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Review of ‘Turbulence and Shell Models’ by Peter D. Ditlevsen

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The problem of hydrodynamic Navier-Stokes turbulence is considered to be one of the last open problems of classical mechanics, and also one of the most tantalizing ones. According to an often-told tale, Sir Haroce Lamb once said: “I am an old man now, and when I die and go to Heaven, there are two matters on which I hope enlightenment. One is quantum electrodynamics and the other is turbulence of fluids. About the former, I am really rather optimistic” [1].

Throughout the 20th century, a series of significant breakthroughs by Taylor, Kolmogorov, Batchelor, Kraichnan, and many others, have began to shed light on this old and important problem. Our current understanding is that the defining feature of turbulence is the existence of an energy cascade which transfers the energy of the velocity field from large scales, via local nonlinear interactions, towards small scales, where it is dissipated by the viscous term. The intermediate range of length scales, where this energy transfer occurs, is known as the *inertial range*. Kolmogorov originally proposed that, during this cascade process, the forcing mechanism via which energy is injected into the system is forgotten within the inertial range [2–4]. From this assumption, the energy spectrum and the scaling exponents ζ_n of the structure functions (defined as the statistical moments of Eulerian velocity differences) can be predicted. Kolmogorov’s theory was first confirmed experimentally in 1962 [5, 6], but during the same year, Kolmogorov [7] and Oboukhov [8] argued that the scaling exponents for the higher-order structure functions should deviate from his original prediction. These deviations are expected to be negligible for low-order structure functions and to increase for higher-order structure functions. Known as *intermittency corrections*, their existence was confirmed by experiments [9] and various theories have been proposed to calculate these scaling exponents from first principles [10–14].

This is where shell models come into play. Shell models are simple nonlinear systems of ordinary differential equations that are intended to merely model the scale by scale local transfer of energy by the energy cascade of three-dimensional turbulence. The main reason why these models have captured our imagination is because, not only do they correctly replicate the Kolmogorov energy spectrum, but they also reproduce the intermittency scaling exponents for the high-order structure functions, consistently with the values found by experiments [15]. Getting the same numbers from both the three-dimensional Navier-Stokes equations and a stripped-down system of ordinary differential equations is a hint that there is a very basic and fundamental principle underlying the phenomenology of intermittency, and one that has yet to be rigorously understood.

The book has been written with the intention that it may serve as a first introduction to both shell models and turbulence, from the standpoint of a researcher primarily interested in shell-model research. As such, it can be an excellent textbook for a special topics graduate course intended to bring young graduate students up to speed with both shell models and turbulence in a timely manner.

Chapter 1 begins with a brief and standard exposition of the Kolmogorov theory for three-dimensional Navier-Stokes turbulence. The author discusses the spectral form of the Navier-Stokes equation at some length, as it is the launching point for the formulation of shell models, and usual topics such as the closure problem, and $4/5$-law, intermittency, and the dissipation anomaly. An unusual feature here is a deep discussion of self-similarity. The proof of the $4/5$-law is the standard one by Landau. The author does not cover more modern proofs, or discuss local homogeneity, as they are not relevant to shell models.

Chapter 2 focuses on two-dimensional Navier-Stokes turbulence and atmospheric turbulence, which can also be modeled via shell models. The treatment is again standard and easy to understand, but for the case of two-dimensional turbulence it perpetuates two misconceptions: First, the author claims that “cascade of energy to small scales is impossible in two-dimensional turbulence”. In recent papers [16, 17], we argued that for finite Reynolds number, two-dimensional turbulence does have a subdominant downscale energy cascade associated with a small amount of energy cascading to small scales. Second, and more importantly, the author uses the Fjørtøft argument [18] (without naming it as such) to justify the direction of the dominant inverse energy cascade and the downscale enstrophy cascade. Merilee and Warn [19] first noted that Fjørtøft’s argument is not rigorous, this was reviewed in Ref. [20], and a simple alternative to the Fjørtøft argument was given by Gkioulekas and Tung [21]. In a future edition, the fallacy of the Fjørtøft argument would make for an excellent exercise for the end of the chapter. The remaining sections 2.2-2.6 give a detailed and very enjoyable introduction to atmospheric turbulence and its governing equations. Section 2.7 discusses the Nastrom-Gage spectrum of the atmosphere, and it is already out of date with respect to recent developments in the literature, but for the purpose of this book, it is fair enough that the author presents the essence of the problem without engaging into the details of the controversies involved.

With chapter 3 and chapter 4, the author introduces the basic theory of shell models, the raison d’être for the book. The Oboukhov, Gledger, GOY, and Sabra shell models are introduced in chapter 3, and their phenomenology is developed further in chapter 4. The discussion is excellent, to the point, and very informative. Chapter 5 is a gentle introduction to chaos theory. It is also very well written, the fundamen-

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tal concepts are clearly explained and illustrated numerically on the shell models that were introduced in chapter 3. Chapter 6 discusses helicity conservation and helicity cascades in three-dimensional Navier-Stokes turbulence. This is an exciting area of research in which the author has published significant research papers, and his exposition will be of great interest even to seasoned turbulence researchers in adjacent areas of specialization. The chapter concludes with the development of the generalized helical GOY model.

Chapter 7 tackles the most fascinating aspect of shell models: their ability to reproduce the intermittency scaling exponents of three-dimensional Navier-Stokes turbulence. The author begins with a very detailed presentation of K62 theory (Kolmogorov's 1962 lognormal model [7]) predicting a quadratic dependence of the scaling exponents ζ_n on n. This is a nice treat, as it is not easy to find a pedagogical exposition of the K62 theory elsewhere in the literature. The author discusses also the β-model and the multifractal model. With respect to the latter, I really appreciate the detailed proof that establishes the relationship between the scaling exponents ζ_n and the multifractal dimension spectrum $\mathscr{D}(h)$. I have not seen this proof before in previous textbooks. Unfortunately, the author's discussion of intermittency in shell models is too short and does not do justice to shell models. The most amazing feature of shell-models is that they reproduce the same numbers for the scaling exponents ζ_n as one may measure in three-dimensional turbulence experiments. This is, of course, known to most turbulence researchers, but since the intended audience for the book includes graduate students, this is a missed opportunity to amaze them.

The book concludes in chapter 8 with an introduction to equilibrium statistical mechanics. The author presents many important concepts that admit rigorous definitions in a simple, accessible, and careful language and provides detailed proofs concerning fundamental concepts such as the partition function, phase-space geometry, and statistical equilibrium. The main reason why one is interested in equilibrium statistical mechanics, in the context of shell model research, is because reasonable models of turbulence should not only reproduce the cascade solutions, but also the statistical equilibrium solutions. Although chapter 8 is only introductory, it will still be of considerable interest even to active turbulence researchers. The appendices cover various folklore-type technical details that will surely be appreciated by graduate students.

Overall, this is a nice, short, and very accessible introduction to shell models and the background knowledge needed to understand shell models and their relevance to turbulence research. As a textbook, supplemented with a few papers from the research literature at the instructor's discretion, it is ideal for a special topics graduate course. The best aspects of the book are: the well-thought-out selection of topics, the detailed explanation of the foundations, and, of course, the very accessible introduction to shell models. At the end of each chapter the book has exercises that instructors can assign to students as homework. All of the exercises are very interesting and range from easy to very challenging. The only shortcoming of the book is that it omits a few useful topics, such as the numerical methods for simulating shell models and extracting the scaling exponents ζ_n, as well as the various points raised above. On the other hand, that's a fair price to pay for keeping the book short, and it is sufficient that the book contains very detailed explanations of the fundamentals. Any graduate student finishing this book and having done the exercises will be able to follow the shell model research literature and get the rest of the story.

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