{x}'}^2)W_\alpha + F_\alpha, \quad (26)$$

where $\mathcal{V}_{\alpha\beta\gamma}$ is a bilinear integrodifferential operator of the form

$$\mathcal{V}_{\alpha\beta\gamma} W_\beta W_\gamma \equiv \iint d\mathbf{X}_\beta d\mathbf{X}_\gamma V_{\alpha\beta\gamma}(\mathbf{x}_0 | \mathbf{X}_\alpha, \mathbf{X}_\beta, \mathbf{X}_\gamma) W_\beta(\mathbf{X}_\beta) W_\gamma(\mathbf{X}_\gamma). \quad (27)$$

All the terms, and especially the nonlinear term, are written in terms of velocity differences!

Quasi-Lagrangian transformation. III. The theory

- Consider the Dyson-Wyld equations with

$$F_{\alpha\beta}(\mathbf{x}_0, t_0 | \mathbf{x}_1, \mathbf{x}_2, t) = \langle w_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}_1, t) w_\beta(\mathbf{x}_0, t_0 | \mathbf{x}_2, t) \rangle \quad (28)$$

$$G_{\alpha\beta}(\mathbf{x}_0, t_0 | \mathbf{x}_1, \mathbf{x}_2, t) = \left\langle \frac{\delta w_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}_1, t)}{\delta f_\beta(\mathbf{x}_0, t_0 | \mathbf{x}_2, t)} \right\rangle \quad (29)$$

- Main results of L'vov and Procaccia theory.

- Individual diagrams in $\Sigma_{\alpha\beta}$ and $\Phi_{\alpha\beta}$ converge when $\ell_0 \rightarrow \infty$ and $\eta \rightarrow 0$.
- Thus, to n-loop order approximation we obtain K41 prediction $\zeta_n = n/3$.
- This is a generalization of Kraichnan's LHDIA theory.
- Diagram locality and rigidity \implies Fusion Rules \implies Anomalous energy sink.
- Intermittency emerges via a multi-interaction effect involving all diagrams
- Scheme for perturbative calculation of scaling exponents ζ_n

Outline of my argument

- The L'vov-Procaccia theory aims to weaken the assumption of self-similarity used by Frisch (H2) while tolerating the other two assumptions: (H1) and (H3)
- A homogeneity assumption stronger than the assumption of local homogeneity, as envisioned by Frisch is required for
 - the elimination of the sweeping interactions
 - the derivation of the 4/5-law
- The quasi-Lagrangian formulation to eliminate the sweeping interactions uses an even stronger homogeneity assumption which involves many-time correlations instead of one-time correlations.
- Local homogeneity is in fact a consistent framework provided that the sweeping interactions can be eliminated in a more rigorous manner.

Definitions of local homogeneity. I.

- The random velocity field \mathbf{u} , is a member of the homogeneity class $\mathcal{H}_m(\mathcal{A})$ where $\mathcal{A} \subseteq \mathbb{R}^d$ a region in \mathbb{R}^d , if and only if $\forall n \in \mathbb{N}^*, \forall \mathbf{x}_l, \mathbf{y}_k, \mathbf{y}'_k \in \mathcal{A}$ we have

$$\left(\sum_{l=1}^m \partial_{\alpha_l, \mathbf{x}_l} + \sum_{k=1}^n (\partial_{\beta_l, \mathbf{y}_l} + \partial_{\beta_l, \mathbf{y}'_l}) \right) \left\langle \left[\prod_{l=1}^m u_{\alpha_l}(\mathbf{x}_l, t) \right] \left[\prod_{k=1}^n w_{\beta_k}(\mathbf{y}_k, \mathbf{y}'_k, t) \right] \right\rangle = 0$$

- The random velocity field \mathbf{u} is a member of the homogeneity class $\mathcal{H}_m^*(\mathcal{A})$ where $\mathcal{A} \subseteq \mathbb{R}^d$ a region in \mathbb{R}^d , if and only if $\forall n \in \mathbb{N}^*, \forall \mathbf{x}_l, \mathbf{y}_k, \mathbf{y}'_k \in \mathcal{A}$ we have

$$\left(\sum_{l=1}^m \partial_{\alpha_l, \mathbf{x}_l} + \sum_{k=1}^n (\partial_{\beta_l, \mathbf{y}_l} + \partial_{\beta_l, \mathbf{y}'_l}) \right) \left\langle \left[\prod_{l=1}^m u_{\alpha}(\mathbf{x}_l, t_l) \right] \left[\prod_{k=1}^n w_{\beta_k}(\mathbf{y}_k, \mathbf{y}'_k, t) \right] \right\rangle = 0$$

- We also write $\mathcal{H}_m \equiv \mathcal{H}_m(\mathbb{R}^d)$ and $\mathcal{H}_m^* \equiv \mathcal{H}_m^*(\mathbb{R}^d)$ and define

$$\mathcal{H}_{\omega}(\mathcal{A}) = \bigcap_{k \in \mathbb{N}} \mathcal{H}_k(\mathcal{A}) \quad \text{and} \quad \mathcal{H}_{\omega}^*(\mathcal{A}) = \bigcap_{k \in \mathbb{N}} \mathcal{H}_k^*(\mathcal{A}). \quad (30)$$

Definitions of local homogeneity. II.

- The homogeneity classes are hierarchically ordered, according to the following relations

$$\mathcal{H}_\omega(\mathcal{A}) \subseteq \mathcal{H}_k(\mathcal{A}), \quad \forall k \in \mathbb{N}, \quad (31)$$

$$\mathcal{H}_\omega^*(\mathcal{A}) \subseteq \mathcal{H}_k^*(\mathcal{A}), \quad \forall k \in \mathbb{N}, \quad (32)$$

$$\mathcal{H}_a(\mathcal{A}) \subseteq \mathcal{H}_b(\mathcal{A}) \wedge \mathcal{H}_a^*(\mathcal{A}) \subseteq \mathcal{H}_b^*(\mathcal{A}), \quad \forall a, b \in \mathbb{N} : a > b, \quad (33)$$

$$\mathcal{H}_a(\mathcal{A}) \subseteq \mathcal{H}_a^*(\mathcal{A}), \quad \forall a \in \mathbb{N}. \quad (34)$$

- Local homogeneity, in the sense of Frisch: $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$.
- The homogeneity condition sufficient to
 - eliminate the sweeping interactions over the domain \mathcal{A} : $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.
 - prove the 4/5-law over the domain \mathcal{A} : $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.
 - employ the Belinicher-L'vov quasi-Lagrangian transformation: $\mathbf{u} \in \mathcal{H}_\omega^*$.

Balance equations and sweeping. I.

- The clearest way to understand the sweeping interactions is by employing the balance equations introduced by L'vov and Procaccia (1996).
- The Navier-Stokes equations, where the pressure term has been eliminated, read

$$\frac{\partial u_\alpha}{\partial t} + \mathcal{P}_{\alpha\beta} \partial_\gamma (u_\beta u_\gamma) = \nu \nabla^2 u_\alpha + \mathcal{P}_{\alpha\beta} f_\beta, \quad (35)$$

where $\mathcal{P}_{\alpha\beta} = \delta_{\alpha\beta} - \partial_\alpha \partial_\beta \nabla^{-2}$ is the projection operator

- The Eulerian generalized structure function is defined as

$$F_n^{\alpha_1 \alpha_2 \dots \alpha_n} (\{\mathbf{x}, \mathbf{x}'\}_n, t) = \left\langle \left[\prod_{k=1}^n w_{\alpha_k} (\mathbf{x}_k, \mathbf{x}'_k, t) \right] \right\rangle, \quad (36)$$

where $\{\mathbf{x}, \mathbf{x}'\}_n$ is shorthand for a list of n position vectors.

Balance equations and sweeping. II.

- The balance equations are obtained by differentiating the definition of F_n with respect to time t and substituting the Navier-Stokes equations:

$$\frac{\partial F_n}{\partial t} + D_n = \nu J_n + Q_n, \quad (37)$$

where D_n represents the contributions from the nonlinear term and

$$Q_n^{\alpha_1 \alpha_2 \dots \alpha_n}(\{\mathbf{X}\}_n, t) = \sum_{k=1}^n \left\langle \left[\prod_{l=1, l \neq k}^n w_{\alpha_l}(\mathbf{x}_l, \mathbf{x}'_l, t) \right] \mathcal{P}_{\alpha_k \beta} (f_\beta(\mathbf{x}_k, t) - f_\beta(\mathbf{x}'_k, t)) \right\rangle. \quad (38)$$

$$J_n^{\alpha_1 \alpha_2 \dots \alpha_n}(\{\mathbf{X}\}_n, t) = \mathcal{D}_n F_n^{\alpha_1 \alpha_2 \dots \alpha_n}(\{\mathbf{X}\}_n, t) \quad (39)$$

$$= \sum_{k=1}^n (\nabla_{\mathbf{x}_k}^2 + \nabla_{\mathbf{x}'_k}^2) F_n^{\alpha_1 \alpha_2 \dots \alpha_n}(\{\mathbf{X}\}_n, t), \quad (40)$$

Balance equations and sweeping. III.

- L'vov and Procaccia (1996) showed that the contribution of the nonlinear term D_n can be rewritten as $D_n = \mathcal{O}_n F_{n+1} + I_n$ where \mathcal{O}_n is a linear integrodifferential operator, and I_n is given by

$$I_n^{\alpha_1 \alpha_2 \cdots \alpha_n}(\{\mathbf{X}\}_n, t) = \sum_{k=1}^n (\partial_{\beta, \mathbf{x}_k} + \partial_{\beta, \mathbf{x}'_k}) \left\langle \mathcal{U}_\beta(\{\mathbf{X}\}_n, t) \left[\prod_{l=1}^n w_{\alpha_l}(\mathbf{X}_l, t) \right] \right\rangle, \quad (41)$$

where $\mathcal{U}_\beta(\{\mathbf{X}\}_n, t)$ is defined as

$$\mathcal{U}_\alpha(\{\mathbf{X}\}_n, t) = \frac{1}{2n} \sum_{k=1}^n (u_\alpha(\mathbf{x}_k, t) + u_\alpha(\mathbf{x}'_k, t)). \quad (42)$$

- The second term, I_n , represents exclusively the effect of the sweeping interactions.
- To set $I_n = 0$ we need the homogeneity assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.

Dropping sweeping: The 4/5-law proof

- If we assume $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ and set $I_2 = 0$, then one can calculate ζ_3 is from the solvability condition of the homogeneous equation $\mathcal{O}_2 F_3 = 0$, as shown by L'vov and Procaccia (1996).
- Use the conservation of energy to show that

$$\begin{aligned}\mathcal{O}_2 F_3(\mathbf{x}_1, \mathbf{x}'_1, \mathbf{x}_2, \mathbf{x}'_2) &= \frac{1}{2} \frac{d[S_3(r_{12}) - S_3(r_{12'})]}{dr_1} + \frac{1}{2} \frac{d[S_3(r_{1'2'}) - S_3(r_{1'2})]}{dr_{1'}} \\ &= A[r_{12}^{\zeta_3-1} - r_{12'}^{\zeta_3-1} + r_{1'2'}^{\zeta_3-1} - r_{1'2}^{\zeta_3-1}].\end{aligned}\tag{43}$$

where $r_{12} = \|\mathbf{x}_1 - \mathbf{x}_2\|$, etc.

- It follows that $D_n \approx \mathcal{O}_2 F_3 = 0 \iff \zeta_3 = 1$
- Dropping I_2 cannot be justified under $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$, i.e. local homogeneity in the sense of Frisch.

Dropping sweeping: The multifractal formalism.

- The homogeneous equations $\mathcal{O}_n F_{n+1} = 0$ are invariant with respect to the following group of transformations

$$\mathbf{r} \mapsto \lambda \mathbf{r}, \quad F_n \mapsto \lambda^{nh + \mathcal{Z}(h)} F_n. \quad (44)$$

- Thus, in an inertial range, solutions $F_{n,h}$ that satisfy the self-similarity property

$$F_{n,h}(\{\lambda \mathbf{x}_k, \lambda \mathbf{x}'_k\}_{k=1}^n, t) = \lambda^{nh + \mathcal{Z}(h)} F_{n,h}(\{\mathbf{x}_k, \mathbf{x}'_k\}_{k=1}^n, t), \quad (45)$$

are admissible.

- The correct solution is the linear combination of these solutions, given by

$$F_n = \int d\mu(h) F_{n,h}. \quad (46)$$

- This conclusion also needs the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.

The Bottom Line

If we assume $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$,
then we can simply “exterminate” the sweeping term,
with no further worries.

Quasi-Lagrangian to Eulerian. I. The claim

- Consider the definitions

$$F_n^{\alpha_1 \alpha_2 \cdots \alpha_n}(\{\mathbf{x}, \mathbf{x}'\}_n, t) = \left\langle \left[\prod_{k=1}^n w_{\alpha_k}(\mathbf{x}_k, \mathbf{x}'_k, t) \right] \right\rangle, \quad (47)$$

$$\mathcal{F}_n^{\alpha_1 \alpha_2 \cdots \alpha_n}(\mathbf{x}_0, t_0 | \{\mathbf{x}, \mathbf{x}'\}_n, t) = \left\langle \left[\prod_{k=1}^n W_{\alpha_k}(\mathbf{x}_0, t_0 | \mathbf{x}_k, \mathbf{x}'_k, t) \right] \right\rangle. \quad (48)$$

- The claim of L'vov and Procaccia was that it can be shown that

$$\mathcal{F}_n(\mathbf{x}_0, t_0 | \{\mathbf{x}, \mathbf{x}'\}_n, t) = F_n(\{\mathbf{x}, \mathbf{x}'\}_n, t), \quad \forall n \in \mathbb{N}^* \quad (49)$$

- The claim can be rewritten equivalently as

$$W_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, \mathbf{x}', t) \stackrel{\mathbf{x}, \mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t). \quad (50)$$

Quasi-Lagrangian to Eulerian. II. Proofs

🔴 Proof was re-examined recently by Gkioulekas.

🟢 The claim holds if and only if

$$W_\alpha(\mathbf{x}_0, t_0 + \Delta t | \mathbf{x}, \mathbf{x}', t) \stackrel{\mathbf{x}, \mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, \mathbf{x}', t), \quad \forall \Delta t \in \mathbb{R} - \{0\} \quad (51)$$

🟢 If $u \in \mathcal{H}_\omega^*$, and u_α is incompressible, then

$$W_\alpha(\mathbf{x}_0, t_0 + \Delta t | \mathbf{x}, \mathbf{x}', t) \stackrel{\mathbf{x}, \mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0 | \mathbf{x}, \mathbf{x}', t). \quad (52)$$

🔴 $u \in \mathcal{H}_\omega^*$ is a sufficient but perhaps not necessary assumption

🔴 However, the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ is **not** sufficient

🔴 The artifact introduced by the quasi-Lagrangian formulation is that the turbulent velocity field is being perceived from the viewpoint of an arbitrary fluid particle whose own motion is also stochastic.

Quasi-Lagrangian to Eulerian. III. Another proof

- Introduce the conditional correlation tensor defined as

$$\mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y} | \{\mathbf{X}\}_n, t) = \left\langle \prod_{k=1}^n W_{\alpha_k}(\mathbf{x}_0, t_0 | \mathbf{x}_k, \mathbf{x}'_k, t) \middle| \rho(\mathbf{x}_0, t_0 | t) = \mathbf{y} \right\rangle \quad (53)$$

$$= \left\langle \prod_{k=1}^n w_{\alpha_k}(\mathbf{x}_k + \mathbf{y}, \mathbf{x}'_k + \mathbf{y}, t) \middle| \rho(\mathbf{x}_0, t_0 | t) = \mathbf{y} \right\rangle \quad (54)$$

- The random velocity field \mathbf{u} is a member of the homogeneity class \mathcal{H}_0^c , if and only if $\forall \mathbf{y} \in \mathbb{R}^d, \forall \mathbf{x}_k, \mathbf{x}'_k \in \mathbb{R}^d$ we have

$$\sum_{k=1}^n (\partial_{\beta_k, \mathbf{y}_k} + \partial_{\beta, \mathbf{y}'_k}) \mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y} | \{\mathbf{X}\}_n, t) = 0, \forall n \in \mathbb{N}, n > 1 \quad (55)$$

- The condition $\mathbf{u} \in \mathcal{H}_0^c$ implies **The claim**

Elimination of the sweeping interactions. I.

- One may conjecture that the sweeping interactions act as a large-scale forcing term whose effect is forgotten in the inertial range.
- It is possible to use the theoretical work based on the quasi-Lagrangian transformation in a way that requires only the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.
 - The quasi-Lagrangian formulation modifies the Navier-Stokes equations by redefining the material derivative.
 - The modified equation remains mathematically equivalent to the Navier-Stokes equation if the velocity field is reinterpreted from an Eulerian field into a quasi-Lagrangian field.
 - This reinterpretation necessitates the stronger assumption $\mathbf{u} \in \mathcal{H}_\omega^*$ to enable a return back to the Eulerian representation.
 - If we accept the hypothesis that the sweeping interactions act as a large-scale forcing, we can just modify the equation of motion in precisely the same way without interpreting the velocity field as quasi-Lagrangian, but rather as Eulerian.

Elimination of the sweeping interactions. II.

- To prove that the sweeping interactions act as a large-scale forcing it is sufficient to calculate the scaling exponent Δ_n associated with the ratio

$$\frac{I_n(R\{\mathbf{X}\}_n)}{(\mathcal{O}_n F_{n+1})(R\{\mathbf{X}\}_n)} \sim \left(\frac{R}{\ell_0}\right)^{\Delta_n}, \quad (56)$$

- Then, provided that one starts with the assumption $\mathbf{u} \in \mathcal{H}_0$, proving $\Delta_n > 0$ is also a proof that $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ which is sufficient to eliminate the sweeping interactions.
- If we assume that the generalized structure functions $F_n(R\{\mathbf{X}\}_n)$ satisfy the fusion rules, then the scaling exponent of $\mathcal{O}_n F_{n+1}(R\{\mathbf{X}\}_n)$ is $\zeta_{n+1} - 1$ and it follows that $\Delta_n = \lambda_n - (\zeta_{n+1} - 1)$.
- The problem of calculating the scaling exponents λ_n needs to be investigated primarily with numerical simulations and the analysis of experimental data.

Elimination of the sweeping interactions. III.

- Commit the following crimes against reality:
 - Assume that the velocity field $u_\alpha(\mathbf{x}, t)$ can be modeled as a random gaussian delta-correlated (in time) stochastic field acting at large scales.
 - Assume that the velocity field $u_\alpha(\mathbf{x}, t)$ has an effect on the velocity differences $w_\alpha(\mathbf{x}, \mathbf{x}', t)$ via the sweeping interactions
 - Disregard that $u_\alpha(\mathbf{x}, t)$ and $w_\alpha(\mathbf{x}, \mathbf{x}', t)$ are obviously constrained by the definition of $w_\alpha(\mathbf{x}, \mathbf{x}', t)$.

- Using the multifractal formulation, the contribution that supports the Holder exponent h gives $\zeta_n = nh + \mathcal{Z}(h)$, which gives the following evaluation:

$$\Delta_n(h) = -2h + \lambda + 2 \quad (57)$$

- The window for positive scaling exponents Δ_n covers the entire range $h \in (0, 1)$ of local scaling exponents.
- The real challenge is to determine what happens in reality

Conclusion

- Analytical theories are an extension of the Frisch reformulation of K41
- The main stumbling block is the elimination of the sweeping interactions
- Lagrangian methods do not prove that the sweeping interactions are negligible in the inertial range.
- The open question: prove that sweeping is negligible in the inertial range.