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Review of "Mathematics of two-dimensional turbulence" by S.Kuksin and A. Shirikyan

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Two-dimensional turbulence has been a very active and intriguing area of research over the last five decades, since the publication of Robert Kraichnan's seminal paper [1] postulating the dual cascade theory. Some reviews are given in [2–4]. The original motivation for studying two-dimensional turbulence was the belief that it would prove to be an easier problem than three-dimensional turbulence and that mathematical techniques developed for the two-dimensional problem would then be used for the three-dimensional problem. It was also believed that two-dimensional turbulence theory could explain flows in very thin domains, such as the large-scale phenomenology of turbulence in planetary atmospheres.

In general terms, theoretical studies of turbulence use a wide range of strategies, including phenomenological theories, analytic theories that depend on hypotheses established experimentally or via numerical simulations, and mathematically rigorous theorems on the Navier-Stokes equations. With a phenomenological approach, one makes a series of hypotheses based on experimental evidence and physical intuition from which conclusions can be drawn about universal features of turbulence. The Kolmogorov theory of three-dimensional turbulence [5–7] and Kraichnan's theory of two-dimensional turbulence [1] typify this approach. In both cases, minimal contact is made with the Navier-Stokes equations. Nevertheless, a lot of successful numerical and experimental work has been motivated by phenomenological theories. With the more rigorous strategy of formulating analytical theories of turbulence, one uses the governing equations as a point of departure to formulate perturbative closure models or nonperturbative strategies. The mathematical foundation for the most advanced of these theories is the Martin–Siggia–Rose formalism [8, 9] (hereafter MSR formalism), with reviews given in [10, 11]. These theories cannot be completely rigorous on their own since the use of the MSR formalism entails certain assumptions: (a) existence and uniqueness of a deterministic solution for the velocity field given a choice of deterministic forcing field; (b) the assumption that the system was initialized at time $t \to -\infty$ and has already converged to statistical steady state. Furthermore, some lack of rigor stems from the dependence on Feynman path integrals. Finally, to connect theoretical predictions about ensemble averages with numerical simulations and experiments requires the additional assumption of ergodicity. Having made these assumptions, the payoff is that it is possible to make considerable inroads toward clarifying, explaining, and predicting the phenomenological behavior of turbulence for the two-dimensional as well as the three-dimensional case. Finally, another strategy is to prove mathematically rigorous theorems about the Navier-Stokes equations using functional analysis and dynamical systems theory techniques. This approach, pioneered by distinguished mathematicians like Leray, Foias, Temam, and many others, has successfully yielded solid results. The price is that it is too difficult to venture as far as one can go using less rigorous strategies that incorporate hypotheses evidenced by experiment or numerical simulations.

Ultimately, all of the above strategies have strengths and weaknesses that complement one another. A curious irony of two-dimensional turbulence research is that whereas the phenomenology of two-dimensional turbulence is richer and poses many more challenges than that of three-dimensional turbulence, two-dimensional turbulence has turned out to be far more amenable to the pure mathematician's toolbox. The current book under review by Kuksin and Shirikyan surveys recent developments in the mathematical theory of the two-dimensional Navier-Stokes equations that are, with no exaggeration, quite breathtaking. The authors use the randomly forced two-dimensional Navier-Stokes equation with a regular Laplacian dissipation term at small scales as their ansatz. Three types of random forcing are considered: (a) kick forcing, consisting of, equispaced in time, delta function spikes with random amplitudes; (b) white noise, i.e. random Gaussian delta-correlated in time forcing, commonly used in MSR theories; (c) compound Poisson processes, which are random kick forces where both the amplitude and the temporal separation between the delta function spikes are randomized.

The authors begin in Chapter 1 with a very terse yet comprehensive review of essential concepts, needed for the proofs of the main results, from the areas of function spaces, measure theory, and Markov random dynamical systems. A solid graduate education in functional analysis is necessary to follow the chapter, but the authors provide citations to many other books that explain underlying concepts in more detail. Chapter 2 begins with a review of the classical Leray results on the existence, uniqueness, and regularity of solutions for the case of the two-dimensional Navier-Stokes equations with deterministic forcing. For the case of stochastic forcing, a series of important general results are proved that culminate in proving the existence of at least one stationary measure. In physical terms, a stationary measure describes the steady-state solution to the randomly forced Navier-Stokes equations when a dynamical balance has been established between forcing and dissipation and the ensemble averages for all observables become constant with respect to time.

The argument continues in Chapter 3 with an array of results establishing the uniqueness of the stationary measure as well as the property of exponential mixing, both for periodic flows on an infinite domain and for flows on a bounded domain for various random forcing configurations. In physical terms, the property of exponential mixing means that regardless of the initial condition, the randomly forced two-

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dimensional Navier-Stokes equation will statistically converge to the steady-state solution at an exponential rate. This convergence has been established for both the velocity field itself and for relevant observables, dependent on the velocity field, such as the energy spectrum. The authors also establish that if the random force is homogeneous, then the velocity field at steady state will also be homogeneous. The chapter concludes with a literature review as well as a physical summary of the main results.

Chapter 4 establishes ergodic theorems as well as some interesting limiting theorems. In particular, the authors establish that the time average of observables, dependent on the velocity field, quickly converge to the ensemble average as one extends the time interval over which the time average is taken. The authors also establish a central limit theorem that shows that the velocity probability distribution is close to Gaussian, in agreement with experiments and numerical simulations (see [3] for a review). Furthermore, the authors prove that the statistical properties of the velocity field at steady state will vary continuously as one varies the statistical parameters of random forcing. Finally, the authors prove that the steady state solution of a system forced by random kicks will converge to the steady-state of the system forced by white noise if the time gap between kicks is shrunk by a factor ε , taking the limit $\varepsilon \to 0^+$, as long as the amplitude of the kicks is also decreased by a factor of $\sqrt{\varepsilon}$.

Having established the existence and uniqueness of a stationary measure for the case of finite viscosity, in Chapter 5, the authors investigate the stationary measure under the limit $\nu \to 0^+$ of viscosity approaching zero. For technical reasons, instead of using a continuous limit it is necessary to work with the discrete limit $\nu_k \to 0^+$ with $k \in \mathbb{N}$ for some chosen viscosity sequence that converges to zero. The authors prove that for every such viscosity sequence, the corresponding stationary measures have a nontrivial limit, as long as forcing is moderated by an $\sqrt{\nu_k}$ factor. It remains an open question whether all possible sequences such that $\nu_k \to 0^+$ with $k \in \mathbb{N}$ lead to a unique limit for the stationary measure. However, it is proved that all stationary measures obtained from any viscosity sequence limit to zero will satisfy certain universal properties from which a phenomenology of two-dimensional turbulence can be deduced. From these universal properties, if we introduce the assumption that the energy spectrum follows a power law, downscale from the forcing range, it is predicted that the energy spectrum will scale as k^{-a} with $a \geq 5$, where k is the wavenumber. The authors also identify an unproven conjecture that would rigorously imply $a = 5$.

Finally, in Chapter 6 the authors outline without proof a number of incomplete results whose development is the subject of current active research. A special highlight is a result that shows that the stationary measure of the three-dimensional Navier-Stokes equations, defined in a quasi-two-dimensional domain in which the vertical direction is very thin, and also randomly forced by random kicks, will converge to the corresponding stationary measure of the two-dimensional Navier-Stokes equations. However, it remains an open question whether this result can be extended for the case of white noise forcing.

In light of the foregoing discussion, the significance of these results is clear. In every well-known theory of two-dimensional and three-dimensional turbulence, one takes for granted the existence and uniqueness of the statistical steady-state solution, that a forced dissipative system will always converge to the steady-state solution, that the ensemble average can be exchanged with a time average, and that the discrete kick forcing typically used by numerical simulations, where time is discretized, properly approximates the case of continuous white noise forcing. These are all assumptions that underlie every theoretical effort to understand the phenomenology of turbulence, but they are also assumptions that are not easy to prove. It is very reassuring to see that during the last decade, at least for the case of two-dimensional Navier-Stokes turbulence, all of these assumptions have been proved rigorously. This is a major achievement, and the authors are leading experts who have played a key role in the development of many of these results.

It is also worth commenting on the phenomenology of the k^{-5} energy spectrum predicted in Chapter 5. This is not an entirely new result. It was first proposed by Tran and Shepherd [12] and Tran and Bowman [13], who predicted a k^{-5} spectrum downscale from the forcing range and a k^{-3} spectrum upscale from the forcing range. This phenomenology is inconsistent with Kraichnan's theory [1] of a downscale enstrophy cascade with k^{-3} scaling and an upscale inverse energy cascade with $k^{-5/3}$ scaling. As was explained by Tran and Shepherd [12], the Kraichnan cascades will fail to materialize in the absence of a dissipation term at large scales in a bounded domain flow. On an infinite domain, energy can simply cascade forever to larger and larger scales, and enstrophy can cascade to small scales and be dissipated by the small-scale diffusion term. However, on a finite domain, if there is no mechanism to dissipate the upscale energy cascade before it hits the largest possible scales, then the cascade configuration will collapse and transition to the conjectured joint k^{-3} and k^{-5} configuration. The results by the authors vindicate the work of Tran et al. [12–14] by eliminating unproven assumptions that they made in order to establish their predictions. As important as this development is, the greater challenge of understanding the robustness of the Kraichnan cascades remains an open question.

Finally, I should like to make some comments about the book itself. It has been written primarily for an audience of pure mathematicians who wish to familiarize themselves with this research area so they can make further contributions. The writing style is very concise; however, the authors provide complete proofs for almost all of their results. An extensive array of very general preliminary results needs to be established before the main results can be proved. The preliminary results are useful in and of themselves and can be used for the future investigation of systems other than the randomly forced two-dimensional Navier-Stokes equations. The authors mention the complex Ginzburg–Landau equation as a possible area of exploration. An extensive bibliography of more than 200 references is given, and I very much appreciate the reverse citation system in which for each item in the bibliography the authors give the page numbers where the given item is cited in the text. Much heavy notation is used throughout the book; however, the authors provide a very useful summary of notation conventions at the end. Last but not least, in Chapters 3, 4, and 5 where the main results are discussed, the authors conclude each chapter with a very clear discussion of the physical implications of their results. These sections are essential to making this work accessible to a more applied audience. A very detailed literature review is also given at the end of every chapter for those who wish to consult the original research papers.

In summary, this is an excellent book presenting and proving a body of results that are of fundamental importance in the development of theories of two-dimensional turbulence. For pure mathematicians, there is much to be learned from the techniques used to prove the theorems that can be applied to a wider range of problems. For applied mathematicians, it is certainly useful to have some understanding of what has been proved rigorously for two-dimensional Navier-Stokes. The results themselves are very interesting and their physical implications are clearly explained. While this is not a book for the faint of heart, I find it an excellent addition to my library and strongly recommend it to everyone engaged in theoretical research on two-dimensional turbulence.

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