Review of "Turbulence, coherent structures, dynamical systems and symmetry", by P. Holmes, J.L. Lumley, G. Berkooz, and C.W. Rowley

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Whenever I embark upon reading a challenging book, I suffer from the irresistible habit of starting out by flipping to the last couple of pages to see how it all turns out at the end. I was not at all disappointed to read a delightful section 13.5 in which the authors offer their final thoughts on how the work detailed in the book can lead to an elucidation of some of the open questions of current turbulence research, and a vivid sense of the computational challenges involved in formulating and using their proposed models. The section concludes by citing Freeman Dyson [1], who noted, quoting the book, "the ironic fact that some of the world's greatest physicists, Einstein and Oppenheimer in particular, spent fruitless years searching for unified theories, overarching laws, and equations that model everything, while ignoring solutions of particular problems". This quote captures the spirit of the book, where the authors undertake to review an extensive body of research aimed towards a deeper understanding of the large-scale coherent structures of turbulence, a problem with immediate relevance to practical engineering applications.

From a broader point of view, turbulence is a very fascinating problem, intersecting across many disciplines. One aspect of turbulence that has interested physicists and mathematicians alike is the existence of universal principles governing the behavior of turbulent flows at small scales, away from boundaries. The study of universal features has proven very amenable to theoretical studies that use the Navier-Stokes equations as a point of departure, with or without a modest set of symmetry-based hypotheses. However, precisely because of their universal nature, the small-scale structures of turbulence cannot be easily controlled, and practical engineering applications are therefore limited. Of equal importance are the large-scale coherent structures of turbulence which emerge from the interaction of the Navier-Stokes equations with specific boundary conditions. As the boundary conditions vary from problem to problem, these coherent structures are not universal, but vary considerably across various flow configurations. This makes it virtually impossible to tackle them via a theoretical first-principles approach, and yet it is the understanding of coherent structures that is most relevant to engineering applications. A direct numerical simulation approach is also out of the question, with the achievement of realistic Reynolds numbers beyond the capabilities of our greatest supercomputers.

The authors review a relatively recent body of research that proposes a systematic methodology for studying the large-scale coherent structures of turbulence. The main idea is to reduce the Navier-Stokes equations into a low-order empirical dynamical model that can then be studied both numerically and via theoretical dynamical systems methods. The reduction occurs as follows. First, experimental data are used to define, via an empirical basis of eigenfunctions, a subspace of $L_2(\Omega)$ that captures most of the energy associated with the large-scale coherent structures. Here, $\Omega \subseteq \mathbb{R}^3$ represents the flow domain. Then the original Navier-Stokes equations are projected into the empirical subspace, leading to a low-order dynamical model for the coherent structures. These models are then further adjusted to account for the neglected effects of small-scale turbulence via stochastic terms. The authors make a strong case that a dynamical systems analysis of the attractor, bifurcations, etc. of the resulting low-order models captures the main features of turbulent coherent structures. Furthermore, these models could have some predictive power via extrapolation towards higher Reynolds numbers, beyond the range of the Reynolds numbers used by the experiments that defined the empirical subspace. It is worth emphasizing that, although the choice of subspace is entirely empirical, and therefore only as good as the information extracted from the underlying experimental data, it is the original Navier-Stokes equations themselves being projected into that subspace. Consequently, whereas this is not a first principles approach for the most part, it is also not entirely empirical either. As it turns out, the overall idea is very general, and the authors report that in addition to fluid turbulence, this approach is now being adopted by researchers in mechanical vibrations, laser dynamics, nonlinear optics, chemical processes, and even in studies of the mental activity of the brain.

The book is organized into four parts. The first part is concerned with the theory underlying the formulation of the low-order empirical models. The second part reviews elements of dynamical systems theory that will be used to analyze the proposed models. The third part focuses on analyzing the coherent structures associated with the boundary layer problem. The book concludes in part 4 with a broader review of the literature. More specifically, Chapter 1 does a good job of setting the stage by giving an overview of the book and introducing needed mathematical notation, with a charming amount of apologies for having the audacity to use mathematical notation that every scientist should know. Chapter 2 gives an overview of classical work on coherent structures, focusing on the mixing layer and the boundary layer for wall-bounded turbulence. Chapters 3 and 4 are the most general and interesting part of the book as they develop very clearly the mathematical theory for selecting an empirical subspace from experimental data and for projecting the governing equations into the chosen subspace. Chapter 5 develops the balanced proper orthogonal decomposition method, which is in many ways superior to the method outlined in Chapter 3. However, although it is well-presented and very interesting in and of itself, it seems

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to be limited mainly to linear systems, and it was not clear how useful it can be with respect to the Navier-Stokes equations. According to the authors, extending the balanced proper orthogonal decomposition method to nonlinear systems is a subject of current research. Chapters 6 and 7 go over the basics of dynamical systems analysis. As the authors themselves warn us in the introduction, the treatment can be perceived as somewhat superficial, by a mathematician's standards, but it is still detailed and informative. Chapter 8 illustrates an application of the overall method to the Kuramoto–Shivashinsky equation, and Chapter 9 gives a cursory overview of randomly perturbed dynamical systems, the study of which is made relevant by the use of random noise terms as models of the small-scale effects left out by the low-order empirical modeling. The book comes back full circle in Chapters 10 and 11 with a detailed re-examination of the problems introduced in Chapter 2 via the new techniques. A broader overview of the field is given in chapters 12 and 13.

The book commands an impressive bibliography of 396 references, making it an invaluable reference for any researcher who wishes to get into this area of research. The first four chapters are an excellent introduction to the proper orthogonal decomposition and the Galerkin projection and could be readily used to prep for a graduate course incorporating these topics. The dynamical systems section (Chapters 6 to 9) is also interesting, but it has been written from the standpoint of introducing this material to engineers, and mathematicians might find that somewhat aggravating. The treatment given in Chapters 10 and 11 is detailed and tedious, and one has to also read all of the referenced papers to get the whole picture. The authors have done the best job possible to present this work as carefully and clearly as possible; however, the reader will require an ample amount of time to assimilate every detail of the argument thoroughly. Chapters 12 and 13 give a comprehensive review of the literature which will be incredibly useful for all adventurous souls that decide to read all 396 references cited by the book. For the most part, except for Chapters 1 to 5, most of the book reads as a book-length review paper of the very extensive literature on a pioneering and very fertile area of research. The book does not make for easy reading, mostly due to the very challenging nature of the topic itself, and the need to read some of the cited literature along with the book for a more comprehensive in-depth understanding. Nevertheless, I strongly recommend the book to everyone who wishes to master this research area, as well as everyone who wants to learn more about the proper orthogonal decomposition method. The research program detailed by the authors is a very promising approach to the problem of the coherent structures of turbulence. Active turbulence researchers, especially researchers who are mainly focused in the engineering applications of turbulence, will find this book a welcome addition to their library.

[1] F. Dyson, *The scientist as a rebel*, New York Review of Books **XLII (9)** (1995), 31–33.