A theoretical study of the cascades of 3D, 2D, and QG turbulence

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This is to certify that I have examined this copy of a doctoral dissertation by

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Abstract

A theoretical study of the cascades of 3D, 2D, and QG turbulence

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Chair of the Supervisory Committee:
Professor and Chair Ka-Kit Tung
Applied Mathematics

This thesis is concerned with two foundational challenges in our understanding of turbulence: the elimination of sweeping interactions from theories of three-dimensional turbulence, and reconciling 2D models with atmospheric turbulence.

The problem of eliminating the sweeping interactions has been a thorn on all efforts aimed at developing analytical theories of three-dimensional hydrodynamic turbulence for many decades. Proposals for addressing this problem have been given by Kraichnan, Yakhot and Giles, and more recently the quasi-Lagrangian formulation of Belinicher and L'vov. In this thesis we will show that the quasi-Lagrangian formulation does not prove that the sweeping interactions can be neglected in the inertial range; it only introduces implicitly the assumption that the sweeping interactions are negligible.

We then turn our attention to the foundational issues of two-dimensional turbulence and the efforts by the atmospheric sciences community to reconcile two-dimensional turbulence and quasi-geostrophic turbulence with the Nastrom-Gage energy spectrum. We suggest that double cascades are a phenomenon shared by two-dimensional turbulence and atmospheric turbulence. However, in two dimensions, the downscale energy cascade is hidden. We conclude that a two-layer quasi-geostrophic model with asymmetric Ekman dissipation can produce a double cascade with an observable downscale energy cascade.

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DEDICATION

I would like to dedicate this dissertation to my high-school mathematics teacher Alexandros Pistofides (on the left bellow) from whom I have learned how to understand and how to teach mathematics.

Chapter 1

INTRODUCTION

Turbulence is a universal phenomenon that manifests itself from human length scales all the way to galactic distances. Turbulent flows can be readily found in simple things such as water flowing out of the kitchen faucet, the trail of vortices behind a fast car, and the flow of water through a river bed. On a planetary scale there is particular interest in atmospheric turbulence which is influenced by planetary differential rotation, solar heating, the planet's radius and gravitational field, and the nearly two dimensional nature of the atmosphere. The variation of these factors accounts for the differences between atmospheric turbulence on Earth and gas giant planets such as Jupiter. Inside the Earth, nuclear reactions at the Earth's core produce heat that excites the Earth's mantle to a hydromagnetic turbulent circulation that is responsible for the Earth's magnetic field. The stability of this circulation is maintained by the nonlinear interaction between turbulence and the electromagnetic field induced by it. Without a magnetic field our planet wouldn't deflect certain types of cosmic radiation, and that would render it uninhabitable. Beyond the planetary scales, our list of turbulent flows includes the motion of plasma inside stars, interstellar dust and plasma in galaxies and intergalaxy space, and the collective motion of the "gas" of stars that forms each galaxy. With such a broad range of phenomena can the problem of turbulence be tractable?

The first step for each case is to formulate a deterministic theory that describes the fluid and how it is governed by the forcing mechanisms, and if applicable, the feedback of the fluid's response to forcing. For most flows encountered in everyday life, the Navier-Stokes equations will suffice. In other situations, appropriate generalizations are available. For example, to study mixing and chemical reactions, we simply add passive and reactive scalars that are advected by the underlying velocity field. For atmospheric turbulence we would have to relinquish the incompressibility condition and include the thermodynamic properties of the gas into our theory (i.e. a temperature equation and a density equation). For hydromagnetic turbulence we need to include Maxwell's theory of electromagnetism. Finally, for astronomical flows it is necessary to formulate the governing equations with care from first principles.

The second step is to describe the solution of the governing equations, and there we're confronted

with the following challenges. First, there is no stable stationary solution. Even with the simplest case, the Navier-Stokes equations, when the flow is sufficiently intense the velocity field continuously fluctuates around its mean value. This means that we may only describe turbulence by treating the velocity field as a random variable. This leads to the second challenge: turbulence is an open dissipative system and therefore very far from thermodynamic equilibrium. The usual approach towards constructing a statistical theory, statistical mechanics, is applicable only to systems that fluctuate around thermodynamic equilibrium. With turbulence a theoretical attack has to address this problem before solid results can be within our reach.

From an experimental viewpoint, turbulent flows can be measured in experiments or with direct numerical simulations. Either option involves a trade-off. In an experiment we have access to very realistic turbulence but we have limited choice in what can be measured. In a direct numerical simulation we have full access to the entire data set and may therefore measure anything we want but it is very difficult to simulate realistic turbulence without making compromises. Experimental studies are often combined with the formulation of phenomenological theories that attempt to explain their results. With phenomenology the goal is to describe what turbulence does with experimental insight and modeling. This insight is essential and sometimes enough to lead us to a convincing explanation about why turbulence does what it does. The bulk of the accumulated literature on turbulence is phenomenological.

From a theoretical viewpoint, progress has been possible only by limiting the scope of the problem. As a result, the bulk of theoretical research focuses on the problem of three-dimensional incompressible hydrodynamic turbulence. A fortunate characteristic of turbulence is that its behavior away from boundaries and at sufficiently small scales is independent of the mechanism via which it is generated. This is a stability property which is commonly called universality. The study of universal features then allow us to obtain results that apply to turbulence at any scale as long as the underlying model is applicable. The question of why turbulence exhibits this universality is another very interesting problem that needs to be understood theoretically.

Recently, in a remarkable book, Frisch [94] reviewed fully developed three-dimensional incompressible hydrodynamic turbulence, and presented an axiomatic framework within which the prevailing phenomenological theories can be incorporated. He also reviewed, rather briefly, the analytical approaches that attempted to explain turbulence from first principles. However, shortly after the book's publication and during the 1995-2001 period, L'vov and Proccacia and their collaborators resolved the remaining obstacles that prevented analytical methods from being effective [16, 17, 174–176]. As a result, the universal self similarity properties of three-dimensional turbulence are now

well understood, and a mathematical framework for understanding universality is now available for the first time.

The work which will be presented in this thesis is based on the idea that the mathematical techniques that have been shown to be effective in understanding three-dimensional turbulence can be applied to elucidate open questions about two-dimensional turbulence. As a first step in that direction, we investigated the elimination of the sweeping interactions from analytical theories, which is discussed in chapter 3. During the course of our investigation, we discovered the cascade superposition principle, and decided to pursue that further and investigate its application to explaining the Nastrom-Gage energy spectrum of the atmosphere. Our work has been presented in two published papers [107, 108], three submitted papers [104–106], and summarized in an invited review paper [109].

An outline of the thesis is as follows. In chapter 2, we review recent breakthroughs in our understanding of three-dimensional turbulence. The problem of the elimination of the sweeping interactions is discussed in chapter 3. Chapter 4 reviews two-dimensional turbulence and the problem of the Nastrom-Gage energy spectrum. It also gives a self-contained conceptual outline of our argument. The technical details of the argument are given in the subsequent chapters. Specifically, in chapter 5 we prove the flux inequalities that constrain the downscale in upscale energy and enstrophy flux in two-dimensional turbulence and more generally in one-layer advection-diffusion models. Chapter 6 is devoted to the double cascade superposition principle of two-dimensional turbulence. In chapter 7 we identify the essential mathematical difference between two-dimensional turbulence and the two-layer quasi-geostrophic model which enables the latter to reproduce the spectrum.

Chapter 2

REVIEW OF 3D TURBULENCE

2.1 The road to Kolmogorov's theory

The history of fluid mechanics begins in ancient Greece with Archimedes, best known for his discovery of the principle of boyancy, who is essentially the father of hydrostatics. During the 15th century, fluid dynamics and aerodynamics captivated the interest of Leonardo da Vinci, who recorded numerous experimental observations, and noted for the first time the existence of coherent structures in turbulence.

The mathematical study of turbulence became possible only after the discovery of calculus by Newton and Leibnitz, and the formulation of classical mechanics by Newton in the 16th century. In 1755 Euler employed classical mechanics to derive an equation of motion for an idealized fluid. He introduced, what we now call, the Eulerian representation in which the state of the fluid is specified by a vector field $u_\alpha(x, y, z, t)$ that gives the fluid velocity at a point (x, y, z) at a given time t. Then he wrote the following equation:

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p \tag{2.1}$$

We use Greek indices to denote vector components, and employ the convention, introduced by Einstein, that repeated indices imply summation over all components. The left-hand side of the equation is the rate of change of the momentum of the infinitesimal fluid particle that happens to be in the given location at the given time. The right hand side is the force, which Euler wrote as the gradient of the pressure p applied on the fluid.

This equation is still interesting to mathematicians who want to analyze it as a partial differential equation. One aspect that makes it interesting to some physicists is that it describes a conservative dynamical system which is amenable to Hamiltonian formalism. As far as nature is concerned, however, Euler's equation is very unphysical. Feynman [89], has characteristically described Eulerian fluid mechanics as the flow of "dry water".

What Euler forgot to account for was the friction between adjacent fluid particles which dissipates kinetic energy into heat. The correction term was added by Navier in 1827 and Stokes in 1845, leading

to the following equation:

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p + \nu \nabla^2 u_\alpha + f_\alpha \tag{2.2}$$

Here, ν is the kinematic viscosity, which measures the strength of the fluid's internal friction. We have also added the term f_α to indicate external forcing separate from the pressure.

The viscous term accounts for the thermodynamic irreversibility of fluid motion that emerges at length scales larger than the mean free path. Euler's equation inherits the reversibility of classical mechanics by overlooking the physical reality of molecules which continuously collide with each other. The eulerian velocity field $u_\alpha(x, y, z, t)$ averages out the actual motion of the molecules. At this point we must model the transfer of energy to the finer motions, which are ignored, via the viscous term $\nu \nabla^2 u_\alpha$. The reason for the positive sign in front of the viscous term is the prediction, by the second law of thermodynamics, that energy is most likely to transfer to the finer molecular motions and stay there, than the other way around. This is a highly non-trivial result that was proven by Boltzmann, who derived the Navier-Stokes equations from the underlying microscopic physics of molecular collisions.

The pressure term may be modeled thermodynamically with an equation of state that relates the pressure to the fluid's temperature and density. In theoretical turbulence we simplify the equations of motion by assuming that the fluid's density is constant. It follows, from the local conservation of mass, that the velocity field satisfies the following constraint:

$$\partial_\alpha u_\alpha = 0 \tag{2.3}$$

Early 20th century ideas about turbulence were very influenced by the experimental work of Taylor in 1935 [242, 243]. Taylor conducted experiments in which turbulence is excited in a wind-stream when it passes through a regular array of rods in a wind tunnel, and then decays when it moves downstream. He treated the velocity field as a random variable and established with extensive measurements that it is approximately homogeneous and isotropic. In other words, the probability density function of the velocity field was shown to be almost invariant with respect to translation and rotation of the frame of reference. The approximation, of course, becomes invalid when the frame of reference is translated downstream over a substantial distance, due to turbulence decay in that direction. On the other hand, if we choose a frame of reference that moves downstream with the mean flow, then the statistics of the velocity field at a fixed point can be modeled by decaying homogeneous and isotropic turbulence. This is why the study of the decay of homogeneous and isotropic turbulence attracted a lot of interest [10–12]. The implied context was that the temporal evolution of the mathematical problem corresponds to the spatial variation of the statistics of the

velocity field in Taylor's experiments.

The theoretical study of homogeneous and isotropic turbulence began with von Karman and Howarth [127] in 1937. In 1940 Robertson employed the mathematical theory of isotropic tensors to systematize the kinematic theory of correlation tensors of the velocity field in homogeneous and isotropic turbulence [215]. The relation between the energy spectrum and the correlation tensor, which is based on a mathematical result attributed to Wiener, was first noticed by Taylor [244] in 1938, and was systematically treated for the first time in 1949 by Batchelor [6].

As a framework for studying turbulence, the assumptions of homogeneity and isotropy are very convenient. However, this didn't change the fact that homogeneous and isotropic turbulence is a mathematical idealization that is never realized in nature. The idea was tolerated because, in light of Taylor's work, a connection with experiment was possible. Also, after Batchelor published his landmark monograph [7] in 1953, it became a fashionable topic. As a result, all of the early attempts to develop an analytic theory of turbulence were done within this framework.

2.2 The Kolmogorov 1941 theory

In 1941 Kolmogorov introduced an alternative framework which, we now know, correctly describes turbulence at small scales and away from the boundary [130]. Instead of imposing restrictions on the velocity field itself, he required that the velocity differences be statistically homogeneous, isotropic, and stationary. As Kolmogorov himself pointed out, his definition is wider because it is mathematically weaker than requiring homogeneity and isotropy in the sense of Taylor, but at the same time narrower because he required stationarity. Furthermore, it seems that Kolmogorov recognized the inadequacy of the Eulerian representation because he presented his definitions using a rudimentary quasi-lagrangian framework. This idea was reintroduced [142] later by Kraichnan, who used it [143, 144] in his LHDIA theory, and by Belinicher and L'vov [15] to renormalize the diagrammatic MSR theory.

A characteristic of all turbulent flows that distinguishes them from their non-turbulent counterparts is the transfer of energy from the forced wavenumbers, by non-linear mode interactions, to other modes where it is dissipated. Kolmogorov's objective was to describe this process mathematically for three-dimensional hydrodynamic turbulence.

The first step had already been taken by Richardson [213] in 1922 who described turbulence qualitatively as a cascade of eddies. He proposed that the large eddies, that are formed by the forcing mechanism, break into smaller eddies because they're dynamically unstable. These smaller eddies continue to break into even smaller eddies for the same reason. When the eddies become

sufficiently small, they are dynamically stable and decay very fast, dissipating the incoming energy into heat.

From a mathematical viewpoint this process is characterized by three variables. The length scale ℓ_0 of the large eddies formed directly by the forcing mechanism is called the integral length scale. The length scale η of the small eddies responsible for dissipation is called the Kolmogorov microscale. We say that the length scale r is in the inertial range when $\eta \ll r \ll \ell_0$. There is also the well-known ε which equals the energy injection rate, the energy flux in the inertial range, and the rate with which energy is being dissipated at small scales. In his first paper [130] Kolmogorov proposed that the statistical properties of quasi-lagrangian velocity differences in the inertial range are independent of both the integral length scale ℓ_0 and the Kolmogorov microscale η. In his second paper [129] he presented a plausibility argument to support his proposal. He called his hypothesis the similarity principle.

During the second World War, access to the Soviet literature was very difficult in western countries. As a result, Kolmogorov's work went unnoticed until 1947 when it was popularized in an expository paper by Batchelor [5]. In his monograph, Batchelor presented the similarity hypothesis under Taylor's homogeneous and isotropic turbulence framework as a dimensional analysis argument, and predicted that the structure functions $S_n(\mathbf{x}, r\mathbf{e})$ will be independent of ℓ_0 and η, and will satisfy the following power law

$$S_n(\mathbf{x}, r\mathbf{e}) = \langle \{[\mathbf{u}(\mathbf{x} + r\mathbf{e}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \mathbf{e}\}^n \rangle = C_n(\varepsilon r)^{n/3}, \tag{2.4}$$

under the limit $\ell_0 \to \infty$ and $\eta \to 0$. Here $\mathbf{e}$ is a unit vector, and ε equals the rate of energy injection into the fluid, the energy flux in the cascade of energy from large scales to small scales, and the rate of energy dissipation at small scales. The constant C_n was believed to be universal, but in fact it is not (except for $n = 3$) and it is dependent on the forcing spectrum. From the above, the energy spectrum $E(k)$ for $\ell_0^{-1} \ll k \ll \eta^{-1}$ can be shown to satisfy

$$E(k) = C\varepsilon^{2/3} k^{-5/3}. \tag{2.5}$$

under the limit $\ell_0 \to \infty$ and $\eta \to 0$, where C is a constant which was initially believed to be independent of the underlying forcing mechanism.

The same prediction for the energy spectrum was also proposed by Heisenberg [36, 115, 208]. He arrived at the same results by modeling the function describing the transfer of energy between modes. Similar modeling attempts were proposed since then. The most interesting one was a proposal by Pao [197] which predicted the form of the energy spectrum not only in the inertial range but also in the dissipation range. The main shortcoming of these approaches is that they are based on specialized

ad-hoc assumptions which lack the simplicity and elegance of the similarity hypothesis.

Kolmogorov's argument is often identified with dimensional analysis. However, dimensional analysis is not the same thing as the similarity principle. The dimensional analysis explanation is a complete argument that is applied directly on the energy spectrum. The similarity principle, on the other hand, applies only on the statistical properties of velocity differences in the inertial range. To argue about the energy spectrum, we need to integrate from real space to Fourier space. The required integrals include length scales outside the inertial range, therefore we need additional considerations before we convince ourselves that the similarity principle implies a $k^{-5/3}$ spectrum.

It has since come to light [94, 236] that there exist departures from Kolmogorov scaling laws for the higher order structure functions (known as intermittency corrections), and Kolmogorov (with Oboukhov) [131, 195] was in fact the first to propose revisions of his original theory. The correct expression for $S_n(r)$ has the form

$$S_n(r) = C_n(\varepsilon r)^{n/3}(r/\ell_0)^{\zeta_n - n/3}, \tag{2.6}$$

where the ζ_n are the corresponding scaling exponents. Experiments have measured ζ_n for small values of n [13, 14, 18, 19, 22, 23, 35, 192, 238]. Naturally, there were many phenomenological models proposed to replace the Kolmogorov 1941 theory. The most interesting ones are Kolmogorov's 1962 theory [131, 195], and the model proposed by She and Leveque [39, 66, 152, 226, 227] which stirred up a fascinating controversy [28, 29, 41, 190, 194]. But like the original Kolmogorov theory, these models are also conjectures. Furthermore, the challenge is not merely to calculate the scaling exponents ζ_n. The robustness of the scaling of the energy spectrum needs to be explained, and the universality of the scaling exponents ζ_n is in fact still an open question.

2.3 Kraichnan's theories

Kraichnan, one of the last postdoctoral students of Einstein, is a remarkable individual who has single-handedly investigated and clarified some of the most fundamental issues of hydrodynamic turbulence. Less well-known is the fact that he made his debut in research by discovering that the general theory of relativity can be obtained as the classical approximation of a spin-2 quantum field theory [134–136]. This result was later independently discovered by Feynman [87, 90] (see the introduction to Ref.[90] for the whole story). One of his greatest insights, which in my opinion has now been vindicated, is that the methods of quantum field theory can be applied and yield fruit to the problem of hydrodynamic turbulence.

Early attempts to formulate an analytical theory of turbulence relied on Taylor's framework of homogeneous and isotropic turbulence as well as crude assumptions about the statistical properties

of the velocity field [9, 37, 209]. The idea was to introduce a relation between the fourth order correlation tensors and lower order correlation tensors of the velocity field in order to obtain a closed statistical theory. In 1957 Kraichnan pointed out that this approach is a dead end [137]; such models are unphysical because they give negative values for the energy spectrum. He counter-proposed the direct interaction approximation theory [138, 139]. The novel features of DIA, compared to closure models, is the introduction of response functions and space-time correlation tensors. He also introduced, for the first time, the idea of modeling the generating mechanism of turbulence with large-scale stochastic forcing. Kraichnan showed that his theory is the exact statistical theory for a model of randomly coupled Navier-Stokes equations, and therefore that the solution of his theory is realizable [140, 141]. Moreover, Wyld provided further justification of DIA when he showed that it is mathematically equivalent with the 1-loop approximation of a diagrammatic perturbation theory [260].

Kraichnan's DIA theory predicted that the energy spectrum in the inertial range is given by

$$E(k) = C\varepsilon^{2/3}k^{-5/3}(k\ell_0)^{1/6}f(k\ell_0) \tag{2.7}$$

when $\eta \to 0$ and where $f(k\ell_0) \to 1$ when $\ell_0 \to \infty$.

In his paper, Kraichnan compared this prediction with Kolmogorov's and explained why he doesn't believe in the similarity principle [139]: The eddy cascade mechanism is realized mainly by vortex stretching. We should anticipate then a web of thin elongated vortex filaments whose total length is of the order of the integral length scale. As these vortex filaments are swept by the mean flow, they disrupt other vortices that are still stretching into smaller length scales. This disruption introduces the integral length scale ℓ_0 into the vortex cascade process and changes the slope of the energy spectrum to $k^{-3/2}$. The similarity hypothesis, on the other hand, implies that this doesn't happen.

In 1962, experiments by Grant [113] and Gibson [102] measured the energy spectrum in the inertial range and found it consistent with Kolmogorov's prediction, not Kraichnan's. This prompted Kraichnan to reevaluate his theory. He proposed that DIA overestimates the sweeping effect because it is formulated with an eulerian velocity field [142]. So, he reformulated his theory using a Lagrangian velocity field, and found that the new theory, to first order, predicts the Kolmogorov energy spectrum [143, 144]. The new theory was called the Lagrangian history direct interaction approximation (LHDIA). A review of Kraichnan's work was given by Leslie [151]. Unfortunately, it was not clear how to generalize LHDIA, which was a first order approximation, to higher orders.

The LHDIA theory did not prove the similarity principle; it only confirmed it to first order. Furthermore, experiments suggested that the similarity hypothesis may not be true after all. Al-

though the energy spectrum follows Kolmogorov scaling, higher order structure functions show a substantial deviation. In fact, recent measurements [13, 238] suggest that the energy spectrum also deviates from Kolmogorov scaling by 0.03. In an ironic way, Kraichnan's mistrust of the similarity principle was correct. In fact, the vortex filaments he anticipated were recently observed by Orzag [225]. The reason why this effect is almost negligible with the energy spectrum and is only observed on higher order structure functions is because the disruptions of the cascade process by the vortex filaments are not frequent. As a result, this effect is called intermittency .

2.4 Beyond Kraichnan's theories

Parallel to these efforts, there have also been attempts to construct exact mathematical theories of turbulence based on functional calculus. The first such formulation was given by Hopf [123], and an equivalent reformulation in terms of path integrals by Rosen [216, 217]. Novikov [193] modified the Hopf formalism to include a gaussian delta correlated stochastic forcing, intended to model the hydrodynamic instability responsible for turbulence. An interesting application of this formalism is the more rigorous and powerful reformulations of the original dimensional analysis arguments used by Kolmogorov [183, 222]. Its main disadvantage is that it restricts the statistical description to one-time velocity correlations. A generalization to include many-time velocity correlations was given by Lewis and Kraichnan [153]; however even that is inadequate because it does not include response functions.

The essential idea of the definitive approach was introduced by Wyld [260]. The main result is that Feynman diagrams can be used to generalize DIA to higher orders, and that DIA itself is essentially a one-loop line-renormalized diagrammatic theory. A generalization of this scheme to a wider range of dynamical systems was given by Martin, Siggia, and Rose [178], although, as they themselves explained, without a sufficiently rigorous justification. Phythian [203] used Feynman path integrals to reformulate the MSR theory, and showed that it can be justified for dynamical systems that are local in time and first-order in time. An assumption implicit in this argument is that the dynamical system has a unique solution for all time. This claim has not been proven for the Navier-Stokes equations in three dimensions, however it is expected to hold on physical grounds. A pedagogical introduction to MSR theory was given recently by L'vov and Procaccia [166] and Eyink [76], and a careful review of the mathematical foundations of the theory itself is given in the paper by Andersen [2] (also see references therein).

Unfortunately, this formalism could not be applied to generalize Kraichnan's more successful LHDIA theory because the Navier-Stokes equations in the Lagrangian representation are not local

in time. Eventually, a way was discovered around this difficulty. It involves combining the MSR formalism with renormalization schemes that eliminate the sweeping interactions. The first such scheme was introduced by Yakhot [262], and another by Belinicher and L'vov [15, 161]. Combined with the MSR formalism, one has a rather solid foundation for further theoretical work.

L'vov and Procaccia have used the quasi-lagrangian renormalization scheme [15, 161] to formulate a diagrammatic theory [167, 168, 170] that generalized Kraichnan's DIA to all orders. It was shown that as long as the theory is truncated to finite order, it predicts agreement with Kolmogorov's theory and the absence of intermittency corrections [167]. It was also shown that if the theory is not truncated, there is a critical divergence that *does* lead to intermittency corrections [168]. L'vov and Procaccia *et al* also formulated a nonperturbative theory [164, 171–173] based on the fusion rules which are predicted by the underlying diagrammatic theory. This theory has been used to derive a nonperturbative method [16, 17, 175] and a perturbative method [176] for calculating the scaling exponents ζ_n. The perturbative method [176] has been used successfully to calculate ζ_n for all n accessible to experimental measurement, but it requires that the deviation of ζ_2 from the Kolmogorov prediction 2/3, which is the small parameter, be already known. A partial review of these developments was given in [174]. In recent papers with Tung [107, 108], I have suggested that the nonperturbative theory can also prove useful in clarifying a number of longstanding questions about the energy and enstrophy cascade of two-dimensional turbulence.

It is worth mentioning that there exists an entirely different theoretical approach to the problem based on renormalization group methods. A detailed review is given in [94, 179, 233] and some relevant criticism in Refs. [72, 148]. There are two interesting points of convergence between renormalization group methods and the theories reviewed previously. First, Eyink [70, 71] employed the renormalization group method to derive the fusion rules under certain assumptions both for shell models of turbulence and for hydrodynamic turbulence itself. The fusion rules are a crucial element in both the perturbative and the non-perturbative theories of L'vov and Procaccia. Second, Giles [103] used the renormalization group method to calculate the scaling exponents ζ_n, without relying on any experimental input, contrary to the paper [176]. In this calculation, the sweeping interactions were eliminated using the scheme by Yakhot [262]. A comparative study of the two approaches would help further progress.

In all the theoretical work that has been reviewed above, it is assumed that the Navier-Stokes equations have a unique solution, that there exists hydrodynamic instability leading to turbulence, and that this instability can be modeled with stochastic forcing acting at large scales. These assumptions are introduced implicitly simply by employing the MSR formalism. Although they are widely

accepted on physical grounds, there has also been substantial effort to deal with them rigorously.

An overview of the mathematical results on the existence and uniqueness of solutions to the Navier-Stokes equations is given in ref. [65, 100] and references therein. Briefly, in two dimensions the existence and uniqueness of strong solutions has been shown rigorously. In three dimensions it has been shown that weak solutions exist, but not that they are unique. It has also been shown that if strong solutions exist, they will have to be unique, but it has not been shown that such strong solutions do in fact exist. The underlying physical issue is whether the velocity field will develop singularities by vortex stretching as it is evolved by the Navier-Stokes equations. It has been shown [52, 53], in a two-dimensional model that includes the vortex stretching process, that such singularities do not develop in finite time. However the relevant question, in three dimensions, remains open.

It is fortunate that this issue does not arise in numerical simulations because the finiteness of the resolution prevents singularities from developing. As long as the smallest resolved length scale is smaller by order of magnitudes than the Kolmogorov microscale, the finite resolution approximation of the Navier-Stokes equations models hydrodynamic turbulence quite adequately. Furthermore, the energy cascade, which is very robust, will not allow any of the Fourier modes to blow out, since all the incoming energy will be transferred to the dissipation range, where it will be disposed of efficiently, given adequate numerical resolution. Another benefit of the finite resolution model is that the path integrals of the corresponding MSR theory are mathematically rigorous.

It should be noted that the Navier-Stokes equations themselves are *not* obviously *more* realistic than the finite resolution model because a "finite resolution" *is* imposed on fluid dynamics by Nature herself at the point where the existence of discrete molecules is important. In our view, disregarding the mathematical issues with the existence, uniqueness, and regularity of the solutions of the Navier-Stokes equations, can be justified if one introduces the assumption that the finite resolution approximation of the Navier-Stokes equations is a satisfactory physical model all by itself. This is not an unreasonable assumption *in the inertial range* of three-dimensional turbulence. We are on less solid ground with the cascades of two-dimensional turbulence, but the underlying mathematical issues do not arise in two dimensions. We do not wish to underestimate the importance of the mathematical issues of existence and uniqueness that remain open for 3D Navier-Stokes; we merely want to highlight the implicit assumption that one makes when one sidesteps these issues, as is done by every theory published to date.

Another very important issue which is "hidden under the rug" is proving the existence of turbulence itself as a consequence of the Navier-Stokes equations. Unfortunately, the theoretical framework

prescribed by the MSR theory cannot account, even in principle, for the existence of the hydrodynamic instability that causes turbulence. In the MSR framework, it is implicitly assumed that the effect of hydrodynamic instability can be *modeled* by a stochastic forcing term. The assumption can be justified if one demonstrates that the resulting stochastic behavior of the velocity field in the inertial range is invariant with respect to large-scale perturbations to the statistics of the forcing term.

There is in fact an extension of MSR theory in terms of a supersymmetric path integral that includes two additional fermionic ghost fields [1, 110, 112]. The surprising result is that correlations involving these additional fields are related to the Lyapunov exponents [111] that quantify hydrodynamic instability. It is therefore possible, in principle, to obtain statistical predictions from this framework with a deterministic forcing as input [246]. Whether this is in fact a practical approach remains to be seen.

The assumptions described so far are needed to bring in the machinery of the MSR formalism. In order to employ the formalism to explain the universality of the direct energy cascade and calculate the intermittency corrections, it is necessary to hypothesize a mathematical description of the energy cascade and use that to narrow down the specific solution which is self-consistent. Frisch [93, 94] proposed a set of hypothesis consisting of assumptions of statistical symmetry (such as homogeneity, isotropy, self-similarity) and the additional assumption of anomalous dissipation. The nature of the theoretical argument is to show that there is only a unique solution that can be admitted that satisfies the hypothesized statistical symmetries. A critical review of the assumed statistical symmetries, and local homogeneity in particular, is part of what concerns us in this paper.

To summarize, we accept the following assumptions on physical or experimental grounds: first, there exists a unique solution to the Navier-Stokes equations that develops hydrodynamic instability for large Reynolds numbers; second, in the limit of fully developed turbulence, local homogeneity and local isotropy (as defined by Frisch) are reinstated statistically, even if only asymptotically, for the velocity field; third, we accept the hypothesis that there exists an anomalous energy sink at small scales. These assumptions are a reasonable starting point for analytical theories of turbulence in three dimensions.

Chapter 3

ELIMINATION OF THE SWEEPING INTERACTIONS

The energy cascade from large scales to small scales is driven by the nonlinear term of the Navier-Stokes equations, and it is often explained as an effect of the vortex stretching and tilting caused by that term. However, the same term is also responsible for a sweeping interaction whereby a vortex is swept altogether from one location to another with minimal distortion. Implicit in the idea of an energy cascade is the assumption that these sweeping interactions have a negligible effect on the structure functions. It has therefore been necessary to use theoretical schemes that "eliminate" sweeping [174]. The goal of this chapter is to call attention to the fact that these schemes do not *prove* that sweeping is negligible; they only introduce the assumption that it is so. Recent doubts [95] concerning the consistency of the local homogeneity framework are directly linked with this problem of rigorously eliminating the sweeping interactions. We will also formulate a conjecture, and explore its plausibility, which, if shown to be true, would establish the assumption that the sweeping interactions are negligible in the inertial range.

It should be noted that a strictly rigorous mathematical theory based exclusively on the Navier-Stokes equations is a very difficult task. In fact, in the three-dimensional case, proving that these equations have a unique solution is still an open problem [65, 100]. For this reason, it is necessary to tolerate unproven assumptions as hypotheses, as long as such assumptions can be reasonably supported by physical arguments, or by experiment. It is within a specific framework of reasonable assumptions, which will be defined in a moment, that we claim that sweeping elimination procedures still do not prove that sweeping interactions are negligible.

The argument of this chapter, summarily, is the following. First, we show that the elimination of the sweeping interactions as well as the derivation of the 4/5-law requires a homogeneity assumption stronger than the assumption of local homogeneity, as envisioned by Frisch [94]. Second, we show that using the quasi-Lagrangian formulation to eliminate the sweeping interactions requires an even stronger homogeneity assumption which involves many-time correlations instead of one-time correlations. We conclude with a discussion of the implications of this argument on the utility of the quasi-Lagrangian formulation. Specifically, we will show that despite this apparent shortcoming, the theoretical work based on the quasi-Lagrangian formulation can still be used as a foundation for a

physically useful theory, along the lines of the Frisch framework, provided that certain considerations are taken into account. Furthermore, local homogeneity is in fact a consistent framework provided that the sweeping interactions can be eliminated in a more rigorous manner.

3.1 Homogeneity and sweeping interactions

The background on homogeneity is as follows: Taylor, Batchelor, Kraichnan, and others, have been willing to tolerate the assumption that turbulence is globally homogeneous and isotropic. However, it was suggested by Kolmogorov himself [130] that a far more realistic approach is to assume local homogeneity and local isotropy. Both frameworks have been reviewed by Monin and Yaglom [185]. Kolmogorov also emphasized the importance of studying stationary turbulence, corresponding to the forced-dissipative case, instead of the free decaying case.

In recent work, Frisch [93, 94] proposed that Kolmogorov's second paper [129] leads to a reformulation of his theory along three assumptions: first, the assumption of local homogeneity and local isotropy; second, an assumption of self-similarity; third, the assumption of an anomalous energy sink. Using the first and third assumption, according to Frisch, one derives the 4/5 law from which we obtain $\zeta_3 = 1$. From the second assumption we have $\zeta_n = nh$. Combined, we obtain the prediction $\zeta_n = n/3$. The assumption of self-similarity, used by Frisch, axiomatically excludes intermittency corrections to the scaling exponents ζ_n. Consequently, the theoretical efforts to calculate the scaling exponents from "first principles" essentially aim to weaken this assumption while tolerating the other two assumptions.

Some faith in the assumption of an anomalous energy sink, in particular, is based on recent evidence from numerical simulations [125] and theoretical evidence from the fusion rules [171, 172]. The assumption of local isotropy can be understood from the principle of linear superposition of the isotropic and anisotropic sectors of the symmetry group $SO(3)$ [4, 25]. Finally, the assumption of self-similarity can be understood via $\mathcal{Z}(h)$ covariance of the statistical theory [16, 17, 175]. This leaves then the assumption of local homogeneity.

3.1.1 Hierarchical definitions of homogeneity

Let $u_\alpha(\mathbf{x}, t)$ be the Eulerian velocity field, and introduce the Eulerian velocity differences w_α:

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) = u_\alpha(\mathbf{x}, t) - u_\alpha(\mathbf{x}', t). \tag{3.1}$$

The Eulerian generalized structure function is defined as the ensemble average of the product of such velocity differences

$$F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t)=\left\langle\left[\prod_{k=1}^{n}w_{\alpha_k}(\mathbf{x}_k,\mathbf{x}'_k,t)\right]\right\rangle, \tag{3.2}$$

where $\{\mathbf{x},\mathbf{x}'\}_n$ is shorthand for a list of n position vectors.

Originally, Frisch [93, 94] wrote his definitions of local homogeneity, local isotropy, and local stationarity using an "equivalence in law" relation. It should be noted that one should distinguish between *many-time equivalence*, that extends to many-time correlations, and *one-time equivalence* that applies only to one-time correlations. The clearest way to bring out this distinction is by defining the equivalence relation in terms of characteristic functionals defined as

$$Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}'}[\mathbf{p},t]=\left\langle\exp\left(i\int d\mathbf{x}\int d\mathbf{x}'\;w_\alpha(\mathbf{x},\mathbf{x}',t)p_\alpha(\mathbf{x},\mathbf{x}'))\right)\right\rangle \tag{3.3}$$

$$Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}',t}[\mathbf{p}]=\left\langle\exp\left(i\int d\mathbf{x}\int d\mathbf{x}'\int dt\;w_\alpha(\mathbf{x},\mathbf{x}',t)p_\alpha(\mathbf{x},\mathbf{x}',t))\right)\right\rangle. \tag{3.4}$$

The structure functions can be evaluated from the characteristic functional by variational differentiation and setting $\mathbf{p}=\mathbf{0}$. For example,

$$F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t)=\left[\prod_{k=1}^{n}\frac{1}{i}\frac{\delta}{\delta p_{\alpha_k}(\mathbf{x}_k,\mathbf{x}'_k)}\right]Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}'}[\mathbf{p},t]\Big|_{\mathbf{p}=0}. \tag{3.5}$$

The difference between $Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}'}[\mathbf{p}]$ and $Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}',t}[\mathbf{p}]$, is that $Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}'}[\mathbf{p}]$ contains information only about one-time correlations, whereas $Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}',t}[\mathbf{p}]$ contains information about many-time correlations as well. This is exploited to distinguish between many-time equivalence and one-time equivalence.

Definition 1 *Consider two stochastic fields $v_\alpha(\mathbf{x},\mathbf{x}',t)$ and $w_\alpha(\mathbf{x},\mathbf{x}',t)$. The "equivalence in law" relations are defined as*

$$v_\alpha(\mathbf{x},\mathbf{x}',t)\overset{\mathbf{x},\mathbf{x}'}{\sim}w_\alpha(\mathbf{x},\mathbf{x}',t)\Longleftrightarrow Z_{\mathbf{v}}^{\mathbf{x},\mathbf{x}'}[\mathbf{p},t]=Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}'}[\mathbf{p},t]\;\forall\mathbf{p}\;\textit{analytic} \tag{3.6}$$

$$v_\alpha(\mathbf{x},\mathbf{x}',t)\overset{\mathbf{x},\mathbf{x}',t}{\sim}w_\alpha(\mathbf{x},\mathbf{x}',t)\Longleftrightarrow Z_{\mathbf{v}}^{\mathbf{x},\mathbf{x}',t}[\mathbf{p}]=Z_{\mathbf{w}}^{\mathbf{x},\mathbf{x}',t}[\mathbf{p}]\;\forall\mathbf{p}\;\textit{analytic}. \tag{3.7}$$

Here, $\overset{\mathbf{x},\mathbf{x}'}{\sim}$ represents one-time equivalence, and $\overset{\mathbf{x},\mathbf{x}',t}{\sim}$ represents many-time equivalence. Thus, we can distinguish between one-time global homogeneity $\mathbf{u}\in\mathcal{H}$ and many-time global homogeneity $\mathbf{u}\in\mathcal{H}^*$:

$$\mathbf{u}\in\mathcal{H}\Longleftrightarrow u_\alpha(\mathbf{x},t)\overset{\mathbf{x}}{\sim}u_\alpha(\mathbf{x}+\mathbf{y},t),\;\forall\mathbf{y}\in\mathbb{R}^d \tag{3.8}$$

$$\mathbf{u}\in\mathcal{H}^*\Longleftrightarrow u_\alpha(\mathbf{x},t)\overset{\mathbf{x},t}{\sim}u_\alpha(\mathbf{x}+\mathbf{y},t),\;\forall\mathbf{y}\in\mathbb{R}^d. \tag{3.9}$$

A detailed review of previous definitions of *local* homogeneity has been given by Hill [120]. To discuss local homogeneity more carefully, we introduce the following definitions:

Definition 2 *The velocity field* $\mathbf{u}$*, as a stochastic field, is a member of the homogeneity class* $\mathcal{H}_m(\mathcal{A})$ *where* $\mathcal{A} \subseteq \mathbb{R}^d$ *a region in* $\mathbb{R}^d$*, if and only if the ensemble average defined as*

$$F_{m,n} \equiv \left\langle \left[\prod_{l=1}^{m} u_{\alpha_l}(\mathbf{x}_l, t)\right] \left[\prod_{k=1}^{n} w_{\beta_k}(\mathbf{y}_k, \mathbf{y'}_k, t)\right] \right\rangle, \tag{3.10}$$

is invariant with respect to a space shift of its arguments $\mathbf{x}_l, \mathbf{y}_k, \mathbf{y'}_k$ *for all* $n > 0$ *in the domain* $\mathcal{A}$*, i.e.*

$$\left(\sum_{l=1}^{m} \partial_{\alpha_l, \mathbf{x}_l} + \sum_{k=1}^{n} (\partial_{\beta_l, \mathbf{y}_l} + \partial_{\beta_l, \mathbf{y'}_l})\right) F_{m,n} = 0, \quad \forall \mathbf{x}_l, \mathbf{y}_k, \mathbf{y'}_k \in \mathcal{A},\ 1 \le l \le m,\ 1 \le k \le n \tag{3.11}$$

Definition 3 *The velocity field* $\mathbf{u}$ *is a member of the homogeneity class* $\mathcal{H}^*_m(\mathcal{A})$ *where* $\mathcal{A} \subseteq \mathbb{R}^d$ *a region in* $\mathbb{R}^d$*, if and only if the ensemble average defined as*

$$F^*_{m,n} \equiv \left\langle \left[\prod_{l=1}^{m} u_{\alpha}(\mathbf{x}_l, t_l)\right] \left[\prod_{k=1}^{n} w_{\beta_k}(\mathbf{y}_k, \mathbf{y'}_k, t)\right] \right\rangle, \tag{3.12}$$

is invariant with respect to a space shift of its arguments $\mathbf{x}_l, \mathbf{y}_k, \mathbf{y'}_k$ *for all* $n > 0$ *in the domain* $\mathcal{A}$*, i.e.*

$$\left(\sum_{l=1}^{m} \partial_{\alpha_l, \mathbf{x}_l} + \sum_{k=1}^{n} (\partial_{\beta_l, \mathbf{y}_l} + \partial_{\beta_l, \mathbf{y'}_l})\right) F^*_{m,n} = 0, \quad \forall \mathbf{x}_l, \mathbf{y}_k, \mathbf{y'}_k \in \mathcal{A},\ 1 \le l \le m,\ 1 \le k \le n \tag{3.13}$$

We also write $\mathcal{H}_m \equiv \mathcal{H}_m(\mathbb{R}^d)$ and $\mathcal{H}^*_m \equiv \mathcal{H}^*_m(\mathbb{R}^d)$. The distinction between $\mathcal{H}_m(\mathcal{A})$ and $\mathcal{H}^*_m(\mathcal{A})$ is that the former requires translational invariance on the one-time correlation tensor $F_{m,n}$, whereas the latter requires translational invariance on the many-time correlation tensor $F^*_{m,n}$, both over the domain $\mathcal{A}$. It should be noted that self-similarity conditions imposed on Eulerian velocity differences correlations have to be one-time, otherwise they would axiomatically rule out the possibility of intermittency corrections to the scaling exponents ζ_n [163, 175]. Thus, experience suggests that we should be cautious with many-time symmetries.

We also define the following transfinite homogeneity classes:

$$\mathcal{H}_\omega(\mathcal{A}) = \bigcap_{k \in \mathbb{N}} \mathcal{H}_k(\mathcal{A}) \quad \text{and} \quad \mathcal{H}^*_\omega(\mathcal{A}) = \bigcap_{k \in \mathbb{N}} \mathcal{H}^*_k(\mathcal{A}). \tag{3.14}$$

In these homogeneity classes the ensemble average of any product of velocities multiplied with any product of velocity differences will be invariant under spatial shifting. Note that even this homogeneity class is weaker than *global homogeneity*.

Remark 1 *An immediate consequence of these definitions is that the homogeneity classes are hier-*

archically ordered, according to the following relations

$$\mathcal{H} \subseteq \mathcal{H}_\omega(\mathcal{A}) \subseteq \mathcal{H}_k(\mathcal{A}),\ \forall k \in \mathbb{N}, \tag{3.15}$$

$$\mathcal{H}^* \subseteq \mathcal{H}^*_\omega(\mathcal{A}) \subseteq \mathcal{H}^*_k(\mathcal{A}),\ \forall k \in \mathbb{N}, \tag{3.16}$$

$$\mathcal{H}_a(\mathcal{A}) \subseteq \mathcal{H}_b(\mathcal{A}) \wedge \mathcal{H}^*_a(\mathcal{A}) \subseteq \mathcal{H}^*_b(\mathcal{A}),\ \forall a, b \in \mathbb{N} : a > b, \tag{3.17}$$

$$\mathcal{H}_a(\mathcal{A}) \subseteq \mathcal{H}^*_a(\mathcal{A}),\ \forall a \in \mathbb{N}. \tag{3.18}$$

3.1.2 Remarks on Kolmogorov's and Frisch's definition of local homogeneity

The definitions of local symmetries "in the sense of Frisch" can be written as

Definition 4 *The Eulerian velocity field is defined to be*

1. *Locally stationary if and only if*

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t + \Delta t)\ , \forall \Delta t \in \mathbb{R}. \tag{3.19}$$

2. *Locally homogeneous if and only if*

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x} + \mathbf{y}, \mathbf{x}' + \mathbf{y}, t)\ , \forall \mathbf{y} \in \mathbb{R}^d. \tag{3.20}$$

3. *Locally isotropic if and only if*

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}_0 + A(\mathbf{x} - \mathbf{x}_0), \mathbf{x}_0 + A(\mathbf{x}' - \mathbf{x}_0), t)\ , \forall A \in SO(d). \tag{3.21}$$

These definitions are also known as: incremental stationarity, incremental homogeneity, and incremental isotropy. It should be stressed that Frisch postulated that these symmetries are valid *asymptotically* for space shifts and time shifts up to a relevant order of magnitude and proposed them as reasonable hypotheses to be used as the basis for a modern reformulation of Kolmogorov's 1941 theory [93].

Remark 2 *The condition of local homogeneity, in the sense of Frisch, is written as* $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$.

To motivate his hypotheses, Frisch argues that homogeneity, isotropy, and time invariance are satisfied by the Navier-Stokes equations and they are violated only by the boundary conditions or any other relevant means of generating turbulence. However, he suggests that for high Reynolds numbers, when the turbulent motion is governed by a strange attractor, the symmetries of the

governing equation are restored asymptotically for small scales. Velocity differences are used to localize the symmetry to small scales.

The paradox inherent in this argument is that we cannot write governing equations for the velocity differences, exclusively in terms of velocity differences. A nonlinear term involving the velocity field, representing the sweeping interactions, is inevitable. As we shall argue below, the stronger homogeneity assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ is required to drop this term. Furthermore, we will argue that $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ is in fact required to derive the 4/5-law, which is the first step in Frisch's argument. Similar concerns were raised recently by Frisch [95] who questioned the consistency of local homogeneity.

As for Kolmogorov, in his first paper [130], he defined homogeneity in a most peculiar way. Instead of using the Eulerian velocity differences $w_\alpha(\mathbf{x}, \mathbf{x}', t)$, he used the following quantity:

$$\mathbf{Y}(\mathbf{x}_0, t_0|\mathbf{x}, t) = \mathbf{x} - \mathbf{x}_0 - (t - t_0)\mathbf{u}(\mathbf{x}_0, t_0) \tag{3.22}$$

$$\mathbf{w}_{Kol}(\mathbf{x}_0, t_0|\mathbf{x}, t) = \mathbf{u}(\mathbf{Y}(\mathbf{x}_0, t_0|\mathbf{x}, t), t) - \mathbf{u}(\mathbf{x}_0, t_0). \tag{3.23}$$

Here, $\mathbf{Y}$ represents the approximate displacement of a fluid particle that is being used as an non-inertial frame of reference. Because of its dependence on the velocity field, it is itself a stochastic variable. Kolmogorov employed structure functions defined in terms of $\mathbf{w}_{Kol}$ in his definitions. Furthermore, he included the requirement of local stationarity in his definition of local homogeneity. As will become apparent in section 4, Kolmogorov's representation of velocity differences is in fact a precursor of the quasi-Lagrangian representation. Although Kolmogorov does not discuss explicitly the problem of sweeping interactions, it is interesting that he foresaw to this extent the need for an non-Eulerian representation of the velocity field.

Another curious feature of the Kolmogorov definition is that it uses a conditional ensemble average conditioned on the statement $\mathbf{u}(\mathbf{x}_0, t_0) = \mathbf{v}$ instead of the usual unconditional ensemble average. This makes the definition stronger. For the case $t = t_0$ it is easy to see that the Kolmogorov definition is as strong as $\mathbf{u} \in \mathcal{H}_0$. However, more generally, the velocity differences used by Kolmogorov are evaluated at two different times t and t_0. For this reason, I find it very unlikely that Kolmogorov's definition can be shown to be as strong as $\mathbf{u} \in \mathcal{H}_1$. On the other hand, the use of the conditional ensemble average and the assumption that that average is independent of $\mathbf{v}$ strengthens the definition in unforeseen ways and may have some interesting consequences. In section 4, we will consider a very similar definition, which *does* rule out the sweeping interactions, where the conditional average is defined in terms of fixing the location of the wandering fluid particle. A detailed discussion of Kolmogorov's definition of local homogeneity is also given by Frisch [95].

3.1.3 Balance equations and sweeping

The clearest way to analyze the effect of the sweeping interactions on the theory of hydrodynamic turbulence is by employing the balance equations of the Eulerian generalized structure functions. These balance equations were introduced by L'vov and Procaccia [172] in a landmark paper, and they are derived as follows.

The Navier-Stokes equations, where the pressure term has been eliminated, read

$$\frac{\partial u_\alpha}{\partial t} + \mathcal{P}_{\alpha\beta}\partial_\gamma(u_\beta u_\gamma) = \nu\nabla^2 u_\alpha + \mathcal{P}_{\alpha\beta}f_\beta, \tag{3.24}$$

where $\mathcal{P}_{\alpha\beta}$ is the projection operator defined as

$$\mathcal{P}_{\alpha\beta} = \delta_{\alpha\beta} - \partial_\alpha\partial_\beta\nabla^{-2}, \tag{3.25}$$

and ∂_α represents spatial differentiation with respect to x_α. Repeated indices imply summation of components. The balance equations are obtained by differentiating the definition of F_n with respect to time t and substituting the Navier-Stokes equations. This leads to exact equations of the form

$$\frac{\partial F_n}{\partial t} + D_n = \nu J_n + Q_n, \tag{3.26}$$

where D_n represents the contributions from the nonlinear term, J_n the contributions of the dissipation term, and Q_n the contribution from the forcing term. To write the terms concisely, we use the following abbreviations to represent aggregates of arguments:

$$\begin{aligned} \mathbf{X} &= (\mathbf{x}, \mathbf{x}') \\ \{\mathbf{X}\}_n &= \{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\} \\ \{\mathbf{X}_\alpha\}_n^k &= \{\mathbf{X}_1, \ldots, \mathbf{X}_{k-1}, \mathbf{X}_{k+1}, \ldots, \mathbf{X}_n\}. \end{aligned} \tag{3.27}$$

The terms themselves read as follows.The forcing contribution is given by

$$Q_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t) = \sum_{k=1}^{n} \left\langle \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \mathcal{P}_{\alpha_k\beta}(f_\beta(\mathbf{x}_k, t) - f_\beta(\mathbf{x'}_k, t)) \right\rangle. \tag{3.28}$$

The dissipation term is given by

$$J_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t) = \mathcal{D}_n F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t) = \sum_{k=1}^{n} (\nabla^2_{\mathbf{x}_k} + \nabla^2_{\mathbf{x'}_k}) F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n, t), \tag{3.29}$$

where $\nabla^2_{\mathbf{x}_k}$ differentiates with respect to $\mathbf{x}_k$, and similarly $\nabla^2_{\mathbf{x'}_k}$ differentiates with respect to $\mathbf{x'}_k$.

The remarkable result, shown in [172], is that the term D_n that represents the contribution of the nonlinear term can be rewritten as $D_n = \mathcal{O}_n F_{n+1} + I_n$ where $\mathcal{O}_n$ is a linear integrodifferential

operator, and I_n is given by

$$I_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t)=\sum_{k=1}^{n}(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x'}_k})\left\langle \mathcal{U}_\beta(\{\mathbf{X}\}_n,t)\left[\prod_{l=1}^{n}w_{\alpha_l}(\mathbf{X}_l,t)\right]\right\rangle, \tag{3.30}$$

where $\mathcal{U}_\beta(\{\mathbf{X}\}_n,t)$ is defined as

$$\mathcal{U}_\alpha(\{\mathbf{X}\}_n,t)=\frac{1}{2n}\sum_{k=1}^{n}\left(u_\alpha(\mathbf{x}_k,t)+u_\alpha(\mathbf{x'}_k,t)\right). \tag{3.31}$$

The first term, $\mathcal{O}_nF_{n+1}$, includes the effect of pressure and part of the advection term. The second term, I_n, represents exclusively the effect of the sweeping interactions.

This decomposition makes rigorous the notion that the nonlinear interactions in the Navier-Stokes equations consist of local interactions that are responsible for the energy cascade and sweeping interactions which would disrupt the energy cascade if they contaminated the inertial range. It also exposes the conditions under which the sweeping interactions can be neglected. We learn that if the ensemble average of the velocity product that appears in the definition of I_n is invariant under a spatial shift, then the derivatives of that ensemble average will add up to zero. And here lies the heart of the problem. The assumption $\mathbf{u}\in\mathcal{H}_0$ by itself is not sufficient to set $I_n=0$. Global homogeneity $\mathbf{u}\in\mathcal{H}$ is sufficient, but it is a stronger assumption than what is required.

Remark 3 *The homogeneity condition needed to eliminate the sweeping interactions over the domain $\mathcal{A}$ is $\mathbf{u}\in\mathcal{H}_1(\mathcal{A})$.*

It should be noted that the local term $\mathcal{O}_nF_{n+1}$ and the dissipation term J_n preserve the local homogeneity condition $\mathbf{u}\in\mathcal{H}_0(\mathcal{A})$ in the sense of Frisch. The two terms in the balance equations that can potentially violate local homogeneity, are the sweeping term I_n and the forcing term Q_n. Asymptotic local homogeneity cannot be disrupted in the inertial range by the forcing term *if* the forcing spectrum is confined to large scales. The uncontrolled quantity is the sweeping term I_n. Recently, Frisch [95] questioned the consistency of local homogeneity as a framework for studying hydrodynamic turbulence. My viewpoint is that local homogeneity can be a consistent framework on the condition that the sweeping term I_n is dominant only at large scales with its influence forgotten as the energy cascades to smaller scales. If that is the case, then none of the other terms in the balance equations violate local homogeneity. This is discussed in further detail in section 5.

3.1.4 Remarks on the 4/5-law proof

In his second paper, Kolmogorov [129] employed an argument that is distinct from dimensional analysis to explain the claim that $\zeta_n=n/3$. He derived the 4/5-law from which he obtained $\zeta_3=1$,

and used a scaling assumption to obtain $\zeta_2 = 2/3$. Frisch's [93, 94] contribution was his observation that the scaling argument can be extended to account for all the scaling exponents ζ_n. With this extension, Kolmogorov's second paper [129] is then an equivalent reformulation of the dimensional analysis argument of his first paper [130]. The superiority of the extended argument is that at least one of the scaling exponents is established rigorously. One also bypasses the universality criticism, appearently attributed to Landau, of the original similarity hypothesis of Kolmogorov. In his book, Frisch [94] gave a more detailed account of his argument, but he didn't derive the 4/5-law on the basis of local homogeneity and local isotropy as prescribed by his framework; he used instead global homogeneity and global isotropy. The same holds for the alternative proof by Rasmussen [210]. An old proof by Monin [184] and Monin and Yaglom [185] claimed to prove the 4/5-law on the basis of local homogeneity and local isotropy, but it was criticized by Lindborg [155]. The criticism was addressed by Hill [119] who gave a corrected proof.

In particular, the criticism of Lindborg [155] was that it was not proved that the correlations involving the pressure field gradient and the velocity field can be eliminated on the basis of local isotropy from the equation that governs the time derivative of the second order structure function tensor. Hill [119] resolved this objection by supplying the needed proof. The principle behind the proof is reflected, in a wider sense, by the mathematical form of the general sweeping term I_n where there is only a local differential operator. The elimination of the nonlocal integral operator from I_n represents the elimination of any contributions by the pressure gradient term to I_n that would break local homogeneity. The pressure gradient *does* contribute to the term $\mathcal{O}_n F_{n+1}$ a non-local integrodifferential operator. However, because $\mathcal{O}_n F_{n+1}$ can be expressed exclusively in terms of the velocity differences, it preserves local homogeneity.

Nevertheless, the proof by Monin and Yaglom [185], as far as our intentions are concerned, has an additional shortcoming, which has also been noticed independently by Frisch [95]: it concerns the elimination of the terms associated with the sweeping interactions. If we refer to the part of the discussion leading to equation (22.14) of Monin and Yaglom [185], we learn that *they are using the quasi-Lagrangian transformation to eliminate the sweeping interaction term*! This can be made more clear if the reader compares the argument involving the two unnumbered equations that precede equation (22.14) of Monin and Yaglom [185] with section 4 and appendix B. The intention of this argument, according to Monin and Yaglom [185], is to "... transform the Navier-Stokes equations so that they contain only the velocity differences and their derivatives". This is precisely what the quasi-Lagrangian formulation does. As we shall argue in the next section of this paper, applying the inverse transformation back to the Eulerian representation requires an assumption of

homogeneity stronger than local homogeneity, but this time $\mathbf{u} \in \mathcal{H}^*_\omega$. Furthermore, the assumption of local stationarity in the quasi-Lagrangian representation can be shown to be equivalent to the assumption $\mathbf{u} \in \mathcal{H}^*_\omega$, in the Eulerian representation. Consequently, even if one does not want to return back to the Eulerian representation and just wants to show the validity of the 4/5-law in the quasi-Lagrangian representation, it is still necessary to assume local stationarity in the quasi-Lagrangian representation, and this assumption reintroduces $\mathbf{u} \in \mathcal{H}^*_\omega$ in the Eulerian frame. We might conjecture that Monin and Yaglom [185] was not concerned with the need to convert back to the Eulerian representation, probably because he intended to be faithful to his understanding of Kolmogorov's definition of local homogeneity.

As far as the theory of the scaling exponents is concerned, it is only necessary to know ζ_3. An elegant way to calculate ζ_3 is from the solvability condition of the homogeneous equation $\mathcal{O}_2 F_3 = 0$ [4, 171]. The idea here is to use the conservation of energy to show that

$$\begin{aligned} \mathcal{O}_2 F_3(\mathbf{x}_1, \mathbf{x'}_1, \mathbf{x}_2, \mathbf{x'}_2) &= \frac{1}{2}\frac{d[S_3(r_{12}) - S_3(r_{12'})]}{dr_1} + \frac{1}{2}\frac{d[S_3(r_{1'2'}) - S_3(r_{1'2})]}{dr_{1'}} \\ &= A[r_{12}^{\zeta_3 - 1} - r_{12'}^{\zeta_3 - 1} + r_{1'2'}^{\zeta_3 - 1} - r_{1'2}^{\zeta_3 - 1}]. \end{aligned} \tag{3.32}$$

where $r_{12} = \|\mathbf{x}_1 - \mathbf{x}_2\|$, etc. It follows that the equation $\mathcal{O}_2 F_3 = 0$ will be satisfied for any configuration of velocity differences if and only if $\zeta_3 = 1$. The homogeneous equation can be obtained from the balance equations in the limit of infinite Reynolds number. For the case of finite Reynolds number, there is a homogeneous and particular solution to the generalized structure functions that are linearly superimposed [107, 108]. Then the calculation of ζ_3 is relevant only for the homogeneous solution. Aside from this issue, this argument too requires that we set $I_2 = 0$. Dropping I_2 cannot be justified under local homogeneity, in the sense of Frisch, and it requires the condition $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$. We arrive then to the following conclusion.

Remark 4 *The homogeneity condition needed to establish $\zeta_3 = 1$ over the domain $\mathcal{A}$ is $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$.*

It should be noted that even though Hill [119] has claimed to show the 4/5-law on the basis of local homogeneity and local isotropy, his definition of local homogeneity is mathematically stronger than the definition $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$ used in the Frisch framework, and it is in fact very similar to $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ (also see section 4.1 of [121]). Consequently, while his proof correctly follows from his stated assumptions, it cannot be used within the Frisch framework of hypotheses to prove the 4/5-law without invoking additional assumptions.

It is possible to derive a rigorous version of the 4/5-law that does not require assumptions of homogeneity, isotropy, stationarity, and not even an ensemble average [67, 78, 245]. This is done by

rephrasing the statement to be proven. Specifically, it has been shown that

$$\lim_{\Delta t\to 0}\lim_{r\to 0}\lim_{\nu\to 0}\int_t^{t+\Delta t} d\tau \int_{SO(3)} \frac{d\Omega(A)}{4\pi}\int_{\mathcal{B}} \frac{d\mathbf{x}}{V(\mathcal{B})}\frac{S_3(\mathbf{x}, rA\mathbf{e})}{r} = -\frac{4}{5}\varepsilon_{\mathcal{B}}, \tag{3.33}$$

for almost every (Lebesgue) point t in time, where $\mathbf{e}$ is a unit vector, $\mathcal{B}\subseteq\mathbb{T}^3$ is a local region in a periodic boundary domain $\mathbb{T}^3$ (topologically equivalent to a torus) with volume $V(\mathcal{B})$, and $\varepsilon_{\mathcal{B}}$ is the local dissipation rate over the region $\mathcal{B}$ given by

$$\varepsilon_{\mathcal{B}} \equiv \lim_{\nu\to 0}\frac{1}{V(\mathcal{B})}\int d\mathbf{x}\,\varepsilon(\mathbf{x},t), \tag{3.34}$$

where $\varepsilon(\mathbf{x},t) = (1/2)\nu\,\langle s_{\alpha\beta}(\mathbf{x},t)s_{\alpha\beta}(\mathbf{x},t)\rangle$ is the dissipation rate density at $(\mathbf{x},t)$ and $s_{\alpha\beta}\equiv\partial_\alpha u_\beta + \partial_\beta u_\alpha$ is the local strain tensor. A similar result was obtained earlier by Nie and Tanveer [191].

It should be noted that this result does not contradict our previous remark. Although the need to make assumptions appears to have been eliminated, this is done so at the price of proving a statement that is mathematically weaker. In the original formulation of the 4/5-law, aside from an ensemble average, all the integrals are absent. These integrals represent an interesting way of obviating the symmetry assumptions needed to prove the 4/5-law in its original formulation.

Recently, there has been considerable interest in extending the 4/5-law to account for deviations from the theoretical prediction caused by the violation of local isotropy [54–57, 157]. From the viewpoint of the experimentalist these extensions make it possible to confirm the validity of the 4/5-law against experimental data. From the viewpoint of the theorist, deviations from local isotropy can be accounted for with the $SO(3)$ group decomposition method [4, 25].

3.2 The quasi-Lagrangian formulation

The essence of the quasi-Lagrangian formulation (also called the *Belinicher-L'vov transformation*) is to look at turbulence using a fluid particle as a non-inertial frame of reference. The representation is Lagrangian because we involve fluid particles, but it is not completely Lagrangian because the fluid particle trajectory is only used to define a new frame of reference, and we continue to look at the velocity field in an Eulerian manner. It is understood, of course, that the only interesting statistics are those involving points within a sphere centered on the moving fluid particle with radius on the order of the integral length scale.

Let $u_\alpha(\mathbf{x},t)$ be the Eulerian velocity field, and let $\rho_\alpha(\mathbf{x}_0,t_0|t)$ be the position of the unique fluid particle initiated at $(\mathbf{x}_0,t_0)$ at time t relative to its initial position at time t_0. The transformation is done in two steps. First, we introduce $v_\alpha(\mathbf{x}_0,t_0|\mathbf{x},t)$ as the Eulerian velocity with respect to the

original inertial frame of reference with a space shift that follows the fluid particle:

$$\begin{aligned} \rho_\alpha(\mathbf{x}_0, t_0|t) &= \int_{t_0}^{t} d\tau\ u_\alpha(\mathbf{x}_0 + \rho(\mathbf{x}_0, t_0|\tau), \tau) \\ v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) &= u_\alpha(\mathbf{x} + \rho(\mathbf{x}_0, t_0|t), t). \end{aligned} \tag{3.35}$$

Then, to complete the transformation we must subtract the velocity of the fluid particle uniformly, so that the particle itself will appear to be motionless:

$$\begin{aligned} w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) &= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - \frac{\partial}{\partial t}\rho_\alpha(\mathbf{x}_0, t_0|t) \\ &= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}_0, t) \\ &= u_\alpha(\mathbf{x} + \rho(\mathbf{x}_0, t_0|t), t) - u_\alpha(\mathbf{x}_0 + \rho(\mathbf{x}_0, t_0|t), t). \end{aligned} \tag{3.36}$$

We define $w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t)$ as the quasi-Lagrangian velocity field, and introduce the quasi-Lagrangian velocity difference $W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t)$ given by

$$W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) \equiv w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - w_\alpha(\mathbf{x}_0, t_0|\mathbf{x}', t) \tag{3.37}$$

$$= v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, t) - v_\alpha(\mathbf{x}_0, t_0|\mathbf{x}', t). \tag{3.38}$$

Differentiating with respect to time, and substituting the Navier-Stokes equations, gives an equation of the form

$$\frac{\partial W_\alpha}{\partial t} + \mathcal{V}_{\alpha\beta\gamma} W_\beta W_\gamma = \nu(\nabla^2_{\mathbf{x}} + \nabla^2_{\mathbf{x}'}) W_\alpha + F_\alpha, \tag{3.39}$$

where $F_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t)$ is the quasi-lagrangian forcing, and $\mathcal{V}_{\alpha\beta\gamma}$ is a bilinear integrodifferential operator of the form

$$\mathcal{V}_{\alpha\beta\gamma} W_\beta W_\gamma \equiv \iint d\mathbf{X}_\beta d\mathbf{X}_\gamma\ V_{\alpha\beta\gamma}(\mathbf{x}_0|\mathbf{X}_\alpha, \mathbf{X}_\beta, \mathbf{X}_\gamma) W_\beta(\mathbf{X}_\beta) W_\gamma(\mathbf{X}_\gamma), \tag{3.40}$$

with $V(\mathbf{x}_0|\mathbf{X}_\alpha, \mathbf{X}_\beta, \mathbf{X}_\gamma)$ the corresponding kernel (see appendix B for more details). The remarkable feature of this equation is that all the terms, and most especially the nonlinear term, are written in terms of velocity differences. Fundamentally, this is the reason why the quasi-Lagrangian transformation eliminates the sweeping interactions and renormalizes the MSR diagrammatic theory.

The key issue is whether it is possible to switch back to the Eulerian representation without reintroducing the sweeping interactions. In a short appendix to their paper, L'vov and Procaccia [167] showed that *in stationary turbulence the ensemble average of the same time quasi-Lagrangian velocity differences is equal to the ensemble average of the corresponding Eulerian velocity differences.* The same appendix is also found in a previous unpublished paper [162]. The proof requires stationarity of the Eulerian velocity field, and incompressibility. A homogeneity condition is also used, which is described as "translational invariance".

In this section, we would like to carefully re-examine this proof, the assumptions needed to make it work, and the relationship between this result and other claims that one might reasonably make about the quasi-Lagrangian velocity differences. Part of our motivation is the crucial importance of this result; an enormous amount of work hinges on it, and as discussed previously, the quasi-Lagrangian formulation is essential in developing analytical theories. Our main interest is to show that the proof requires that we assume $\mathbf{u} \in \mathcal{H}^*_\omega$, which is a stronger condition than what is actually needed to eliminate the sweeping interactions or to prove the 4/5-law ($\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$).

3.2.1 Characterizations of the claim

Let $\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{x}, \mathbf{x}'\}_n, t)$ be the generalized structure function in the quasi-Lagrangian representation, defined as

$$\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{x}, \mathbf{x}'\}_n, t) = \left\langle \left[\prod_{k=1}^{n} W_{\alpha_k}(\mathbf{x}_0, t_0|\mathbf{x}_k, \mathbf{x}'_k, t) \right] \right\rangle. \tag{3.41}$$

The claim of L'vov and Procaccia was that it can be shown that

$$\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{x}, \mathbf{x}'\}_n, t) = F_n(\{\mathbf{x}, \mathbf{x}'\}_n, t), \tag{3.42}$$

which can be rewritten equivalently as

$$W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t). \tag{3.43}$$

As a first step, consider the following easy-to-prove propositions which give equivalent characterizations of the claim (3.43):

Proposition 1 *The claim* (3.43) *holds if and only if the quasi-Langrangian velocity is locally stationary with respect to* t_0*:*

$$W_\alpha(\mathbf{x}_0, t_0 + \Delta t|\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t), \ \forall \Delta t \in \mathbb{R} - \{0\} \tag{3.44}$$

Proof: ($\Rightarrow$): Assume that the claim (3.43) holds. Then, it follows that

$$\begin{aligned} W_\alpha(\mathbf{x}_0, t_0 + \Delta t|\mathbf{x}, \mathbf{x}', t) &\overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t) && \text{by (3.43)} \\ &\overset{\mathbf{x},\mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) && \text{by (3.43).} \end{aligned}$$

($\Leftarrow$): Now assume that the quasi-Lagrangian velocity field is locally stationary with respect to t_0. Using the evaluation

$$\lim_{t_0 \to t} \rho_\alpha(\mathbf{x}_0, t_0|t) = 0, \tag{3.45}$$

it follows that

$$\begin{aligned} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) &\overset{\mathbf{x},\mathbf{x}'}{\sim} \lim_{t_0 \to t} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) && \text{by assumption} \\ &\overset{\mathbf{x},\mathbf{x}'}{\sim} \lim_{t_0 \to t} w_\alpha(\{\mathbf{x}, \mathbf{x}'\} + \rho(\mathbf{x}_0, t_0|t), t) && \text{definition} \\ &\overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t) && \text{continuity on } w_\alpha, \end{aligned}$$

and that concludes the proof □

Proposition 2 *Assume local stationarity on the Eulerian velocity field:*

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t + \Delta t),\ \forall \Delta t \in \mathbb{R}. \tag{3.46}$$

Then, the claim (3.43) *holds if and only if the quasi-Langrangian velocity is locally stationary*

$$W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x}}{\sim} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t + \Delta t),\ \forall \Delta t \in \mathbb{R} - \{0\} \tag{3.47}$$

Proof: (⇒): Assume that the claim (3.43) holds.

$$\begin{aligned} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t + \Delta t), &\overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t + \Delta t) && \text{by (3.43)} \\ &\overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t) && \text{by (3.46)} \\ &\overset{\mathbf{x},\mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) && \text{by (3.43)} \end{aligned}$$

(⇐): Now assume that the quasi-Lagrangian velocity field is locally stationary. Using the evaluation

$$\lim_{t \to t_0} \rho_\alpha(\mathbf{x}_0, t_0|t) = 0, \tag{3.48}$$

it follows that

$$\begin{aligned} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) &\overset{\mathbf{x},\mathbf{x}'}{\sim} \lim_{t \to t_0} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t) && \text{by assumption} \\ &\overset{\mathbf{x},\mathbf{x}'}{\sim} \lim_{t \to t_0} w_\alpha(\{\mathbf{x}, \mathbf{x}'\} + \rho(\mathbf{x}_0, t_0|t), t) && \text{definition} \\ &= w_\alpha(\mathbf{x}, \mathbf{x}', t_0) && \text{continuity on } w_\alpha \\ &\overset{\mathbf{x},\mathbf{x}'}{\sim} w_\alpha(\mathbf{x}, \mathbf{x}', t) && \text{Eulerian stationarity} \end{aligned}$$

and that concludes the proof □

The implication of these propositions is that the relationship between the Eulerian in the quasi-Lagrangian formulations can be established as an immediate consequence of stationarity of the quasi-Lagrangian velocity field with respect to t_0. As Lebedev and L'vov [162] noted, the variable t_0 does not appear anywhere in the quasi-Lagrangian Navier-Stokes equations (3.39), consequently *the form*

of the governing equations allows stationary solutions with respect to t_0. However, to assert that the quasi-Lagrangian velocity field *is* stationary, it is necessary to assume that the quasi-Lagrangian forcing field is also stationary. Since the definition of the quasi-Lagrangian forcing field entangles the Eulerian forcing field f_α with the trajectory field ρ_α, and since the trajectory field itself is not time invariant (due to the initial condition $\rho_\alpha(\mathbf{x}_0, t_0|t_0) = 0$), we cannot make this assumption without justification. This was the reason, cited by Lebedev and L'vov [162], for the rigorous proof which is the topic of this section. Proposition 2 shows that assuming stationarity in the quasi-Lagrangian representation is sufficient to prove the claim (3.43), and thus this assumption implicitly introduces in the Eulerian frame the conditions needed to prove the claim.

3.2.2 MSR theory for Lagrangian trajectories

The governing equation for ρ_α is

$$\frac{\partial \rho_\alpha(\mathbf{x}_0, t_0|t)}{\partial t} = u_\alpha(\mathbf{x}_0 + \rho(\mathbf{x}_0, t_0|t), t) \tag{3.49}$$

with initial condition $\rho_\alpha(\mathbf{x}_0, t_0|t_0) = 0$. Deriving the stationarity condition (3.44) requires an MSR theory where the velocity field u_α can be thought of as the forcing field with known statistical properties, and the Lagrangian trajectories field ρ_α as the governed field whose properties we wish to deduce. Unfortunately we may not apply the standard MSR theory because the equation itself does not assume the standard form $N_\alpha[\rho] = u_\alpha$ with u_α independent of ρ and furthermore the initial condition is set at a finite time t_0 and not at $t_0 \to -\infty$. We need to develop the statistical theory from scratch, and for that purpose the path integral formulation is most expedient.

Note that every value of t_0 corresponds to a distinct initial value problem. We may therefore treat, for the purposes of the statistical theory, the field ρ_α as a function only of $\mathbf{x}_0, t$ and let t_0 to be taken at a fixed value. We can also go a step further and note that for every value of $\mathbf{x}_0$ the governing equation is an ordinary differential equation. It follows that in constructing an MSR theory for ρ_α we have two options: We may construct a statistical theory for the *restricted problem* in which $\mathbf{x}_0$ is also fixed and the field ρ_α is taken as a function of only t, or a theory for the *full problem* in which only t_0 remains fixed and ρ_α is taken as a function of $\mathbf{x}_0$ and t. In the restricted case we cannot calculate correlations between fields ρ_α with different values of $\mathbf{x}_0$. In the full case, we can. For our needs, the restricted statistical theory will be sufficient.

Introduce an operator $\mathcal{Q}_{\mathbf{x}_0}[\rho]$ via the kernel

$$Q^{\mathbf{x}_0}_{\alpha\beta}[\rho](t, \mathbf{y}, \tau) \equiv \delta_{\alpha\beta}\delta(t - \tau)\delta(\mathbf{y} - \mathbf{x}_0 - \rho(t)) \tag{3.50}$$

such that

$$\mathcal{Q}_{\mathbf{x}_0}[\rho]u_\alpha \equiv \int d\mathbf{y} \int d\tau \; Q^{\mathbf{x}_0}_{\alpha\beta}[\rho](t,\mathbf{y},\tau)u_\beta(\mathbf{y},\tau) = u_\alpha(\mathbf{x}_0+\rho(t),t). \tag{3.51}$$

We also introduce a functional $\mathcal{L}_{\mathbf{x}_0,t_0}[u]$ that constructs ρ_α from the velocity field. This operation is of course admissible in both the restricted and the full theory.

Since $\rho = \mathcal{L}_{\mathbf{x}_0,t_0}[u]$ is equivalent to $\dot{\rho}_\alpha = Q_{\mathbf{x}_0}[\rho]u_\alpha$, it follows that there exists a functional $J[u]$ such that $\delta[\dot{\rho}_\alpha - \mathcal{Q}_{\mathbf{x}_0}[\rho]u_\alpha] = J[u]\delta[\rho - \mathcal{L}_{\mathbf{x}_0,t_0}[u]]$ which can be evaluated by integrating both sides over ρ:

$$J[u] = \int_{\mathcal{P}(t_0)} \mathcal{D}\rho \; \delta[\dot{\rho}_\alpha - \mathcal{Q}_{\mathbf{x}_0}[\rho]u_\alpha]. \tag{3.52}$$

Here, $\mathcal{P}(t_0)$ is the domain of integration and it is defined as the set of all $\rho_\alpha(t)$ that satisfy the initial condition $\rho_\alpha(t_0) = 0$. The integral is a Feynman path integral [88] (a pedagogical introduction is given in Ref. [237]). Suppose we would like to evaluate the ensemble average $\langle M[u,\rho]\rangle$ where M is some arbitrary functional of ρ_α and u_α. We treat the velocity field u_α as a forcing field with known statistics. We assume then that we know how to evaluate the ensemble average of any expression in terms of the velocity field. We have:

$$\begin{aligned}
\langle M[u,\rho]\rangle = \langle M[u,\mathcal{L}_{\mathbf{x}_0,t_0}[u]]\rangle &= \left\langle \int_{\mathcal{P}(t_0)} \mathcal{D}\varrho \; M[u,\varrho]\delta[\varrho - \mathcal{L}_{\mathbf{x}_0,t_0}[u]] \right\rangle \\
&= \int_{\mathcal{P}(t_0)} \mathcal{D}\varrho \; \langle M[u,\varrho]\delta[\varrho - \mathcal{L}_{\mathbf{x}_0,t_0}[u]]\rangle \\
&= \int_{\mathcal{P}(t_0)} \mathcal{D}\varrho \; \langle M[u,\varrho]J^{-1}[u]\delta[\dot{\varrho}_\alpha - \mathcal{Q}_{\mathbf{x}_0}[\varrho]u_\alpha]\rangle \\
&= \int_{\mathcal{P}(t_0)} \mathcal{D}\varrho \; \left\langle M[u,\varrho]J^{-1}[u] \int \mathcal{D}\beta \; \exp(i\beta_\alpha(\dot{\varrho}_\alpha - \mathcal{Q}_{\mathbf{x}_0}[\varrho]u_\alpha)) \right\rangle \\
&= \iint_{\mathcal{P}(t_0)} \mathcal{D}\varrho\mathcal{D}\beta \; \exp(i\beta_\alpha\dot{\varrho}_\alpha)\langle M[u,\varrho]J^{-1}[u]\exp(-i\beta_\alpha\mathcal{Q}_{\mathbf{x}_0}[\varrho]u_\alpha)\rangle.
\end{aligned}$$

Here, we have used the convention that repeated Greek indices imply integrating temporal coordinates throughout their domain in addition to summation of vector components. For example, the expression $\beta_\alpha\dot{\rho}_\alpha$ is an implicit abbreviation for

$$\beta_\alpha\dot{\rho}_\alpha = \int dt \; \beta_\alpha(t)\dot{\rho}_\alpha(t) \tag{3.53}$$

We also use the formal representation for the delta function

$$\delta[u] = \int \mathcal{D}\beta \; \exp(i\beta_\alpha u_\alpha) \tag{3.54}$$

which is valid in the sense of generalized functional distributions.

If the velocity field is incompressible, it can be shown that $J[u] = 1$. A detailed proof of this result is given in appendix C. We conclude that the stochastic theory simplifies to:

$$\langle M[u,\rho]\rangle = \iint_{\mathcal{P}(t_0)} \mathcal{D}\varrho\mathcal{D}\beta \ \exp(i\beta_\alpha\dot{\varrho}_\alpha)\langle M[u,\varrho]\exp(-i\beta_\alpha\mathcal{Q}_{\mathbf{x}_0}[\varrho]u_\alpha)\rangle. \tag{3.55}$$

This statement is a concise expression of the statistical theory for Lagrangian trajectories.

3.2.3 Transform back to Eulerian representation

We now use the statistical theory to derive the relationship between the quasi-Lagrangian correlation and the Eulerian correlation. The proof given here follows the one given by L'vov and Procaccia [167], but it is presented in more detail to show the underlying assumptions. The proof makes an essential use of eq. (3.55) derived above.

The argument is essentially based on the following identity:

$$\rho_\alpha(\mathbf{r}_0, t_0 + \Delta t|t) = \rho_\alpha(\mathbf{r}_0, t_0|t) - \rho_\alpha(\mathbf{r}_0, t_0|t_0 + \Delta t). \tag{3.56}$$

To see why this is true, note that the expression on the right-hand side satisfies the governing equation for the $t_0 + \Delta t$ problem, and it also satisfies the initial condition. Therefore, by uniqueness, it has to be equal to the left-hand side.

To facilitate with calculations, we define $\mathcal{M}_{\mathbf{x}_0}[\beta,\rho]$ as

$$\mathcal{M}_{\mathbf{x}_0}[\beta,\rho] = \langle M[u,\rho]\exp(-i\beta_\alpha Q_{\mathbf{x}_0}[\rho]u_\alpha)\rangle \tag{3.57}$$

$$= \left\langle M[u,\rho]\exp\left(-i\int dt\ \beta_\alpha(t)u_\alpha(\mathbf{x}_0+\rho(t),t)\right)\right\rangle, \tag{3.58}$$

and we also use the notation $\mathcal{M}(\mathbf{x}_0,t_0)$ for the ensemble average $\langle M[u,\rho]\rangle$ evaluated under a given choice of $\mathbf{x}_0$ and t_0. Consequently, we may write

$$\mathcal{M}(\mathbf{x}_0,t_0) = \iint_{\mathcal{P}(t_0)} \mathcal{D}\rho\mathcal{D}\beta\exp(i\beta_\alpha\dot{\rho}_\alpha)\mathcal{M}_{\mathbf{x}_0}[\beta,\rho] \tag{3.59}$$

$$= \iint_{\mathcal{P}(t_0)} \mathcal{D}\rho\mathcal{D}\beta\exp\left(i\int dt\ \beta_\alpha(t)\frac{\partial\rho_\alpha(t)}{\partial t}\right)\mathcal{M}_{\mathbf{x}_0}[\beta,\rho]. \tag{3.60}$$

The key statement to be proven is the following proposition, that shows the connection between stationarity in the quasi-Lagrangian representation and homogeneity in the Eulerian representation.

Proposition 3 *The condition* $\mathcal{M}_{\mathbf{x}_0}[\beta,\rho+\mathbf{x}] = \mathcal{M}_{\mathbf{x}_0}[\beta,\rho],\ \forall\beta\forall\mathbf{x}\in\mathbb{R}^d$ *implies* $\mathcal{M}(\mathbf{x}_0,t_0+\Delta t) = \mathcal{M}(\mathbf{x}_0,t_0)\ \forall\Delta t\in\mathbb{R}$*, provided that the velocity field* u *is incompressible.*

Proof:

To facilitate our argument, introduce a new field λ_α defined as equal to the right hand side of (3.56).

$$\lambda_\alpha(\mathbf{r}_0, t_0|t) = \rho_\alpha(\mathbf{r}_0, t_0 + \Delta t|t) \tag{3.61}$$

$$= \rho_\alpha(\mathbf{r}_0, t_0|t) - \rho_\alpha(\mathbf{r}_0, t_0|t_0 + \Delta t) = B_{\alpha\beta}\rho_\beta. \tag{3.62}$$

The connection between λ_α and ρ_α is *linear*, in the sense that we can construct an appropriate kernel $B_{\alpha\beta}$ made of delta functions that transforms one field into the other. The functional determinant of B is equal to 1, so a change in variables under the path integral does not introduce an additional factor, namely $\mathcal{D}\lambda = \mathcal{D}\rho$. This is usually true with simple transformations, such as space shifting and rotations, because they merely reshuffle the order in which we integrate over all possible histories. In this case, we need to take into account that the permissible histories are constrained by the initial condition $\lambda_\alpha(\mathbf{r}_0, t_0|t_0 + \Delta t) = 0$ which is different from the initial condition of the field ρ_α. It follows that, while there is no need to introduce a functional determinant, the domain of integration has to change from $\mathcal{P}(t_0)$ to $\mathcal{P}(t_0 + \Delta t)$. Finally, it is easy to see that $\partial\lambda_\alpha/\partial t = \partial\rho_\alpha/\partial t$ and the hypothesis implies that $\mathcal{M}_{\mathbf{x}_0}[\beta, \rho] = \mathcal{M}_{\mathbf{x}_0}[\beta, \lambda]$. We may then write:

$$\mathcal{M}(\mathbf{x}_0, t_0) = \iint_{\mathcal{P}(t_0)} \mathcal{D}\rho\mathcal{D}\beta \exp\left(i \int dt\ \beta_\alpha(t) \frac{\partial\rho_\alpha(t)}{\partial t}\right) \mathcal{M}_{\mathbf{x}_0}[\beta, \rho] \tag{3.63}$$

$$= \iint_{\mathcal{P}(t_0+\Delta t)} \mathcal{D}\lambda\mathcal{D}\beta \exp\left(i \int dt\ \beta_\alpha(t) \frac{\partial\lambda_\alpha(t)}{\partial t}\right) \mathcal{M}_{\mathbf{x}_0}[\beta, \lambda] \tag{3.64}$$

$$= \mathcal{M}(\mathbf{x}_0, t_0 + \Delta t)\ \forall \Delta t \in \mathbb{R}. \tag{3.65}$$

□

Proposition 4 *If $u \in \mathcal{H}^*_\omega$, and u_α is incompressible, then*

$$W_\alpha(\mathbf{x}_0, t_0 + \Delta t|\mathbf{x}, \mathbf{x}', t) \overset{\mathbf{x},\mathbf{x}'}{\sim} W_\alpha(\mathbf{x}_0, t_0|\mathbf{x}, \mathbf{x}', t). \tag{3.66}$$

Proof: Let $n \in \mathbb{N}^*$ be given, and define the functional $M[u, \rho]$ as

$$M[u, \rho] = \prod_{k=1}^{n} W_{\alpha_k}(\mathbf{x}_0, t_0|\mathbf{x}_k, \mathbf{x'}_k, t) \tag{3.67}$$

$$= \prod_{k=1}^{n} [u_{\alpha_k}(\mathbf{x}_k + \rho(\mathbf{x}_0, t_0|t), t) - u_{\alpha_k}(\mathbf{x'}_k + \rho(\mathbf{x}_0, t_0|t), t)]. \tag{3.68}$$

Consequently, the functional $\mathcal{M}_{\mathbf{x}_0}[\beta,\rho]$ reads

$$\mathcal{M}_{\mathbf{x}_0}[\beta,\rho] = \left\langle M[u,\rho]\exp\left(-i\int dt\ \beta_\alpha(t)u_\alpha(\mathbf{x}_0+\rho(t),t)\right)\right\rangle \tag{3.69}$$

$$= \sum_{m=0}^{+\infty}\frac{(-i)^m}{m!}\int\cdots\int dt_1\cdots dt_m\ \beta_{\alpha_1}(t_1)\cdots\beta_{\alpha_m}(t_m)$$
$$\times\left\langle\left[\prod_{l=1}^{n}W_{\beta_l}(\mathbf{x}_0,t_0|\mathbf{x}_l,\mathbf{x'}_l,t)\right]\left[\prod_{k=1}^{m}u_{\alpha_k}(\mathbf{x}_0+\rho(t_k),t_k)\right]\right\rangle. \tag{3.70}$$

From the assumption $u\in\mathcal{H}^*_\omega$ we see that the ensemble average in the equation above is invariant with respect to a uniform spatial shift. It follows that $\mathcal{M}_{\mathbf{x}_0}[\beta,\rho+\mathbf{x}] = \mathcal{M}_{\mathbf{x}_0}[\beta,\rho],\forall\mathbf{x}\in\mathbb{R}^d$, and using proposition 3, this implies that $\mathcal{M}(\mathbf{x}_0,t_0+\Delta t)=\mathcal{M}(\mathbf{x}_0,t_0),\ \forall\Delta t\in\mathbb{R}$. Consequently, we have

$$\mathcal{F}_n(\mathbf{x}_0,t_0|\{\mathbf{X}\}_n,t) = \mathcal{M}(\mathbf{x}_0,t_0) = \mathcal{M}(\mathbf{x}_0,t_0+\Delta t) \tag{3.71}$$

$$= \mathcal{F}_n(\mathbf{x}_0,t_0+\Delta t|\{\mathbf{X}\}_n,t),\ \forall n\in\mathbb{N}^* \tag{3.72}$$

□

The claim (3.43) follows by combining proposition 4 with proposition 1. Note that the many-time homogeneity assumption $\mathbf{u}\in\mathcal{H}^*_\omega$ is in fact also a *necessary* condition because it is necessary that the ensemble average in (3.70) be invariant with respect to uniform spatial shifting for all functions $\beta_{\alpha_k}(t)$. If one chooses to use for $\beta_{\alpha_k}(t)$ very narrow normalized Gaussian curves that approach the limit of the delta function, then the integrals in (3.70) will constrain the evaluation of the ensemble average to the many-time case. Finally, the homogeneity condition has to be a global condition; the asymptotic condition $\mathbf{u}\in\mathcal{H}^*_\omega(\mathcal{A})$ is not sufficient.

It should be noted that once the relationship between quasi-Lagrangian correlation functions and Eulerian correlation functions is established, it can be easily extended to response functions as well without making any further assumptions. Starting from the stationarity condition (3.46), we deduce from the quasi-Lagrangian formulation of the Navier-Stokes equations that the quasi-Lagrangian forcing field is also stationary. Then, we may use an MSR theory on the quasi-Lagrangian Navier-Stokes equations to obtain stationarity on the response functions. From there, the relationship between the quasi-Lagrangian response functions and the Eulerian response functions can be easily established.

3.2.4 Why the quasi-Lagrangian transformation requires $\mathbf{u}\in\mathcal{H}^*_\omega$

We would now like to discuss is why the assumption $\mathbf{u}\in\mathcal{H}^*_\omega$ is required in order to employ the quasi-Lagrangian formulation, and consider an interesting alternative. The artifact introduced by the quasi-Lagrangian formulation is that the turbulent velocity field is being perceived from the viewpoint

of an arbitrary fluid particle whose own motion is also stochastic. Consequently, to relate the quasi-Lagrangian correlation tensor $\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{X}\}_n, t)$ with the Eulerian correlation tensor $F_n(\{\mathbf{X}\}_n, t)$ a certain sense of homogeneity is required to ensure that the velocity field is being perceived by the fluid particle in the same way regardless of the actual position of the particle. Our analysis of the proof, given previously, has shown that the required homogeneity condition is stronger than what is required to eliminate the sweeping interactions. What is particularly interesting about the stronger condition $\mathbf{u} \in \mathcal{H}^*_\omega$ is that it requires translational invariance from a group of the *many-time* correlation tensors $F^*_{m,n}$.

Let us now consider an alternative approach. Introduce the conditional correlation tensor defined as

$$\mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y}|\{\mathbf{X}\}_n, t) = \left\langle \prod_{k=1}^{n} W_{\alpha_k}(\mathbf{x}_0, t_0|\mathbf{x}_k, \mathbf{x'}_k, t) \middle| \rho(\mathbf{x}_0, t_0|t) = \mathbf{y} \right\rangle \tag{3.73}$$

$$= \left\langle \prod_{k=1}^{n} w_{\alpha_k}(\mathbf{x}_k + \mathbf{y}, \mathbf{x'}_k + \mathbf{y}, t) \middle| \rho(\mathbf{x}_0, t_0|t) = \mathbf{y} \right\rangle \tag{3.74}$$

This definition is identical to the definition of the quasi-Lagrangian correlation tensor $\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{X}\}_n, t)$, except that the ensemble average is replaced with the conditional average predicated on the fluid particle being located at position $\mathbf{y}$ at a given time t.

Definition 5 *The velocity field* $\mathbf{u}$*, as a stochastic field, is a member of the homogeneity class* $\mathcal{H}^c_0$*, if and only if the ensemble average defined as*

$$\sum_{k=1}^{n} (\partial_{\beta_k, \mathbf{y}_k} + \partial_{\beta, \mathbf{y'}_k}) \mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y}|\{\mathbf{X}\}_n, t) = 0, \forall \mathbf{y} \in \mathbb{R}^d, \forall \mathbf{x}_k, \mathbf{x'}_k \in \mathbb{R}^d, \forall n \in \mathbb{N}, n > 1 \tag{3.75}$$

The physical interpretation of this definition is that the local homogeneity symmetry of the turbulent velocity field over an ensemble constrained by the condition that the fluid particle being somewhere should be preserved if the condition changes to the fluid particle being somewhere else. Because we have to cover all possibilities for the fluid particle's location, including locations far away from the area of interest, it is not hard to see that this is a very strong assumption.

We will now show that the coveted relationship $\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{X}\}_n, t) = F_n(\{\mathbf{X}\}_n, t)$ can be obtained if only the conditional average $\mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y}|\{\mathbf{X}\}_n, t)$ is itself homogeneous.

Proposition 5 *The condition* $\mathcal{H}^c_0$ *implies that*

$$\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{X}\}_n, t) = F_n(\{\mathbf{X}\}_n, t) \tag{3.76}$$

Proof: Let $p(\mathbf{x}_0, t_0|\mathbf{x}, t)$ be the probability that a fluid particle originating at $(\mathbf{x}_0, t_0)$ will be located at $\mathbf{x}$ at time t. It follows that

$$\mathcal{F}_n(\mathbf{x}_0, t_0|\{\mathbf{X}\}_n, t) = \int d\mathbf{y}\ \mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y}|\{\mathbf{X}\}_n, t) p(\mathbf{x}_0, t_0|\mathbf{y}, t) \tag{3.77}$$

$$= \int d\mathbf{y}\ \mathcal{F}_n(\mathbf{x}_0, t_0, \mathbf{y}|\{\mathbf{X}\}_n - \mathbf{y}, t) p(\mathbf{x}_0, t_0|\mathbf{y}, t) \tag{3.78}$$

$$= F_n(\{\mathbf{X}\}_n, t) \tag{3.79}$$

□

Since the condition $\mathcal{H}_0^c$ can eliminate the sweeping interactions, it immediately follows that

$$\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_0^c. \tag{3.80}$$

An interesting question would be to determine the relative strength of the condition $\mathcal{H}_0^c$ versus $\mathcal{H}_\omega^*$.

This result is interesting, in part, because it shows that defining local homogeneity with a conditional ensemble average dramatically strengthens the power of the definition. As we have mentioned previously, in his first paper, Kolmogorov [130] also defined local homogeneity using a conditional ensemble average conditioned on the velocity field. Because of the approximate nature of the quasi-Lagrangian transformation used by Kolmogorov (which is not identical to the quasi-Lagrangian transformation of Belinicher and L'vov), his definition can be rephrased in terms of a conditional average on the location of the fluid particle. We may conjecture that Kolmogorov had the sweeping interactions in mind when he formulated his definition, but there is no such explicit indication in his papers.

3.3 Further remarks on the sweeping interactions

We have seen that when using the quasi-Lagrangian transformation we are required to make the homogeneity assumption $\mathbf{u} \in \mathcal{H}_\omega^*$ which is much stronger than the assumption of local homogeneity $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$, in the sense of the Frisch framework, otherwise we cannot return back to the Eulerian representation. Furthermore, this homogeneity assumption is introduced implicitly just by assuming stationarity in the quasi-Lagrangian representation, even if we don't wish to go back to the Eulerian representation (see proposition 2). The question that we would like to consider now is whether the utility of the theoretical work that relies on the transformation itself is reduced. We would like to claim that this is not the case, and define a line of investigation that can clarify this outstanding issue.

The first thing to consider is that the stronger homogeneity assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ is needed to establish the 4/5-law. Given that the validity of this law has been confirmed both in numerical

simulations and experiments validates from an experimental point of view the hypothesis that the sweeping interactions can be disregarded with impunity. The main two question that needs to be addressed then is: How does one justify *theoretically* the elimination of the sweeping interactions?

3.3.1 Elimination of the sweeping interactions

Formulating a rigorous mathematical argument that can justify why the sweeping interactions are not relevant in the inertial range of the energy cascade is a very difficult challenge. In this paper we will give some ad hoc plausibility arguments and point, in a qualitative sense, to the principle involved. However the underlying theoretical problem remains open.

It is widely accepted that the behavior of the structure functions in the inertial range does not depend on the statistical properties of forcing, as long as the spectrum of the forcing term is confined to large length scales. In a sense, as the energy cascades toward smaller length scales, the characteristic features of the forcing term are "forgotten". One may conjecture that the sweeping interactions behave in a similar way as a large-scale forcing term whose effect is forgotten in the inertial range. We may base this conjecture on the fact that even though the required homogeneity symmetry $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ may not hold exactly, it can be expected to hold asymptotically at small scales. Consequently, even though we cannot set $I_n(\{\mathbf{X}\}_n) = 0$ exactly, we might expect this term to become rapidly small when the average separation R behind the points $\{\mathbf{X}\}_n$ goes to zero. But does it vanish rapidly enough? A rigorous argument would have to estimate how fast I_n, as a function of R, is approaching zero in the small-scale limit $R/\ell_0 \to 0$, and then calculate the scaling exponent Δ_n associated with the ratio

$$\frac{I_n(R\{\mathbf{X}\}_n)}{(\mathcal{O}_n F_{n+1})(R\{\mathbf{X}\}_n)} \sim \left(\frac{R}{\ell_0}\right)^{\Delta_n}, \tag{3.81}$$

where R is the scaling parameter and ℓ_0 the forcing scale. Then, provided that one starts with the assumption $\mathbf{u} \in \mathcal{H}_0$, proving $\Delta_n > 0$ is also a proof that $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ which is sufficient to eliminate the sweeping interactions. Let λ_n be the scaling exponent of $I_n(R\{\mathbf{X}\}_n)$. If we assume that the generalized structure functions $F_n(R\{\mathbf{X}\}_n)$ satisfy the fusion rules [171, 172], then the scaling exponent of $\mathcal{O}_n F_{n+1}(R\{\mathbf{X}\}_n)$ is $\zeta_{n+1} - 1$ and it follows that $\Delta_n = \lambda_n - (\zeta_{n+1} - 1)$. The challenge, then, is to calculate the scaling exponents λ_n which are not likely to be universal.

It is easy to see that this argument cannot be extended to the inverse energy cascade of two-dimensional turbulence. In that case, the forcing term operates at large wavenumbers. Given that we can be quite certain that the inverse energy cascade is local, we can assume that the forcing term is forgotten in the inertial range. The problem is that the energy is going towards small wavenumbers. As we have noted, I_n essentially measures how much homogeneity is violated at a given length scale

R. At large length scales, the flow will begin to sense the violation of homogeneity caused by the boundary conditions which will in turn make the sweeping term I_n larger in magnitude. If it becomes comparable to $\mathcal{O}_n F_{n+1}$, it will probably disrupt the inverse energy cascade.

Numerical simulations have shown that it is possible to obtain an inverse energy cascade under certain conditions [27], but it can also be disrupted under other conditions [31, 59–61]. Physically, this disruption arises from the spontaneous generation of long-lived coherent vortices that carry a significant amount of enstrophy. An explanation of this effect was given by Boffetta *et al* [27], in terms of the "bottleneck" effect [82]. The general idea is that the behavior of the energy spectrum in the inertial range is modified at wavenumbers near the dissipation range because some of the triad interactions at these length scales are disrupted by the dissipation term, thus making the transfer of energy less efficient. It is reasonable to anticipate the same effect in a high-resolution simulation of the inverse energy cascade, where the cascade has manifested successfully, without being arrested by coherent structures. However, we would like to suggest that the deviations observed by Danilov and Gurarie [59–61] and Borue [31] are more likely to be caused by a similar effect where the triad interactions are disrupted *by the sweeping term I_n rather than the dissipation term* at large scales. From a theoretical perspective, one can say that I_n excites anisotropic sectors of the $SO(3)$ decomposition of the structure functions F_n which combine linearly with the locally isotropic contribution. It follows then that to obtain an inverse energy cascade in the forced-dissipative setting, one requires a dissipation term at large scales which will not only dispose of the incoming energy, but will also damp out the sweeping term I_n over the entire range of length scales where it is comparable to $\mathcal{O}_n F_{n+1}$.

We have referenced the inverse energy cascade of two-dimensional turbulence as an example where it is not safe to "eliminate" the sweeping interactions. The criticism of the quasi-Lagrangian formulation by Mou and Weichman [186] is essentially that it has not been demonstrated that it is safe in the same sense to eliminate the sweeping interactions in the downscale energy cascade of three-dimensional turbulence.

3.3.2 Estimating the scaling exponent Δ_n

The problem of calculating the scaling exponents λ_n needs to be investigated primarily with numerical simulations and the analysis of experimental data. However, it is possible to make a speculative theoretical calculation, if we are willing to commit the following crimes against reality: First, we assume that the velocity field $u_\alpha(\mathbf{x}, t)$ can be modeled as a random gaussian delta-correlated (in time) stochastic field acting at large scales. Furthermore, we assume that the velocity field $u_\alpha(\mathbf{x}, t)$

has an effect on the velocity differences $w_\alpha(\mathbf{x},\mathbf{x}',t)$ via the sweeping interactions, but completely disregard the reverse effect of the velocity differences on the velocity field via eddy viscosity, and the fact that $u_\alpha(\mathbf{x},t)$ and $w_\alpha(\mathbf{x},\mathbf{x}',t)$ are obviously constrained by the definition of $w_\alpha(\mathbf{x},\mathbf{x}',t)$. Under these assumptions we will show that the sweeping interactions can be neglected in the inertial range.

We have shown in Appendix D that under these assumptions the sweeping term $I_n(\{\mathbf{X}\}_n,t)$ can be decomposed into three contributions:

$$I_n(\{\mathbf{X}\}_n,t) = I_{n,(1)}(\{\mathbf{X}\}_n,t) + I_{n,(2)}(\{\mathbf{X}\}_n,t) + I_{n,(3)}(\{\mathbf{X}\}_n,t), \tag{3.82}$$

which are given by

$$I^{\alpha_1\alpha_2\cdots\alpha_n}_{n,(1)}(\{\mathbf{X}\}_n,t) = \langle \mathcal{U}_\beta(\{\mathbf{X}\}_n,t)\rangle\, H^{\alpha_1\alpha_2\cdots\alpha_n\beta}_n(\{\mathbf{X}\}_n,t), \tag{3.83}$$

$$I^{\alpha_1\alpha_2\cdots\alpha_n}_{n,(2)}(\{\mathbf{X}\}_n,t) = \sum_{l=1}^{n}\sum_{m=1}^{n} F^{\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n}_{n-1}(\{\mathbf{X}\}^m_n,t) I_{\alpha_m}(\mathbf{X}_m,\mathbf{X}_l,t), \tag{3.84}$$

$$I^{\alpha_1\alpha_2\cdots\alpha_n}_{n,(3)}(\{\mathbf{X}\}_n,t) = \sum_{l=1}^{n}\sum_{m=1}^{n} B^{\alpha_m\beta}_1(\mathbf{X}_m,\mathbf{X}_l,t) H^{\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n\beta}_{n-1}(\{\mathbf{X}\}^m_n). \tag{3.85}$$

Here, $H^{\alpha_1\alpha_2\cdots\alpha_n}_n$, $B^{\alpha\beta}_1$, and I_α are defined as

$$H^{\alpha_1\alpha_2\cdots\alpha_n\beta}_n(\{\mathbf{X}\}_n,t) = \left[\sum_{k=1}^{n}(\partial_{\beta,\mathbf{x}_k} + \partial_{\beta,\mathbf{x}'_k}) F^{\alpha_1\alpha_2\cdots\alpha_n}_n(\{\mathbf{X}\}_n,t)\right], \tag{3.86}$$

$$B^{\alpha\beta}_1(\mathbf{X},\mathbf{Y},t) = \langle \mathcal{U}_\beta(\mathbf{Y},t) w_\alpha(\mathbf{X},t)\rangle - \langle \mathcal{U}_\beta(\mathbf{Y},t))\rangle \langle w_\alpha(\mathbf{X},t)\rangle\,, \tag{3.87}$$

$$I_\alpha(\mathbf{X}_1,\mathbf{X}_2,t) = \sum_{k=1}^{2}(\partial_{\beta,\mathbf{x}_k} + \partial_{\beta,\mathbf{x}'_k}) B^{\alpha\beta}_1(\mathbf{X}_1,\mathbf{X}_2,t). \tag{3.88}$$

It is worth noting that the assumption $\mathbf{u} \in \mathcal{H}_0$ implies that $H_n(\{\mathbf{X}\}_n,t) = 0$ and therefore $I_{n,(1)}(\{\mathbf{X}\}_n,t) = I_{n,(3)}(\{\mathbf{X}\}_n,t) = 0$. However, we will retain generality and keep all three terms. It should also be stressed that we are not assuming statistical independence between the velocity field $u_\alpha(\mathbf{x},t)$ and the velocity differences $w_\alpha(\mathbf{x},\mathbf{x}',t)$. On the contrary, we assume that the two are related to each other in the sense that the velocity field is forcing the velocity differences via the sweeping interactions. However, the case of total statistical independence gives exactly $I_n(\{\mathbf{X}\}_n,t) = I_{n,(1)}(\{\mathbf{X}\}_n,t)$, so it is covered by our argument bellow.

Let λ be the scaling exponent of $B^{\alpha\beta}_1$ such that

$$B^{\alpha\beta}_1(R\mathbf{X},R\mathbf{Y},t) \sim (R/\ell_0)^\lambda. \tag{3.89}$$

It immediately follows that I_α scales as

$$I_\alpha(R\mathbf{X}_1,R\mathbf{X}_2,t) \sim (R/\ell_0)^\lambda (1/R) g(R). \tag{3.90}$$

Here, $g(R)$ is a smooth function which represents departure from local homogeneity in the sense $\mathbf{u} \in \mathcal{H}_1$. Without loss of generality, we associate the scaling exponent b to the function $g(R)$. The contribution $(1/R)$ arises from the derivatives. Using a similar line of argument we see that H_n scales as

$$H_n^{\alpha_1\alpha_2\cdots\alpha_n}(R\{\mathbf{X}\}_n, t) \sim (R/\ell_0)^{\zeta_n}(1/R)f_n(R), \tag{3.91}$$

where $f_n(R)$ is also a smooth function representing departure from local homogeneity in the sense $\mathbf{u} \in \mathcal{H}_0$. We associate the scaling exponent a_n to the function $f_n(R)$. The three contributions to $I_n(\{\mathbf{X}\}_n, t)$ then scale as

$$I_{n,(1)}(R\{\mathbf{X}\}_n, t) \sim (R/\ell_0)^{\zeta_n}(1/R)f_n(R), \tag{3.92}$$

$$I_{n,(2)}(R\{\mathbf{X}\}_n, t) \sim (R/\ell_0)^{\zeta_{n-1}}(R/\ell_0)^{\lambda}(1/R)g(R), \tag{3.93}$$

$$I_{n,(3)}(R\{\mathbf{X}\}_n, t) \sim (R/\ell_0)^{\lambda}(R/\ell_0)^{\zeta_{n-1}}(1/R)f_{n-1}(R), \tag{3.94}$$

and from power counting we find that the corresponding scaling exponents are

$$\lambda_{n,1} = \zeta_n - 1 + a_n, \quad \lambda_{n,2} = \zeta_{n-1} + \lambda - 1 + b, \quad \text{and } \lambda_{n,3} = \zeta_{n-1} + \lambda - 1 + a_{n-1}. \tag{3.95}$$

Using the multifractal formulation, the contribution that supports the Hölder exponent h gives $\zeta_n = nh + \mathcal{Z}(h)$, which gives the following evaluation for the scaling exponents Δ_n:

$$\Delta_{n,1}(h) = (\zeta_n - 1 + a_n) - (\zeta_{n+1} - 1) = -h + a_n, \tag{3.96}$$

$$\Delta_{n,2}(h) = (\zeta_{n-1} + \lambda - 1 + b) - (\zeta_{n+1} - 1) = -2h + \lambda + \beta, \tag{3.97}$$

$$\Delta_{n,3}(h) = (\zeta_{n-1} + \lambda - 1 + a_{n-1}) - (\zeta_{n+1} - 1) = -2h + \lambda + a_{n-1}. \tag{3.98}$$

Because the functions $f_n(R)$ and $g(R)$ are smooth, we can Taylor-expand them around $R = 0$ and get, to first order, $a_n = b = 1$. It is also reasonable to assume that $\lambda > 0$. From these evaluations we find that the window for positive scaling exponents Δ_n is at least $h \in (0, 1/2)$. Admittedly, this is a rather narrow interval, even though it is sufficient for the downscale energy cascade of three-dimensional turbulence. However, the situation is probably a lot better than that. If we allow negative evaluations of R , which can be defined by reflecting the points $\{\mathbf{X}\}_n$ around their center of mass, we may expect that $R = 0$ is an extremum and therefore $f_n'(R) = g'(R) = 0$. It is easy to show that, using the evaluation $a_n = b = 2$, we find that the window for positive scaling exponents Δ_n covers the entire range $h \in (0, 1)$ of local scaling exponents. Although this is somewhat encouraging, the real challenge is to determine what happens in reality and make a comparison of that against the speculative predictions given above.

3.3.3 An alternative to Lagrangian transformations

It is possible to use the theoretical work based on the quasi-Lagrangian transformation in a way that requires only the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$. This can be done via the following line of argument: The quasi-Lagrangian formulation modifies the Navier-Stokes equations by redefining the material derivative. The modified equation remains mathematically equivalent to the Navier-Stokes equation because the velocity field is reinterpreted from an Eulerian field into a quasi-Lagrangian field. It is precisely this reinterpretation which necessitates the stronger assumption $\mathbf{u} \in \mathcal{H}^*_\omega$ to enable a return back to the Eulerian representation. On the other hand, if we accept the hypothesis that the sweeping interactions can be absorbed into the statistical forcing term, we can just modify the equation of motion in precisely the same way without interpreting the velocity field as quasi-Lagrangian, but rather as Eulerian. From there, one can derive the same balance equations (3.26) except that $I_n = 0$, and consequently the only assumption that is being made implicitly is just $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$. One may then proceed from this Eulerian modified Navier-Stokes equation and work through the argument of the papers [16, 17, 164, 167, 168, 170, 175, 176] with impunity, since the governing equation would have the same mathematical form as the quasi-Lagrangian Navier-Stokes equation.

In connection with this argument, it is interesting to note that the idea of just modifying the Navier-Stokes equation was considered by Kraichnan [142] in 1964 who suggested a more crude modification. This modification brute-forces locality in Fourier space by simply discarding triad interactions accross a wide wavenumber interval and retaining only the local triad interactions. From the same paper we learn that Kraichnan suspected that there was a relationship between the quasi-Lagrangian transformation of Kolmogorov and the general idea of modifying the Navier-Stokes equation in such a way but noted that bringing that out rigorously is difficult. In my view the quasi-Lagrangian transformation of Belinicher and L'vov, which is different from the Lagrangian transformation used by Kraichnan in his theories, is the key to finding possibly the best way to modify the Navier-Stokes equations in the way that Kraichnan intended.

An alternative argument that was proposed by Yakhot [262] and used by Giles [103] to calculate a perturbation expansion for the scaling exponents ζ_n eliminates the sweeping interactions by modifying the statistical theory itself. This is different from the quasi-Lagrangian formulation and this proposal where the change is made on the governing equation and *then* propagated into the statistical theory. Again, to justify why one can modify the statistical theory requires the assumption $\mathbf{u} \in \mathcal{H}_1(\mathcal{A})$ or an argument justifying the hypothesis that the sweeping interactions can be modeled as large-scale stochastic forcing, which brings us back to the challenge of showing that $\Delta_n > 0$.

3.4 Discussion and Conclusion

In the original formulation of his theory, Kolmogorov assumed local homogeneity, local isotropy, and local stationarity in a non-Eulerian representation very similar to the quasi-Lagrangian representation of Belinicher and L'vov. Frisch [93, 94] revised this argument by stating the same assumptions in the Eulerian representation. This is a decision that we agree with, because the energy spectrum and the structure functions are both Eulerian rather than Lagrangian quantities. Frisch [94] has also chosen to strengthen the assumption of self-similarity to make it possible to deduce all the scaling exponents ζ_n and obtain the prediction $\zeta_n = n/3$. Ultimately, this assumption needs to be replaced with a weaker assumption of self-similarity to permit intermittency corrections, and this is the approach followed in the papers [16, 17, 164, 167, 168, 170, 175, 176]. Frisch himself proposed the multi-fractal hypothesis [94], which converges with the approach of Belinicher *et al.* in the papers [16, 17, 175] in a very interesting way. Finally, Frisch [94] made the very important observation that, in order to carry Kolmogorov's argument through, it is necessary to assume the existence of an anomalous energy sink.

In the present paper we have shown that the assumptions of the Frisch framework are still not strong enough to prove the 4/5-law in the Eulerian representation. We have also argued that a rigorous proof that establishes $\Delta_n > 0$ would rectify this problem. A positive response to the question raised recently by Frisch [95] concerning the self-consistency of local homogeneity in the $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$ sense would also follow from the validity of the conjecture $\Delta_n > 0$. This would be a fundamental breakthrough finally putting to rest the problem of the sweeping interactions that has concerned the community for the last 60 years. It would essentially establish that the sweeping interactions can be modeled as stochastic forcing acting only at large scales. Then we can simply drop from the Navier-Stokes equations the portion of the nonlinearity associated with the sweeping interactions, and build the entire statistical theory on the modified Navier-Stokes equations. We have also explained why we cannot use the quasi-Lagrangian formulation to go around the problem. The reason is that using the formulation requires the even stronger assumption $\mathbf{u} \in \mathcal{H}^*_\omega$.

We would also like to emphasize that our conclusions are *not* a criticism of the extensive and very important theoretical work *based* on the quasi-Lagrangian formulation [16, 17, 164, 167, 168, 170, 175, 176]. In fact, as long as one's intention is to solve the problem of *globally* homogeneous turbulence, there is no issue whatsoever with respect to the sweeping interactions and the quasi-Lagrangian transformation, provided that the assumption of global homogeneity is many-time rather than one-time. On the other hand, it is desirable to move away from the assumptions of global homogeneity and global isotropy, which cannot be physically realized, and take steps towards building a theory

based on the assumptions of asymptotic local homogeneity $\mathbf{u} \in \mathcal{H}_0(\mathcal{A})$ and local isotropy in an Eulerian framework, as envisioned by Frisch. Our paper implies that the results in the previous work [16, 17, 164, 167, 168, 170, 175, 176] can be readily carried over and applied towards this goal provided that the hypothesis $\Delta_n > 0$ is proved, and our proposal rather than the quasi-Lagrangian transformation is employed.

Chapter 4

UNDERSTANDING 2D TURBULENCE AND THE NASTROM-GAGE SPECTRUM

Large-scale flows in thin fluid shells, such as planetary atmospheres and the ocean, tend to be quasi-two-dimensional. Two-dimensional flows differ from three-dimensional turbulence in that there are usually two closely related conservative quantities exchanged by nonlinear triad interactions. Furthermore, the cascades of two-dimensional turbulence do not exhibit universal behavior with the same degree of consistency that we have come to expect from three-dimensional turbulence. Also interesting is the inverse energy cascade, unique in "2d-like" systems, where an initially noisy velocity field continuously forced by white noise small-scale forcing will nonetheless evolve into coherent vortical structures. The striking resemblence between the pattern formation of two-dimensional turbulence and similar patterns in the atmospheres of gas-giant planets, like Jupiter, tickles the imagination and raises interesting but hard questions. [264]

When Kraichnan [145], Leith [150] and Batchelor [8] first pioneered the study of two-dimensional turbulence, it was thought that it would be easier to handle theoretically and simpler to simulate numerically than three-dimensional turbulence. The fact that no convincing simulation of the dual cascades predicted by KLB, with an upscale energy cascade and a downscale enstrophy cascade, has been achieved during the ensuing three decades is a hint that the problem of two-dimensional turbulence is richer than was thought, perhaps even richer than three-dimensional isotropic homogeneous turbulence. In addition, because geophysical fluids behave more like two-dimensional fluids than three-dimensional isotropic homogeneous fluids, it is not possible to simply ignore the theoretical and numerical problems of two-dimensional turbulence on the grounds that it is a fictitious fluid.

4.1 Dynamics of two-dimensional turbulence

Let $u_\alpha(\mathbf{r}, t)$ be the Eulerian velocity field. The governing equations of two-dimensional turbulence are:

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p + \mathcal{D} u_\alpha + f_\alpha, \tag{4.1}$$

$$\partial_\alpha u_\alpha = 0, \tag{4.2}$$

where f_α is the forcing term, and $\mathcal{D}$ is the dissipation operator given by

$$\mathcal{D} \equiv (-1)^{\kappa+1}\nu_\kappa \nabla^{2\kappa} + (-1)^{m+1}\beta\nabla^{-2m} \tag{4.3}$$

Here the integers κ and m describe the order of the dissipation mechanisms, and the numerical coefficients ν_κ and β are the corresponding viscosities. $\mathcal{D}$ is the overall dissipation operator. The term f_α represents stochastic forcing that injects energy into the system at a range of length scales in the neighborhood of the integral length scale ℓ_0. The term $\beta\nabla^{-2m}u_\alpha$ describes a dissipation mechanism that operates on large-scale motions. The operator ∇^{-2m} represents applying the inverse Laplacian ∇^{-2} repeatedly m times. In Fourier space it is diagonalized, and its definition may therefore be extended to fractional values for m. The same holds for κ.

The case $\kappa = 1$ corresponds to standard molecular viscosity. The interaction of the atmosphere with the viscous Ekman boundary layer introduces an energy sink to the interior fluid, known as Ekman damping, that corresponds to the case $m = 0$ [202]. The same case also seems to describe an energy dissipation mechanism in soap film experiments [214]. In this sense, one may claim that the case $m = 0$ is "physical" and the case $m > 0$ is "artificial", or numerical.

4.1.1 Reformulations of governing equations

To eliminate pressure we multiply both sides of the Navier-Stokes equation with the operator $\mathcal{P}_{\alpha\beta} \equiv \delta_{\alpha\beta} - \partial_\alpha\partial_\beta\nabla^{-2}$ and we employ $\mathcal{P}_{\alpha\beta}u_\beta = u_\beta$ and $\mathcal{P}_{\alpha\beta}\partial_\beta = 0$ to obtain

$$\frac{\partial u_\alpha}{\partial t} + \mathcal{P}_{\alpha\beta}\partial_\gamma(u_\beta u_\gamma) = \mathcal{D}u_\alpha + \mathcal{P}_{\alpha\beta}f_\beta \tag{4.4}$$

The operator $\mathcal{P}_{\alpha\beta}$ can be expressed in terms of a kernel $P_{\alpha\beta}(\mathbf{x})$ as

$$\mathcal{P}_{\alpha\beta}v_\beta(\mathbf{x}) = \int d\mathbf{y}P_{\alpha\beta}(\mathbf{x}-\mathbf{y})v_\beta(\mathbf{y}) = \int d\mathbf{y}P_{\alpha\beta}(\mathbf{y})v_\beta(\mathbf{x}-\mathbf{y}) \tag{4.5}$$

For two-dimensional turbulence $P_{\alpha\beta}(\mathbf{x})$ is given by

$$P_{\alpha\beta}(\mathbf{x}) = \delta_{\alpha\beta}\delta(\mathbf{x}) - \frac{1}{2\pi}\left[\frac{\delta_{\alpha\beta}}{r^2} - 2\frac{x_\alpha x_\beta}{r^4}\right] \tag{4.6}$$

The scalar vorticity ζ is given by $\zeta = \varepsilon_{\alpha\beta}\partial_\alpha u_\beta$ with $\varepsilon_{\alpha\beta}$ being the Levi-Civita tensor in two dimensions. From the incompressibility condition $\partial_\alpha u_\alpha = 0$, it follows that there is a function ψ, called the streamfunction, such that $u_\alpha = \varepsilon_{\alpha\beta}\partial_\beta\psi$. Using the identity $\varepsilon_{\alpha\beta}\varepsilon_{\beta\gamma} = \delta_{\alpha\gamma}$ one then shows that $\zeta = \varepsilon_{\alpha\beta}\varepsilon_{\beta\gamma}\partial_\alpha\partial_\gamma\psi = \nabla^2\psi$ from which we get $\psi = \nabla^{-2}\zeta$ and $u_\alpha = \varepsilon_{\alpha\beta}\partial_\beta\nabla^{-2}\zeta$.

The vorticity equation is obtained by differentiating ζ with respect to time and employing the Navier-Stokes equations:

$$\frac{\partial\zeta}{\partial t} + J(\psi,\zeta) = \mathcal{D}\zeta + g \tag{4.7}$$

where $J(\psi, \zeta)$ is the Jacobian defined as

$$J(A, B) = \varepsilon_{\alpha\beta}(\partial_\beta A)(\partial_\alpha B) \tag{4.8}$$

and $g = \varepsilon_{\alpha\beta}\partial_\alpha f_\beta$ is the forcing term. The nonlinear term $J \equiv J(\psi, \zeta)$ is obtained by employing the following argument

$$J = \varepsilon_{\alpha\beta}\partial_\alpha \mathcal{P}_{\beta\gamma}\partial_\delta(u_\gamma u_\delta) = \varepsilon_{\alpha\beta}\partial_\alpha[u_\gamma \partial_\gamma u_\beta] \tag{4.9}$$

$$= u_\gamma \partial_\gamma \zeta + (\varepsilon_{\alpha\beta}\partial_\alpha u_\gamma)(\partial_\gamma u_\beta) = u_\gamma \partial_\gamma \zeta = J(\psi, \zeta) \tag{4.10}$$

The term $(\varepsilon_{\alpha\beta}\partial_\alpha u_\gamma)(\partial_\gamma u_\beta)$ represents vortex stretching, but in two dimensions it can be shown that

$$(\varepsilon_{\alpha\beta}\partial_\alpha u_\gamma)(\partial_\gamma u_\beta) = 0 \tag{4.11}$$

by direct substitution of the vector components.

4.1.2 Conservation laws

The critical feature that distinguishes two-dimensional turbulence from three-dimensional turbulence is that there are two relevant conservation laws rather than just one.

It can be shown that if two arbitrary fields $a(\mathbf{x}, t)$ and $b(\mathbf{x}, t)$ satisfy a homogeneous (Dirichlet or Neumann) boundary condition, then $\langle\langle J(a, b)\rangle\rangle = 0$, where we use the notation

$$\langle\langle f\rangle\rangle \equiv \iint \langle f(x, y)\rangle \; dxdy, \tag{4.12}$$

for the combined spatial and ensemble average. It follows from the product rule of differentiation that

$$\langle\langle J(ab, c)\rangle\rangle = \langle\langle aJ(b, c)\rangle\rangle + \langle\langle bJ(a, c)\rangle\rangle = 0, \tag{4.13}$$

from which we obtain the identity

$$\langle\langle aJ(b, c)\rangle\rangle = \langle\langle bJ(c, a)\rangle\rangle = \langle\langle cJ(a, b)\rangle\rangle, \tag{4.14}$$

which was also shown previously by Tran and Shepherd [250]. This identity can be used to derive the conservation laws.

The equation $\partial\zeta/\partial t + J(\psi, \zeta) = 0$ conserves the enstrophy $G = (1/2)\langle\langle\zeta^2\rangle\rangle$ because

$$\langle\langle\dot{G}\rangle\rangle = \langle\langle\zeta\dot{\zeta}\rangle\rangle = \langle\langle-\zeta J(\psi, \zeta)\rangle\rangle = \langle\langle-\psi J(\zeta, \zeta)\rangle\rangle = 0. \tag{4.15}$$

To derive the conservation of energy we first note that the Laplacian operator ∇^2 and the inverse Laplacian operator ∇^{-2} are both self-adjoint in the sense that they satisfy $\langle\langle f(\nabla^2 g)\rangle\rangle = \langle\langle g(\nabla^2 f)\rangle\rangle$

and $\langle\!\langle f(\nabla^{-2}g)\rangle\!\rangle = \langle\!\langle g(\nabla^{-2}f)\rangle\!\rangle$for any fields $f(x,y)$ and $g(x,y)$. The self-adjoint property of the inverse Laplacian operator ∇^{-2} implies that the energy $E \equiv (1/2)\langle\!\langle -\psi\zeta\rangle\!\rangle$ is also conserved via the following argument:

$$\langle\!\langle \dot{E}\rangle\!\rangle = (1/2)\langle\!\langle -\psi\dot{\zeta} - \zeta\dot{\psi}\rangle\!\rangle = (1/2)[\langle\!\langle \psi J(\psi,\zeta)\rangle\!\rangle + \langle\!\langle \zeta\nabla^{-2}J(\psi,\zeta)\rangle\!\rangle] \tag{4.16}$$

$$= (1/2)[\langle\!\langle \psi J(\psi,\zeta)\rangle\!\rangle + \langle\!\langle \nabla^{-2}\zeta)J(\psi,\zeta)\rangle\!\rangle] = \langle\!\langle \psi J(\psi,\zeta)\rangle\!\rangle = \langle\!\langle \zeta J(\psi,\psi)\rangle\!\rangle = 0. \tag{4.17}$$

The energy spectrum $E(k)$ and the enstrophy spectrum $G(k)$ are defined as

$$E(k) = \frac{1}{2}\frac{d}{dk}\langle\!\langle -\psi^{<k}\zeta^{<k}\rangle\!\rangle, \tag{4.18}$$

$$G(k) = \frac{1}{2}\frac{d}{dk}\langle\!\langle (\zeta^{<k})^2\rangle\!\rangle, \tag{4.19}$$

with $\psi^{<k}$ and $\zeta^{<k}$ the streamfunction and vorticity fields with all the Fourier wavenumbers greater in magnitude than k filtered out. The spectral equations are obtained by differentiating $E(k)$ and $G(k)$ with respect to t, and employing the Fourier transform of the governing equation (4.7):

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi_E(k)}{\partial k} = -D_E(k) + F_E(k) \tag{4.20}$$

$$\frac{\partial G(k)}{\partial t} + \frac{\partial \Pi_G(k)}{\partial k} = -D_G(k) + F_G(k). \tag{4.21}$$

It is understood that ensemble averages have been taken in the above quantities. Here $\Pi_E(k)$ is the energy flux transfered from $(0,k)$ to $(k,+\infty)$ per unit time by the nonlinear term in (4.7), $D_E(k)$ the energy dissipation, and $F_E(k)$ the energy forcing spectrum, and likewise for the enstrophy (G) equation. The conservation laws imply for the viscous case that $\Pi_E(0) = \lim_{k\to\infty}\Pi_E(k) = 0$ and $\Pi_G(0) = \lim_{k\to\infty}\Pi_G(k) = 0$. For the inviscid case, this condition can be violated, in principle, by anomalous dissipation for solutions that have singularities. The energy and enstrophy spectrum are related as $G(k) = k^2E(k)$, and likewise it is easy to show that $D_B(k) = k^2D_A(k)$ and $F_B(k) = k^2F_A(k)$. Combining these equations with (4.20) and (4.21) we obtain the so-called Leith constraint [150]:

$$\frac{\partial \Pi_G(k)}{\partial k} = k^2\frac{\partial \Pi_E(k)}{\partial k}. \tag{4.22}$$

4.2 Direction of fluxes

It was recognized by Fjørtøft [92] and Charney [38] that the direction of net energy transfer in 2D and QG turbulence may be different from that for 3D isotropic and homogeneous turbulence and that the cause for this different behavior should be attributed to the former's twin conservation of energy and enstrophy. It is often claimed that Fjørtøft has shown that if a unit of energy is

moved downscale, many more units of it have to be moved upscale in order to preserve the twin energy and enstrophy conservation. However, Fjørtøft's analysis of triadic transfers was flawed [106, 180, 254]. His proof made use of the simultaneous conservation of energy and enstrophy in 2D and QG turbulence, and the fact that the enstrophy spectrum $G(k)$ is related to the energy spectrum $E(k)$ by $G(k) = k^2 E(k)$.

In his paper, Fjørtøft [92] gives two distinct proofs. The first proof does show that the only admissible triad interactions are those that spread energy from the middle wavenumber to the outer wavenumbers (and vice versa, the ones that bring in energy to the central wavenumber from the outer wavenumbers). These are the triad interactions defined by Waleffe [256] as class "R". An alternative set of triad interactions are the ones where energy is transfered from the smallest wavenumber to the two largest ones; these are the class "F" triad interactions, and they are dominant in three-dimensional turbulence. Fjørtøft's proof can be employed to rule these out in two-dimensional turbulence. *However*, as was pointed out by Merilees and Warn [180], there exist also class "R" triad interactions that transfer more energy downscale than upscale. Thus, eliminating the class "F" interactions is not sufficient to constrain the direction of the energy flux or the enstrophy flux.

To see this more clearly, let us carefully reconsider Fjørtøft's [92] argument, using the notation of Kraichnan [145]. Let $T(k,p,q)$ be the energy transfer to the wavenumber shell k from the wavenumber shells p and q. Detailed conservation of energy and enstrophy implies that

$$T(k,p,q) + T(p,q,k) + T(q,k,p) = 0 \tag{4.23}$$

$$k^2 T(k,p,q) + p^2 T(p,q,k) + q^2 T(q,k,p) = 0. \tag{4.24}$$

Solving for $T(p,q,k)$ and $T(q,k,p)$ in terms of $T(k,p,q)$ we find:

$$T(p,q,k) = \frac{k^2 - q^2}{q^2 - p^2}\, T(k,p,q) \quad \text{and} \quad T(q,k,p) = \frac{p^2 - k^2}{q^2 - p^2}\, T(k,p,q). \tag{4.25}$$

Let us assume that $p < k < q$. We see that $T(k,p,q) < 0$ implies that $T(p,q,k) > 0$ and $T(q,k,p) > 0$. Thus, class "F" triad interactions can be ruled out. The transfer difference towards the "outer" wavenumbers p and q from the inner wavenumber k is given by:

$$\Delta T(k,p,q) \equiv T(p,q,k) - T(q,k,p) = \frac{2k^2 - p^2 - q^2}{q^2 - p^2}\, T(k,p,q). \tag{4.26}$$

For the traditional example $p = k/2$ and $q = 2k$, we have $\Delta T(k,p,q) = -(9/15) T(k,p,q)$ which is in the upscale direction when $T(k,p,q) < 0$ (i.e. when energy goes to the outer wavenumbers). However, for $q^2 = \lambda(2k^2 - p^2)$ we have

$$\Delta T(k,p,q) = \frac{(2k^2 - p^2)(1-\lambda)}{q^2 - p^2}\, T(k,p,q). \tag{4.27}$$

We see that the transfer difference is upscale only when $\lambda > 1$ and is in fact downscale when $\lambda < 1$. The constraint $q > k$ implies $\lambda > k^2/(2k^2 - p^2)$, however it is easy to show that $p < k$ implies $1 > k^2/(2k^2 - p^2)$. So there is a critical region

$$\frac{k^2}{2k^2 - p^2} < \lambda < 1, \tag{4.28}$$

for the parameter λ where the outgoing triad interactions transfer more energy downscale than upscale for all $p < k$. Despite this problem, Fjørtøft's proof has been popularized in textbooks [221] and review articles [241] as a rigorous proof that constrains the direction of the fluxes in two-dimensional turbulence, thereby becoming a misunderstood "folklore" argument.

The second result of Fjørtøft [92] is an upper bound on the total energy accumulated on wavenumbers larger than some given k. This result however applies only to initial value problems without forcing, where energy unsurprisingly has to be bounded. This inequality was later taken by Charney [38] as a proof that energy cannot go downscale, since the energy $E^{>k}(t)$ accumulated at wavenumbers larger than k is bounded by

$$E^{>k}(t) = \int_k^\infty E(q)\, dq \leq \frac{1}{k^2} \int_k^\infty q^2 E(q)\, dq \leq \frac{1}{k^2} G(t) \leq \frac{1}{k^2} G(0), \tag{4.29}$$

where $G(t)$ is the total enstrophy at time t. Thus, the energy spectrum $E(k,t)$ is bounded by $E(k,t) \leq ck^{-3}$ for some constant c. Tung and Welch [254] pointed out that this behavior of the energy spectrum is merely a consequence of the requirement for convergence of the Fourier representation of the enstrophy spectrum $G(k)$, which implies that $G(k)$ must decay faster than k^{-1} as $k \to \infty$. Therefore the energy spectrum $E(k)$ must decay faster than k^{-3} as $k \to \infty$. It says nothing about the direction of energy cascade, thus it does not help Fjørtøft's "proof" in the first half of the paper.

Other proofs [74, 211, 212, 221, 224] have been reviewed recently by Gkioulekas and Tung [106]. In that paper [106], and also chapter 5 of this thesis, we have also given a very interesting unified proof both for the forced-dissipatice case and for the decaying case: Assume that the forcing spectrum $F_E(k)$ is confined to a narrow interval of wavenumbers $[k_1, k_2]$. Then, we have

$$F_E(k) = 0 \text{ and } F_G(k) = 0, \forall k \in (0, k_1) \cup (k_2, +\infty), \tag{4.30}$$

and it can be shown [106] for the forced dissipative case, without making any ad hoc assumptions, that under stationarity, the fluxes $\Pi_E(k)$ and $\Pi_G(k)$ will satisfy the inequalities

$$\int_0^k q\Pi_E(q)\, dq < 0,\ \forall k > k_2 \tag{4.31}$$

$$\int_k^{+\infty} q^{-3}\Pi_G(q) > 0,\ \forall k < k_1. \tag{4.32}$$

The constraint (4.31) holds trivially for $k < k_1$, since $\Pi_E(k) < 0$ for all $k < k_1$. For $k > k_2$, the integration range also includes the energy injection interval $[k_1, k_2]$ and both the upscale and the downscale cascade range. The inequality (4.31) implies that the negative flux in the $(0, k_1)$ interval is more intense than the positive flux in the $(k_2, +\infty)$ because the weighted average of $\Pi_E(k)$ gives more weight to the large wavenumbers. Thus, (4.31) implies that energy fluxes upscale in the net. Similarly, (4.32) implies that enstrophy fluxes downscale in the net. These results can be extended [106] to the decaying case provided that there exists a small wavenumber $\varepsilon_1 > 0$ and a large wavenumber $\varepsilon_2 > 0$ such that

$$D_A(q) + \frac{\partial A(k)}{\partial t} \geq 0, \ \forall k \in (0, \varepsilon_1) \tag{4.33}$$

$$D_A(q) + \frac{\partial A(k)}{\partial t} \geq 0, \ \forall k \in (\varepsilon_2, +\infty). \tag{4.34}$$

Note that (4.33) implies (4.31) and (4.34) implies (4.32).

It should be noted that, unlike previous proofs, in both the forced-dissipative and the decaying case, the inequalities (4.31) and (4.32) have the same mathematical form. Our argument then is a unified proof that covers all cases, and specialized results can be deduced from our inequalities for special cases. Note that none of these results forbids energy from being transferred downscale even when it is shown that the net flux should be directed upscale; they merely say that in those cases the energy going upscale in the upscale range should be larger than that going downscale in the downscale range.

4.3 Phenomenology of 2D turbulence

Based on Kolmogorov's [5, 129, 130] concept of an downscale energy cascade in three-dimensional turbulence, Kraichnan [145], Leith [150], and Batchelor [8] (KLB) proposed that in two-dimensional turbulence there is an upscale energy cascade and a downscale enstrophy cascade, when the stochastic forcing injects energy and enstrophy in a narrow band of intermediate length scales. Assuming that all the energy flows upscale and that all of the enstrophy flows downscale, KLB invoke a dimensional analysis argument, similar to Kolmogorov's, to show that the energy spectrum in the upscale energy range is

$$E(k) = C_{ir}\varepsilon^{2/3}k^{-5/3}, \tag{4.35}$$

and in the downscale enstrophy range is

$$E(k) = C_{uv}\eta^{2/3}k^{-3}. \tag{4.36}$$

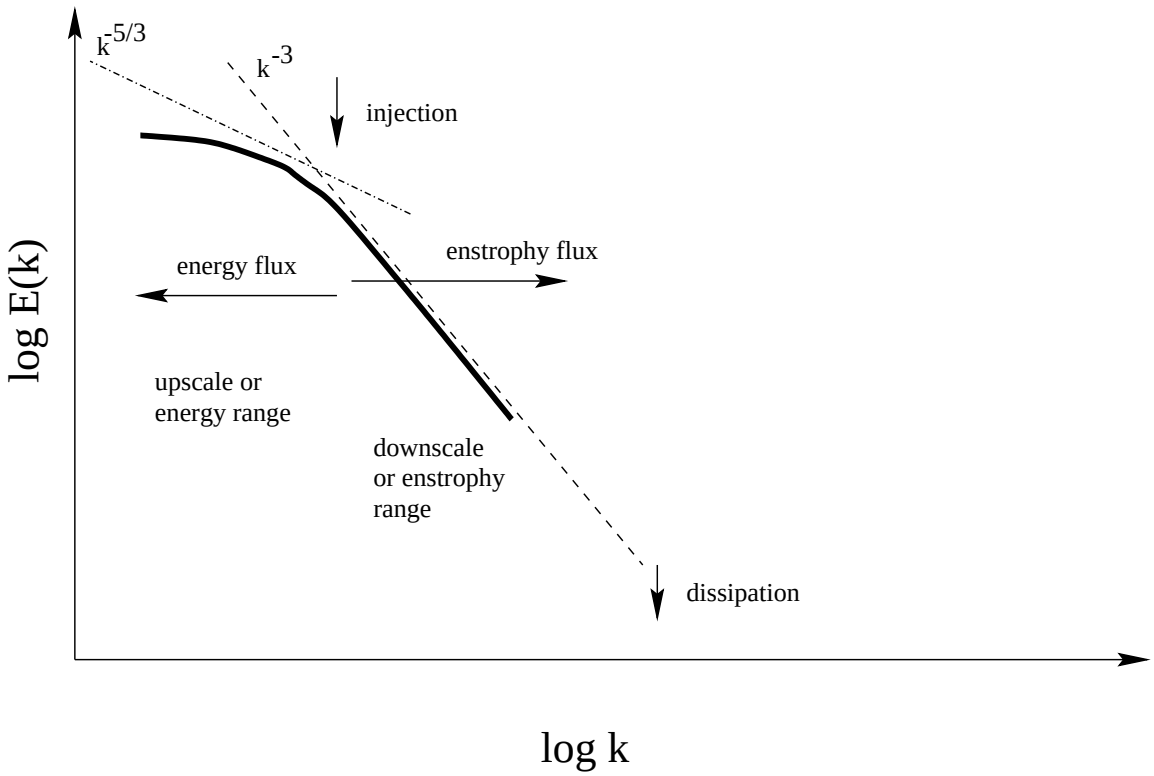

Figure 4.1: The Kraichnan-Leith-Batchelor scenario of a dual-pure cascade. There is a pure energy upscale cascade upscale of the injection and a pure downscale enstrophy cascade downscale of the injection.

Anticipating the objection that the dimensional analysis arguments cannot be applied to the enstrophy cascade, because of nonlocality, Kraichnan, in a subsequent paper [147], proposed that the enstrophy cascade energy spectrum is given by

$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\ln(k\ell_0)]^{-1/3}, \tag{4.37}$$

and showed, using a one-loop closure model [146], that this logarithmic correction is consistent with constant enstrophy flux. Furthermore, to eliminate the singularity at $k\ell_0 = 1$, Bowman [32] showed that a constant term needs to be introduced along with the logarithmic correction, and the improved equation reads

$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}, \tag{4.38}$$

where χ is a constant that may or may not be universal.

In the KLB idealization there is only a single flux in each inertial range: a pure energy upscale

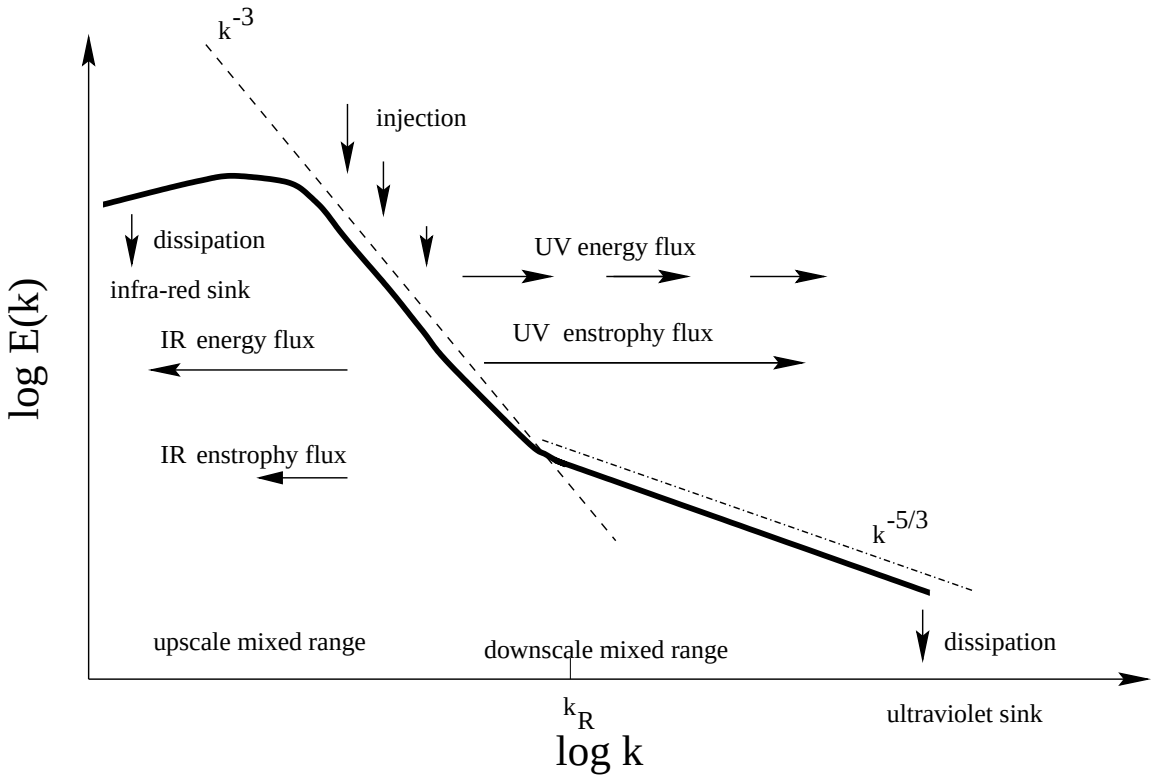

Figure 4.2: The scenario of a dual-double cascade. There is a double cascade of energy and enstrophy upscale of the injection scale and a double cascade of energy and enstrophy downscale of the injection scale.

cascade on the upscale side of injection and a pure downscale enstrophy cascade on the downscale side of injection (see Fig. 4.1). This situation is called the *dual-pure* cascade. The dimensional analysis argument used by previous authors would fail if there is more than one flux in each inertial range. There is no large-scale (infra-red) sink of energy in the KLB picture, but it requires an infinite domain: all injected energy is transferred to larger and larger scales and thus escapes dissipation altogether. This picture is clearly unrealizable in numerical experiments and in nature, where the domain is finite. In reality the finite size of the numerical or natural domain contains the energy, which then piles up at the largest scales.

Eyink [74] has shown that although in the limit $\nu \to 0$, the total energy, under steady state, is infinite, for finite ν, it is bounded. It has also been shown by Tran and Shepherd [249] that in a finite domain with finite viscosity, a cascade of enstrophy on the downscale side of injection is not allowed without the presence of an infrared dissipation term that can provide an energy sink at large scales. Further considerations by Tran and Bowman [247] have shown that in the absense of

a sufficient infra-red sink, most energy and enstrophy are dissipated at the forcing scale, leading to steep spectra both upscale and downscale of injection. Tran and Bowman [248] have also shown that the inverse energy cascade could be realized temporarily as a quasi-steady state even in the absense of an infrared sink, but not the enstrophy cascade. Since Kraichnan's conjecture of a dual cascade was intended for the problem of turbulence driven by the Navier-Stokes equations *without the infrared dissipation sink*, these papers highlight a number of serious problems with this conjecture.

Here, we are concerned with the case where there is a significant dissipation sink at large scales, and the viscosity coefficients for both sinks are finite. The presence of such an infrared sink unavoidably dissipates some enstrophy, and the finite viscosity enstrophy sink at small scales will inevitably also dissipate some energy. It is therefore necessary to scrutinize, in a similar manner, the conjecture that in turbulence driven by the infrared-damped Navier-Stokes equations there is a dual cascade that consists of an upscale double cascade of energy and enstrophy, where the energy flux is dominant, and a downscale double cascade of enstrophy and energy, where the enstrophy flux is dominant (see Fig. 4.2). Because there are two fluxes present in each cascade, it is not obvious that the form of the energy spectrum can be deduced from dimensional analysis. Furthermore, it is alternatively possible that non-local interactions may dominate that transfer energy and enstrophy directly to the dissipation range, or that the cascades may fail to form altogether.

Recent numerical simulations [42, 124, 160, 201] have validated the prediction k^{-3} for the energy spectrum of the downscale enstrophy cascade. Prior to the groundbreaking paper by Lindborg and Alvelius [160], numerical simulations had failed to reproduce the k^{-3} energy spectrum of the downscale range. As a result, a number of alternative theories had been proposed that predict steeper scaling [181, 205, 218]. It is now understood that the presence of a dissipation sink at large scales is required for a successful simulation of the enstrophy cascade [247–249].

The inverse energy cascade presents with a more paradoxical situation. From a theoretical standpoint there is no obvious reason why the inverse energy cascade should not be local. From the standpoint of numerical simulations, there are many positive reports of the predicted $k^{-5/3}$ energy spectrum [27, 96, 118, 199, 200, 228]. The most convincing simulation of the inverse energy cascade has been reported in the paper by Boffetta *et al.*[27], where in addition to the $k^{-5/3}$ prediction, the 3/2 law has also been confirmed. On the other hand, the locality of the inverse energy cascade has been challenged [59] on the grounds of other numerical simulations giving conflicting results [31, 60, 61]. The current understanding is that under certain conditions there are coherent structures that spontaneously form while the inverse energy cascade converges to stationarity. Apparently, the inverse energy cascade, as a physical process, continues to take place but it is hidden by the

coherent structures which give the dominant contribution to the energy spectrum. Removing the coherent structures artificially by postprocessing simulation data recovers the $k^{-5/3}$ energy spectrum [31, 60, 91]. This aspect of the inverse energy cascade is not well understood. Furthermore, the phenomenon of these coherent structures is of considerable interest to oceanographers. Despite some interesting theoretical work [165, 206, 263], a comprehensive theory of the two-dimensional inverse energy cascade still remains an unfinished task.

4.4 The Nastrom-Gage spectrum

According to Kraichnan [145], the study of two-dimensional turbulence was motivated by the hope that it would prove a useful model for atmospheric turbulence. This idea was later encouraged by Charney [38] who claimed that quasi-geostrophic turbulence is isomorphic to two-dimensional turbulence. The question that was then posed was whether the energy spectrum of the atmosphere at length scales that are orders of magnitude larger than the thickness of the atmosphere can be explained in terms of the theory of two-dimensional turbulence. This question continues to be debated today.

Early observations by Wiin-Nielsen [259] suggested that the energy spectrum of the atmosphere follows a k^{-3} power law behavior consistent with an enstrophy cascade. Because of the sparseness of observational stations, only results for the planetary scales ($\sim$ 10000km) and synoptic scales ($\sim$ 1000km) were shown. Wiin-Nielsen's data at the time appeared to fit this picture, with approximately a -3 power law for wavenumbers between 8 and 16, and a (less defined) -0.4 power law for wavenumbers smaller than 8. The break in the slopes was identified [43, 259] as the location of energy injection by baroclinic instability, which was assumed to occur in a narrow wavenumber band around 8.

An analysis by Nastrom and Gage [188, 189] of high resolution wind and temperature measurements, collected using commercial airplane flights in the upper troposphere and lower stratosphere in the late 70's, showed that there is a robust k^{-3} spectrum extending from approximately 3000 km to 800 km in wavelength (the "synoptic scales") and a robust $k^{-5/3}$ spectrum extending from 600 km down to a few kilometers (the "mesoscales"). The transition from one slope to the other occurs gradually between 600km and 800km. Recent measurements [50, 51, 177] have confirmed the $k^{-5/3}$ part of the atmospheric energy spectrum. This remarkably robust spectrum is widely known as the Nastrom-Gage spectrum (see Figure 4.3). General Circulation models have been shown to be capable of reproducing the Nastrom-Gage spectrum in agreement with observations [132, 133, 229].

Naively one could interpret the -3 slope as two-dimensional (2D) turbulence, for which Kraichnan

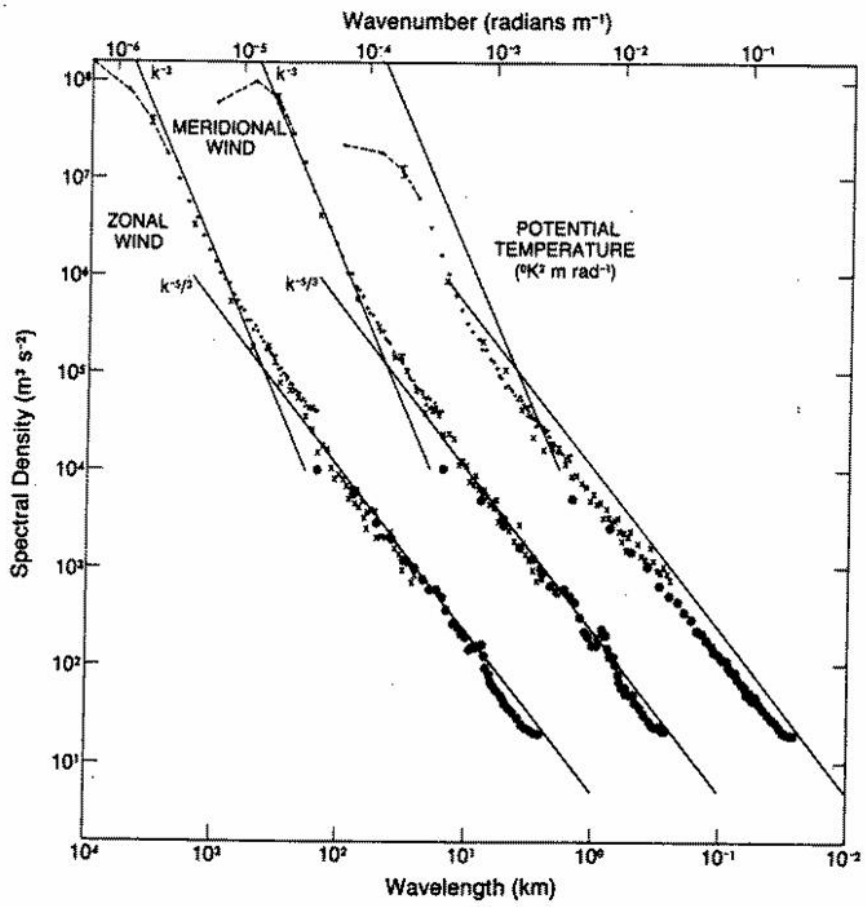

Figure 4.3: The Nastrom-Gage spectrum of atmospheric turbulence [98, 99, 188, 189]. For the purpose of showing the individual spectrum, the spectrum for meridional wind is shifted one decade to the right, while that for potential temperature is shifted two decades to the right.

predicted the -3 slope for downscale enstrophy cascade. One could also interpret the $-5/3$ sloped portion of the spectrum for scales less than 600 km as three-dimensional (3D) turbulence, for which Kolmogorov predicted a $-5/3$ spectral slope for homogeneous isotropic 3D turbulence with downscale energy cascade, and attribute the transition from the -3 slope to the $-5/3$ slope as due to the motion being predominantly 2D for the large scales in the thin shell of the troposphere and the shorter scales as being 3D. However, results of classical 3D turbulence cannot be applied here to motions whose horizontal scales range up to 600km in wavelength, while its vertical scale is about 10km (which is the scale of the depth of the troposphere). The -3 sloped portion could conceivably be explained by the Quasi-Geostrophic (QG) turbulence of Charney [38], which bears much resemblance to 2D turbulence. However caution should be exercised over most of the scales involved (the synoptic scales), which are in the forcing range where energy and enstrophy injection occurs and therefore are not in an inertial range [254, 257].

Some of the earliest theories for the $-5/3$ part of the spectrum involved internal gravity waves [63, 255]. As pointed out by Gage and Nastrom [99], observational information on the vertical velocity spectrum revealed that there is a basic inconsistency between the observed spectra and theories of internal waves as the cause of the mesoscale spectrum. There is "simply too much energy" in the horizontal spectrum compared to the vertical spectrum to be consistent with the idea that both are due to a common spectrum of internal waves.

A proposed theory still being considered is due to Lilly [154], who suggested that thunderstorms at the short-wave end (at km scales) of the spectrum have enough energy to generate stratified turbulence which then collapses into 2D turbulence. This 2D turbulence cascades upscale to form the $-5/3$ portion of the 2D spectrum due to the negative energy flux, in the same way as positive energy flux in Kolmorov's theory for 3D turbulence generate its $-5/3$ spectrum (since the dimensional argument for the $-5/3$ spectral slope is independent of the sign of the energy flux). According to this theory, in addition to the small-scale source there is also a large-scale source injecting potential enstrophy and thereby giving rise to the k^{-3} spectrum at large scales. A variation on this theme is the theory of Falkovich [81] where the k^{-3} portion is explained as a condensate and not as an enstrophy cascade.

How much of the thunderstorm's energy, which consists of gravity waves, which radiate away, and in the form of 3D turbulence, which naturally tends to cascade into still smaller scales, can be converted into 2D turbulence and cascaded up three decades of scales is questionable. Extensive numerical calculations of stratified turbulence show that only 2% of the energy is converted into 2D turbulence [232]. That may or may not be sufficient to generate the observed spectrum without

further study. What is more difficult to explain with this theory of thunderstorm source of energy is the fact that the spectrum, in particular the transition wavenumber between the shallow portion and the steeper portion of the spectrum, appears to be approximately the same whether it is in winter or summer, and whether the airplane flew over storms or not.

At first Lilly's theory of small-scale and large-scale source was favored because Lindborg [156], using third order structure functions, appeared to have deduced from observed data that the energy flux is upscale. Later, however, Cho and Linborg [47, 48] corrected a sign error which then led them to conclude that their analyses of data at "mesoscales in both the upper troposphere and lower stratosphere provide no support for an inverse energy cascade 2D turbulence." There is now even analysis [49] of aircraft data which gives the magnitude of the finite dissipation rate at the small scales (or equivalently, the downscale energy flux).

4.5 The Tung-Orlando double cascade theory

The "remarkable degree of universality" (Gage and Nastrom 1986) in spectral amplitude and in spectral shape over the entire range of wavelengths encompassing both the -3 and the $-5/3$ parts of the spectrum is "hard to explain" if it were due to forcing on two ends of the spectrum by two unrelated physical processes. The spectrum is perhaps best explained by a source only at the large scales being responsible for both parts of the spectrum, as first proposed by Tung and Orlando [252]. There is a natural source for this energy injection and it is of sufficient amplitude: baroclinic instability caused by the north-south temperature of the lower atmosphere (ultimately due to the sun heating the tropics more than the high latitudes, making the atmosphere potentially unstable). Such injection of energy and potential enstrophy occurs at the synoptic scales (10000km to 2000km).

If η_{uv} is the downscale enstrophy flux and ε_{uv} is the downscale energy flux, it was suggested by Tung and Orlando [252] that they would coexist on the downscale side of injection in the same inertial range and their separate contributions to the energy spectrum would give the latter a compound spectral shape, with a -3 slope transitioning to a shallower $-5/3$ slope as the wavenumber increases. So, in a sense, the Tung-Orlando theory is that the entire Nastrom-Gage spectrum can be conceptualized as a double cascade of both energy and potential enstrophy.

Using a dimensional argument, the transition from -3 slope to $-5/3$ slope is expected to occur at the transition wavenumber k_t with order of magnitude estimated by $k_t \approx \sqrt{\eta_{uv}/\varepsilon_{uv}}$. Recent measurements and data analysis [47] estimate $\eta_{uv} \approx 2\times 10^{-15}\mathrm{s}^{-3}$ and $\varepsilon_{uv} \approx 6\times 10^{-11}\mathrm{km}^2\mathrm{s}^{-3}$. From these estimates we find the mean value of the transition scale $k_t = \sqrt{\eta_{uv}/\varepsilon_{uv}} \approx 0.57 \times 10^{-2}\mathrm{km}^{-1}$ and $\lambda_t = 2\pi/k_t \approx 1 \times 10^3\mathrm{km}$ which has the correct order of magnitude.

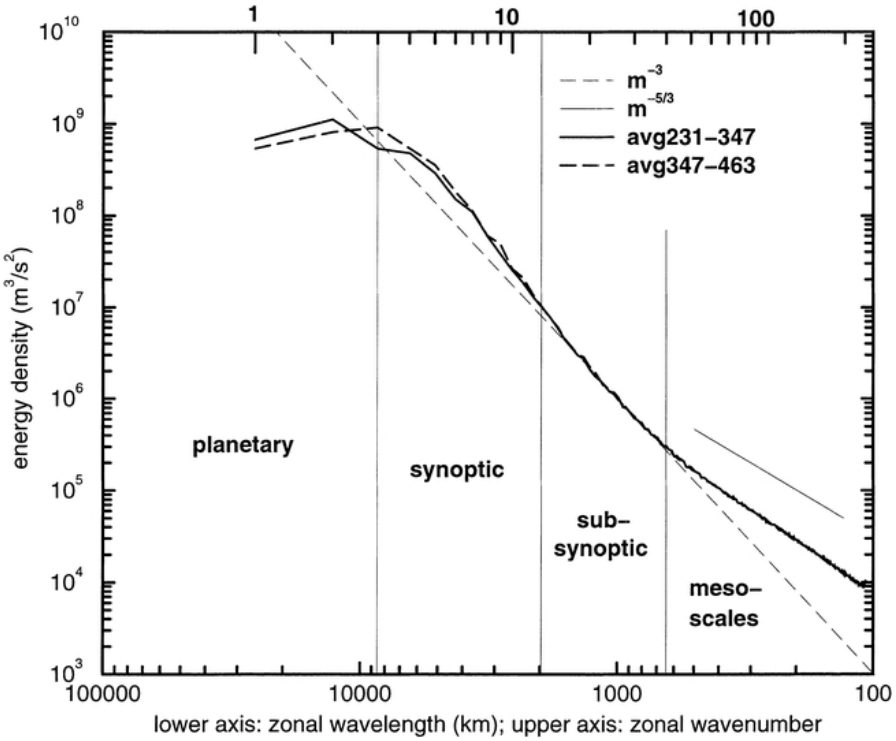

Figure 4.4: The energy spectrum predicted by the model of Tung and Orlando [252]. Note the -3 sloped spectrum over the subsynoptic scales transitioning at around 700km to a -5/3 sloped spectrum over the mesoscales. These features compare very favorably to the observed spectrum. For larger scales, the synoptic and planetary scales, there were not enough long distanced flight segments in the Nastrom-Gage data, hence the drop off in power. Nevertheless, station data can be used to supplement the aircraft data for the larger scales. Our model results compare favorably at those scales to these data as well, including the steeper than -3 slope over the synoptic scales, which are located in the forcing region [257].

Tung and Orlando [252] have demonstrated numerically that a two-layer quasi-geostrophic channel model with thermal forcing, Ekman damping, and hyperdiffusion can reproduce this compound spectrum (see Figure 4.4). The resolution of these simulations goes down to 100km in wavelength. The diagnostic shown in figure 7 of Tung and Orlando [252], shows both the constant downscale energy and enstrophy fluxes coexisting in the same inertial range. Furthermore, Tung and Orlando [252] confirmed the dimensional estimate $k_t \approx \sqrt{\eta_{uv}/\varepsilon_{uv}}$ for the transition wavenumber k_t.

The Tung-Orlando theory is contrary to the widely accepted misconception in the atmospheric science community that the argument by Fjørtøft [92] forbids a downscale energy flux in two-dimensional turbulence altogether, and through the isomorphism theorem of Charney [38] also in quasi-geostrophic turbulence. Various aspects of this misconception have been clarified in recent papers [106, 180, 254] and in the present thesis.

Although the nature of the nonlinear interactions which give rise to the downscale energy flux changes from quasi-geostrophic to stratified three-dimensional in the mesoscales, as far as the energy spectrum $E(k)$ is concerned, it is the existence of a downscale energy flux from the largest scales (10000km) to the smallest scales (1km) which gives rise to the $k^{-5/3}$ slope, regardless of the character of the motion. The recent interest [158, 159] in understanding the $k^{-5/3}$ slope in terms of three-dimensional stratified turbulence is well motivated, since it is necessary to account for length scales less than 100km in wavelength where the quasi-geostrophic assumption fails. It is the view of the author that it is equally important to understand why the quasi-geostrophic model is capable of supporting a downscale energy cascade with $k^{-5/3}$ scaling, because one also has to account for the existence of $k^{-5/3}$ scaling in the vicinity of the transition range (800km to 600km) where the quasi-geostrophic assumption is presumably valid. Estimates for the breakdown of the validity of the QG constraint range from 500km to 100km.

4.6 Double cascades in two-dimensional turbulence

In recent papers [107, 108], and in chapter 6 of this thesis, we have suggested that the double cascade phenomenon takes place in pure 2D turbulence too, where it is possible, in principle to have a transition from k^{-3} scaling to $k^{-5/3}$ scaling with increasing k. As has been pointed out by previous authors [31, 74], as long as the dissipation terms at large scales and small scales have finite viscosity coefficients and the inertial ranges exist, the downscale enstrophy flux will be accompanied by a small downscale energy flux, and the upscale energy flux will be accompanied by a small upscale enstrophy flux. Dimensional analysis arguments are premised on the assumption that these additional fluxes can be ignored, consequently the energy spectrum predictions obtained by such arguments are valid

only to leading order. We have argued [105] that although subleading effects can be ignored with impunity for strictly two-dimensional turbulence, for models of quasi-geostrophic turbulence, such as the two-layer model, the subleading contributions can be important in the inertial range and cannot be safely ignored.

For the case of two-dimensional turbulence, we argued [107, 108] that the subleading fluxes are associated with a subleading downscale energy cascade and a subleading inverse enstrophy cascade that contribute *linearly* to the total energy spectrum in addition to the dominant contributions. The two contributions are homogeneous solutions of the underlying statistical theory, which is in fact *linear*. Furthermore, the two homogeneous solutions are independent of each other, so the downscale energy cascade is independent of the downscale enstrophy flux η_{uv} and the downscale enstrophy cascade is independent of the downscale energy flux ε_{uv}. As a result, in the downscale inertial range, the total energy spectrum $E(k)$ has the following three contributions:

$$E(k) = E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k) + E_{uv}^{(p)}(k),\ \forall k\ell_0 \gg 1, \tag{4.39}$$

where $E_{uv}^{(\varepsilon)}(k)$, $E_{uv}^{(\eta)}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E_{uv}^{(\varepsilon)}(k) &= a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3}\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)}) \\ E_{uv}^{(\eta)}(k) &= b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}), \end{aligned} \tag{4.40}$$

with $\mathcal{D}_{uv}^{(\varepsilon)}$ and $\mathcal{D}_{uv}^{(\eta)}$ describing the dissipative corrections. Here we use the logarithmic correction of Kraichnan [147], adjusted by the constant χ of Bowman [32] for the contribution of the enstrophy cascade. We have also assumed without explicit justification that we may ignore the possibility of intermittency corrections to the subleading downscale energy cascade. For the downscale enstrophy cascade intermittency corrections have been ruled out by Eyink [77]. For the downscale energy cascade we conjecture that intermittency corrections are small for the same reasons as in three-dimensional turbulence. The scales $\ell_{uv}^{(\varepsilon)}$, $\ell_{uv}^{(\eta)}$ are the dissipation length scales for the downscale energy and enstrophy cascade. Finally, $E_{uv}^{(p)}(k)$ is the contribution from the effect of forcing and the sweeping interactions and it represents a particular solution to the statistical theory. The latter can become significant via the violation of statistical homogeneity caused by the boundary conditions [104]. Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrum will take the simple form in the downscale range:

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3} + b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{4.41}$$

We see that the energy spectrum will take the slope of -3 for small k , and $-5/3$ for large k . The transition from one slope to the other occurs at k_t , given by $\varepsilon_{uv}k_t^2 \sim \eta_{uv}$.

It should be emphasized that the formation of cascades observable in the energy spectrum is by no means guaranteed. There are two prerequisites that need to be satisfied: first, the contribution of the particular solution $E_{uv}^{(p)}(k)$ has to be negligible both downscale and upscale of the injection scale, i.e.

$$\begin{aligned} E_{uv}^{(p)}(k) &\ll E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k), \ \forall k\ell_0 \gg 1 \\ E_{ir}^{(p)}(k) &\ll E_{ir}^{(\varepsilon)}(k) + E_{ir}^{(\eta)}(k), \ \forall k\ell_0 \ll 1. \end{aligned} \tag{4.42}$$

Second, the dissipative adjustment $\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)})$ and $\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)})$ of the homogeneous solution has to be such that it does not destroy the power law scaling in the inertial range. Furthermore, the dissipation scales $\ell_{uv}^{(\eta)}$ and $\ell_{uv}^{(\varepsilon)}$ have to be positioned so that the incoming energy and enstrophy can be dissipated.

The idea of two cascades in the same wavenumber region has an interesting precedent in the case of three-dimensional turbulence, where there is interest in understanding the double cascade of helicity and energy [4, 40, 182]. There, the situation is more straightforward because the helicity cascade and the energy cascade reside in separate isotropic sectors of the $SO(3)$ group [4, 25]. This makes it easier to argue in support of a superposition principle.

In two-dimensional turbulence the situation is more interesting because both cascades reside in the same isotropic sectors. The main argument in support of our conjecture was given in section 2 of Ref. [107]. Additional evidence is given in section 3 of the same paper. It should be noted that our main argument exploits the linearity of the exact statistical theory of two-dimensional turbulence (i.e. the complete infinite system of equations governing the relevant fully-unfused structure functions). Nonlinear results, such as the one proposed by Lilly [154], follow from closure models instead of the exact theory. Likewise, phenomenological arguments with the eddy-turnover rate, such as that by Kraichnan [147], are essentially coming out of a 1-loop nonlinear closure theory, and would also lead to nonlinear expressions for the energy spectrum. Most closure arguments miss the point that the *exact* statistical theory is in fact *linear*.

4.7 The difference between 2D turbulence and the two-layer model

In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2\Pi_E(k) - \Pi_G(k) < 0, \tag{4.43}$$

for all wavenumbers outside of the forcing range. This inequality was communicated to us by Danilov [105, 108] and it implies that the contribution of the downscale energy cascade to the energy spectrum

is overwhelmed by the contribution of the downscale enstrophy cascade and cannot be seen visually on a plot. This result was conjectured earlier by Smith [230] who debated the Tung-Orlando theory [252] by arguing that the downscale energy cascade can never have enough flux to move the transition wavenumber k_t into the inertial range. The obvious counterargument is that the two-layer model is a different dynamical system than the two-dimensional Navier-Stokes equations, and that it is not obvious that the Danilov inequality cannot be violated in the two-layer model [108, 251]. After the debate with Smith [230, 251], we identified [105] the essential mathematical difference between two-dimensional turbulence and the two-layer quasi-geostrophic model.

In the two-layer model forcing is due to thermal heating, which injects energy directly into the baroclinic part of the total energy. The two-layer fluid sits atop an Ekman boundary layer near the ground, which introduces Ekman damping in the lower layer [122] but *not* in the upper layer. Following Salmon [221], one may then derive the governing equations for the model, which read:

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1 \tag{4.44}$$

$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2 + 2e_2 \tag{4.45}$$

$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0. \tag{4.46}$$

Here, $\zeta_1 = \nabla^2\psi_1$ is the relative vorticity of the top layer and $\zeta_2 = \nabla^2\psi_2$ is the relative vorticity of the bottom layer, and ω is the vertical velocity. The temperature equation is situated between the two layers and it satisfies the geostrophic condition $T = (2/h)(\psi_1 - \psi_2)$ with h the separation between the two layers. Furthermore, f is the Coriolis term, N is the Brunt-Väisälä frequency, and Q_0 is the thermal forcing on the temperature equation. The dissipation terms include momentum dissipation of relative vorticity, in each layer, and Ekman damping from the lower boundary layer, and they read:

$$d_1 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_1 \tag{4.47}$$

$$d_2 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_2 \tag{4.48}$$

$$e_2 = -\nu_E\zeta_2. \tag{4.49}$$

We have shown [105] that it is the asymmetric presence of Ekman damping $e_2 = -\nu_E\zeta_2$ on the bottom layer but not the top layer which causes the violation of the Danilov inequality (4.43) in the two-layer model. As a result, the top layer has more enstrophy than the bottom layer, as is realistic in the atmosphere, and provided that the difference in enstrophy between the two layers is large enough, the subleading downscale energy cascade will be observable in the energy spectrum. If one artificially adds an identical Ekman damping $e_1 = -\nu_E\zeta_1$ in the upper layer it can be easily

shown that Danilov's inequality (4.43) applies. In that case of symmetric dissipation, the subleading downscale energy cascade will be hidden by the dominant downscale enstrophy cascade.

For the case of asymmetric Ekman damping that we are considering here, we have shown [105] that a *sufficient* condition to *satisfy* the Danilov inequality is

$$\nu_E < 4\nu k_{\max}^{2p} \left(\frac{k_{\max}}{k_R} \right)^2 . \tag{4.50}$$

Here $k_{\max}$ is either the truncation wavenumber in the numerical model, or, in the theoretical case of infinite resolution, is the hyperviscosity dissipation wavenumber, beyond which the spectral enstrophy dissipation rate becomes negligible. Equivalently, a *necessary* condition to *violate* Danilov's inequality is

$$\nu_E > 4\nu k_{\max}^{2p} \left(\frac{k_{\max}}{k_R} \right)^2 . \tag{4.51}$$

We have also derived [105] a *necessary and sufficient condition* for violating the Danilov inequality. However, the price that must be paid for doing so is that the condition is *uncontrolled*. By this, we mean that the condition has the form

$$\nu_E k_R^2 > \Lambda \nu k_{\max}^{2p+2}, \tag{4.52}$$

but it is not possible to find a universal value for Λ that will *always* work.

It should be noted that the simulation of Tung and Orlando [252] has already shown that it is possible to have an observable downscale energy cascade. The only issue that required clarification was to understand why it happens in the two-layer model but not in two-dimensional turbulence, when the value of Ekman damping is larger than the subgrid hyperdiffusion.

4.8 Conclusions

We would now like to summarize the main points of this chapter. We have seen that two-dimensional turbulence has been investigated numerically to considerable detail. Both the downscale enstrophy cascade and the inverse energy cascade have been successfully observed. However, neither cascade is as robust as the downscale energy cascade of three-dimensional turbulence. The disruption of cascades in two-dimensional turbulence is associated physically with the emergence of long-lived coherent structures. It is not really understood why these coherent structures emerge in two-dimensional turbulence but not in three-dimensional turbulence. However, we have reviewed some significant breakthroughs in understanding the particulars of the cascades of two-dimensional turbulence, when these cascades are not disrupted.

The situation becomes even more interesting for the case of flows that are approximately two-dimensional, and especially in the context of understanding the Nastrom-Gage energy spectrum of the atmosphere. We have reviewed some of the proposed theories, and discussed more extensively the Tung-Orlando theory. All that can be said with certainty is that the work to date in this direction raises more questions than it answers! Consequently an open mind is needed to make further progress.

The irony of the situation in this field is that although there is considerable interest in two-dimensional turbulence on the one hand, and in General Circulation Models on the other hand, there is relatively limited interest in the theoretical understanding of simpler models in between these two extremes, such as, for example, the two-layer quasi-geostrophic model. Finite layer quasi-geostrophic models have the advantage that they are possibly within range of theoretical analysis using tools that have proved themselves in studies of two-dimensional turbulence. It is worthwhile to study these models for two reasons: first, "because they are there"; second, for the very good reasons given by Held [117].

Chapter 5

DIRECTION OF FLUXES IN ONE-LAYER ADVECTION-DIFFUSION MODEL

In this chapter we will derive a pair of inequalities for the general case of the one-layer advection-diffusion model that shows that the weighted average of the energy flux is negative and the weighted average of the enstrophy flux is positive. The averages involved are such that the inequalities can be satisfied only when most of the energy goes upscale and most of the enstrophy goes downscale. For example, the energy flux inequality gives more weight to large wavenumbers than small wavenumbers. Consequently, the upscale energy flux at small wavenumbers must be significantly larger than the downscale flux at large wavenumbers to make the average come out negative. A similar consideration applies to the enstrophy flux inequality.

What is remarkable is that in the forced-dissipative case these inequalities can be derived without any assumptions, except for requiring that the forcing spectrum is confined to a finite interval of wavenumbers, which can even be relaxed if necessary. No assumptions on the existence of inertial ranges are necessary, which means that the inequalities are also valid in situations where the inertial ranges fail to exist. For the case of decaying turbulence, it is necessary to make an assumption concerning the time derivative of the energy spectrum, but given that assumption the same inequalities continue to hold. An extensive discussion of the Rhines proof is given at the end of this chapter.

5.1 Preliminaries

We shall first present the general case of the one-layer advection-diffusion model which encompasses 2D turbulence, CHM turbulence, and SQG turbulence, before considering the subcases separately. The governing equation of these systems has the distinguishing form of a conservation law for a vorticity-like quantity ζ:

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = -[\nu(-\Delta)^p + \nu_1(-\Delta)^{-h}]\zeta + F, \tag{5.1}$$

where $\psi(x, y, t)$ is the streamfunction and ζ is related to it through a linear operator $\mathcal{L}$ by $\zeta = -\mathcal{L}\psi$. We assume that $\mathcal{L}$ is a diagonal operator in Fourier space whose Fourier transform $L(k)$ satisfies $L(k) > 0$ and $L'(k) > 0$. The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ, and is defined

as

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial y}. \tag{5.2}$$

We have written the dissipation of ζ in a more general form than normally used. Our proof does not depend on the details of the operator $\mathcal{D} = \nu(-\Delta)^p + \nu_1(-\Delta)^{-h}$, only that it is a positive operator. F is the forcing function; ν is the hyperdiffusion coefficient; ν_1 is the hypodiffusion coefficient. The physical case of molecular diffusion and Ekman damping corresponds to $p = 1$ and $h = 0$.

It can be shown that if a and b satisfy a homogeneous (Dirichlet or Neumann) boundary condition, then $\langle\langle J(a,b)\rangle\rangle = 0$, where we use the notation $\langle\langle f\rangle\rangle \equiv \iint (f(x,y))dxdy$. It follows from the product rule that

$$\langle\langle J(ab,c)\rangle\rangle = \langle\langle aJ(b,c)\rangle\rangle + \langle\langle bJ(a,c)\rangle\rangle = 0, \tag{5.3}$$

from which we obtain the identity

$$\langle\langle aJ(b,c)\rangle\rangle = \langle\langle bJ(c,a)\rangle\rangle = \langle\langle cJ(a,b)\rangle\rangle, \tag{5.4}$$

which was also shown previously in Ref.[250]. We assume that the operator $\mathcal{L}$ is self-adjoint in the sense that it satisfies $\langle\langle f(\mathcal{L}g)\rangle\rangle = \langle\langle g(\mathcal{L}f)\rangle\rangle$ for any fields $f(x,y)$ and $g(x,y)$. This is true, if we assume that $\mathcal{L}$ is diagonal in Fourier space.

The conservation law $\partial\zeta/\partial t + J(\psi,\zeta) = 0$ conserves the "enstrophy"-like quadratic $B = (1/2)\langle\langle\zeta^2\rangle\rangle$ for any arbitrary linear operator $\mathcal{L}$, because

$$\langle\langle\dot{B}\rangle\rangle = \langle\langle\zeta\dot{\zeta}\rangle\rangle = \langle\langle -\zeta J(\psi,\zeta)\rangle\rangle = \langle\langle -\psi J(\zeta,\zeta)\rangle\rangle = 0. \tag{5.5}$$

For self-adjoint operators $\mathcal{L}$, the "energy"-like quadratic $A = (1/2)\| - \psi\zeta\|$ is also conserved. To show that, note that

$$\langle\langle\dot{A}\rangle\rangle = (1/2)\langle\langle -\psi\dot{\zeta} - \zeta\dot{\psi}\rangle\rangle = (1/2)[\langle\langle\psi J(\psi,\zeta)\rangle\rangle + \langle\langle\zeta\mathcal{L}^{-1}J(\psi,\zeta)\rangle\rangle] \tag{5.6}$$

$$= (1/2)[\langle\langle\psi J(\psi,\zeta)\rangle\rangle + \langle\langle\mathcal{L}^{-1}\zeta)J(\psi,\zeta)\rangle\rangle] \tag{5.7}$$

$$= \langle\langle\psi J(\psi,\zeta)\rangle\rangle = \langle\langle\zeta J(\psi,\psi)\rangle\rangle = 0. \tag{5.8}$$

Let $A(k)$ and $B(k)$ be the spectral density of A and B, respectively such that $A = \int_0^{+\infty} A(k)\ dk$ and $B = \int_0^{+\infty} B(k)\ dk$, and k is the isotropic 2D wavenumber. The spectral equations are obtained by differentiating $A(k)$ and $B(k)$ with respect to t, and employing the Fourier transform of the governing equation (5.1):

$$\frac{\partial A(k)}{\partial t} + \frac{\partial \Pi_A(k)}{\partial k} = -D_A(k) + F_A(k) \tag{5.9}$$

$$\frac{\partial B(k)}{\partial t} + \frac{\partial \Pi_B(k)}{\partial k} = -D_B(k) + F_B(k). \tag{5.10}$$

It is understood that ensemble averages have been taken in the above quantities. Here $\Pi_A(k)$ is the spectral density of A transfered from $(0,k)$ to $(k,+\infty)$ per unit time by the nonlinear term in (5.1), $D_A(k)$ the dissipation of A, and $F_A(k)$ the forcing spectrum of A, and likewise for the B equation. The conservation laws imply for the viscous case that $\Pi_A(0) = \lim_{k\to\infty} \Pi_A(k) = 0$ and $\Pi_B(0) = \lim_{k\to\infty} \Pi_B(k) = 0$. For the inviscid case, this condition can be violated, in principle, by anomalous dissipation for solutions that have singularities. The spectra of A and B are related as $B(k) = L(k)A(k)$, and likewise it is easy to show, from the diagonal structure of the $\mathfrak{L}$ operator in Fourier space, that $D_B(k) = L(k)D_A(k)$ and $F_B(k) = L(k)F_A(k)$. Combining these equations with (5.10) and (5.9) we obtain the so-called Leith constraint [150]:

$$\frac{\partial \Pi_B(k)}{\partial k} = L(k)\frac{\partial \Pi_A(k)}{\partial k}, \tag{5.11}$$

which shows that if $\Pi_B(k)$ is strictly constant, then $\Pi_A(k)$ is also strictly constant and vice versa.

5.2 Flux inequalities for the forced-dissipative case

Assume that the forcing spectrum $F_A(k)$ is confined to a narrow interval of wavenumbers $[k_1, k_2]$. Then, we have

$$F_A(k) = 0 \text{ and } F_B(k) = 0, \forall k \in (0,k_1) \cup (k_2, +\infty), \tag{5.12}$$

and we can show, without making any ad hoc assumptions, that under stationarity, the fluxes $\Pi_A(k)$ and $\Pi_B(k)$ will satisfy the inequalities

$$\int_0^k L'(q)\Pi_A(q)\, dq < 0,\ \forall k > k_2 \tag{5.13}$$

$$\int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) > 0,\ \forall k < k_1. \tag{5.14}$$

The $\Pi_A(k)$ inequality is shown as follows: Integrating (5.9) and (5.10) over the $(k,+\infty)$ interval and employing the stationarity conditions $\partial A(k)/\partial t = 0$ and $\partial B(k)/\partial t = 0$ gives:

$$\Pi_A(k) = \int_k^{+\infty} [D_A(q) - F_A(q)]\, dq \tag{5.15}$$

$$\Pi_B(k) = \int_k^{+\infty} [D_B(q) - F_B(q)]\, dq = \int_k^{+\infty} L(q)[D_A(q) - F_A(q)]\, dq. \tag{5.16}$$

Using integration by parts, and the Leith constraint, we have the relation

$$\Pi_B(k) = \int_0^k \frac{\partial \Pi_B(q)}{\partial q}\, dq = \int_0^k L(q)\frac{\partial \Pi_A(q)}{\partial q}\, dq \tag{5.17}$$

$$= L(k)\Pi_A(k) - \int_0^k L'(q)\Pi_A(q)\, dq, \tag{5.18}$$

from which we obtain the inequality itself

$$\int_0^k L'(q)\Pi_A(q)\,dq = L(k)\Pi_A(k) - \Pi_B(k) \tag{5.19}$$

$$= \int_k^{+\infty} [L(k) - L(q)][D_A(q) - F_A(q)]\,dq \tag{5.20}$$

$$< 0,\ \forall k \in (k_2, +\infty). \tag{5.21}$$

Here we use $L(q) - L(k) > 0,\ \forall q \in (k, +\infty)$, and $D_A(q) - F_A(q) \geq 0$ which follows from $D_A(q) \geq 0$ and $F_A(q) = 0, \forall q > k > k_2$.

The corresponding inequality for the flux $\Pi_B(k)$ can be derived similarly. We begin by integrating (5.9) and (5.10), but this time over the $(0, k)$ interval:

$$\Pi_A(k) = -\int_0^k [D_A(q) - F_A(q)]\,dq \tag{5.22}$$

$$\Pi_B(k) = -\int_0^k [D_B(q) - F_B(q)]\,dq = -\int_0^k L(q)[D_A(q) - F_A(q)]\,dq. \tag{5.23}$$

Similarly, to avoid the singularity at $q = 0$, we do the integration by parts over the $(k, +\infty)$ interval:

$$\Pi_A(k) = -\int_k^{+\infty} \frac{\partial \Pi_A(q)}{\partial q}\,dq = -\int_k^{+\infty} \frac{1}{L(q)}\frac{\partial \Pi_B(q)}{\partial q}\,dq \tag{5.24}$$

$$= \frac{\Pi_B(k)}{L(k)} - \int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q)\,dq, \tag{5.25}$$

and consequently, we obtain

$$\int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) = -\frac{L(k)\Pi_A(k) - \Pi_B(k)}{L(k)} \tag{5.26}$$

$$= \frac{1}{L(k)} \int_0^k [L(k) - L(q)][D_A(q) - F_A(q)]\,dq \tag{5.27}$$

$$> 0,\ \forall k \in (0, k_1). \tag{5.28}$$

Here, the inequality changes direction, because $L(k) - L(q) > 0,\ \forall q < k$.

Note that both proofs are based on the inequality

$$L(k)\Pi_A(k) - \Pi_B(k) < 0,\ \forall k \in (0, k_1) \cup (k_2, +\infty), \tag{5.29}$$

which holds both upscale and downscale of the forcing range in the forced-dissipative case discussed here. We called this inequality, in a previous paper [108], the "Danilov inequality" because it was communicated to us by Danilov. It is worth noting that this inequality is the flux analog of a similar but distinct inequality derived by Fjørtøft [92] in terms of the energy spectrum and the enstrophy spectrum. Finally, similar flux inequalities were known to Eyink [74]. Eq. (5.29) is a sharper and more general variation of these previous results.

Also note that, for $k < k_1$, since $D_A(k) > 0$ for all k, it follows immediately from the steady state version of (5.9) and the assumption (5.12) that

$$\Pi_A(k) = -\int_0^k D_A(q)\, dq < 0, \forall k \in (0, k_1). \tag{5.30}$$

In general, one can easily show, for the forced-dissipative case where the forcing spectrum obeys (5.12), under statistical equilibrium, that

$$\Pi_A(k) > 0 \text{ and } \Pi_B(k) > 0,\ \forall k \in (k_2, +\infty) \tag{5.31}$$

$$\Pi_A(k) < 0 \text{ and } \Pi_B(k) < 0,\ \forall k \in (0, k_1). \tag{5.32}$$

It follows that, contrary to some popular misconceptions, both fluxes go downscale on the downscale side of injection, and upscale on the upscale side of injection.

5.3 Flux inequalities for the time-dependent case

We now generalize the proof to time-dependent cases. Since (5.19) and (5.26) are mathematical identities, they hold whether or not the quantities involved are time-dependent.

$$\int_0^k L'(q)\Pi_A(q)\, dq = L(k)\Pi_A(k) - \Pi_B(k),\ \forall k \in (k_2, +\infty) \tag{5.33}$$

$$\int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) = -\frac{L(k)\Pi_A(k) - \Pi_B(k)}{L(k)},\ \forall k \in (0, k_1). \tag{5.34}$$

Equations (5.21) and (5.28), however, should be modified to:

$$L(k)\Pi_A(k) - \Pi_B(k) = \int_k^{+\infty} [L(k) - L(q)][D_A(q) - F_A(q) + \frac{\partial A(q)}{\partial t}]\, dq,\ \forall k \in (0, k_1) \tag{5.35}$$

$$L(k)\Pi_A(k) - \Pi_B(k) = -\int_0^k [L(k) - L(q)][D_A(q) - F_A(q) + \frac{\partial A(q)}{\partial t}]\, dq,\ \forall k \in (k_2, +\infty). \tag{5.36}$$

Choosing k to be outside the forcing range $[k_1, k_2]$, and combining the previous four equations we obtain:

$$\int_0^k L'(q)\Pi_A(q)\, dq = \int_k^{+\infty} [L(k) - L(q)][D_A(q) + \frac{\partial A(q)}{\partial t}]\, dq,\ \forall k \in (k_2, +\infty) \tag{5.37}$$

$$\int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) = \frac{1}{L(k)} \int_0^k [L(k) - L(q)][D_A(q) + \frac{\partial A(q)}{\partial t}]\, dq,\ \forall k \in (0, k_1). \tag{5.38}$$

The equations (5.37) and (5.38) together are a general and remarkable result, because they relate the weighted mean of flux of A in $(0, k)$ to what happens outside this range, and the weighted mean of flux of B in $(k, +\infty)$ to what happens outside $(k, +\infty)$.

(a) *Initial stage:* During the initial development, nonlinear interactions transfer energy from one wavenumber to another. If the initial condition $A_0(k)$ for $A(k)$ is of compact support (which is almost always the case in reality) then we can expect that during the initial stages of decay where A is still in the process of spreading there will be a small wavenumber $\varepsilon_1 > 0$ and a large wavenumber $\varepsilon_2 > 0$ such that

$$A_0(k) = 0 \text{ and } \frac{\partial A(k)}{\partial t} \geq 0, \ \forall k \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty). \tag{5.39}$$

Combining this condition with (5.37) and (5.38), it follows that:

$$\int_0^k L'(q)\Pi_A(q)\, dq \leq \int_k^{+\infty} [L(k) - L(q)] \frac{\partial A(q)}{\partial t}\, dq \leq 0, \ \forall k \in (\varepsilon_2, +\infty) \tag{5.40}$$

$$\int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) \geq \frac{1}{L(k)} \int_0^k [L(k) - L(q)] \frac{\partial A(q)}{\partial t} \geq 0\, dq, \ \forall k \in (0, \varepsilon_1). \tag{5.41}$$

Note that each of the two previous inequalities uses only part of the assumption, i.e.

$$\frac{\partial A(k)}{\partial t} > 0, \ \forall k \in (\varepsilon_2, +\infty) \Longrightarrow \int_0^k L'(q)\Pi_A(q)\, dq < 0, \ \forall k \in (\varepsilon_2, +\infty) \tag{5.42}$$

$$\frac{\partial A(k)}{\partial t} > 0, \ \forall k \in (0, \varepsilon_1) \Longrightarrow \int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) > 0, \ \forall k \in (0, \varepsilon_1). \tag{5.43}$$

Furthermore, for $t = 0$, the assumption

$$A_0(k) = 0, \ \forall k \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty), \tag{5.44}$$

implies that

$$\left.\frac{\partial A(k)}{\partial t}\right|_{t=0} \geq 0, \ \forall k \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty), \tag{5.45}$$

from the positivity of $A(k)$. However for $t > 0$ the latter is an additional hypothesis.

(b) *Intermediate stage:* In the intermediate stage, nonlinear spreading and dissipation are both active at the small scales. Nonlinear transfer still supplies some A to small and large scales by spreading. Therefore

$$D_A(q) + \frac{\partial A(q)}{\partial t} \geq 0, \ \forall k \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty), \tag{5.46}$$

and so from (5.37) and(5.38)) we again obtain

$$D_A(q) + \frac{\partial A(k)}{\partial t} \geq 0, \ \forall k \in (\varepsilon_2, +\infty) \Longrightarrow \int_0^k L'(q)\Pi_A(q)\, dq \leq 0, \ \forall k \in (\varepsilon_2, +\infty) \tag{5.47}$$

$$D_A(q) + \frac{\partial A(k)}{\partial t} \geq 0, \ \forall k \in (0, \varepsilon_1) \Longrightarrow \int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) \geq 0, \ \forall k \in (0, \varepsilon_1). \tag{5.48}$$

(c) *Final decaying stage:* In the final stages of unforced turbulence, A decays due to dissipation. The decay rate of $A(k)$ is the same as the dissipation rate. Therefore,

$$D_A(q) + \frac{\partial A(q)}{\partial t} = 0, \ \forall k \in (0, \varepsilon_1) \cup (\varepsilon_2, +\infty), \tag{5.49}$$

and consequently,

$$\int_0^k L'(q)\Pi_A(q)\, dq = 0, \ \forall k \in (\varepsilon_2, +\infty) \tag{5.50}$$

$$\int_k^{+\infty} \frac{L'(q)}{[L(q)]^2}\Pi_B(q) = 0, \ \forall k \in (0, \varepsilon_1). \tag{5.51}$$

We do not have upscale cascade. During this final stage, nonlinear spreading has already occurred, and dissipation of energy dominates. Nevertheless, this still implies that A is transferred in the net upscale and B in the net downscale.

The implication of these results is that net energy flux is directed in the net upscale for the time-dependent case of 2D and barotropic QG turbulence in the absence of forcing if the initial condition is of compact support and if it is assumed that it subsequently spreads into small scales. For SQG turbulence the result is reversed, in the sense that energy in the $z = 0$ layer is transferred downscale in the net.

5.4 Implications for two-dimensional turbulence

For the case of 2D turbulence, $A(k)$ is the energy spectrum $E(k)$, $B(k)$ is the enstrophy spectrum $G(k)$ and $L(k) = k^2$. The inequality (5.13) simplifies to:

$$\int_0^k 2q\Pi_E(q)\, dq < 0, \ \forall k \in (k_2, +\infty). \tag{5.52}$$

This integral constraint implies that energy fluxes upscale in the net. The constraint (5.52) also holds trivially for $k < k_1$, since $\Pi_A(k) < 0$ for all $k < k_1$. For $k > k_2$, the integration range also includes the energy injection interval $[k_1, k_2]$ and both the upscale cascade range and the downscale cascade range. The inequality (5.52) implies that the negative flux in the $(0, k_1)$ interval is more intense than the positive flux in the $(k_2, +\infty)$ because the weighted average of $\Pi_E(k)$ gives more weight to the large wavenumbers.

Similarly, (5.14) reduces to

$$\int_k^{+\infty} 2q^{-3}\Pi_G(q)\, dq > 0, \ \forall k \in (0, k_1), \tag{5.53}$$

which is a statement that enstrophy fluxes downscale in the net.

The inequalities (5.52) and (5.53) are the two main results in 2D turbulence we were looking for, and they constitute proofs that in forced-dissipative 2D turbulence under statistical equilibrium energy predominantly is transferred upscale while enstrophy downscale.

To understand the implications of these inequalities on two-dimensional turbulence we have to distinguish between the following cases and consider them separately:

(a) *No infrared sink of energy, finite box:* This is the case considered by Tran and Shepherd [249]. The coefficient of hypoviscosity, which provides the sink at the large scales, is zero. i.e. $\nu_1 = 0$. The only dissipation mechanism is a very small molecular viscosity ν, with $p = 1$. Our result of net energy cascade (5.52) still holds. However, without a sink of energy at large scales, the energy which is fluxed upscale piles up until it is dissipated by the small viscosity at the forcing scale [247, 248]. No inertial range exists where the fluxes of energy and enstrophy are constant. Nevertheless, (5.52) implies that there is more energy flux dissipated on the upscale side of the forcing range than on the downscale side of the forcing range, and likewise (5.53) implies that there is more enstrophy dissipated on downscale side of the forcing range than on the upscale side of the forcing range .

(b) *No infrared sink of energy, infinite box:* Same as in case (a) except that the domain is infinite. This is the classical case of 2D turbulence considered by Kraichnan [145], Leith [150], and Batchelor [8]. Although there is no infrared sink of energy, the energy cascaded upscale can keep on cascading to ever larger scales. There is no pile up of energy, but there is always a spectral region at larger and larger scales where steady state cannot be achieved. Let this region be denoted by $0 < k < k_0(t)$. Quasi-steady state can be achieved for $k > k_0$. In this latter spectral region, our inequalities (5.52) and (5.53) do hold. Since energy transferred upscale through k_0 is "lost" to the region downscale from k_0, the infinite domain acts in effect like a perfect infrared sink. Furthermore, in the original formulation of the KLB theory, the molecular viscosity coefficient ν is taken to $\nu \to 0^+$, with the result that the energy dissipated at the ultraviolet end of the spectrum vanishes in the limit. In this configuration, all injected energy is transferred upscale and all injected enstrophy is transferred downscale.These results for the KLB theory have been summarized by Gkioulekas and Tung [107, 108].

(c) *Finite infrared and ultraviolet sinks of energy:* When there is a finite infrared sink of energy upscale of injection and a finite ultraviolet sink of energy downscale of injection, there is in general both an upscale and a downscale flux of energy. This situation has been considered in Eyink [74] and Gkioulekas and Tung [107, 108]. The upscale flux should be larger than the downscale flux, according to (5.52). It should be noted that, because of the inequality (5.29), the contribution of downscale energy flux to the energy spectrum in the inertial range on the downscale side of injection

is always subleading and hidden. This is not true in some baroclinic cases of QG turbulence [105].

5.5 Implications for models of QG turbulence

As derived by Charney (1971), QG turbulence conserves two quantities, total energy, which consists of horizontal components of kinetic energy plus available potential energy, and potential enstrophy. We now discuss briefly the implications of the flux inequalities on one-layer and two-layer simplifications of the quasi-geostrophic turbulence model.

(a) *CHM turbulence:* This model is a two-dimensional version of the quasi-geostrophic model, and represents physically two-dimensional turbulence on a rotating frame of reference. The governing equation is (5.1) with $L(k) = k^2 + \lambda^2$, where λ is the deformation wavenumber. The total energy E and total potential enstrophy G are given by $E = (1/2)\||\nabla\psi|^2 + \lambda^2|\psi|^2\|$ and $G = (1/2)|\zeta|^2$. The flux inequalities are

$$\int_0^k 2q\Pi_E(q)\, dq < 0,\ \forall k \in (k_2, +\infty) \tag{5.54}$$

$$\int_k^{+\infty} \frac{2q}{(q^2+\lambda^2)^2}\Pi_G(q)\, dq > 0,\ \forall k \in (0, k_1), \tag{5.55}$$

and they still imply that the total energy is mainly transferred upscale whereas the potential enstrophy is mainly transfer downscale.

(b) *SQG turbulence:* This model can be derived from the quasi-geostrophic model by assuming that the potential vorticity is zero over the entire three-dimensional domain (see appendix G). Then, it can be shown that the behaviour of the entire system is coupled to its behaviour in the boundary condition at the layer $z = 0$ [253]. At $z = 0$, the potential temperature Θ is governed by (5.1) with $L(k) = k$, where $\Theta = \zeta$. The conserved quadratic B represents the total energy E_{2D} of the system at the layer $z = 0$, whereas the quadratic A is the total energy E_{3D} integrated over the whole domain $z \in (0, +\infty)$ [105]. In this system, there is no enstrophy, since the potential vorticity has been taken equal to zero, and consequently there is no enstrophy cascade. The flux inequalities are

$$\int_k^{+\infty} q^{-2}\Pi_{E_{2D}}(q)\, dq > 0,\ \forall k \in (0, k_1) \tag{5.56}$$

$$\int_0^k \Pi_{E_{3D}}(q)\, dq < 0,\ \forall k \in (k_2, +\infty), \tag{5.57}$$

and they imply that downscale from injection the dominant process is a downscale energy cascade at the layer $z = 0$. Upscale from injection the energy spectrum is dominated by an inverse energy cascade of the total energy over the entire domain. It should be noted that just as in two-dimensional turbulence, a dissipation sink is probably needed both upscale and downscale of injection to allow either cascade to form successfully.

(c) *2-layer model of QG turbulence:* This model consists of two symmetrically coupled layers of two-dimensional turbulence where the deformation wavenumber λ is the coupling constant [219–221]. For the general baroclinic case, specifically with Ekman damping only in the lower layer, Danilov's inequality (5.29) does not necessarily hold for 2-layer models [105]. We therefore do not have a conclusive proof for the case of 2-layer models. However, numerical results [252] show that most of the energy will still go upscale in this system, although some small fraction goes downscale. In particular, the upscale energy cascade in the inertial range upscale of injection is much larger than the downscale flux of energy in the inertial range downscale of injection.

5.6 Remarks on Rhines proof

Rhines [211, 212] starts with the assumption:

$$\frac{d}{dt}\int_0^{+\infty}(k-K)^2E(k)\,dk > 0, \tag{5.58}$$

where $K = E_1/E_0$ is the first moment of $E(k)$ and E_a is defined as

$$E_a = \int_0^{+\infty} k^a E(k)\,dk. \tag{5.59}$$

Here (5.58) is a "postulate that the peak will spread in time" from its current centre of "mass" K, not necessarily in particular realizations, but in a probabilistic sense where an ensemble average over all initial conditions, constrained by the initial energy spectrum, has been taken. Rhines then shows that

$$\frac{dK^2}{dt} < 0, \tag{5.60}$$

which means that the average location of the peak tends to move toward smaller wavenumbers, and concludes from this that the energy has a tendency to be transfered upscale.

In his 1975 paper [211], the details of the proof are not given. In his 1979 paper [212], the following more detailed argument is given which is correct for the inviscid case $\nu = 0$ and $\nu_1 = 0$: Expanding

$$\int_0^{+\infty}(k-K)^2E(k)\,dk = E_2 - 2KE_1 + K^2E_0 = E_2 - K^2E_0, \tag{5.61}$$

and solving for K^2, we obtain

$$E_0K^2 = E_2 - \int_0^{+\infty}(k-K)^2E(k)\,dk. \tag{5.62}$$

Differentiating with respect to t, and writing $E_a' = dE_a/dt$, we have,

$$E_0'K^2 + E_0\frac{dK^2}{dt} = E_2' - \frac{d}{dt}\int_0^{+\infty}(k-K)^2E(k)\,dk < E_2', \tag{5.63}$$

which gives,

$$\frac{dK^2}{dt} < \frac{E_2' - E_0' K^2}{E_0}. \tag{5.64}$$

If we assume $E_0' = 0$ and $E_2' = 0$, which can be deduced from conservation of energy and enstrophy for the case where there are no viscosities, then it follows that

$$\frac{dK^2}{dt} < 0. \tag{5.65}$$

However, Rhines' argument was supposed to work for the viscous case as well (see page 405, last equation, of Ref. [212]) where $E_0' < 0$ and $E_2' < 0$. It appears that the term $E_0' K^2$ in (5.63) was ignored in that derivation. If the term is included, then the right hand side of (5.64) has two terms of opposite sign, and it is not immediately clear which term dominates. Nevertheless, Scott [224] showed, using the Holder inequality, that the proof can still be completed, for the case of Ekman damping and molecular diffusion ($h = 0$ and $p = 1$).

As it stands, this proof is interesting, but it cannot be extended to the forced-dissipative case because it relies on describing the behavior of time-derivatives of the energy spectrum rather than fluxes. Furthermore, it relies on the assumption (5.58), without proof. The difference between the assumption (5.58) and the assumption used in our proof is that, (5.58) is a global condition stated over the entire range of wavenumbers, whereas the assumption needed for our proof in the previous sections is a local condition over the intervals $(0, \varepsilon_1) \cup (\varepsilon_2, +\infty)$. We suspect that the need to make some assumption for proofs covering the decaying case is unavoidable because it is necessary to weed out unusual initial conditions.

It should be noted that the Rhines proof given by Salmon [221] is different from the proof given in the original papers [211, 212]. The difference is that in (5.58) K, which is time dependent, is replaced with a constant wavenumber k_1 representing the initial position of the peak. This modified proof was extended to the general case of α-turbulence [231]. However, we feel that the original assumption (5.58) is more reasonable, on physical grounds, and there is no benefit in modifying (5.58).

Furthermore, it should be stressed that there is an important difference between the proof of Scott [224] and the original Rhines proof. The main difference is that Rhines assumes that the *unnormalized* variance of the energy spectrum is increasing with time (Eq.(5.58)) whereas Scott [224] assumes that the *normalized* variance σ_E^2 is increasing. The definition of σ_E^2 is

$$\sigma_E^2 \equiv \frac{\int_0^{+\infty} (k - K)^2 E(k)\, dk}{\int_0^{+\infty} E(k)\, dk}. \tag{5.66}$$

Because the denominator of σ_E^2 is decreasing with time, it is easy to see that the assumption $d\sigma_E^2/dt > 0$ is mathematically weaker than the assumption (5.58) used in the original formulation of the Rhines proof. Consequently, since it is shown to be possible to arrive to the same conclusion under a weaker assumption, the statement proved by Scott [224] is better than the statement claimed by Rhines. Thus, Scott [224] implicitly also rehabilitates the original Rhines proof. However, Scott [224] did not consider the case of hyperdiffusion and hypodiffusion in his paper.

For the more general case of hyperdiffusion and hypodiffusion, from the conservation laws, we find that E_0' and E_2' read:

$$E_0' = -2\nu E_{2p} - 2\nu_1 E_{-2h} \tag{5.67}$$

$$E_2' = -2\nu E_{2p+2} - 2\nu_1 E_{-2h+2}, \tag{5.68}$$

and the time-derivative of K^2 is now bound by

$$\frac{dK^2}{dt} < \frac{E_2' - E_0' K^2}{E_0} = \frac{2\nu(K^2 E_{2p} - E_{2p+2}) + 2\nu_1(K^2 E_{-2h} - E_{-2h+2})}{E_0} \tag{5.69}$$

$$= \frac{2\nu}{E_0}\left(\frac{E_1^2 E_{2p}}{E_0^2} - E_{2p+2}\right) + \frac{2\nu_1}{E_0}\left(\frac{E_1^2 E_{-2h}}{E_0^2} - E_{-2h+2}\right) \tag{5.70}$$

$$= \frac{2\nu}{E_0}\frac{E_1^2 E_{2p} - E_0^2 E_{2p+2}}{E_0^2} + \frac{2\nu_1}{E_0}\frac{E_1^2 E_{-2h} - E_0^2 E_{-2h+2}}{E_0^2}. \tag{5.71}$$

The first term is again negative because

$$E_0^2 E_{2p+2} = (E_0 E_{2p+2})E_0 \geq (E_2 E_{2p})E_0 = E_{2p}(E_0 E_2) \geq E_{2p} E_1^2, \tag{5.72}$$

for all $p > 0$. Here, we employ the inequality $E_1^2 \leq E_0 E_2$, and the theorem that the function $\mathcal{E}(\kappa, \alpha) \equiv E_{\kappa+\alpha}/E_\kappa$ is an increasing function with respect to κ for $\alpha > 0$ (see appendix E) which implies that $E_0 E_{2p+2} \geq E_2 E_{2p}$ for all real $p > 0$. For $h = 0$, the second term is negative too. To see this, note that the numerator of that term reads:

$$E_1^2 E_0 - E_0^2 E_2 = E_0(E_1^2 - E_0 E_2) \leq 0. \tag{5.73}$$

However, so far as we know, the sign of the second term is indeterminate when $0 < h < 1/2$ and can be shown to be positive when $h > 1/2$. To show this, note that:

$$\frac{E_0^2 E_{-2h+2}}{E_1^2 E_{-2h}} \leq \frac{E_0 E_{-2h+1} E_{-2h+2}}{E_1 E_{-2h+2} E_{-2h}} = \frac{E_0 E_{-2h+1}}{E_1 E_{-2h}} \leq \frac{E_{-2h} E_{-2h+1}}{E_{-2h+1} E_{-2h}} = 1, \tag{5.74}$$

Here we use $E_0/E_1 \leq E_{-2h+1}/E_{-2h+2}$, which is valid for $h > 1/2$ and $E_0/E_1 \leq E_{-2h}/E_{-2h+1}$, which is valid for $h > 0$. It follows that

$$E_1^2 E_{-2h} - E_0^2 E_{-2h+2} \geq 0, \text{ for } h > 1/2. \tag{5.75}$$

Thus, the validity of the Rhines proof continues for the case of hyperdiffusion ($p > 1$), but cannot be extended to the case of hypodiffusion ($h > 0$).

It is interesting to note that when one begins with the weaker hypothesis of Scott [224], it can be shown that the contribution from the hypodiffusion term is *always* positive for $h > 0$. In the argument above we cannot show this unless $h > 1/2$. To see this, we retrace the argument of Scott [224] for the case of hyperdiffusion and hypodiffusion:

$$\frac{dK^2}{dt} = \frac{d}{dt}\left(\frac{E_2}{E_0}\right) - \frac{d\sigma_E^2}{dt} = \frac{E_2' E_0 - E_2 E_0'}{E_0^2} - \frac{d\sigma_E^2}{dt} \tag{5.76}$$

$$= -\frac{2\nu}{E_0^2}(E_{2p+2}E_0 - E_2 E_{2p}) - \frac{2\nu_1}{E_0^2}(E_{-2h+2}E_0 - E_2 E_{-2h}) - \frac{d\sigma_E^2}{dt}. \tag{5.77}$$

The first term is negative because $E_{2p+2}E_0 \geq E_2 E_{2p}$, for all $p > 0$. As Scott[224] noted, the second term vanishes for $h = 0$, however it is *positive* for $h > 0$. Thus, for $h > 0$, the sign of dK^2/dt remains indeterminate.

5.7 Concluding remarks

We have shown two inequalities (5.13) and (5.14), which for the case of two-dimensional turbulence imply that the weighted-average of the energy flux is negative and the weighted-average of the enstrophy flux is positive. This implies that the energy tends to go upscale in the net and the enstrophy tends to go downscale in the net. For the forced-dissipative case, the inequalities can be derived without any ad hoc assumptions. For the decaying case, a sufficient condition for the energy inequality is to assume that there exists a very large wavenumber k such that over the interval $(k, +\infty)$ the energy spectrum is increasing or constant. Likewise, for the enstrophy inequality it is sufficient that we assume that there exists a very small wavenumber k such that over the interval $(0, k)$ the energy spectrum is also increasing or constant. From a physical point of view, these assumptions are slightly more plausible than the assumption (5.58) made by Rhines in his proof. It should be noted that unlike previous proofs in both the forced-dissipative and the decaying case, the inequalities have the same mathematical form. Our argument then is a unified proof that covers all cases, and specialized results can be deduced from our inequalities for special cases. We have also briefly discussed the implications of our results for one-layer and two-layer models of quasi-geostrophic turbulence.

Note that none of the results obtained in this chapter forbids energy from being transferred downscale even when it is shown that the net flux should be directed upscale; they merely say that in those cases the energy going upscale in the upscale range should be larger than that going downscale in the downscale range. In fact, for the case of finite domains with finite viscosity, Gkioulekas and

Tung [107, 108] showed that the downscale flux of energy on the short-wave side of injection must be nonzero. Even in the case of 2-layer model where Tung and Orlando [252] found in their numerical experiment that the downscale energy flux over the mesoscales contributes visibly to the observed energy spectrum, it is still true that there is a larger inverse energy cascade from the synoptic to the planetary scales. The exception is the case of Surface QG turbulence, where most of the energy at the $z = 0$ layer goes downscale, as shown here. We suspect that this may be due to the collapse of temperature gradients on solid surfaces (a model of frontogenesis), and differs from the turbulence in the free atmosphere. In the free troposphere, there is strong observational evidence [26, 239] that energy flux is negative (upscale) from synoptic to planetary scales, and the positive (downscale) flux over the mesoscales [48, 49] is small by comparison.

Chapter 6

DOUBLE CASCADES OF ENERGY AND ENSTROPHY IN 2D TURBULENCE

Both the K41 theory for three-dimensional turbulence, and the KLB theory for two-dimensional turbulence are based on a dimensional analysis argument. However, Frisch [93, 94] has suggested that Kolmogorov's second paper [129] leads to the following more rigorous reformulation of the dimensional analysis argument. First, one postulates local homogeneity, local isotropy, a self-similarity assumption, and the existence of an anomalous energy sink. An anomalous sink is defined as one which remains finite in the limit as the viscous coefficient approaches zero from above. Then, one uses these assumptions to derive the 4/5 law, and from the self-similarity assumption the scaling for all structure functions and the energy spectrum can be deduced. It is possible to formulate a theory for two-dimensional turbulence in the same way. The problem is that the assumptions involved are not obviously true in two-dimensional turbulence. For example, it is not obvious that an anomalous enstrophy sink exists at small scales, and an anomalous energy sink at large scales.

We propose that one way to make progress is to adapt the theoretical work of L'vov and Procaccia [169–173] to two-dimensional turbulence. This amounts to replacing the assumption of the anomalous sinks with the weaker assumption that the energy and enstrophy cascades satisfy the *fusion rules*. We continue to postulate local homogeneity and local isotropy, and we also postulate a weaker assumption of self-similarity than the one used by Frisch.

The historical background of the fusion rules is as follows: they were first introduced by Eyink [70, 71] both for shell models of turbulence and for hydrodynamic turbulence itself. The rule for the case $p = 2$ has been proven by L'vov and Procaccia [167, 168, 170] for the direct energy cascade of three-dimensional turbulence, and there is further support by experiments [20, 21, 62, 79, 80, 114]. Another case is the passive scalar problem of Kraichnan [149]; the fusion rules have been thoroughly proven [3] to hold for all p, and there is also experimental support [45, 46]. The problem of two-dimensional turbulence is similar enough to both problems to make the hypothesis plausible.

From the fusion rules, it is possible to show the locality of interactions in the inertial range. Furthermore, it is also possible to calculate the location of the boundary between the inertial range in the dissipation range, as well as the boundary between the inertial range and the forcing range. In the first case, we obtain the location of the dissipation length scale, and from that we may calculate

whether the anomaly of the enstrophy or energy sink is recovered. In the second case, we find whether the proposed inertial range is stable with respect to perturbations to the statistical forcing or the boundary conditions. These two calculations amount to necessary and sufficient conditions for the formation of an observable inertial range.

In future publications, we will present these developments in detail. In the present thesis we restrict ourselves to the conclusions that are suggested from the structure of the theory, as long as we assume that the aforementioned conditions are satisfied.

6.1 Theoretical background for three-dimensional turbulence

We begin with a brief overview of the corresponding theory [169] for three-dimensional turbulence, before concetrating on two-dimensional turbulence. This review stresses those aspects of the theory that we find relevant for the problem of two-dimensional turbulence, and it reflects, to some extent, our point of view. Let w_α represent the Eulerian velocity difference, defined as

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) = u_\alpha(\mathbf{x}, t) - u_\alpha(\mathbf{x}', t). \tag{6.1}$$

and define the generalized structure function in terms of the following product

$$F_n^{\alpha_1 \alpha_2 \ldots \alpha_n}(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left\langle \left[\prod_{k=1}^{n} w_{\alpha_k}(\mathbf{x}_k, \mathbf{x'}_k, t) \right] \right\rangle, \tag{6.2}$$

where $\langle \cdot \rangle$ denotes ensemble average. For convenience, we omit the tensorial superscripts, unless they are needed for clarity. The inertial range, for the case of three-dimensional turbulence, can be characterized as the region where the generalized structure functions satisfy approximately

$$F_n(\{\lambda \mathbf{x}_k, \lambda \mathbf{x'}_k\}_{k=1}^n, t) = \lambda^{\zeta_n} F_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t), \tag{6.3}$$

where ζ_n are the corresponding scaling exponents and $\lambda = 1 \pm \epsilon$ with ϵ small. The energy spectrum $E(k)$ is related to F_2 via a linear transformation (see appendix F for details). If $0 < \zeta_2 < 2$, then the energy spectrum has scaling $E(k) \sim k^{-1-\zeta_2}$ [94]. If there is a logarithmic correction, then the result also holds for $\zeta_2 = 2$. Kolmogorov's theory [129, 130] predicts that, for the energy range of three-dimensional turbulence, the scaling exponents satisfy $\zeta_n = n/3$. In reality, there are non-trivial corrections, called "intermittency corrections" [94].

Differentiating F_n with respect to time, and substituting into the Navier-Stokes equations, gives a sequence of equations of the form

$$\frac{\partial F_n}{\partial t} + D_n = \nu J_n + \beta H_n + Q_n, \tag{6.4}$$

where D_n is the combined contribution of the pressure and the nonlinear term, Q_n is the contribution of the forcing term, J_n accounts for diffusion or hyperdiffusion, and H_n accounts for damping or hypodiffussion (see appendix A for the detailed form of the terms). Equations of this type have been introduced by L'vov and Procaccia [170, 172]. It has been shown, in section IV-B and appendix B of reference [172], that D_n can be rewritten in the form

$$D_n = \mathcal{O}_n F_{n+1} + I_n, \tag{6.5}$$

where $\mathcal{O}$ is a linear operator, and I_n represents the interaction of velocity differences with the mean flow. Under global homogeneity, where the mean flow is removable by a Gallilean transform, we have $I_n = 0$. For a stationary system, the generalized structure functions will satisfy an infinite chain of balance equations of the form

$$\mathcal{O}_n F_{n+1} + I_n = \nu J_n + \beta H_n + Q_n. \tag{6.6}$$

The essential point is that these equations are a *linear* system with respect to the generalized structure functions F_n. They become nonlinear only when they are truncated by some nonlinear closure schemes, such as the Leith [150] and Pouquet [207] schemes used by Lilly [154]. It follows that they have a homogeneous solution that satisfies the equation $\mathcal{O}_n F_{n+1} = 0$, and a particular solution driven by Q_n and I_n. The viscous terms act to modify the homogeneous solution by introducing a dissipation range, as discussed in the next section.

The distinguishing features of the direct energy cascade are that it exhibits universal scaling, and that it involves a process by which energy is transferred progressively from large scales towards small scales by triad interactions. The fusion rules are intended to describe this cascade process mathematically. The locality of interactions in the integrals of $\mathcal{O}_n F_{n+1}$ can be proven from the fusion rules. However, it is still possible for the non-local interactions to contribute sufficient influence in the inertial range and destroy universal scaling. The extent of this influence is characterized by the magnitude of *the other terms* of the balance equations.

Universal features, such as the energy cascade, are represented by the homogeneous solution, as long as dissipative corrections can be neglected, whereas non-universal features, such as the forcing range, are represented by the particular solution. The extent of the energy cascade is therefore the region where the homogeneous solution dominates the particular solution[1]. This leads to a calculation that enables us to examine the stability of the energy cascade with respect to perturbations

[1]This is in fact the more rigorous definition of an inertial range with a cascade of a conserved quantity in the traditional sense. The self-similarity property of the generalized structure functions in an inertial range is only approximately true in two-dimensional turbulence, and, appearently, even for the case of three-dimensional turbulence.

to the forcing statistics by comparing the order of magnitude of Q_n against $\mathcal{O}_n F_{n+1}$. Similarly, it is also possible to derive the location of the dissipation length scales [173] by comparing the order of magnitude of νJ_n and βH_n against the typical term in $\mathcal{O}_n F_{n+1}$. The fusion rules are used to find the scaling exponents of $\mathcal{O}_n F_{n+1}$, so that it can be compared with the other terms.

It should be stressed that it is not obvious that a non-trivial homogeneous solution should exist in the first place. It can be shown, however, that a consequence of the conservation of energy is that the equation $\mathcal{O}_2 F_3 = 0$ has a non-trivial solution if $\zeta_3 = 1$ [173]. To see this explicitly, note that after some calculations, we find

$$\begin{aligned} \mathcal{O}_2 F_3(\mathbf{x}_1, \mathbf{x'}_1, \mathbf{x}_2, \mathbf{x'}_2) &= \frac{1}{2}\frac{d[S_3(r_{12}) - S_3(r_{12'})]}{dr_1} + \frac{1}{2}\frac{d[S_3(r_{1'2'}) - S_3(r_{1'2})]}{dr_{1'}} \\ &= A[r_{12}^{\zeta_3 - 1} - r_{12'}^{\zeta_3 - 1} + r_{1'2'}^{\zeta_3 - 1} - r_{1'2}^{\zeta_3 - 1}]. \end{aligned} \tag{6.7}$$

where $r_{12} = \|\mathbf{x}_1 - \mathbf{x}_2\|$, etc, and

$$S_3(r) = \langle [u_\alpha(\mathbf{x}_0 + r\mathbf{e}, t) - u_\alpha(\mathbf{x}_0, t)]^3 \rangle \tag{6.8}$$

is the standard 3rd order structure function (assuming local homogeneity and local isotropy) with $\mathbf{e}$ an arbitrary unit vector. Equation (6.7) is zero (the homogeneous equation $\mathcal{O}_2 F_3 = 0$) for all configurations of velocity differences when either the numerical constant $A = 0$ (the trivial solution), or when $\zeta_3 = 1$ (the non-trivial solution). Given $\zeta_3 = 1$, one may show that it leads to a constant energy flux proportional to A.

Further development [17, 175] of this theory has given theoretical grounds in support of the self-similarity property of the generalized structure functions F_n. The argument can be summarized as follows: first, it is observed that the homogeneous equations are invariant with respect to the following group of transformations

$$\mathbf{r} \mapsto \lambda \mathbf{r}, \quad F_n \mapsto \lambda^{nh + \mathcal{Z}(h)} F_n. \tag{6.9}$$

This means that in an inertial range, where the homogeneous equations are a valid approximation, solutions $F_{n,h}$ that satisfy the self-similarity property

$$F_{n,h}(\{\lambda \mathbf{x}_k, \lambda \mathbf{x'}_k\}_{k=1}^n, t) = \lambda^{nh + \mathcal{Z}(h)} F_{n,h}(\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t), \tag{6.10}$$

are admissible. What actually happens in the inertial range of three-dimensional turbulence is that the correct solution is the linear combination of these solutions, given by

$$F_n = \int d\mu(h) F_{n,h}. \tag{6.11}$$

This conclusion also follows from the multifractal model of Frisch [94]. In this case, however, it is established on theoretical grounds without the multifractal assumption. For each n the observable scaling exponent ζ_n is given by

$$\zeta_n = \min_h [nh + \mathcal{Z}(h)]. \tag{6.12}$$

All the other contributions are hidden. In other words, for each F_n, the observable scaling exponent ζ_n is caused by a $F_{n,h}$ with distinct h.

According to L'vov and Procaccia, $\mathcal{Z}(h)$ can be evaluated from a solvability condition applied on the homogeneous equations [17, 175]. This can be used, in principle, to calculate the scaling exponents [16]. Unfortunately, this is a very difficult calculation. The only scaling exponent that is not difficult to evaluate is ζ_3. Recently L'vov and Procaccia [176], proposed an alternative perturbative argument that can calculate all the scaling exponents accessible to experiment, given ζ_2 and ζ_4. An independent calculation by Giles [103] closes the argument by showing how the scaling exponents can be evaluated diagrammatically without any experimental input. For our present purpose, only ζ_2 is needed. Based on the assumption that intermittency corrections are negligeble for ζ_2, we will use $\zeta_2 \approx 2\zeta_3/3$. This yields the well-known $k^{-5/3}$ spectrum of three-dimensional turbulence.

6.2 The case of two-dimensional turbulence

These arguments can be repeated for the case of two-dimensional turbulence. The balance equations have the same form, and the relevant locality and stability arguments can be extended to the two-dimensional case as well. The homogeneous equation $\mathcal{O}_n F_{n+1} = 0$ still has a solution that corresponds to an energy cascade like before. However, two-dimensional turbulence also conserves enstrophy, and it is anticipated that there is a corresponding enstrophy cascade. A homogeneous solution corresponding to the enstrophy cascade cannot be obtained from $\mathcal{O}_n F_{n+1} = 0$. To show how the enstrophy cascade can arise, it is necessary to introduce the generalized structure functions of the *vorticity* differences

$$G_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left\langle \prod_{k=1}^n [\zeta(\mathbf{x}_k, t) - \zeta(\mathbf{x'}_k, t)] \right\rangle. \tag{6.13}$$

These are related with F_n through differentiation as follows:

$$G_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left[\prod_{k=1}^n \varepsilon_{\alpha_k \beta_k} (\partial_{\alpha_k, \mathbf{x}_k} + \partial_{\alpha_k, \mathbf{x'}_k}) \right] F_n^{\beta_1 \beta_2 \ldots \beta_n}(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) \tag{6.14}$$

The relationship between F_n and G_n is a linear transformation, written in abbreviated form as $G_n = \mathcal{T}_n F_n$. For G_n a similar infinite chain of equations can be written by applying $\mathcal{T}_n$ on the

balance equations for F_n

$$\frac{\partial G_n}{\partial t} + \mathcal{T}_n \mathcal{O}_n F_{n+1} + \mathcal{T}_n I_n = \nu \mathcal{T}_n J_n + \beta \mathcal{T}_n H_n + \mathcal{T}_n Q_n. \tag{6.15}$$

Since $\mathcal{T}_n$ is a local differential operator, the relative scaling between the terms remains the same. It follows that the locality and stability proofs don't need to be repeated for this equation. The corresponding homogenous equation

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} = 0 \tag{6.16}$$

admits two non-trivial solutions: one corresponding to the energy cascade (the solution of $\mathcal{O}_n F_{n+1} = 0$), and one corresponding to the enstrophy cascade, which is exclusive to the operator $\mathcal{T}_n \mathcal{O}_n$. Specifically, using the conservation of enstrophy, one may show that

$$\mathcal{T}_2 O_2 F_3(\mathbf{x}_1, \mathbf{x'}_1, \mathbf{x}_2, \mathbf{x'}_2) = B[r_{12}^{\zeta_3 - 3} - r_{12'}^{\zeta_3 - 3} + r_{1'2'}^{\zeta_3 - 3} - r_{1'2}^{\zeta_3 - 3}]. \tag{6.17}$$

and the solvability condition for $\mathcal{T}_2 \mathcal{O}_2 F_3 = 0$ is $\zeta_3 = 3$. It can be shown that this corresponds to constant enstrophy flux proportional to B. Using the axiomatic assumption $\zeta_2 = 2\zeta_3/3$, we obtain the k^{-3} spectrum of the enstrophy cascade. However, since $\mathcal{O}_2 F_3 = 0$ is satisfied when $\zeta_3 = 1$, the necessary and sufficient solvability condition for $\mathcal{T}_2 \mathcal{O}_2 F_3 = 0$ is $\zeta_3 \in \{1, 3\}$. We conclude that two homogeneous solutions are admissible to eq. (6.16): an energy cascade with constant energy flux and no enstrophy flux, or an enstrophy cascade with constant enstrophy flux and no energy flux. These two solutions were shown to exist by Kraichnan [147], who however did not know that they could be superimposed (see discussion in section 3.2). Furthermore, Kraichnan's argument shows that the reason why both solutions, by themselves, correspond to pure cascades is because they are individually self-similar.

It is appropriate to use equation (6.15) because we expect the vorticity differences statistics to reach stationarity. It is the stationarity condition that legitimizes using the balance equations, as modified by the $\mathcal{T}_n$ operator. The enstrophy cascade solution itself is extracted from the conservation of enstrophy. Furthermore, unlike the case of three-dimensional turbulence, in two-dimensional turbulence the generalized structure functions F_n have logarithmic corrections. It is unclear whether these corrections can be accounted for as the effective scaling of the combination of the individual $F_{n,h}$ solutions as a whole, or whether they should be treated as subleading contributions to $F_{n,h}$ themselves. In any event, as discussed in [**?**], we are satisfied with adopting the theory of Falkovich and Lebedev [84, 85] to derive the logarithmic corrections, as long as the existence of separation of scales in the enstrophy range can be shown.

Another point that requires careful clarification for the case of two-dimensional turbulence is the effect of the dissipative terms in controlling the location of the dissipation scales. The dissipation terms in the balance equation can be written as

$$\begin{aligned}\nu J_n + \beta H_n &= \nu \sum_{k=1}^{n} (\nabla_{\mathbf{x}_k}^{2\kappa} + \nabla_{\mathbf{x}'_k}^{2\kappa}) F_n + \beta \sum_{k=1}^{n} (\nabla_{\mathbf{x}_k}^{-2m} + \nabla_{\mathbf{x}'_k}^{-2m}) F_n. \\ &\equiv \mathcal{D}_n F_n,\end{aligned} \tag{6.18}$$

which is a linear transformation $\mathcal{D}_n$ on F_n. It follows that the contribution of the inertial range combined with the dissipation range to F_n is in fact a homogeneous solution to the following modified equation:

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} - \mathcal{T}_n \mathcal{D}_n F_n = 0. \tag{6.19}$$

In other words, the presence of the dissipative terms modifies the linear operator that generates the balance equations, and in doing so *it modifies the homogeneous solutions* responsible both for the leading and subleading cascades both downscale and upscale. The modification amounts to truncating the inertial range with the dissipation range.

To find the length scale where transition to dissipation range takes place, it is sufficient to substitute the inertial range solution F_n for one of the four cascade possibilities, and compare the interaction term with the dissipative term. The point where the dissipative term becomes important is where the dissipation terms modify the homogeneous solution and where the dissipation range begins. It follows from this that the location of the dissipation scale corresponding to one of the homogeneous solutions present is independent of the energy or enstrophy flux corresponding to the other homogeneous solutions. Without this fact it is not possible to estimate the dissipation scales using dimensional analysis, or the method employed by Frisch [94], which Smith [230] attempted to apply to the problem at hand.

6.3 The double cascade energy spectrum

The overall structure of the proposed theory has interesting consequences. We have two distinct homogeneous solutions that correspond to the two conservation laws. Furthermore, because the homogeneous equations (6.16) are an *infinite linear system*, a linear combination of the two solutions is also a solution. We also have an upscale and a downscale spectral region where universal behavior can be situated.

In the limiting case of infinite Reynolds numbers in an infinite domain, we are expecting a pure upscale energy cascade and a pure downscale enstrophy cascade. The pure cascades correspond to

the individual homogeneous solutions discussed earlier. In the more realistic case of finite domain and finite viscosities, we anticipate instead a downscale and upscale double cascade of energy and enstrophy. From the linearity of the statistical theory it is clear that such double cascades are realized as *linear combinations* of the two homogeneous solutions that represent the energy cascade and the enstrophy cascade. As one approaches the limit of infinite Reynolds number, for each double cascade, one solution will be dominant and responsible for the observable scaling of the structure functions and the energy spectrum, whereas the other solution will be hidden.

It should be remembered that each homogeneous solution is individually modified by the dissipative terms independently of the other homogeneous solution. Furthermore, each individual homogeneous solution is itself a bundle of homogeneous solutions $F_{n,h}$ combined in a specific way to guarantee the solvability condition. In two-dimensional turbulence we have two such bundles of solutions coexisting side by side; one for the energy cascade and one for the enstrophy cascade. The self-similarity condition on one of the two homogeneous solutions is applicable to the bundle , approximately, only when all other contributions from the $F_{n,h}$ bundle are effectively hidden, except for the one contributing the dominant scaling exponent ζ_n.

For the downscale double cascade, the linear structure of the theory implies the following expression for the structure functions $S_n(r)$:

$$S_n(r) = S^{(\varepsilon)}_{n,uv}(r) + S^{(\eta)}_{n,uv}(r) + S^{(p)}_{n,uv}(r), \forall r \in (0, \ell_0), \tag{6.20}$$

where $S^{(p)}_{n,uv}(r)$ is the particular solution driven by the forcing and the boundary term, $S^{(\varepsilon)}_{n,uv}(r)$ is the contribution of the homogeneous solution corresponding to the energy cascade, and $S^{(\eta)}_{n,uv}(r)$, similarly, the contribution corresponding to the enstrophy cascade, are given by

$$\begin{aligned} S^{(\varepsilon)}_{n,uv} &= a_{n,1}(\varepsilon_{uv} r)^{n/3} \mathcal{D}^{(\varepsilon)}_{n,uv}(r/\ell^{(\varepsilon)}_{n,uv}) \\ S^{(\eta)}_{n,uv} &= a_{n,2}(\eta_{uv} r^3)^{n/3} (\chi_n + \ln(\ell_0/r))^{2n/3} \mathcal{D}^{(\eta)}_{n,uv}(r/\ell^{(\eta)}_{n,uv}), \end{aligned} \tag{6.21}$$

where ε_{uv} and η_{uv} are the downscale energy and enstrophy fluxes, and $\mathcal{D}^{(\eta)}_{n,uv}$, $\mathcal{D}^{(\varepsilon)}_{n,uv}$ are the dissipative corrections, expected to satisfy $\mathcal{D}^{(\varepsilon)}_{n,uv}(x) \approx 1$ and $\mathcal{D}^{(\eta)}_{n,uv}(x) \approx 1$ when $0 < x < 1$. The energy spectrum is a linear transform of $S_2(r)$ (see appendix A) and it is given by

$$E(k) = E^{(\varepsilon)}_{uv}(k) + E^{(\eta)}_{uv}(k) + E^{(p)}_{uv}(k), \ \forall k\ell_0 \gg 1, \tag{6.22}$$

where $E^{(p)}_{uv}(k)$ is the contribution of the particular solution , and $E^{(\varepsilon)}_{uv}(k)$, $E^{(\eta)}_{uv}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E^{(\varepsilon)}_{uv}(k) &= a_{uv} \varepsilon^{2/3}_{uv} k^{-5/3} \mathcal{D}^{(\varepsilon)}_{uv}(k\ell^{(\varepsilon)}_{uv}) \\ E^{(\eta)}_{uv}(k) &= b_{uv} \eta^{2/3}_{uv} k^{-3} [\chi + \ln(k\ell_0)]^{-1/3} \mathcal{D}^{(\eta)}_{uv}(k\ell^{(\eta)}_{uv}), \end{aligned} \tag{6.23}$$

with $\mathcal{D}_{uv}^{(\varepsilon)}$ and $\mathcal{D}_{uv}^{(\eta)}$ describing the dissipative corrections. The scales $\ell_{uv}^{(\varepsilon)}$,$\ell_{uv}^{(\eta)}$ are the dissipation length scales for the downscale energy and enstrophy cascade. The absence of intermittency corrections, which would introduce additional factors involving the scale ℓ_0, is assumed. Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrun will take the simple form

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3} + b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \tag{6.24}$$

It should be stressed however that without a more detailed development of this theory, one cannot take it for granted that there will exist a spectral region – the inertial range – where this approximation can be justified. Furthermore, the mixed form of the energy spectrum $E(k)$ is obviously not self-similar.

Similar expressions can be written for the upscale range. Of particular interest is the energy spectrum which is written as

$$E(k) = E_{ir}^{(\varepsilon)}(k) + E_{ir}^{(\eta)}(k) + E_{ir}^{(p)}(k),\ \forall k\ell_0 \ll 1, \tag{6.25}$$

with the constituent terms defined similarly, where

$$\begin{aligned} E_{ir}^{(\varepsilon)}(k) &= a_{ir}\varepsilon_{ir}^{2/3}k^{-5/3}\mathcal{D}_{ir}^{(\varepsilon)}(k\ell_{ir}^{(\varepsilon)}) \\ E_{ir}^{(\eta)}(k) &= b_{ir}\eta_{ir}^{2/3}k^{-3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}). \end{aligned} \tag{6.26}$$

It is unclear at this point whether or not the upscale enstrophy cascade should admit a logarithmic correction. Again, $\ell_{ir}^{(\varepsilon)}$, $\ell_{ir}^{(\eta)}$ are the dissipation scales of the upscale energy and enstrophy cascades.

If we postulate that the fusion rules are satisfied by all four cascades, then it is possible to calculate the location of all four dissipation scales $\ell_{uv}^{(\varepsilon)}$,$\ell_{uv}^{(\eta)}$, $\ell_{ir}^{(\varepsilon)}$, $\ell_{ir}^{(\eta)}$. The theoretical basis for such a calculation lies in observing that the dissipation terms act by modifying the homogeneous solutions, as discussed in the previous section. A detailed account of the calculation of all the relevant scales will be given elsewhere. It is based on an upgraded version of the method outlined in [172]. To find the dissipation length scales of the energy spectrum, it is sufficient to find the corresponding dissipation scales of F_2, with which they have the same order of magnitude. For the dominant cascades, we find that the dissipation scales are given by

$$\begin{aligned} \ell_{uv}^{(\eta)} &= \ell_0\left[\frac{\mathcal{R}_{uv}^{(\eta)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right]^{-1/(2\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\eta)}}\frac{\eta_{uv}^{1/3}}{\nu}\right]^{-1/(2\kappa)} \\ \ell_{ir}^{(\varepsilon)} &= \ell_0\left[\frac{\mathcal{R}_{ir}^{(\varepsilon)}}{\mathcal{R}_{0,ir}^{(\varepsilon)}}\right]^{3/(2+6m)} = \left[\frac{1}{\mathcal{R}_{0,ir}^{(\varepsilon)}}\frac{\varepsilon_{ir}^{1/3}}{\beta}\right]^{3/(2+6m)}, \end{aligned} \tag{6.27}$$

where, $\mathcal{R}_{uv}^{(\eta)}$ and $\mathcal{R}_{ir}^{(\varepsilon)}$, are the Reynolds numbers corresponding to the downscale enstrophy cascade and upscale energy cascade, correspondingly, defined as

$$\begin{aligned}\mathcal{R}_{uv}^{(\eta)} &= \frac{\eta_{uv}^{1/3}\ell_0^{2\kappa}}{\nu} \\ \mathcal{R}_{ir}^{(\varepsilon)} &= \frac{\varepsilon_{ir}^{1/3}\ell_0^{-2m-2/3}}{\beta}.\end{aligned} \tag{6.28}$$

The numerical constants $\mathcal{R}_{0,uv}^{(\varepsilon)}, \mathcal{R}_{0,uv}^{(\eta)}$ can be interpreted as critical Reynolds numbers for the corresponding cascades, and they may have some dependence on κ and m. They represent the minimum required Reynolds number for separating the dissipation scale from the injection scale.

These relations are consistent with what is anticipated from dimensional analysis. However, for the relevant case of molecular diffusion $\kappa = 1$, we find that the dissipation scale $\ell_{uv}^{(\eta)}$ is instead obtained by solving the following transcendental equation

$$\left(\frac{\ell_0}{\ell_{uv}^{(\eta)}}\right)^{3/2} \ln\left(\frac{\ell_0}{\ell_{uv}^{(\eta)}}\right) = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\eta)}}\frac{\eta_{uv}^{1/3}\ell_0^2}{\nu}\right]^{3/4}. \tag{6.29}$$

We see that in most cases the result is consistent with dimensional analysis and with "Hypothesis 3" of Eyink [74]. The interesting exception is the case $\kappa = 1$ which is consistent with "Hypothesis 2" of Eyink [74] but not "Hypothesis 3". These results should be modified if there are intermittency corrections to the cascades or if it should turn out that the fusion rules fail. A similar calculation for the subleading cascades gives the other two dissipation scales

$$\begin{aligned}\ell_{uv}^{(\varepsilon)} &= \ell_0\left[\frac{\mathcal{R}_{uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\varepsilon)}}\right]^{3/(2-6\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\varepsilon)}}\frac{\varepsilon_{uv}^{1/3}}{\nu}\right]^{3/(2-6\kappa)} \\ \ell_{ir}^{(\eta)} &= \ell_0\left[\frac{\mathcal{R}_{ir}^{(\eta)}}{\mathcal{R}_{0,ir}^{(\eta)}}\right]^{1/(2m)} = \left[\frac{1}{\mathcal{R}_{0,ir}^{(\eta)}}\frac{\eta_{ir}^{1/3}}{\beta}\right]^{1/(2m)},\end{aligned} \tag{6.30}$$

where, $\mathcal{R}_{uv}^{(\varepsilon)}$ and $\mathcal{R}_{ir}^{(\eta)}$, are defined as

$$\begin{aligned}\mathcal{R}_{uv}^{(\varepsilon)} &= \frac{\varepsilon_{uv}^{1/3}\ell_0^{2\kappa-2/3}}{\nu} \\ \mathcal{R}_{ir}^{(\eta)} &= \frac{\eta_{ir}^{1/3}\ell_0^{-2m}}{\beta}.\end{aligned} \tag{6.31}$$

For the case of Ekman damping $m = 0$, the expression given for $\ell_{ir}^{(\eta)}$ is invalid. If we allow the logarithmic correction, then the expression for $\ell_{ir}^{(\eta)}$ should be replaced by an exponential function. However it is far more plausible that this result means that a constant inverse enstrophy flux may not be possible for this case. That would imply that the energy flux cannot be constant either, as

we will show in the next section. It should be stressed again that these results hinge on the validity of the fusion rules.

For the case where there is the dominant downscale enstrophy flux and the dominant upscale energy flux, the interesting dissipation scales are defined as $\ell_{ir} \equiv \ell_{ir}^{(\varepsilon)}$ and $\ell_{uv} \equiv \ell_{uv}^{(\eta)}$. That is, ℓ_{ir} is the leading upscale energy dissipation scale, and ℓ_{uv} is the leading downscale enstrophy dissipation scale.

To put this theory on solid ground, it is necessary to consider a number of subtle questions, such as locality and stability. The crucial condition that needs to be established to show that energy and enstrophy cascades will form both on the upscale and downscale side of injection is threefold: First, the homogeneous solution shouldn't be hidden by the particular solution. Second, the dissipative corrections to the homogeneous solution should not destroy the inertial range but allow ample room for a cascade to form. Third, the dissipative scales must be positioned so that the incoming energy and enstrophy can be dissipated. Further development of this theory promises to show under what circumstances this condition is satisfied.

6.4 The subleading cascades

Is it possible to see the subleading cascades? The answer is that in numerical simulations we can look for the constant energy flux in the downscale cascade, for example. In a recent comment, Smith [230] reported a small energy flux accompanying the enstrophy flux in the downscale range of a numerical simulation of two-dimensional turbulence. Constant downscale energy flux has been observed before in Danilov and Gurarie [60, 61] (see their Figure 1 in [61], and Figure 1,2 in [60]) and Borue [31] (see his Figure 3)). So far as we know, the subleading inverse enstrophy cascade has not been discussed much in the turbulence literature.

In a hypothetical situation where the fluxes are fixed but the inertial ranges are extended indefinitely, the subleading cascades will eventually be exposed after certain "transition" wavenumbers. These can be obtained by comparing the leading and subleading terms in the energy spectrum equation (6.22), and these transition wavenumbers for the downscale and upscale ranges, respectively, are given by

$$k_t^{(uv)} \approx \sqrt{\frac{\eta_{uv}}{\varepsilon_{uv}}} = \frac{1}{\lambda_{uv}} \qquad k_t^{(ir)} \approx \sqrt{\frac{\eta_{it}}{\varepsilon_{ir}}} = \frac{1}{\lambda_{ir}}. \tag{6.32}$$

It follows that the necessary condition for exposing the subleading cascades is: $\ell_{uv} \ll \lambda_{uv}$ and $\ell_{ir} \gg \lambda_{ir}$.

As has been pointed out by Danilov [58], this condition cannot be satisfied for the case of two-dimensional turbulence because the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained

by

$$k^2\Pi_E(k) - \Pi_G(k) < 0, \tag{6.33}$$

for all wavenumbers outside of the forcing range. To prove this, note that downscale of injection, in a stationary system, the energy and enstrophy flux are given by

$$\begin{aligned}\Pi_E(k) &= 2\nu \int_k^{+\infty} q^{2\kappa} E(q)dq + 2\beta \int_k^{+\infty} q^{-2m} E(q)dq \\ \Pi_G(k) &= 2\nu \int_k^{+\infty} q^{2\kappa+2} E(q)dq + 2\beta \int_k^{+\infty} q^{-2m+2} E(q)dq,\end{aligned} \tag{6.34}$$

as long as the entire forcing spectrum is localized in the $[0, k]$ interval. These relations are an immediate consequence of the observation that all the energy and enstrophy dissipated at the interval $[k, +\infty)$ has to cross the wave number k to come from the $[0, k]$ interval where it is injected. It follows that

$$k^2\Pi_E(k) - \Pi_G(k) = 2\nu \int_k^{\infty} (k^2 - q^2) q^{2\kappa} E(q)dq + 2\beta \int_k^{\infty} (k^2 - q^2) q^{-2m} E(q)dq \leq 0. \tag{6.35}$$

The same argument can be repeated when the wavenumber k is on the upscale side of injection. In that case, the energy flux and the enstrophy flux satisfy

$$\begin{aligned}\Pi_E(k) &= -2\nu \int_0^{k} q^{2\kappa} E(q)dq - 2\beta \int_0^{k} q^{-2m} E(q)dq \\ \Pi_G(k) &= -2\nu \int_0^{k} q^{2\kappa+2} E(q)dq - 2\beta \int_0^{k} q^{-2m+2} E(q)dq,\end{aligned} \tag{6.36}$$

and the same inequality follows.

In fact, it follows from Danilov's inequality (6.33), that the transition scales will be located in the dissipation range both upscale and downscale of injection. A similar claim was given by Smith [230], however his argument was problematic in some respects, as pointed out by Tung [251]. It should be stressed that the same claim is not applicable in quasi-geostrophic turbulence, where the dynamics is different.

Although the subleading cascades remain effectively hidden, it will be shown that as one approaches the KLB limit the transition scale will converge towards the dissipation scale of the contribution to the energy spectrum by the leading cascade. This convergence is essential in justifying the proof given in section 2.1. Furthermore, it can be shown that although the energy and enstrophy flux associated with the subleading cascades is vanishing rapidly in the KLB limit, there will always be a sufficient amount to form subleading cascades with separation of scales proportional to that of the leading cascades.

We begin with the demonstration that in the KLB limit the transition scales coincide with the dissipation scales of the leading cascade. When the system reaches equilibrium, the downscale enstrophy flux η_{uv} , for example, will be equal to the corresponding dissipation rate at small scales. Since the effect of the sink at large scales can be safely ignored, the dominant contribution to the enstrophy flux is given by the integrals

$$\eta_{uv} \approx 2\nu \int_{1/\ell_0}^{1/\ell_{uv}} k^{2\kappa+2} E(k) dk + 2\nu \int_{1/\ell_{uv}}^{+\infty} k^{2\kappa+2} E(k) dk. \tag{6.37}$$

When we substitute[2]

$$E(k) \approx b_{uv} \eta_{uv}^{2/3} k^{-3} \mathcal{D}_{uv}(k\ell_{uv}), \tag{6.38}$$

from (6.22), we find that the first integral diverges in the limit $\nu \to 0$, whereas the second integral stays finite because it is moderated by dissipative corrections to the energy spectrum. The vanishing viscosity eliminates the second integral and moderates the divergence of the first integral. As a result, the dominant contribution comes from the first integral and it follows that

$$\eta_{uv} \approx b_{uv} \nu \eta_{uv}^{2/3} (1/\ell_{uv})^{2\kappa} + \nu C_1, \tag{6.39}$$

In fact, for the case $\kappa \neq 1$, the relevant coefficient

$$\mathcal{A}(\nu) \equiv \nu (1/\ell_{uv})^{2\kappa}, \tag{6.40}$$

turns out to be constant. For the exceptional case $\kappa = 1$, it will vanish in the limit $\nu \to 0$. However, it does so very slowly. To see this, note that for $\kappa = 1$, $\mathcal{A}(\nu)$ can be evaluated analytically as

$$\mathcal{A}(\nu) = \eta_{uv}^{1/3} \left[\ln \left(\frac{\ell_0}{\ell_{uv}} \right) \right]^{-4/3}. \tag{6.41}$$

This means, for example, that increasing the separation of scales ratio all the way up to 10^{10} will only decrease $\mathcal{A}(\nu)$ by a little more than an order of magnitude. Incidentally, this calculation shows that there is an anomalous sink of enstrophy for hyperviscosity $\kappa > 1$, and practically so for molecular viscosity $\kappa = 1$.

Using a similar argument, the downscale energy flux is given by

$$\varepsilon_{uv} \approx 2\nu \int_{1/\ell_0}^{1/\ell_{uv}} k^{2\kappa} E(k) dk + 2\nu \int_{1/\ell_{uv}}^{+\infty} k^{2\kappa} E(k) dk. \tag{6.42}$$

Again, although the first integral does not balance the viscosity ν, it still is the case that the second integral vanishes more rapidly than the first integral. Using a similar argument, as earlier, the

[2]The asymptotically valid assumption of the almost-pure double cascade, enters the argument at this step.

downscale energy flux can be written as:

$$\varepsilon_{uv} \approx b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2} + \nu C_2, \tag{6.43}$$

where again, it is expected that the dominant contribution is the first term.

If the quantities νC_1 and νC_2 are small enough to be negligible, then it follows that

$$\begin{aligned}\lambda_{uv}^2 = \frac{\varepsilon_{uv}}{\eta_{uv}} &\approx \frac{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2} + \nu C_2}{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa} + \nu C_1} \\ &\approx \frac{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2}}{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa}} = \ell_{uv}^2.\end{aligned} \tag{6.44}$$

Using a similar argument, we may show that as $\beta \to 0$, we have $\ell_{ir} \approx \lambda_{ir}$.

The same argument can be repeated for the hypothetical case, which is inconsistent, where the dominant upscale cascade is the enstrophy cascade and the dominant downscale cascade is the energy cascade. The form of the energy spectrum will then be different, but it will be compensated by the laws governing the dominant dissipation scales which will also be different. In particular, using the approximation $E(k) \approx a_{uv}\varepsilon^{2/3}k^{-5/3}\mathcal{D}_{uv}(k\ell_{uv})$ from (6.22), we get

$$\begin{aligned}\eta_{uv} &= a_{uv}\nu\varepsilon_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2/3} + \nu C_1 \\ \varepsilon_{uv} &= a_{uv}\nu\varepsilon_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2/3-2} + \nu C_2.\end{aligned} \tag{6.45}$$

Except for the case $\kappa = 1$, which is problematic in a number of ways, when the quantities νC_1 and νC_2 are negligible, then we still get $\ell_{uv} \approx \lambda_{uv}$.

A consequence of this coincidence is that it enables the calculation of the separation of scales of the subleading cascades as a function of the separation of scales of the leading cascades. First, we write the downscale energy and enstrophy flux in terms of the corresponding dissipation scales

$$\begin{aligned}\eta_{uv} &= \nu^3[\mathcal{R}_{0,uv}^{(\eta)}]^3[\ell_{uv}^{(\eta)}]^{-6\kappa} \\ \varepsilon_{uv} &= \nu^3[\mathcal{R}_{0,uv}^{(\varepsilon)}]^3[\ell_{uv}^{(\varepsilon)}]^{2-6\kappa}.\end{aligned} \tag{6.46}$$

Then, the flux ratio is given by

$$\lambda_{uv}^2 = \frac{\varepsilon_{uv}}{\eta_{uv}} = \left(\frac{\mathcal{R}_{0,uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right)^3 \left[\frac{\ell_{uv}^{(\eta)}}{\ell_{uv}^{(\varepsilon)}}\right]^{6\kappa-2} [\ell_{uv}^{(\eta)}]^2, \tag{6.47}$$

We have shown that in the KLB limit $\lambda_{uv} \approx \ell_{uv}^{(\eta)}$. It follows that the dissipation scale of the subleading downscale energy cascade is given asymptotically by

$$\ell_{uv}^{(\varepsilon)} \approx \ell_{uv}^{(\eta)} \left(\frac{\mathcal{R}_{0,uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right)^{3/(6\kappa-2)}. \tag{6.48}$$

This equation shows that asymptotically the extent of the subleading downscale energy cascade $\ell_{uv}^{(\varepsilon)}$ is proportional to the extent of the leading downscale enstrophy cascade $\ell_{uv}^{(\eta)}$. For a very large order of hyperdiffusion κ the proportionality constant approaches unity. In that case, $\ell_{uv}^{(\varepsilon)} \approx \ell_{uv}^{(\eta)}$. Nevertheless the ratio of the leading and subleading dissipation scales cannot be taken as 1 in (6.47). A small difference in the dissipation scales ratio $\ell_{uv}^{(\eta)}/\ell_{uv}^{(\varepsilon)}$ can still result in a significant adjustment of the transition scale, because in equation (6.47) it is being raised to very large powers. (see [251])

A similar result can be derived for the upscale range. The upscale energy and enstrophy fluxes are given by

$$\begin{aligned} \varepsilon_{ir} &= \beta^3 [\mathcal{R}_{0,ir}^{(\varepsilon)}]^3 [\ell_{ir}^{(\varepsilon)}]^{2+6m} \\ \eta_{ir} &= \beta^3 [\mathcal{R}_{0,ir}^{(\eta)}]^3 [\ell_{ir}^{(\eta)}]^{6m}, \end{aligned} \tag{6.49}$$

and, likewise, the flux ratio reads

$$\lambda_{ir}^2 = \frac{\varepsilon_{ir}}{\eta_{ir}} = \left(\frac{\mathcal{R}_{0,ir}^{(\varepsilon)}}{\mathcal{R}_{0,ir}^{(\eta)}} \right)^3 \left[\frac{\ell_{ir}^{(\varepsilon)}}{\ell_{uir}^{(\eta)}} \right]^{6m} [\ell_{ir}^{(\varepsilon)}]^2, \tag{6.50}$$

and in the KLB limit we find

$$\ell_{ir}^{(\eta)} \approx \ell_{ir}^{(\varepsilon)} \left(\frac{\mathcal{R}_{0,ir}^{(\varepsilon)}}{\mathcal{R}_{0,ir}^{(\eta)}} \right)^{1/2m}. \tag{6.51}$$

The significance of these results is that they highlight that the subleading cascades are not a hypothetical possibility; even far into the KLB limit, there will be sufficient downscale energy flux and upscale enstrophy flux to provide subleading cascades that are proportionally as large as the leading cascades. In fact, if that were not the case it would signal an inconsistency in the overall theory, since the energy flux and the enstrophy flux are constrained to be constant over the same spectral region.

As separation of scales increases, the *percentage* of these counter fluxes decreases rapidly. To see this, use the approximation $\ell_{uv} \approx \lambda_{uv}$ to show that

$$\varepsilon_{uv} = \eta_{uv} \lambda_{uv}^2 \leq \eta \lambda_{uv}^2 = \varepsilon \left(\frac{\lambda_{uv}}{\ell_0} \right)^2 \approx \varepsilon \left(\frac{\ell_{uv}^{(\eta)}}{\ell_0} \right)^2. \tag{6.52}$$

Similarly, for the upscale range we have

$$\eta_{ir} \leq \eta \left(\frac{\ell_0}{\lambda_{ir}} \right)^2 \approx \eta \left(\frac{\ell_0}{\ell_{ir}^{(\varepsilon)}} \right)^2. \tag{6.53}$$

It follows from these inequalities that a separation of scales of one decade is sufficient to reduce the counter fluxes percentagewise to about 1%. However, because the derivation implicitly assumes the KLB limit, it does not follow that a separation of scales of only one decade is sufficient to reach that limit.

6.5 On the coexistence of constant energy and enstrophy flux

The notion of an inertial range where constant (in wavenumber) energy flux and a constant enstrophy flux coexist runs against a common misconception. It is widely believed that Kraichnan [145] showed, in his original paper, that it is not possible for constant energy flux and constant enstrophy flux to coexist in the same inertial range. As a matter of fact, the conclusion itself is false, and a careful examination of the actual argument that Kraichnan made will show that it can be adopted to corroborate the opposite conclusion: that it is possible for constant energy flux and constant enstrophy flux to coexist. We will begin with a presentation of our own simpler proof. Then, we will discuss Kraichnan's argument in detail.

6.5.1 Proof that fluxes can coexist

There are several objections that can be raised against the claim that constant energy flux and constant enstrophy flux cannot coexist. First, we can argue that the claim is inconsistent with numerical simulations. Consider for example the case of the direct enstrophy cascade. Lindborg and Alvelius [160] report that they were able to produce a direct enstrophy cascade where the enstrophy flux is constant. Because the simulation takes place in a finite domain and the dissipation wave number is itself finite, there is inevitably a certain amount of energy dissipated with the enstrophy. This energy has to find its way from the forcing range to the dissipation range. Because the enstrophy flux is observably constant in the inertial range, the energy flux is constrained to be constant as well, from the relationship (6.57), which we will prove in a moment.

In our opinion, nothing more needs to be said to settle the matter. However, because this claim is considered, by many, controversial, it is still useful and interesting to support it with a theoretical proof. We begin with showing that when one of the two fluxes is constant, the other flux is also required to be constant. In other words, the possibility where either the energy or the enstrophy flux is constant and the other flux is not constant is not allowed. Then, we will show that the scenario where both fluxes are constant and non-zero is allowed.

Let $\varepsilon(k)$ be the amount of energy and $\eta(k)$ be the amount of enstrophy transferred by triad interactions from the $(0, k)$ interval to the $(k, +\infty)$ interval. The conservation of energy and enstrophy by the triad interactions implies the following boundary conditions for $\varepsilon(k)$ and $\eta(k)$

$$\begin{aligned} \varepsilon(0) &= \lim_{k\to+\infty} \varepsilon(k) = 0 \\ \eta(0) &= \lim_{k\to+\infty} \eta(k) = 0. \end{aligned} \tag{6.54}$$

Recall that the evolution of the energy and enstrophy spectrum is governed by

$$\begin{aligned} \frac{\partial E(k)}{\partial t} + \frac{\partial \varepsilon(k)}{\partial k} &= -\mathcal{D}(k)E(k) + F(k) \\ \frac{\partial G(k)}{\partial t} + \frac{\partial \eta(k)}{\partial k} &= -\mathcal{D}(k)G(k) + k^2 F(k), \end{aligned} \tag{6.55}$$

where $\mathcal{D}(k)$ is the dissipation operator given by

$$\mathcal{D}(k) = 2\nu k^{2\kappa} + 2\beta k^{-2m}, \tag{6.56}$$

and $F(k)$ is the forcing spectrum. The second equation can be obtained from the first equation by multiplying it with k^2. It follows that the energy flux $\varepsilon(k)$ and enstrophy flux $\eta(k)$ satisfy the following constraint

$$\frac{\partial \eta(k)}{\partial k} = k^2 \frac{\partial \varepsilon(k)}{\partial k}. \tag{6.57}$$

Note that this relationship, originally proven by Leith [150], is exact, and it is an immediate consequence of the relationship $G(k) = k^2 E(k)$ between the energy spectrum and the enstrophy spectrum. This immediately implies that it is not permitted for only one of the two fluxes $\varepsilon(k)$ and $\eta(k)$ to be constant in wavenumber.

To show that it is possible for a constant energy and enstrophy flux to coexist we proceed with the following argument. It can be shown that $\varepsilon(k)$ and $\eta(k)$ are related with the third order standard structure function $S_3(r)$ by linear transformations written as

$$\begin{aligned} \varepsilon(k) &= \int_0^{+\infty} \mathcal{K}_1(k,r) S_3(r) dr \\ \eta(k) &= \int_0^{+\infty} \mathcal{K}_2(k,r) S_3(r) dr, \end{aligned} \tag{6.58}$$

where $\mathcal{K}_1(k,r)$ and $\mathcal{K}_2(k,r)$ are the kernels of appropriate integro-differential operators. A similar relationship of this form for the energy flux $\varepsilon(k)$ was given by Frisch [94] for the case of three-dimensional turbulence. In a double cascade scenario, we anticipate from the linearity of the statistical theory that

$$\begin{aligned} S_{3,uv}(r) &= -a_1 \varepsilon_{uv} r + a_2 \eta_{uv} r^3 \\ S_{3,ir}(r) &= a_3 \varepsilon_{ir} r - a_4 \eta_{ir} r^3. \end{aligned} \tag{6.59}$$

These relationships are expected on the same grounds as the form of the energy spectrum given by (6.22) as long as the corresponding cascades form successfully. Without having detailed knowledge of the kernels $\mathcal{K}_1(k,r)$ and $\mathcal{K}_2(k,r)$, it is natural to suggest that the contributions of the energy and enstrophy homogeneous solutions are mixed together in the evaluation of the energy flux and

enstrophy flux. This would have implied that the presence of the enstrophy solution interferes with the energy flux $\varepsilon(k)$, and the presence of the energy solution interferes with the enstrophy flux $\eta(k)$. If that is the case, then it is plausible to expect that it may be forbidden for both fluxes to be constant simultaneously as a result of this interference effect.

We will show that these interferences do not take place, and in fact we will demonstrate that this result can be established without having full knowledge of the mathematical form of the kernels $\mathcal{K}_1(k,r)$ and $\mathcal{K}_2(k,r)$. It is a direct consequence of the conservation of enstrophy and the relationship between the energy and enstrophy spectrum.

Begin with introducing a potential function $P(k)$, defined from the energy flux $\varepsilon(k)$ as

$$P(k) = \int_0^k 2q\varepsilon(q)\, dq. \tag{6.60}$$

It follows that there is a kernel $\mathcal{K}(k,r)$, that relates the function $P(k)$ with $S_3(r)$ by a linear transformation

$$P(k) = \int_0^{+\infty} \mathcal{K}(k,r) S_3(r) dr. \tag{6.61}$$

The energy flux can be calculated from the following relationship

$$\varepsilon(k) = \frac{1}{2k}\frac{\partial P(k)}{\partial k}. \tag{6.62}$$

Likewise, the enstrophy flux is given by

$$\begin{aligned} \eta(k) &= \int_0^k q^2 \frac{\partial \varepsilon(q)}{\partial q}\, dq = k^2\varepsilon(k) - \int_0^k 2q\varepsilon(q)\, dq \\ &= \frac{k}{2}\frac{\partial P(k)}{\partial k} - P(k). \end{aligned} \tag{6.63}$$

These relationships imply the following corresponding relationships between the kernels $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}$.

$$\begin{aligned} \mathcal{K}_1(k,r) &= \frac{1}{2k}\frac{\partial \mathcal{K}(k,r)}{\partial k} \\ \mathcal{K}_2(k,r) &= \frac{k}{2}\frac{\partial \mathcal{K}(k,r)}{\partial k} - \mathcal{K}(k,r). \end{aligned} \tag{6.64}$$

For an inertial range (k_1, k_2) with a double cascade of energy and enstrophy we require that

$$\begin{aligned} &\varepsilon(k) = \varepsilon,\ \forall k \in (k_1, k_2) \\ &\eta(k) = \eta,\ \forall k \in (k_1, k_2). \end{aligned} \tag{6.65}$$

It is sufficient to show that there is a unique function $P(k)$ that can satisfy both conditions. Indeed, the requirement that the energy flux be constant gives

$$\frac{1}{2k}\frac{\partial P(k)}{\partial k} = \varepsilon \Longleftrightarrow P(k) = c_1 + \varepsilon k^2. \tag{6.66}$$

where c_1 is an integration constant. Likewise, the requirement that the enstrophy flux be constant gives

$$\frac{k}{2}\frac{\partial P(k)}{\partial k} - P(k) = \eta \iff P(k) = -\eta + c_2 k^2. \tag{6.67}$$

where c_2, likewise, is an integration constant. It follows that both conditions can be satisfied by $P(k)$ if $c_1 = -\eta$ and $c_2 = \varepsilon$, yielding

$$P(k) = \varepsilon k^2 - \eta, \ \forall k \in (k_1, k_2). \tag{6.68}$$

The fact that the two forms of the function $P(k)$ are consistent so that adjusting the integration constants c_1 and c_2 can yield a function $P(k)$ consistent with double energy and enstrophy fluxes, provides the evidence that the double cascade is indeed allowed. The permission is also implicit in the constraint given by equation (6.57), however this argument establishes the necessary consistency between equations (6.65) and (6.59). In realistic situations $P(k)$ has an additional term for dissipative adjustments, which can be safely neglected when universal cascades form successfully.

The well-known derivations of the scaling of $S_3(r)$ for the cases of a pure energy cascade and a pure enstrophy cascade, suggest the following implications

$$\begin{aligned} &\int_0^{+\infty} \mathcal{K}(k,r) S_3(r) dr = \varepsilon k^2 \iff S_3(r) = a\varepsilon r \\ &\int_0^{+\infty} \mathcal{K}(k,r) S_3(r) dr = -\eta \iff S_3(r) = b\eta r^3. \end{aligned} \tag{6.69}$$

Since our theory already predicts the form of $S_3(r)$, as a linear combination it is only sufficient to show that

$$\begin{aligned} S_3(r) = a\varepsilon r + b\eta r^3 &\implies P(k) = \varepsilon k^2 - \eta \\ &\implies [\varepsilon(k) = \varepsilon] \wedge [\eta(k) = \eta]. \end{aligned} \tag{6.70}$$

The first step follows from the converse of (6.69), and the second step from (6.66), (6.67). Thus, both steps can be justified without detailed knowledge of the kernels $\mathcal{K}_1, \mathcal{K}_2$.

6.5.2 A review of Kraichnan's argument

There is a widely believed folklore argument, sometimes attributed to Kraichnan. According to this argument, one begins by subdividing the inertial range wavenumber interval into logarithmically spaced little intervals. Then, one argues that, because of locality, there exists a cascade of adjacent pair interactions such that both energy and enstrophy flow from one interval into the next. Consequently, from the point of view of a particular interval, it is being forced only by the preceding

interval, and therefore the energy and enstrophy flux into the interval has to satisfy

$$\eta(k) = \varepsilon(k)k^2. \tag{6.71}$$

It follows, according to this (incorrect) argument, that it is not possible for constant energy flux and constant enstrophy flux to coexist, because (6.71) cannot be satisfied by $\varepsilon(k)$ and $\eta(k)$ if they are both constant.

As a matter of fact, Kraichnan has explicitly argued against the foundation of this argument in the fifth paragraph of section 1 of his paper. He explained that the concept of the dominant interactions as adjacent pair interactions between logarithmically spaced intervals may be valid in three-dimensional turbulence but is not applicable in two-dimensional turbulence, where transfers can only occur between the middle member of a triad and the two triad members, one on the long-wave side and one on the short-wave side.

However, Kraichnan continues in the seventh paragraph, with the following argument. He claims that for "similarity cascades", for the case where the energy flux is constant, for example, the enstrophy flux is given by

$$\eta(k) = A\varepsilon k^2, \tag{6.72}$$

using a "similarity argument". It follows that when the constant A is non-zero, there is a violation of the conservation of enstrophy, therefore it must be that $A = 0$. That would imply that the enstrophy flux itself is zero. A similar argument can be made for the case where the enstrophy flux is constant. This argument appears to prove that constant energy flux and constant enstrophy flux cannot coexist.

Kraichnan takes it for granted that inertial ranges are self-similar, so he does not consider it necessary to define his notion of self-similarity explicitly in section 1 of his paper. The definition he gives in section 2 is

$$\begin{aligned} E(\lambda k) &= \lambda^{-n} E(k) \\ T(\lambda k, \lambda p, \lambda q) &= \lambda^{\zeta} T(k, p, q), \end{aligned} \tag{6.73}$$

where $T(k, p, q)$ are the spherically integrated transfer rates of the triad interactions. As long as we assume that there are no intermittency corrections, the scaling exponent ζ is given by

$$\zeta = -\frac{1+3n}{2} = -\frac{7+3\zeta_2}{2}, \tag{6.74}$$

where $n = 2 + \zeta_2$ is the absolute value of the slope of $E(k) \sim k^{-n}$ and the scaling exponent actually used by Kraichnan. The energy cascade corresponds to $\zeta = -3$ and the enstrophy cascade to $\zeta = -5$.

There is no physical justification behind asserting this similarity condition, and this is precisely where the flaw in Kraichnan's argument is located. We agree with Kraichnan that if one *assumes* this similarity condition, then the cascades have to be pure cascades. However, it is obvious that by asserting a similarity condition on the energy spectrum one axiomatically excludes the energy spectrum of the double cascade, which has two spectral slopes in linear combination. The same problem arises if one imposes a self-similarity condition on the transfer rates $T(k,p,q)$ or the structure functions $S_3(r)$. In our theoretical framework, the self-similarity of the generalized structure functions is applicable only to the individual homogeneous solutions but not to a linear combination of those solutions. The same principle applies to the triad interactions transfer rates used in Kraichnan's argument.

The rest of Kraichnan's argument proceeds as follows. The fluxes can be obtained from the triad interactions transfer rates by the following integrals

$$\begin{aligned}\varepsilon(k) &= \frac{1}{2}\int_k^\infty d\kappa \int_0^k\int_0^k T(\kappa,p,q)\,dp\,dq - \frac{1}{2}\int_0^k d\kappa\int_k^\infty\int_k^\infty T(\kappa,p,q)\,dp\,dq \\ \eta(k) &= \frac{1}{2}\int_k^\infty d\kappa \int_0^k\int_0^k \kappa^2 T(\kappa,p,q)\,dp\,dq - \frac{1}{2}\int_0^k d\kappa\int_k^\infty\int_k^\infty \kappa^2 T(\kappa,p,q)\,dp\,dq.\end{aligned} \tag{6.75}$$

These relations are exact. Using his similarity condition, and the constraint between energy and enstrophy transfer rates, and some algebra, Kraichnan shows that the fluxes are given by

$$\begin{aligned}\varepsilon(k) &= k^{\zeta+3}\int_0^1 dv\int_1^{+\infty} dw\; W_\varepsilon(v,w,\zeta)T(1,v,w) \\ \eta(k) &= k^{\zeta+5}\int_0^1 dv\int_1^{+\infty} dw\; W_\eta(v,w,\zeta)T(1,v,w).\end{aligned} \tag{6.76}$$

This relation is similar, in spirit, to the relation (6.58) between $\varepsilon(k)$, $\eta(k)$ and $S_3(r)$. The relation $\eta(k) = A\varepsilon(k)k^2$, used by Kraichnan in section 1, follows immediately from (6.76), which is correct only when the self-similarity condition on $T(k,p,q)$ holds. The functions W_ε and W_η read

$$\begin{aligned}W_\varepsilon(v,w,\zeta) &= -\frac{1}{w^2-v^2}\left[(1-v^2)\int_1^w u^{-\zeta-4}du - (w^2-1)\int_v^1 u^{-\zeta-4}du\right] \\ W_\eta(v,w,\zeta) &= -\frac{1}{w^2-v^2}\left[(1-v^2)w^2\int_1^w u^{-\zeta-6}du - (w^2-1)v^2\int_v^1 u^{-\zeta-6}du\right].\end{aligned} \tag{6.77}$$

A simple evaluation of the integrals shows that

$$\begin{aligned}W_\eta(v,w,-3) &= 0 \\ W_\varepsilon(v,w,-5) &= 0.\end{aligned} \tag{6.78}$$

The other two relevant combinations $W_\varepsilon(v,w,-3)$ and $W_\eta(v,w,-5)$ are non-zero. From these

Kraichnan derives, implicitly, the following equivalences:

$$\begin{aligned} \zeta = -3 &\Longleftrightarrow [\varepsilon(k) = \varepsilon] \wedge [\eta(k) = 0] \\ \zeta = -5 &\Longleftrightarrow [\varepsilon(k) = 0] \wedge [\eta(k) = \eta]. \end{aligned} \tag{6.79}$$

Therefore it follows that, as long as the similarity condition[3] holds, double cascades are not allowed.

If we assume that Kraichnan's self-similarity condition applies *to the homogeneous solutions only*, then it follows that the energy cascade solution transfers no enstrophy and the enstrophy cascade solution transfers no energy. This is a desirable conclusion since in the case of a pure cascade there will be only one homogeneous solution present. In a double cascade of energy and enstrophy, the transfer rate can be decomposed into two parts

$$T(k,p,q) = T_\varepsilon(k,p,q) + T_\eta(k,p,q), \tag{6.80}$$

with each part being individually self-similar with different scaling exponents as follows

$$\begin{aligned} T_\varepsilon(\lambda k, \lambda p, \lambda q) &= \lambda^{-3} T_\varepsilon(k,p,q) \\ T_\eta(\lambda k, \lambda p, \lambda q) &= \lambda^{-5} T_\eta(k,p,q). \end{aligned} \tag{6.81}$$

The first part $T_\varepsilon(k,p,q)$ is responsible for the energy transfer and transfers no enstrophy. The second part $T_\eta(k,p,q)$ is responsible for the enstrophy transfer and transfers no energy. Therefore, Kraichnan's argument can the leveraged to show that the interactions responsible for the enstrophy transfer do not interfere with the energy transfer and vice versa. We have given a simpler proof of the same result in the preceding section, but some readers might find this argument more convincing.

6.6 Evidence from numerical simulations

The inertial ranges of two-dimensional turbulence have been studied extensively with numerical simulations and experiments as well as theoretically for many decades. In this section we summarize the accumulated theoretical and experimental insight derived from these studies. Our goal is to give the reader a current account of the extent of theoretical and experimental support for the enstrophy and energy cascade of two-dimensional turbulence. Our conclusion is that there is experimental support for the existence of both cascades under favorable conditions. However, unlike the case of three-dimensional turbulence, neither cascade is completely robust. Furthermore, we note that all present theoretical work implicitly assumes that the cascades have formed successfully and proceeds to explain their statistical behavior. This question of robustness remains open, and we believe that

[3]It should be noted that, the self-similarity of the energy spectrum has not been used in this argument, and it is in fact violated for the case of a pure enstrophy cascade by the logarithmic correction. Only the self-similarity of the triad interactions has been invoked.

it can be addressed by our framework. Some initial comments to that effect for the case of inverse energy cascade are given.

6.6.1 The inverse energy cascade

The inverse energy cascade with the $k^{-5/3}$ energy spectrum on the upscale side of injection appeared to be robust. It has been observed in numerous simulations and experiments [96, 118, 199, 228], and we even have experimental indications that there are no intermittency corrections [27, 200]. Yakhot [263] has formulated an interesting theoretical explanation for the lack of intermittency corrections, using a mathematical technique developed by Polyakov [206] for Burger's turbulence. His argument is based on the unproven assumption that the pressure gradients are local.

In the Kolmogorov downscale energy cascade of three-dimensional turbulence, smaller length scales always exist, so we may disregard the presence of boundary conditions and assume that our system is unbounded with impunity. In the two-dimensional inverse energy cascade, on the other hand, the boundary conditions become significant for sufficiently large length scales. If the upscale cascading energy is not dissipated at length scales smaller than the typical length scales of the boundary conditions, then the energy is condensed at large scales leading to steep spectra. This corresponds to the formation of large-scale coherent structures, mentioned by many authors without necessarily elucidating their origin theoretically. The formation of $k^{-5/3}$ scaling, and the subsequent break down due to this condensation effect, have also been observed in numerical simulations [234, 235]. To form a stationary inverse energy cascade it is therefore necessary that large-scale dissipation remove the energy at the length scale much smaller than the typical length scale of the boundary conditions.

Danilov and Gurarie [61] have conducted numerical simulations using $(m, \kappa) = (0, 2)$, and showed that the optimal β yielding an energy spectrum closest to the KLB prediction of $k^{-5/3}$ scaling does not correspond to constant energy flux. Decreasing β improves the energy flux but the slope of the energy spectrum steepens. This behavior is somewhat minimized in simulations using $(m, \kappa) = (0, 8)$, but the reverse relation between optimizing the flux and optimizing the spectrum persists. Sukoriansky *et al* [240] note that using higher order large-scale dissipation ($m > 0$) may produce a constant energy flux, but distorts the spectrum. It has therefore been suggested that the locality of the inverse energy cascade should be called into question.

There are two aspects of this behaviour that call for an explanation. The first is the behavior of the upscale energy flux, which is non-constant. The second is the observed steepening of the energy spectrum. We wish to begin by pointing out that the behaviour of the energy flux, by itself,

is not paradoxical but quite reasonable. It makes sense that decreasing β, or using hypodiffusion, improves the upscale energy flux, in the sense of making it more constant over a larger range. It is also reasonable that the upscale energy flux is not constant under Ekman damping. As we have mentioned previously, the dissipation scale law for the inverse enstrophy cascade breaks down for the case of Ekman damping. It is therefore unlikely that an inverse enstrophy cascade with constant enstrophy flux can be realized. Because the energy flux is linked with the enstrophy flux, the energy flux cannot[4] be constant either.

The interesting phenomenon is the steepening of the energy spectrum for small β. Both Danilov and Gurarie [60], and earlier Borue [31], observed that this steepening is caused by coherent structures. These structures cover a relatively small portion of the domain, but they account for most of the energy. When the energy spectrum of the background flow, in which these structures are situated, is evaluated instead, the regular $-5/3$ scaling is restored. This means that the homogeneous solution for the inverse energy cascade still exists, and therefore, the observed steepening of the energy spectrum is caused either by the homogemeous solution associated with an inverse enstrophy cascade, or the particular solution. Danilov and Gurarie [60] examined this first possibility by a very careful analysis of these coherent structures and established that they receive most of their energy from the forcing term. Furthermore, as Danilov [59] has demonstrated, in a paper published in this volume, the inverse enstrophy flux is very small for all cases, and in fact it is larger in the simulations where Ekman damping is used and smaller in the simulation where hypodiffusion is used. It follows that the homogeneous solution associated with the inverse enstrophy cascade is not likely to be responsible for the observed steepening.

The contribution of the particular solution $E_{ir}^{(p)}(k)$ to the energy spectrum plays a crucial role in explaining this paradoxical behavior. We propose that the particular solution dominates and hides the homogeneous solution which continues to exist. Physically, the particular solution is associated with the contribution to the energy spectrum by the coherent structures. The homogeneous solution, on the other hand, is associated with the contribution of the background turbulent flow. The resulting energy spectrum is a linear combination of the two solutions. The corresponding physical interpretation is that the coherent structures coexist side-by-side with the background flow, and each contributes its share to the energy spectrum. The fact that Danilov and Gurarie [60] can separate the two contributions to the energy spectrum is further evidence that the contributions are linearly superimposed. Danilov [59] noted that the steepening of the energy spectrum can be

[4]It is not strange that it is possible to see $k^{-5/3}$ scaling without constant energy flux. This indicates that there is a fortunate cancellation between the corrections to the inertial range scaling from the forcing range and the dissipation range. The cancellation is not perfect, and that accounts for the "bulge" observed in some of the numerical simulations [61] where Ekman damping is being used.

attributed to the non-locality of the triad interactions. The triad interactions are obtained from a Fourier transform that mixes the homogeneous and particular solution together. Nevertheless, the local interactions associated with the homogeneous solution continue to be there, and they remain responsible for a small part of the energy and enstrophy transfer associated with the background flow.

This point is illustrated with the case of the inverse energy cascade in a simulation that uses hypodiffusion. As Danilov [59] reports, the energy spectrum exhibits a strong deviation from $k^{-5/3}$ scaling, which indicates non-locality. However, in the same simulation a very constant energy flux is reported. This apparent discrepancy can be explained as follows: The non-local interactions associated with the coherent structures transfer energy and enstrophy directly from the forcing range to the dissipation range, and they are in fact responsible for the greater part of energy and enstrophy transfer [58]. It follows that the effect of the particular solution to the energy and enstrophy flux in the inertial range is to simply shift it by a constant amount. The energy and enstrophy flux associated with the homogeneous solution is also constant, and as a result the total fluxes are both constant. That the use of hypodiffusion aggravates the departure from universal scaling in the energy spectrum instead of restoring it, is additional evidence that the non-universality of the inverse energy cascade, and the associated non-locality, result from the particular solution and not from dissipative adjustment of the homogeneous solution.

Eyink [69] recently communicated to us results from recent numerical simulations where it has been shown that it is possible to produce an inverse energy cascade with constant energy flux under hypodiffusion, if the numerical resolution is sufficiently large. The requirement on the resolution probably increases as one increases the order of hypodiffusion. This is significant, because, as we shall argue below, hypodiffusion is required to produce the enstrophy cascade.

Nevertheless, the situation here should be contrasted with the case of the downscale energy cascade in three-dimensional turbulence. In the latter case, the scaling behavior of the energy cascade is robust even under very low numerical resolutions. The inverse energy cascade, on the other hand, is only obtained within carefully defined parameter ranges for the infrared sink and the length scale associated with the size of the domain.

6.6.2 The enstrophy cascade

Numerical simulations do not reproduce the k^{-3} energy spectrum of the downscale range consistently. As a result, this prediction of Kraichnan's is considered by many an unproven conjecture. Alternative theories have been proposed that predict steeper scaling [181, 205, 218]. Kraichnan [145] himself

noted that the non-locality of the direct enstrophy cascade makes the application of dimensional analysis inconsistent. However, using a 1-loop closure model [146], he showed that introducing a logarithmic correction to the energy spectrum restores the constant enstrophy flux [147]. The same result can be obtained with other 1-loop models [126].

If higher order closures yield additional higher powers of logarithmic corrections, they may add up to a power law renormalization leading to a steeper spectrum. Eyink has shown recently [77] that such a renormalization does not take place when a downscale enstrophy cascade manifests with constant enstrophy flux, although logarithmic corrections are not excluded. This result is a refinement of an earlier argument [73] that only ruled out energy spectra steeper than $k^{-11/3}$.

Falkovich and Lebedev [84, 85] used a Lagrangian approach [44, 83] to confirm Kraichnan scaling with the logarithmic correction. They also predict that the vorticity structure functions have regular (the scaling exponents form a straight line) logarithmic scaling given by

$$\langle [\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]^n \rangle \sim [\eta \ln(\ell_0/r_{12})]^{2n/3}. \tag{6.82}$$

Eyink [75] noted that this theory does not follow from first principles and that it rests on an unproven regularity for the velocity field. However, should this regularity condition be proven, it would then follow that the Kraichnan scaling scenario is the only one that is statistically stable [86].

Although Eyink's most recent result [77] shows that intermittency corrections are excluded in the enstrophy cascade, the argument still relies on the successful formation of an enstrophy cascade under given configurations of dissipation at large scales and small scales. An additional argument is then needed to show that the homogeneous solution corresponding to the enstrophy cascade is not hidden by the particular solution and is not destroyed by dissipative adjustments. Given this argument, the results of Eyink [77] combined with the theory by Falkovich and Lebedev [84, 85] give a satisfying theory for the downscale enstrophy cascade.

Indeed, the crucial step is assuming that one may expand a velocity increment $u_\alpha(\mathbf{r}) - u_\alpha(0) = \sigma_{\alpha\beta} r_\beta$, when the point separation is within the enstrophy inertial range. In the limit of infinite Reynolds number, the velocity field may not be differentiable but only Holder continuous. In that case, the Taylor expansion is not justified.

A conspicuous characteristic of the theoretical argument of Falkovich and Lebedev is that it is based on the Euler equations instead of the Navier-Stokes equations; the effect of the viscosity terms is ignored. We suspect that the regularity assumption is also linked with the assumption that it is safe to ignore interactions with the forcing range from inside the inertial range of the enstrophy cascade. The reason is that a steeper energy spectrum would result from a breakdown in the separation of scales in the Batchelor argument employed by Falkovich and Lebedev, as was pointed out by Eyink

[75]. Our assessment is that the theory of Falkovich and Lebedev describes accurately the scaling behavior of the homogeneous solution corresponding to the downscale enstrophy cascade, but an additional argument is required to establish that the homogeneous solution is not hidden by the particular solution and that it is not destroyed by dissipative adjustments.

The qualitative description of the enstrophy range implicit in this theory is that the enstrophy acts as an almost passive scalar advected by a large-scale velocity field. Because straining is suppressed in regions of high vorticity, through their relationship between the velocity field and vorticity, we obtain the logarithmic correction. If the velocity field was independent of the vorticity, then the logarithmic correction would have been absent. The physical notion of "large-scale velocity field" corresponds to the underlying assumption of regularity. If the assumption is correct, then it can be shown that perturbations to the vorticity caused by changes in the forcing mechanism are advected by the leading order velocity field. Consequently, the vorticity perturbations act as a passive scalar and they provide a k^{-3} correction to the energy spectrum. Because that would dominate a steeper energy spectrum, it follows that all steeper energy spectra are structurally unstable. We conclude from this argument, that observing a steeper energy spectrum indicates either that an inertial range has not formed or that the regularity assumption is violated.

It is well known that in the presence of coherent structures, the enstrophy range does not follow k^{-3} scaling. Borue [30] showed that using hypodiffusion ($m = 8$ and $k = 1, 8$) disrupts the coherent structures and with increasing Reynolds number the scaling of the enstrophy range approaches asymptotically Kraichnan scaling. As pointed out by Tran and Shepherd [249], all the successful simulations of the k^{-3} spectral range are done so far with the hypodiffusion device. For example, Lindborg and Alvelius [160] showed that the downscale enstrophy range with a k^{-3} energy spectrum can be created in a high resolution simulation, if one uses hyperdiffusion and hypodiffusion $\kappa = m = 2$ and 4096^2 resolution. According to Pasquero and Falkovich [201], the logarithmic correction can be also observed if the simulation is allowed to run for a very long time. The presence of the logarithmic correction has also been confirmed by Ishihira and Kaneda [124]. An experiment by Paret, Jullien and Tabeling [198] has measured the slope of the enstrophy range to -3.0 ± 0.2 and indicates that there is no substantial small-scale intermittency correction in the higher order structure functions, consistent with the prediction by Falkovich and Lebedev. Bowman, Shadwick, and Morrison [34] used a reduced statistical description of turbulence, called *spectral reduction* [33], to calculate both the energy spectrum of the enstrophy cascade as well as higher order statistics, and they found agreement with the scaling proposed by Falkovich and Lebedev.

Although these results favor the Kraichnan scaling scenario, no numerical simulation has re-

produced clean scaling for the physically relevant case of Ekman damping and molecular diffusion. As a matter of fact, Bernard [24] has given an elementary proof that under Ekman damping it is not possible for the energy spectrum of the downscale cascade to scale as k^{-3} with or without the logarithmic correction. A steeper energy spectrum is predicted instead. From our viewpoint, Ekman damping acts by modifying the operator of the balance equations, consequently it changes the homogeneous solutions responsible for the enstrophy cascade.

Nam *et al.* [187] have derived a law governing the steepening of the enstrophy cascade by Ekman damping, however it cannot be used directly to predict the slope of the energy spectrum from the viscosity parameters without additional experimental input. That the behavior of the spectrum at the downscale range is so dependent on the nature of the energy sink at the largest scales is also one of the surprising and important messages from the work of Tran and Shepherd [249], and later Tran and Bowman [247]. A numerical simulation by Schorghofer [223] using Ekman damping and molecular diffusion showed that the enstrophy range approaches Kraichnan scaling with increasing Reynolds number, but failed to yield the k^{-3} slope with the same precision as simulations employing hyperdiffusion and hypodiffusion. Note that the simulation by Ishihira and Kaneda [124] uses a filtered Ekman damping that acts only on wavenumbers smaller than the injection wavenumber. Such a dissipation filter is effectively a hypodiffusion, and Bernard's argument does not apply in this case.

The accumulated evidence of numerical experiments suggests that the enstrophy cascade can be observed in the energy spectrum, but its existence is fragile and heavily dependent on the dissipation mechanisms. It is quite clear that the enstrophy cascade cannot be realized under Ekman damping, and that hypodiffusion is needed. In light of the numerical results communicated to us by Eyink, it is possible to obtain an inverse energy cascade under hypodiffusion, provided that there is sufficient resolution. It may therefore be possible to obtain the dual cascade with careful tuning of the relevant parameters under even larger numerical resolutions. It remains an open question whether the enstrophy cascade can be realized under hypodiffusion and molecular viscosity.

6.7 Conclusions

Unlike the very idealized and clearly unrealizable scenario of pure upscale energy cascade and pure downscale enstrophy cascade envisaged by Kraichnan, Leith and Batchelor for two-dimensional turbulence, in realistic situations there are double cascades, as long as universality does not fail. Upscale of energy and enstrophy injection, there are upscale fluxes of both energy and enstrophy. Downscale of injection there are downscale fluxes of both energy and enstrophy. We establish theoretically that

constant energy flux and constant enstrophy flux can co-exist in the same inertial range. When both are present, dimensional analysis which was used by Kraichnan to yield the energy spectrum is no longer valid. A correct form for the energy spectrum is derived here, which turns out to be a linear superposition of the $k^{-5/3}$ spectrum associated with the energy flux (regardless of the direction), the k^{-3} spectrum associated with the enstrophy flux, and a particular solution contributed by forcing and the boundary conditions. This follows from the linearity of the infinite system governing the inertial range behavior for the structure functions. In the limiting case where the separation of scales in the double cascades is very large, the subleading downscale energy cascade and the subleading upscale enstrophy cascade continue to exist. Although they are shown to be hidden, their extent is asymptotically proportional to the extent of the corresponding leading cascades.

The experimental evidence from numerical simulations shows that the inverse energy cascade is not structurally stable. The direct enstrophy cascade can be reproduced in numerical simulations that use hypodiffusion and hyperviscosity. Hypodiffusion is known to be necessary, but it has not been determined whether hyperviscosity is required.

Chapter 7

DANILOV INEQUALITY AND THE TWO-LAYER MODEL

In two-dimensional turbulence, the energy flux ε_{uv} and the enstrophy flux η_{uv} are constrained by an inequality that was communicated to us by Sergey Danilov [108]. This constraint implies that the contribution of the downscale energy cascade to the energy spectrum is overwhelmed by the contribution of the downscale enstrophy cascade and cannot be seen visually on a plot. This result was conjectured earlier by Smith [230] who claimed that the downscale energy cascade can never have enough flux to move the transition wavenumber k_t into the inertial range. The two-layer model is a different dynamical system than the two-dimensional Navier-Stokes equations, and the validity of the Danilov inequality in the two-layer model is not obvious [108, 251]. The other claim of Smith [230], that the simulations of Tung and Orlando [252] were not well-resolved numerically, has not been shown beyond reasonable doubt due to arithmetic uncertainty in his dimensional analysis estimates, and has been discussed at length in section 3 of Tung [251].

In the present chapter we will show that in the two-layer model, when the Ekman dissipation coefficient ν_E is below a critical value, then the Danilov inequality will be satisfied. We will also argue that the asymmetric presence of Ekman damping on the bottom layer but not the top layer may cause the violation of the Danilov inequality for larger values of the Ekman dissipation coefficient. In this case, the top layer has more enstrophy than the bottom layer, as is realistic in the atmosphere, and provided that the difference in enstrophy between the two layers is large enough, the downscale energy cascade will be made observable in the energy spectrum. We derive specific conditions on how large this difference needs to be in order for the Danilov inequality to be violated for some wavenumbers k. The simulation of Tung and Orlando [252] has shown that it is possible to have an observable downscale energy cascade, which implies a violation of the Danilov inequality. The role of the argument in this chapter is to explain how and why this can happen, given that it is a surprising and very unexpected result. We also hope that our argument here will raise renewed interest in understanding the phenomenology of the two-layer quasi-geostrophic model.

7.1 The formal setup

Recall that the governing equations for the two-layer model read:

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1 + f) = -\frac{2f}{h}\omega + d_1 \tag{7.1}$$

$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2 + f) = +\frac{2f}{h}\omega + d_2 + 2e_2 \tag{7.2}$$

$$\frac{\partial T}{\partial t} + \frac{1}{2}[J(\psi_1, T) + J(\psi_2, T)] = -\frac{N^2}{f}\omega + Q_0. \tag{7.3}$$

Here, $\zeta_1 = \nabla^2\psi_1$ is the relative vorticity of the top layer and $\zeta_2 = \nabla^2\psi_2$ is the relative vorticity of the bottom layer, and ω is the vertical velocity. The temperature equation is situated between the two layers and it satisfies the geostrophic condition $T = (2/h)(\psi_1 - \psi_2)$ with h the separation between the two layers. Furthermore, f is the Coriolis term, N is the Brunt-Väisälä frequency, and Q_0 is the thermal forcing on the temperature equation. The dissipation terms include momentum dissipation of relative vorticity, in each layer, and Ekman damping from the lower boundary layer, and they read:

$$d_1 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_1 \tag{7.4}$$

$$d_2 = (-1)^{\kappa+1}\nu\nabla^{2\kappa}\zeta_2 \tag{7.5}$$

$$e_2 = -\nu_E\zeta_2. \tag{7.6}$$

This model can be reduced to a coupled 2D-like system by employing the temperature equation to eliminate the vertical velocity ω. This leads to the definition of the potential vorticity q_1 and q_2 as

$$q_1 = \nabla^2\psi_1 + f + \frac{k_R^2}{2}(\psi_2 - \psi_1) \tag{7.7}$$

$$q_2 = \nabla^2\psi_2 + f - \frac{k_R^2}{2}(\psi_2 - \psi_1), \tag{7.8}$$

with $k_R \equiv 2\sqrt{2}f/(hN)$ the Rossby radius of deformation wavenumber. The governing equations for q_1 and q_2 are shown to be

$$\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) = f_1 + d_1 \tag{7.9}$$

$$\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) = f_2 + d_2 + e_2, \tag{7.10}$$

Here f_1 and f_2 is the thermal forcing on each layer given by $f_1 = -(k_R^2 Q)/(2f)$ and $f_2 = (k_R^2 Q)/(2f)$ where $Q = (1/4)k_R^2 h Q_0$.

The two inviscid quadratic invariants are the energy E and the total layer potential enstrophies

G_1 and G_2 given by

$$E \equiv \langle\langle \psi_1\zeta_1 + \psi_2\zeta_2 \rangle\rangle \tag{7.11}$$

$$G_1 \equiv \langle\langle \zeta_1^2 \rangle\rangle, \quad G_2 \equiv \langle\langle \zeta_2^2 \rangle\rangle. \tag{7.12}$$

The energy and enstrophy spectra are defined as

$$E(k) \equiv \frac{d}{dk}\langle\langle \psi_1^{<k}\zeta_1^{<k} \rangle\rangle + \frac{d}{dk}\langle\langle \psi_2^{<k}\zeta_2^{<k} \rangle\rangle, \tag{7.13}$$

$$G_1(k) \equiv \frac{d}{dk}\langle\langle \zeta_1^{<k}\zeta_1^{<k} \rangle\rangle, \tag{7.14}$$

$$G_2(k) \equiv \frac{d}{dk}\langle\langle \zeta_2^{<k}\zeta_2^{<k} \rangle\rangle, \tag{7.15}$$

and the total enstrophy spectrum $G(k)$ is $G(k) = G_1(k) + G_2(k)$. We define the streamfunction spectra

$$U_1(k) \equiv \frac{d}{dk}\langle\langle \psi_1^{<k}\psi_1^{<k} \rangle\rangle, \tag{7.16}$$

$$U_2(k) \equiv \frac{d}{dk}\langle\langle \psi_2^{<k}\psi_2^{<k} \rangle\rangle, \tag{7.17}$$

$$C(k) \equiv \frac{d}{dk}\langle\langle \psi_1^{<k}\psi_2^{<k} \rangle\rangle, \tag{7.18}$$

and also $U(k) \equiv U_1(k) + U_2(k)$. It is easy to show the triangle inequality $2C(k) \leq U(k)$. The energy and enstrophy spectra are related with the streamfunction spectra via

$$E(k) = (k^2 + k_R^2/2)U(k) - k_R^2 C(k) \tag{7.19}$$

$$G(k) = (k^4 + k^2k_R^2 + k_R^4/2)U(k) - k_R^2(2k^2 - k_R^2)C(k). \tag{7.20}$$

From the standpoint of examining the Danilov inequality, it is most convenient to work with the streamfunction spectra. However, following Salmon [219–221], for physical understanding it is useful to work with the energy and enstrophy spectra. Furthermore, it is helpful to distinguish between barotropic energy and baroclinic energy as follows: Let $\psi \equiv (\psi_1 + \psi_2)/2$ and $\tau \equiv (\psi_1 - \psi_2)/2$. So, $\psi_1 = \psi + \tau$ and $\psi_2 = \psi - \tau$. Now we define three spectra $E_K(k)$, $E_P(k)$, and $E_C(k)$ in terms of ψ and τ:

$$E_K(k) \equiv 2k^2\frac{d}{dk}\langle\langle \psi^{<k}\psi^{<k} \rangle\rangle, \tag{7.21}$$

$$E_P(k) \equiv 2(k^2 + k_R^2)\frac{d}{dk}\langle\langle \tau^{<k}\tau^{<k} \rangle\rangle, \tag{7.22}$$

$$E_C(k) \equiv 2k^2\frac{d}{dk}\langle\langle \psi^{<k}\tau^{<k} \rangle\rangle. \tag{7.23}$$

Here $E_K(k)$ is the barotropic energy spectrum and $E_P(k)$ the baroclinic energy spectrum. It is easy to show that the definitions are self-consistent, i.e. $E(k) = E_K(k) + E_P(k)$. The relation between the

energy spectra and the enstrophy spectra can be now written in terms of two constraint equations:

$$G_1(k) = (1/2)[k^2 E_K(k) + (k^2 + k_R^2)E_P(k)] + (k^2 + k_R^2)E_C(k) \tag{7.24}$$

$$G_2(k) = (1/2)[k^2 E_K(k) + (k^2 + k_R^2)E_P(k)] - (k^2 + k_R^2)E_C(k). \tag{7.25}$$

Thus we see that the physical interpretation of $E_C(k)$ is that it represents the difference in potential enstrophy distribution between the two layers, and it is given by

$$E_C(k) = \frac{G_1(k) - G_2(k)}{2(k^2 + k_R^2)}. \tag{7.26}$$

The right-hand-side of the Danilov inequality reads

$$k^2\Pi_E(k) - \Pi_G(k) = \int_k^{+\infty} dq\ (k^2 D_E(q) - D_G(q)) \equiv \int_k^{+\infty} dq\ \Delta(k,q), \tag{7.27}$$

so the key question is whether $\Delta(k,q)$ is positive or negative for wavenumbers $k < q < k_{max}$. Here $k_{\max}$ is either the truncation wavenumber in the numerical model, or, in the theoretical case of infinite resolutions, is the hyperviscosity dissipation wavenumber, beyond which the spectral enstrophy dissipation rate becomes negligible.

With some straightforward but tedious calculations, it can be shown that the dissipation rate $D_E(k)$ for the total energy, and the dissipation rate $D_G(k)$ for the total potential enstrophy are given by

$$D_E(k) = 2\nu k^{2p+2} U(k) + 2\nu_E k^2 U_2(k) \tag{7.28}$$

$$D_G(k) = 2\nu k^{2p+2} E(k) + \nu_E k^2 [(2k^2 + k_R^2)U_2(k) - k_R^2 C(k)], \tag{7.29}$$

thus it follows that

$$\Delta(k,q) = k^2 D_E(q) - D_G(q) \tag{7.30}$$

$$= 2\nu q^{2p+2}[k^2 U(q) - E(q)] + \nu_E k_R^2 q^2 C(q) + \nu_E q^2 (2k^2 - 2q^2 - k_R^2)U_2(q) \tag{7.31}$$

$$= 2\nu q^{2p+2}[(k^2 - q^2)U(q) + (k_R^2/2)(2C(q) - U(q))] \tag{7.32}$$

$$+ \nu_E k_R^2 q^2 (C(q) - U_2(q)) + 2\nu_E q^2 (k^2 - q^2)U_2(q). \tag{7.33}$$

The first and third terms in this expression are always negative. Consequently, a *necessary* condition for *violating* the Danilov inequality is that the second term has to be positive, i.e. $C(q) - U_2(q) \geq 0$. Otherwise, if $C(q) - U_2(q) \leq 0$, then the Danilov inequality will be satisfied. A physical interpretation of this condition will be given in section 7.3.1.

7.2 Controlled necessary condition

A controlled sufficient condition to satisfy the Danilov inequality can be derived in terms of the physical parameters of the problem by noting that $C(q) - U_2(q) \leq (1/2)(U_1(q) - U_2(q))$. It follows that

$$\begin{aligned}
\Delta(k,q) &= 2\nu q^{2p+2}[(k^2 - q^2)U(q) + (k_R^2/2)(2C(q) - U(q))] + \nu_E k_R^2 q^2 (C(q) - U_2(q)) \\
&\quad + 2\nu_E q^2 (k^2 - q^2) U_2(q) \\
&\leq 2\nu q^{2p+2}[(k^2 - q^2)U(q) + (k_R^2/2)(2C(q) - U(q))] + \nu_E k_R^2 q^2 (1/2)(U_1(q) - U_2(q)) \\
&\quad + 2\nu_E q^2 (k^2 - q^2) U_2(q) \\
&= [2\nu q^{2p+2}(k^2 - q^2) + (1/2)\nu_E k_R^2 q^2] U_1(q) \\
&\quad + [2(\nu q^{2p+2} + \nu_E q^2)(k^2 - q^2) - (1/2)\nu_E k_R^2 q^2] U_2(q) \\
&\leq [2\nu q^{2p+2}(k^2 - q^2) + (1/2)\nu_E k_R^2 q^2] U_1(q).
\end{aligned}$$

Here, we have used the inequality $2C(k) \leq U(k)$ to eliminate the $(k_R^2/2)(2C(q) - U(q))$ term. We have also eliminated the $U_2(q)$ term because it is unconditionally negative.This leads to the following controlled *sufficient* condition to *satisfy* Danilov's inequality:

$$\nu_E < 4\nu k_{\max}^{2p} \left(\frac{k_{\max}}{k_R} \right)^2. \tag{7.34}$$

Equivalently, a *necessary* condition to *violate* Danilov's inequality is

$$\nu_E > 4\nu k_{\max}^{2p} \left(\frac{k_{\max}}{k_R} \right)^2. \tag{7.35}$$

It is interesting to note that in the numerical simulation of the two-layer model the algorithm adopted by Tung and Orlando [252] for determining the magnitude of the hyperviscosity coefficient is $\nu_E \gg \nu k_{\max}^{2p}$, for all but the last twenty wavenumbers k in the dissipation range. Tung and Orlando [252] obtained an energy spectrum with the compound slope configuration and the transition wavenumber k_t occured in the inertial range downscale from injection in agreement with the condition $k_t \approx \sqrt{\eta_{uv}/\epsilon_{uv}}$, thus implying a violation of Danilov's inequality.

7.3 An uncontrolled necessary and sufficient condition

The question now arises: is it possible to derive a sufficient condition to violate the Danilov inequality of the form $\nu_E k_R^2 \geq \Lambda \nu k_{\max}^{2p+2}$ for some universal constant Λ? So far as we know, this is not possible. However, it is possible to derive an *uncontrolled* necessary and sufficient condition for violating the Danilov inequality.

We begin with defining

$$2C(q) = \lambda(q)U(q) \tag{7.36}$$

$$U_2(q) = u(q)U(q) \tag{7.37}$$

Here $0 \leq u(q) \leq 1$ and $-1 \leq \lambda(q) \leq 1$. We may thus rewrite everything in terms of $U(q)$ by employing

$$C(q) - U_2(q) = (\lambda(q)/2 - u(q))U(q) \tag{7.38}$$

$$2C(q) - U(q) = (\lambda(q) - 1)U(q) \tag{7.39}$$

Then we can rewrite $\Delta(k, q)$, as follows:

$$\Delta(k,q) = 2\nu q^{2p+2}[(k^2 - q^2)U(q) + (k_R^2/2)(2C(q) - U(q))] + \nu_E k_R^2 q^2 (C(q) - U_2(q)) \tag{7.40}$$

$$+ 2\nu_E q^2 (k^2 - q^2) U_2(q) \tag{7.41}$$

$$= q^2 U(q)[-\nu q^{2p}(2(q^2 - k^2) + k_R^2(1 - \lambda(q)) \tag{7.42}$$

$$+ \nu_E k_R^2 (\lambda(q)/2 - u(q) + 2(k/k_R)^2 u(q) - 2(q/k_R)^2 u(q))]. \tag{7.43}$$

It easy to see that a sufficient condition to get $\Delta(k, q) \geq 0$ is

$$\nu_E k_R^2 (\lambda(q)/2 - u(q) + 2(k/k_R)^2 u(q) - 2(q/k_R)^2 u(q)) \geq \nu q^{2p}(2(q^2 - k^2) + k_R^2(1 - \lambda(q))). \tag{7.44}$$

The necessary condition to violate the Danilov inequality $C(q) - U_2(q) \geq 0$, which was derived previously, implies that $\lambda(q)/2 - u(q) \geq 0$. Provided that we assume the stronger condition

$$\lambda(q)/2 - u(q) + 2(k/k_R)^2 u(q) - 2(q/k_R)^2 u(q) \geq 0, \tag{7.45}$$

we may rewrite our sufficient condition as:

$$\frac{\nu_E k_R^2}{\nu q^{2p}} \geq \frac{2(q^2 - k^2) + k_R^2(1 - \lambda(q))}{\lambda(q)/2 - u(q) + 2(k/k_R)^2 u(q) - 2(q/k_R)^2 u(q)}. \tag{7.46}$$

To violate the Danilov inequality at wavenumber k, this sufficient condition must hold for all q such that $k < q < k_{max}$. Since the numerator is always positive, the condition (7.45) is in fact a stronger necessary condition for violating the Danilov inequality.

7.3.1 Physical interpretation of necessary conditions

We would like now to discuss the plausibility of the necessary condition $\lambda(q)/2 - u(q) > 0$ and the stronger necessary condition (7.45). To this end, we rewrite these conditions equivalently in terms of the physical energy and enstrophy spectra.

It is easy to write $U_1(k)$, $U_2(k)$, and $C(k)$ in terms of $E_K(k)$, $E_P(k)$, and $E_C(k)$:

$$U_1(k) = \frac{d}{dk}\langle\langle(\psi+\tau)^{<k}(\psi+\tau)^{<k}\rangle\rangle \tag{7.47}$$

$$= \frac{d}{dk}\langle\langle\psi^{<k}\psi^{<k}\rangle\rangle + 2\frac{d}{dk}\langle\langle\psi^{<k}\tau^{<k}\rangle\rangle + \frac{d}{dk}\langle\langle\tau^{<k}\tau^{<k}\rangle\rangle \tag{7.48}$$

$$= \frac{E_K(k)}{2k^2} + \frac{E_P(k)}{2(k^2+k_R^2)} + \frac{E_C(k)}{k^2}, \tag{7.49}$$

and with a similar argument we find

$$U_2(k) = \frac{E_K(k)}{2k^2} + \frac{E_P(k)}{2(k^2+k_R^2)} - \frac{E_C(k)}{k^2} \tag{7.50}$$

$$C(k) = \frac{E_K(k)}{2k^2} - \frac{E_P(k)}{2(k^2+k_R^2)}. \tag{7.51}$$

We may thus write $\lambda(k)$ and $u(k)$ in terms of $E_K(k)$, $E_P(k)$, and $E_C(k)$:

$$\lambda(k) = \frac{2C(k)}{U(k)} = 2\,\frac{\dfrac{E_K(k)}{2k^2} - \dfrac{E_P(k)}{2(k^2+k_R^2)}}{\dfrac{E_K(k)}{k^2} + \dfrac{E_P(k)}{(k^2+k_R^2)}} \tag{7.52}$$

$$= \frac{(k^2+k_R^2)E_K(k) - k^2E_P(k)}{(k^2+k_R^2)E_K(k) + k^2E_P(k)}, \tag{7.53}$$

$$\tag{7.54}$$

and

$$u(k) = \frac{U_2(k)}{U(k)} = \frac{\dfrac{E_K(k)}{2k^2} + \dfrac{E_P(k)}{2(k^2+k_R^2)} - \dfrac{E_C(k)}{k^2}}{\dfrac{E_K(k)}{k^2} + \dfrac{E_P(k)}{(k^2+k_R^2)}} \tag{7.55}$$

$$= \frac{1}{2}\frac{(k^2+k_R^2)E_K(k) + k^2E_P(k) - 2(k^2+k_R^2)E_C(k)}{(k^2+k_R^2)E_K(k) + k^2E_P(k)}. \tag{7.56}$$

and the necessary condition $\lambda(q)/2 - u(q) \geq 0$ can now be rewritten as

$$\lambda(q)/2 - u(q) = \frac{(q^2+k_R^2)E_C(q) - q^2E_P(q)}{(q^2+k_R^2)E_K(q) + q^2E_P(q)} > 0. \tag{7.57}$$

The denominator is obviously positive, consequently the condition is equivalent to

$$G_1(q) - G_2(q) = 2(q^2+k_R^2)E_C(q) > 2q^2E_P(q), \tag{7.58}$$

for all q such as $k < q < k_{max}$.

This is a very interesting result. The requirement, in part, is that there should be more enstrophy on the top layer than the bottom layer, i.e. $G_1(q) > G_2(q)$. It is reasonable to expect this if there is more dissipation on the bottom layer than the top layer. This is the case for the model we are

considering where there is Ekman damping at the bottom layer but not at the top layer. However, the actual condition also requires that the difference should be larger than the potential energy $E_P(q)$ multiplied with $2q^2$. In our model, as has been pointed out by Salmon [219–221], the energy is injected into the system at large scales as baroclinic energy, and most of this energy is converted into barotropic energy near the Rossby wavenumber k_R. It is therefore reasonable to expect that this necessary condition will be satisfied for wavenumbers $q > k_R$.

It should be noted that this is only a necessary condition. A sufficient condition would require furthermore that equation (7.46) be valid. The necessary precondition for that is the stronger requirement

$$\frac{\lambda(q)/2 - u(q)}{u(q)} > 2\left(\frac{q}{k_R}\right)^2 - 2\left(\frac{k}{k_R}\right)^2. \tag{7.59}$$

This condition can be rewritten equivalently as

$$\frac{G_1(q) - G_2(q) - 2q^2E_P(q)}{2G_2(q) + k_R^2(E_K(q) - E_P(q))} > 2\left(\frac{q}{k_R}\right)^2 - 2\left(\frac{k}{k_R}\right)^2. \tag{7.60}$$

and it can be simplified further to give

$$G_1(q) - \left(1 + 4\,\frac{q^2 - k^2}{k_R^2}\right) G_2(q) > 2q^2E_K(q) - 2k^2(E_K(q) - E_P(q)). \tag{7.61}$$

The constraint $u(q) \geq 0$ implies that

$$2G_2(q) + k_R^2(E_K(q) - E_P(q)) \geq 0, \tag{7.62}$$

and subtracting this inequality from the necessary condition above gives the following simplification:

$$G_1(q) - (1 + 4(q/k_R)^2)G_2(q) > 2q^2E_K(q), \tag{7.63}$$

for all q such that $k < q < k_{max}$.

This condition, which is also a necessary condition for violating the Danilov inequality, places an even stronger constraint on the difference between the enstrophy between the two layers. The fly in the ointment is the $4(q/k_R)^2G_2(q)$ term. If we want to go from the transition scale at about 700km in wavelength down to length scales of a few kilometers, then $4(q/k_R)^2$ increases at least by four orders of magnitude. In the numerical simulations of Tung and Orlando [252], the variability of $4(q/k_R)^2$ is relatively small because they only go down to 100km in wavelength. We may therefore conjecture that a violation of the Danilov inequality requires that the ratio $(k_d/k_R)^2$, where k_d is the dissipation wavenumber, must be small. In other words, the subleading downscale energy cascade is expected to be observable in a numerical simulation with small separation of scales between k_R and

k_d, but not in a numerical simulation where the separation between these scales is large. A trend of diminishing downscale energy flux with increasing numerical resolution, and thus with increasing separation of scales, has already been seen in the simulations of Tung and Orlando [252].

It should be noted, of course, that the quasi-geostrophic model is not valid at length scales much smaller than 100km in wavelength. When three-dimensional effects become relevant the conservation of enstrophy is violated by the nonlinearity itself which reflects itself in additional nonlinear dissipative contributions to $D_G(k)$. These terms make it all the more easier to violate the Danilov inequality for wavenumbers k that correspond to scales less than 100km in wavelength, thereby preserving the very extensive downscale energy cascade observed in the Nastrom-Gage spectrum.

7.4 Conclusions and Discussion

We have shown that in the two-layer quasi-geostrophic model, the violation of the Danilov inequality is possible only as a result of asymmetric Ekman damping operating on only one of the two-layers. This creates an imbalance between the amount of enstrophy accumulated in one layer versus the amount accumulated in the other layer, and the downscale energy cascade will become observable on the condition that this imbalance is sufficiently large. We have derived in the present chapter a sufficient condition for *not* violating the Danilov inequality which explains why the $k^{-5/3}$ spectrum has not been observed in some of the previous simulations of the two-layer model. We have also derived a necessary and sufficient condition for violating the Danilov inequality, but it is an uncontrolled condition. The numerical simulation by Tung and Orlando [252] has confirmed that a double cascade with the transition wavenumber located in the inertial range can be realized. This can only occur when the Danilov inequality is violated for some wavenumbers k in the inertial range. The parameterization of the Ekman damping in that simulation does in fact satisfy the necessary condition derived in this paper.

Our work in the present chapter explains why it can be reproduced in numerical simulations that use baroclinic models, while the same effect cannot be realized in simulations of two-dimensional turbulence. On the other hand our work here does not rule out the possibility that the shallower part of the spectrum observed over the mesoscales [188] can be due to dynamics other than QG, whether it is barotropic or baroclinic, especially for scales of 100 km or less (see e.g. Refs. [158, 159] with Bousinesq dynamics). Our present work serves to point out that over the larger scales ($\gtrapprox$ 600km), where the transition to a shallower spectrum occurs, baroclinic QG theory by itself is a viable mechanism for explaining the transition from -3 to $-5/3$ slopes.

Furthermore, as proposed first by Tung and Orlando [252], the downscale energy flux, which is

important in explaining the $k^{-5/3}$ energy spectrum over the mesoscales in most theories, originates at larger scales (the synoptic scales). Its contribution to the energy spectrum is hidden for smaller wavenumbers under the k^{-3} part of the spectrum, and then emerges for larger k past the transition scale. It remains an open question, one that is beyond the scope of this thesis, to explain how this downscale energy flux can be continued into length scales too small for QG theory to describe, and how it is eventually dissipated.

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Appendix A

THE GENERALIZED BALANCE EQUATIONS

We briefly review the theory of the generalized balance equations. These equations were first derived by L'vov and Procaccia [172] and they are the foundation of previous work [104, 107, 108] as well as this thesis. The main feature of the balance equations that we would like to stress here is the separation of the interaction term into local interactions and sweeping interactions. We also derive the balance equations that govern the generalized structure functions of the vorticity.

A.1 The balance equations

Let $w_\alpha(\mathbf{x}, \mathbf{x}', t)$ be the Eulerian velocity differences

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) = u_\alpha(\mathbf{x}, t) - u_\alpha(\mathbf{x}', t). \tag{A.1}$$

To write equations concisely, we introduce the following notation to represent aggregates of position vectors

$$\mathbf{X} = (\mathbf{x}, \mathbf{x}') \tag{A.2}$$

$$\{\mathbf{X}\}_n = \{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\} \tag{A.3}$$

$$\{\mathbf{X}_\alpha\}_n^k = \{\mathbf{X}_1, \ldots, \mathbf{X}_{k-1}, \mathbf{X}_{k+1}, \ldots, \mathbf{X}_n\}. \tag{A.4}$$

We use the notation $\{\mathbf{X}\}_n + \Delta\mathbf{x}$ as a shorthand to represent shifting all the constituent vectors of $\{\mathbf{X}\}_n$ by the same displacement $\Delta\mathbf{x}$. Similarly, $\lambda\{\mathbf{X}\}_n$ represents taking the scalar product of λ with every vector in $\{\mathbf{X}\}_n$. Finally, the notation $\|\{\mathbf{X}\}_n\| \sim R$ means that all point to point distances in the geometry of velocity differences $\{\mathbf{X}\}_n$ have the same order of magnitude R. And, the notation $\|\{\mathbf{X}\}_n\| \ll \|\{\mathbf{Y}\}_n\|$ means that all the point to point distances in $\{\mathbf{Y}\}_n$ are much larger than all the point to point distances in $\{\mathbf{X}\}_n$.

The eulerian one-time fully unfused correlation tensors are formed by multiplying n velocity differences evaluated at $2n$ distinct points

$$F_n(\{\mathbf{X}\}_n, t) = \left\langle \left[\prod_{k=1}^{n} w_{\alpha_k}(\mathbf{X}_k, t) \right] \right\rangle. \tag{A.5}$$

When all velocity differences share one point in common, that is $\mathbf{x}'_k = \mathbf{x}_0$, we say that the correlation F_n is *partially fused*.

The generalized balance equations can be derived by differentiating F_n with respect to t and substituting the Navier-Stokes equations:

$$\frac{\partial F_n(t)}{\partial t} = \sum_{k=1}^{n} \left\langle \frac{\partial w_{\alpha_k}(\mathbf{x}_k, \mathbf{x'}_k, t)}{\partial t} \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \right\rangle \tag{A.6}$$

$$= \sum_{k=1}^{n} [-N_{kn} + Q_{kn}] + \nu J_n + \beta H_n = -N_n + Q_n + \nu J_n + \beta H_n \tag{A.7}$$

The terms νJ_n and βH_n are the contributions of the small-scale and large-scale sinks with

$$J_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x}, \mathbf{x'}\}_n, t) = \sum_{k=1}^{n} (\nabla_k^{2\kappa} + \nabla_{k'}^{2\kappa}) F_n(\{\mathbf{x}, \mathbf{x'}\}_n, t) \tag{A.8}$$

$$H_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x}, \mathbf{x'}\}_n, t) = \sum_{k=1}^{n} (\nabla_k^{-2m} + \nabla_{k'}^{-2m}) F_n(\{\mathbf{x}, \mathbf{x'}\}_n, t) \tag{A.9}$$

where ∇_k^2 is the Laplacian with respect to $\mathbf{x}_k$; $\nabla_{k'}^2$ is the Laplacian with respect to $\mathbf{x'}_k$, N_{kn} represents the contribution of $\mathcal{P}_{\alpha\beta}\partial_\gamma(u_\beta u_\gamma)$, and Q_{kn} represents the contribution of $\mathcal{P}_{\alpha\beta} f_\beta$ which read

$$Q_{kn}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x}, \mathbf{x'}\}_n, t) = \left\langle \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \mathcal{P}_{\alpha_k\beta}(f_\beta(\mathbf{x}_k, t) - f_\beta(\mathbf{x'}_k, t)) \right\rangle \tag{A.10}$$

$$N_{kn}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x}, \mathbf{x'}\}_n, t) = \left\langle \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \mathcal{P}_{\alpha_k\beta}[\partial_{\gamma,\mathbf{x}_k}(u_{\beta,\mathbf{x}_k} u_{\gamma,\mathbf{x}_k}) - \partial_{\gamma,\mathbf{x'}_k}(u_{\beta,\mathbf{x'}_k} u_{\gamma,\mathbf{x'}_k})] \right\rangle \tag{A.11}$$

$$= \left\langle \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \mathcal{P}_{\alpha_k\beta} \mathcal{N}_\beta(\mathbf{x}_k, \mathbf{x'}_k, t) \right\rangle \tag{A.12}$$

Here we use the abbreviations $u_{\alpha,\mathbf{x}_k} = u_\alpha(\mathbf{x}_k, t)$ and $u_{\alpha,\mathbf{x'}_k} = u_\alpha(\mathbf{x'}_k, t)$, and $\partial_{\alpha,\mathbf{x}_k}$ is the spatial derivative in the α direction with respect to $\mathbf{x}_k$. Also, $\mathcal{N}_\beta(\mathbf{x}_k, \mathbf{x'}_k, t)$ is the non-linear factor above and using the additional abbreviation $w_{\alpha,k} = w_\alpha(\mathbf{x}_k, \mathbf{x'}_k, t)$, and incompressibility, it can be rewritten as:

$$\mathcal{N}_{\beta,k} = \mathcal{N}_\beta(\mathbf{x}_k, \mathbf{x'}_k, t) = \partial_{\gamma,\mathbf{x}_k}(u_{\beta,\mathbf{x}_k} u_{\gamma,\mathbf{x}_k}) - \partial_{\gamma,\mathbf{x'}_k}(u_{\beta,\mathbf{x'}_k} u_{\gamma,\mathbf{x'}_k}) \tag{A.13}$$

$$= u_{\gamma,\mathbf{x}_k} \partial_{\gamma,\mathbf{x}_k}(u_{\beta,\mathbf{x}_k} - u_{\beta,\mathbf{x'}_k}) + u_{\gamma,\mathbf{x'}_k} \partial_{\gamma,\mathbf{x'}_k}(u_{\beta,\mathbf{x}_k} - u_{\beta,\mathbf{x'}_k}) \tag{A.14}$$

$$= \partial_{\gamma,\mathbf{x}_k}(u_{\gamma,\mathbf{x}_k} w_{\beta,k}) + \partial_{\gamma,\mathbf{x'}_k}(u_{\gamma,\mathbf{x'}_k} w_{\beta,k}) \tag{A.15}$$

A.2 Decomposition of the interaction term

It is easy to see that the nonlinear terms N_{kn} cannot be written exclusively in terms of velocity differences. The remarkable characteristic of the derivation of the balance equations by L'vov and Procaccia [172] is that the nonlinear term N_{kn} is rearranged as the sum of a local term D_{kn} and the

sweeping term I_{kn} such that the local term can be expressed as a linear operator on F_{n+1}. Although L'vov and Procaccia [172] eliminated the sweeping term on the grounds of global homogeneity, we believe it is appropriate to retain it here in its simplified form.

To isolate the sweeping term, we define the mean velocity $\mathcal{U}_\alpha(\{\mathbf{x},\mathbf{x}'\}_n,t)$ as:

$$\mathcal{U}_\alpha(\{\mathbf{x},\mathbf{x}'\}_n,t)=\frac{1}{2n}\sum_{k=1}^{n}(u_\alpha(\mathbf{x}_k,t)+u_\alpha(\mathbf{x}'_k,t)) \tag{A.16}$$

and the velocity fluctuation

$$v_\alpha(\mathbf{x},\{\mathbf{y},\mathbf{y}'\}_n,t)=u_\alpha(\mathbf{x},t)-\mathcal{U}_\alpha(\{\mathbf{y},\mathbf{y}'\}_n,t) \tag{A.17}$$

$$=\frac{1}{2n}\sum_{k=1}^{n}[w_\alpha(\mathbf{x},\mathbf{y}_k)+w_\alpha(\mathbf{x},\mathbf{y}'_k)] \tag{A.18}$$

We may then decompose $\mathcal{N}_{\alpha,k}$ to $\mathcal{N}_{\alpha,k}=\mathcal{S}_{\alpha,k}+\mathcal{L}_{\alpha,k}$ where $\mathcal{S}_{\alpha,k}$ represents sweeping interactions and $\mathcal{L}$ represents local interactions with

$$\begin{aligned}\mathcal{S}_{\alpha,k}&=\mathcal{S}_\alpha(\mathbf{x}_k,\mathbf{x}'_k,\{\mathbf{y},\mathbf{y}'\}_n,t)=\mathcal{U}_\beta(\{\mathbf{y},\mathbf{y}'\}_n,t)(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x}'_k})w_\alpha(\mathbf{x}_k,\mathbf{x}'_k,t)\\ \mathcal{L}_{\alpha,k}&=\mathcal{L}_\alpha(\mathbf{x}_k,\mathbf{x}'_k,\{\mathbf{y},\mathbf{y}'\}_n,t)=(v_\beta(\mathbf{x}_k,\{\mathbf{y},\mathbf{y}'\}_n,t)\partial_{\beta,\mathbf{x}_k}+v_\beta(\mathbf{x}'_k,\{\mathbf{y},\mathbf{y}'\}_n,t)\partial_{\beta,\mathbf{x}'_k})w_\alpha(\mathbf{x}_k,\mathbf{x}'_k,t)\end{aligned} \tag{A.19}$$

These give two contributions: $N_{kn}=D_{kn}+I_{kn}$ from $\mathcal{L}$ and $\mathcal{S}$ which read

$$D_{kn}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t)=\left\langle\left[\prod_{l=1,l\neq k}^{n}w_{\alpha_l}(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\int d\mathbf{y}P_{\alpha_k\beta}(\mathbf{y})\mathcal{L}_\beta(\mathbf{x}_k-\mathbf{y},\mathbf{x}'_k-\mathbf{y},\{\mathbf{x},\mathbf{x}'\}_n,t)\right\rangle \tag{A.20}$$

$$I_{kn}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t)=\left\langle\left[\prod_{l=1,l\neq k}^{n}w_{\alpha_l}(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\int d\mathbf{y}P_{\alpha_k\beta}(\mathbf{y})\mathcal{S}_\beta(\mathbf{x}_k-\mathbf{y},\mathbf{x}'_k-\mathbf{y},\{\mathbf{x},\mathbf{x}'\}_n,t)\right\rangle \tag{A.21}$$

and represent local and sweeping interactions, correspondingly.

A.3 The local and sweeping interaction terms

The sweeping term I_{kn} can be simplified as follows: We use the decomposition $P_{\alpha\beta}(\mathbf{x})=\delta_{\alpha\beta}\delta(\mathbf{x})-P^{\parallel}_{\alpha\beta}(\mathbf{x})$ to split I_{kn} to two terms: $I_{kn}=I^{(1)}_{kn}+I^{(2)}_{kn}$. We also use $\mathcal{P}^{\parallel}_{\alpha\beta}u_\beta=0$. The integral inside the ensemble average of I_{kn} splits to two parts: I_1 and I_2. The second part is shown to be zero by

incompressibility:

$$I_2 = \int d\mathbf{y} P^{\parallel}_{\alpha\beta}(\mathbf{y}) \mathcal{S}_\beta(\mathbf{x}_k - \mathbf{y}, \mathbf{x'}_k - \mathbf{y}, \{\mathbf{x}_k, \mathbf{x'}_k\}_n, t) \tag{A.22}$$

$$= \int d\mathbf{y} P^{\parallel}_{\alpha\beta}(\mathbf{y}) \mathcal{U}_\gamma(\{\mathbf{x}_k, \mathbf{x'}_k\}_n, t)(\partial_{\gamma,\mathbf{x}_k} + \partial_{\gamma,\mathbf{x'}_k}) w_\beta(\mathbf{x}_k - \mathbf{y}, \mathbf{x'}_k - \mathbf{y}, t) \tag{A.23}$$

$$= \mathcal{U}_\gamma(\{\mathbf{x}_k, \mathbf{x'}_k\}_n, t)(\partial_{\gamma,\mathbf{x}_k} + \partial_{\gamma,\mathbf{x'}_k}) \int d\mathbf{y} P^{\parallel}_{\alpha\beta}(\mathbf{y}) w_\beta(\mathbf{x}_k - \mathbf{y}, \mathbf{x'}_k - \mathbf{y}, t) = 0 \tag{A.24}$$

Because $P^{\parallel}_{\alpha\beta}$ is the nonlocal part of the projection operator, this result implies that the pressure effect does not contribute to the sweeping interactions or to the violation of local homogeneity, in the sense of Frisch [93, 94]. Thus, I_{kn} is determined by the first integral I_2 and it simplifies to

$$I^{\alpha_1\alpha_2\cdots\alpha_n}_{kn}(\{\mathbf{x}, \mathbf{x}'\}_n, t) = \int d\mathbf{y} \delta_{\alpha_k\beta}\delta(\mathbf{y}) \left\langle \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \mathcal{S}_\beta(\mathbf{x}_k - \mathbf{y}, \mathbf{x'}_k - \mathbf{y}, \{\mathbf{x}_k, \mathbf{x'}_k\}_n, t) \right\rangle \tag{A.25}$$

$$= \left\langle \left[\prod_{l=1, l\neq k}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \mathcal{U}_\gamma(\{\mathbf{x}_k, \mathbf{x'}_k\}_n, t)(\partial_{\gamma,\mathbf{x}_k} + \partial_{\gamma,\mathbf{x'}_k}) w_\beta(\mathbf{x}_k, \mathbf{x'}_k, t) \right\rangle \tag{A.26}$$

$$= (\partial_{\gamma,\mathbf{x}_k} + \partial_{\gamma,\mathbf{x'}_k}) \left\langle \mathcal{U}_\gamma(\{\mathbf{x}_k, \mathbf{x'}_k\}_n, t) \left[\prod_{l=1}^{n} w_{\alpha_l}(\mathbf{x}_l, \mathbf{x'}_l, t) \right] \right\rangle \tag{A.27}$$

This result was given previously by L'vov and Procaccia in section IV-B and appendix B of Ref.[172].

The local interaction term D_{kn} can be written as a linear transformation of F_{n+1}. First, note that

$$\mathcal{L}_\alpha(\mathbf{x}_k, \mathbf{x'}_k, \{\mathbf{y}, \mathbf{y}'\}_n, t) = (v_\beta(\mathbf{x}_k, \{\mathbf{y}, \mathbf{y}'\}_n, t)\partial_{\beta,\mathbf{x}_k} + v_\beta(\mathbf{x'}_k, \{\mathbf{y}, \mathbf{y}'\}_n, t)\partial_{\beta,\mathbf{x'}_k}) w_\alpha(\mathbf{x}_k, \mathbf{x'}_k, t) \tag{A.28}$$

$$= \frac{1}{2n} \sum_{l=1}^{n} [\partial_{\beta,\mathbf{x}_k}(w_\beta(\mathbf{x}_k, \mathbf{y}_l, t) + w_\beta(\mathbf{x}_k, \mathbf{y'}_l, t)) w_\alpha(\mathbf{x}_k, \mathbf{x'}_k, t)] \tag{A.29}$$

$$+ \sum_{l=1}^{n} [\partial_{\beta,\mathbf{x'}_k}(w_\beta(\mathbf{x}_k, \mathbf{y}_l, t) + w_\beta(\mathbf{x}_k, \mathbf{y'}_l, t)) w_\alpha(\mathbf{x}_k, \mathbf{x'}_k, t)] \tag{A.30}$$

It follows that D_{kn} is given by

$$D^{\alpha_1\alpha_2\cdots\alpha_n}_{kn}(\{\mathbf{x}, \mathbf{x}'\}_n, t) = \sum_{l=1}^{n} \int d\mathbf{y} P_{\alpha_k\beta}(\mathbf{y}) D^{\alpha_1\alpha_2\cdots\alpha_{k-1}\beta\cdots\alpha_n}_{knl}(\{\mathbf{x}, \mathbf{x}'\}_n, \mathbf{y}, t) \tag{A.31}$$

with $D_{knl} = D_{knl1} + D_{knl2} + D_{knl3} + D_{knl4}$, and

$$D_{knl1}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t) = \partial_{\alpha_{n+1},\mathbf{x}_k} F_{n+1}(\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=1}^{k-1},\mathbf{x}_k-\mathbf{y},\mathbf{x}'_k-\mathbf{y},\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=k+1}^{n},\mathbf{x}_k-\mathbf{y},\mathbf{x}_l) \tag{A.32}$$

$$D_{knl2}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t) = \partial_{\alpha_{n+1},\mathbf{x}_k} F_{n+1}(\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=1}^{k-1},\mathbf{x}_k-\mathbf{y},\mathbf{x}'_k-\mathbf{y},\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=k+1}^{n},\mathbf{x}_k-\mathbf{y},\mathbf{x}'_l) \tag{A.33}$$

$$D_{knl3}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t) = \partial_{\alpha_{n+1},\mathbf{x}'_k} F_{n+1}(\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=1}^{k-1},\mathbf{x}_k-\mathbf{y},\mathbf{x}'_k-\mathbf{y},\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=k+1}^{n},\mathbf{x}'_k-\mathbf{y},\mathbf{x}_l) \tag{A.34}$$

$$D_{knl4}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t) = \partial_{\alpha_{n+1},\mathbf{x}'_k} F_{n+1}(\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=1}^{k-1},\mathbf{x}_k-\mathbf{y},\mathbf{x}'_k-\mathbf{y},\{\mathbf{x}_m,\mathbf{x}'_m\}_{m=k+1}^{n},\mathbf{x}'_k-\mathbf{y},\mathbf{x}'_l) \tag{A.35}$$

We define, thus, the linear integrodifferential operator $\mathcal{O}_n$ such that

$$D_n(\{\mathbf{x},\mathbf{x}'\}_n,t) = \sum_{k=1}^{n} D_{kn}(\{\mathbf{x},\mathbf{x}'\}_n,t) \tag{A.36}$$

$$= \int O_n(\{\mathbf{X}\}_n,\{\mathbf{Y}\}_{n+1}) F_{n+1}(\{\mathbf{Y}\}_{n+1},t)\, d\{\mathbf{Y}\}_{n+1} \tag{A.37}$$

$$= (\mathcal{O}_n F_{n+1})(\{\mathbf{x},\mathbf{x}'\}_n,t). \tag{A.38}$$

A.4 Summary of the balance equations

Putting everything together, the generalized balance equations for the velocity field can be written as

$$\frac{\partial F_n}{\partial t} + \mathcal{O}_n F_{n+1} + I_n = \mathcal{D}_n F_n + Q_n \tag{A.39}$$

Here $\mathcal{D}_n$ is the differential operator representing dissipation, given by

$$\mathcal{D}_n = \sum_{k=1}^{n} [\nu(\nabla_{\mathbf{x}_k}^{2\kappa} + \nabla_{\mathbf{x}'_k}^{2\kappa}) + \beta(\nabla_{\mathbf{x}_k}^{-2m} + \nabla_{\mathbf{x}'_k}^{-2m})] \tag{A.40}$$

and $\mathcal{O}_n$ is the linear integrodifferential operator such that

$$D_n(\{\mathbf{x},\mathbf{x}'\}_n,t) = \sum_{k=1}^{n} D_{kn}(\{\mathbf{x},\mathbf{x}'\}_n,t) \tag{A.41}$$

$$= \int O_n(\{\mathbf{X}\}_n,\{\mathbf{Y}\}_{n+1}) F_{n+1}(\{\mathbf{Y}\}_{n+1},t)\, d\{\mathbf{Y}\}_{n+1} = (\mathcal{O}_n F_{n+1})(\{\mathbf{x},\mathbf{x}'\}_n,t). \tag{A.42}$$

where D_{kn} was defined previously by (A.31), (A.32), (A.33), (A.34), (A.35). The term I_n represents the sweeping interactions, and it is given by

$$I_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t) = \sum_{k=1}^{n} (\partial_{\gamma,\mathbf{x}_k} + \partial_{\gamma,\mathbf{x}'_k}) \left\langle \mathcal{U}_\gamma(\{\mathbf{x}_k,\mathbf{x}'_k\}_n,t) \left[\prod_{l=1}^{n} w_{\alpha_l}(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\right\rangle \tag{A.43}$$

The forcing term Q_n is given by

$$Q_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x},\mathbf{x}'\}_n,t)=\sum_{k=1}^{n}\left\langle\left[\prod_{l=1,l\neq k}^{n}w_{\alpha_l}(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\mathcal{P}_{\alpha_k\beta}(f_\beta(\mathbf{x}_k,t)-f_\beta(\mathbf{x}'_k,t))\right\rangle \tag{A.44}$$

A.5 Balance equations for the vorticity

A similar set of equations can be derived for the generalized structure functions of the vorticity. Let $\xi(\mathbf{x},\mathbf{x}',t)$ be the vorticity difference defined as

$$\xi(\mathbf{x},\mathbf{x}',t)=\zeta(\mathbf{x},t)-\zeta(\mathbf{x}',t) \tag{A.45}$$

$$=\varepsilon_{\alpha\beta}(\partial_{\alpha,\mathbf{x}}+\partial_{\alpha,\mathbf{x}'})w_\beta(\mathbf{x},\mathbf{x}',t), \tag{A.46}$$

and let $V_n(\{\mathbf{X}\}_n,t)$ be the generalized structure function of the vorticity defined as

$$V_n(\{\mathbf{X}\}_n,t)=\left\langle\left[\prod_{k=1}^{n}\xi(\mathbf{X}_k,t)\right]\right\rangle \tag{A.47}$$

It is easy to see that the vorticity generalized structure functions are related to the velocity generalized structure functions by

$$V_n(\{\mathbf{X}\}_n,t)=\prod_{k=1}^{n}[\varepsilon_{\alpha_k\beta_k}(\partial_{\alpha_k,\mathbf{x}_k}+\partial_{\alpha_k,\mathbf{x}'_k})]F_n^{\alpha_1\cdots\alpha_n}(\{\mathbf{X}\}_n,t) \tag{A.48}$$

Let $\mathcal{T}_n$ be an abbreviation for the differential operator that transforms F_n to V_n such that $V_n=\mathcal{T}_nF_n$.

The balance equations for V_n and be derived easily by applying the operator $\mathcal{T}_n$ on the balance equations for F_n. The result is

$$\frac{\partial V_n}{\partial t}+\mathcal{T}_n\mathcal{O}_nF_{n+1}+\mathcal{I}_n=\mathcal{D}_nV_n+\mathcal{Q}_n \tag{A.49}$$

Here $\mathcal{Q}_n$ is the forcing term and $\mathcal{I}_n$ is the sweeping term. The forcing term reads.

$$\mathcal{Q}_n(\{\mathbf{X}\}_n,t)=\sum_{k=1}^{n}\mathcal{Q}_{kn}(\{\mathbf{X}\}_n^k,\mathbf{X}_k,t) \tag{A.50}$$

$$\mathcal{Q}_{kn}(\{\mathbf{X}\}_n^k,\mathbf{Y},t)=\left\langle\left[\prod_{k=1}^{n-1}\xi(\mathbf{X}_k,t)\right]g(\mathbf{Y},t)\right\rangle \tag{A.51}$$

To calculate the sweeping term we use (4.11) to cancel the vortex tilting contributions. With a little bit of algebra we find that

$$\mathcal{I}_n(\{\mathbf{X}\}_n,t)=\prod_{j=1}^{n}[\varepsilon_{\alpha_j\beta_j}(\partial_{\alpha_j,\mathbf{x}_j}+\partial_{\alpha_j,\mathbf{x}'_j})]\sum_{k=1}^{n}(\partial_{\gamma,\mathbf{x}_k}+\partial_{\gamma,\mathbf{x}'_k})\left\langle\mathcal{U}_\gamma(\{\mathbf{x}_k,\mathbf{x}'_k\}_n,t)\left[\prod_{l=1}^{n}w_{\beta_l}(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\right\rangle \tag{A.52}$$

$$=\sum_{k=1}^{n}(\partial_{\gamma,\mathbf{x}_k}+\partial_{\gamma,\mathbf{x}'_k})\left\langle\mathcal{U}_\gamma(\{\mathbf{x}_k,\mathbf{x}'_k\}_n,t)\left[\prod_{l=1}^{n}\xi(\mathbf{x}_l,\mathbf{x}'_l,t)\right]\right\rangle \tag{A.53}$$

The exact mathematical form of the term $\mathcal{T}_n \mathcal{O}_n F_{n+1}$ is not required. It is only sufficient to note that once it is shown that the operator $\mathcal{O}_n$ is local, then it easily follows that the operator $\mathcal{T}_n \mathcal{O}_n$ is also local since $\mathcal{T}_n$ is a differential operator, thus local.

Appendix B

QUASI-LAGRANGIAN REPRESENTATION OF THE NAVIER-STOKES EQUATIONS

We show how the quasi-Lagrangian transformation makes it possible to write the Navier-Stokes equations in terms of velocity differences, thereby eliminating the sweeping interactions. The reader should compare this argument with the derivation of the 4/5-law by Monin and Yaglom [185], to see that they are doing the same thing.

We begin by noting that the Eulerian velocity field $u_\alpha(\mathbf{x},t)$ can be reconstructed from $v_\alpha(\mathbf{x}_0,t_0|\mathbf{x},t)$ as:

$$u_\alpha(\mathbf{x},t) = u_\alpha(\mathbf{x}-\rho(\mathbf{x}_0,t_0|t)+\rho(\mathbf{x}_0,t_0|t),t) \tag{B.1}$$

$$= v_\alpha(\mathbf{x}_0,t_0|\mathbf{x}-\rho(\mathbf{x}_0,t_0|t),t). \tag{B.2}$$

To eliminate $\rho(\mathbf{x}_0,t_0|t)$ we use

$$\rho_\alpha(\mathbf{x}_0,t_0|t) = \int_{t_0}^{t} d\tau\ u_\alpha(\mathbf{x}+\rho(\mathbf{x}_0,t_0|\tau),\tau) \tag{B.3}$$

$$= \int_{t_0}^{t} d\tau\ v_\alpha(\mathbf{x}_0,t_0|\mathbf{x},\tau), \tag{B.4}$$

and therefore, $u_\alpha(\mathbf{x},t)$ reads

$$u_\alpha(\mathbf{x},t) = v_\alpha\left(\mathbf{x}_0,t_0|\mathbf{x}-\int_{t_0}^{t} d\tau\ \mathbf{v}(\mathbf{x}_0,t_0|\mathbf{x},\tau),t\right). \tag{B.5}$$

Let $R_\alpha(\mathbf{x}_0,t_0|\mathbf{x},t)$ be defined as

$$R_\alpha(\mathbf{x}_0,t_0|\mathbf{x},t) = (\mathbf{x})_\alpha - \int_{t_0}^{t} d\tau\ v_\alpha(\mathbf{x}_0,t_0|\mathbf{x},\tau), \tag{B.6}$$

such that $u_\alpha(\mathbf{x},t) = v_\alpha(\mathbf{x}_0,t_0|\mathbf{R}(\mathbf{x}_0,t_0|\mathbf{x},t),t)$. Also, define $v^0_\alpha(\mathbf{x}_0,t_0|t) \equiv v_\alpha(\mathbf{x}_0,t_0|\mathbf{x}_0,t)$ such that we may write concisely $w_\alpha = v_\alpha - v^0_\alpha$. It is easy to see that $\partial_{\alpha,\mathbf{x}} R_\beta = \delta_{\alpha\beta}$ and $\partial R_\alpha/\partial t = -v^0_\alpha$, and we may use these relations to show that the quasi-Lagrangian transformation preserves incompressibility, as follows:

$$\partial_{\alpha,\mathbf{x}} w_\alpha = \partial_{\alpha,\mathbf{x}}(v_\alpha - v^0_\alpha) = \partial_{\alpha,\mathbf{x}} v_\alpha = (\partial_{\beta,\mathbf{x}} v_\alpha)\delta_{\alpha\beta} \tag{B.7}$$

$$= (\partial_{\beta,\mathbf{x}} v_\alpha)(\partial_{\alpha,\mathbf{x}} R_\beta) = \partial_{\alpha,\mathbf{x}} u_\alpha = 0. \tag{B.8}$$

The key result is that the sweeping interactions are eliminated in the transformation of the material derivative itself. The show this, consider an arbitrary field $U(\mathbf{x},t)$ and its quasi-Lagrangian transformation $\mathcal{U}(\mathbf{x}_0,t_0|\mathbf{x},t)$ (where the fluid particle follows the Eulerian velocity field $u_\alpha(\mathbf{x},t)$). From the relation

$$U(\mathbf{x},t)=\mathcal{U}(\mathbf{x}_0,t_0|\mathbf{R}(\mathbf{x}_0,t_0|\mathbf{x},t),t), \tag{B.9}$$

we find that

$$\frac{\partial U}{\partial t}=\frac{\partial \mathcal{U}}{\partial t}+(\partial_{\alpha,\mathbf{x}}\mathcal{U})\frac{\partial R_\alpha}{\partial t}=\frac{\partial \mathcal{U}}{\partial t}+(\partial_{\alpha,\mathbf{x}}\mathcal{U})(-v_\alpha^0), \tag{B.10}$$

and

$$\partial_{\alpha,\mathbf{x}}U=(\partial_{\beta,\mathbf{x}}\mathcal{U})(\partial_{\alpha,\mathbf{x}}R_\beta)=(\partial_{\beta,\mathbf{x}}\mathcal{U})\delta_{\alpha\beta}=(\partial_{\alpha,\mathbf{x}}\mathcal{U}), \tag{B.11}$$

and it follows that:

$$\frac{DU}{Dt}=\frac{\partial U}{\partial t}+u_\alpha\partial_{\alpha,\mathbf{x}}U=\frac{\partial \mathcal{U}}{\partial t}+w_\alpha\partial_{\alpha,\mathbf{x}}\mathcal{U}. \tag{B.12}$$

This equation is identical to the unnumbered equation preceding equation (22.14) in Monin and Yaglom [185]. It is easy to see that since the material derivative is written in terms of velocity differences, if it is applied on $w_\alpha(\mathbf{x}_0,t_0|\mathbf{x},t)$ we shall obtain an equation written exclusively in terms of velocity differences.

Appendix C

EVALUATION OF $J[U]$

In this appendix, we provide a detailed evaluation of the functional determinant $J[u]$ that we encounter in the derivation of the MSR theory for Lagrangian trajectories. The procedure was outlined briefly in L'vov and Procaccia [167]. However it is not as trivial as it seems. We have followed the outline and rederived the following more complete version of the proof:

First, we discretize time in Δt intervals and introduce the following notation:

$$\begin{aligned} t_n &= t_0 + n\Delta t \\ \rho_\alpha^n &= \rho_\alpha(\mathbf{x}_0, t_0|t_n) \\ u_\alpha^n &= u_\alpha(\mathbf{x}_0 + \rho_\alpha^n, t_n) \\ u_\alpha^{n,m} &= u_\alpha(\mathbf{x}_0 + \rho_\alpha^n, t_m). \end{aligned} \tag{C.1}$$

Each of the objects $\rho_\alpha^n, u_\alpha^n, u_\alpha^{n,m}$ is a field that is a function of $\mathbf{x}_0$ only. Note that $\rho_\alpha^0 = 0$. The governing equation for the Lagrangian trajectories field is equivalent to a set of the following discretized equations:

$$\frac{\rho_\alpha^{n+1} - \rho_\alpha^n}{\Delta t} = u_\alpha^n. \tag{C.2}$$

There are, of course, many alternative discretizations to choose from. The rule is that, once we have chosen a discretization, we have to stay with it. We cannot switch to another scheme in the middle of the computations, for the sake of convenience. To evaluate $J[u]$ we proceed from the path integral definition:

$$\begin{aligned} J[u] &= \int_{\mathcal{P}(t_0)} \mathcal{D}\rho\ \delta[\dot{\rho}_\alpha - Q_{\alpha\beta}^{\mathbf{x}_0}[\rho]u_\beta] = \lim_{\Delta t\to 0} \prod_{n\in\mathbb{Z}-\{0\}} \int \frac{d\rho^n}{a} \delta\left(\frac{\rho_\alpha^{n+1} - \rho_\alpha^n}{\Delta t} - u_\alpha^n\right) \\ &= \lim_{\Delta t\to 0} \prod_{n=1}^{+\infty} \frac{A_n(\Delta t)}{a} \prod_{n=-\infty}^{-1} \frac{B_n(\Delta t)}{a}, \end{aligned} \tag{C.3}$$

where $A_n(\Delta t)$ and $B_n(\Delta t)$ are defined as

$$\begin{aligned} A_n(\Delta t) &= \int d\rho_\alpha^{n+1}\ \delta\left(\frac{\rho_\alpha^{n+1} - \rho_\alpha^n}{\Delta t} - u_\alpha^n\right), \\ B_n(\Delta t) &= \int d\rho_\alpha^n\ \delta\left(\frac{\rho_\alpha^{n+1} - \rho_\alpha^n}{\Delta t} - u_\alpha^n\right). \end{aligned} \tag{C.4}$$

Here, a is a normalization constant such that the product in (C.3) converges. Obviously, if such a constant exists, it will be unique.

The A_n integral is easy to evaluate:

$$A_n(\Delta t) = \int d\rho^{n+1}\ \Delta t \delta(\rho_\alpha^{n+1} - \rho_\alpha^n - \Delta t u_\alpha^n) = \Delta t. \tag{C.5}$$

To evaluate B_n we need to rewrite the discretized governing equation so that it is explicit with respect to ρ_α^n:

$$\begin{aligned}
\rho_\alpha^{n+1} - \rho_\alpha^n &= \Delta t u_\alpha(\mathbf{x}_0 + \rho_\alpha^n, t_n) \\
&= \Delta t u_\alpha(\mathbf{x}_0 + \rho_\alpha^{n+1} - \dot{\rho}_\alpha^{n+1}\Delta t + O(\Delta t^2), t_n) \\
&= \Delta t(u_\alpha(\mathbf{x}_0 + \rho_\alpha^{n+1}, t_n) - \dot{\rho}_\beta^{n+1}\Delta t \partial_\beta(u_\alpha(\mathbf{x}_0 + \rho_\alpha^{n+1}, t_n)) + O(\Delta t^2) \\
&= u_\alpha^{n+1,n}\Delta t - (\rho_\beta^{n+1} - \rho_\beta^n)\partial_\beta u_\alpha^{n+1,n}\Delta t + O(\Delta t^2),
\end{aligned} \tag{C.6}$$

therefore

$$(\delta_{\alpha\beta} + \partial_\beta u_\alpha^{n+1,n}\Delta t)(\rho_\beta^{n+1} - \rho_\beta^n) = u_\alpha^{n+1,n}\Delta t. \tag{C.7}$$

We proceed by employing the following change of variables:

$$R_\alpha^n = (\delta_{\alpha\beta} + \partial_\beta u_\alpha^{n+1,n}\Delta t)\rho_\beta^n. \tag{C.8}$$

The integral differentials are transformed according to a determinant as follows:

$$dR_\alpha^n = \det(\delta_{\alpha\beta} + \partial_\beta u_\alpha^{n+1,n}\Delta t) d\rho_\beta^n. \tag{C.9}$$

We will now show that incompressibility implies that the determinant is equal to 1. For brevity, introduce

$$M_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\beta u_\alpha^{n+1,n}\Delta t. \tag{C.10}$$

In the determinant expansion, every term other than $M_{11}M_{22}M_{33}$ is $O(\Delta t^2)$ because it includes at least two off-diagonal factors each of which contributes a factor of Δt . It follows that:

$$\begin{aligned}
\det M &= M_{11}M_{22}M_{33} + O(\Delta t^2) \\
&= (1 + \Delta t\partial_1 u_1^{n+1,n})(1 + \Delta t\partial_2 u_2^{n+1,n})(1 + \Delta t\partial_3 u_3^{n+1,n}) + O(\Delta t^2) \\
&= 1 + \Delta t(\partial_1 u_1^{n+1,n} + \partial_2 u_2^{n+1,n} + \partial_3 u_3^{n+1,n}) + O(\Delta t^2) \\
&= 1 + O(\Delta t^2).
\end{aligned} \tag{C.11}$$

Note that in the last step we employed the incompressibility condition. It follows that $d\rho^n = dR^n$. We may now proceed and evaluate the integral B_n.

$$\begin{aligned} B_n(\Delta t) &= \int d\rho^n \ \Delta t \delta[\rho_\alpha^{n+1} - \rho_\alpha^n - \Delta t u_\alpha^n] \\ &= \Delta t \int d\rho^n \ \delta((\delta_{\alpha\beta} + \partial_\beta u_\alpha^{n+1,n}\Delta t)(\rho_\beta^{n+1} - \rho_\beta^n) - u_\alpha^{n+1,n}\Delta t) \\ &= \Delta t(1 + O(\Delta t^2)) \int dR^n \ \delta((\delta_{\alpha\beta} + \partial_\beta u_\alpha^{n+1,n}\Delta t)\rho_\beta^{n+1} - R_\alpha^n - \Delta t u_\alpha^{n+1,n}) \\ &= \Delta t + O(\Delta t^3). \end{aligned} \tag{C.12}$$

In the last step, the crucial requirement is that ρ_β^{n+1} and $u_\alpha^{n+1,n}$ should not depend on ρ_α^n and therefore R_α^n. If we set the normalization constant $a = \Delta t$, then $J[u]$ evaluates as:

$$J[u] = \lim_{\Delta t \to 0} \prod_{n=1}^{+\infty} \frac{A_n(\Delta t)}{a} \prod_{n=-\infty}^{-1} \frac{B_n(\Delta t)}{a} = 1. \tag{C.13}$$

Note that $O(\Delta t^2)$ contributions to the integrals $A_n(\Delta t)$ and $B_n(\Delta t)$, which we have disregarded, would vanish anyway after taking the limit $\Delta t \to 0$, so they can be safely ignored with impunity.

Appendix D

SWEEPING INTERACTIONS UNDER A GAUSSIAN MEAN FIELD

We exploit the following mathematical result: if $f_\alpha(\mathbf{x}_1, t_1)$ is a Gaussian stochastic field, the ensemble averages of the form $\langle f_\alpha(\mathbf{x}_1, t_1) R[f]\rangle$ can be evaluated for any analytic functional $R[f]$ by the following integral

$$\langle f_\alpha(\mathbf{x}_1, t_1) R[f]\rangle = \langle f_\alpha(\mathbf{x}_1, t_1)\rangle \langle R[f]\rangle + \int d\mathbf{x}_2 dt_2 \ \langle f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_2, t_2)\rangle_c \left\langle \frac{\delta R[f]}{\delta f_\beta(\mathbf{x}_2, t_2)} \right\rangle, \quad \text{(D.1)}$$

where

$$\langle f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_2, t_2)\rangle_c \equiv \langle f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_2, t_2)\rangle - \langle f_\alpha(\mathbf{x}_1, t_1)\rangle \langle f_\beta(\mathbf{x}_2, t_2)\rangle . \quad \text{(D.2)}$$

This a generalization of Gaussian integration by parts, a technique attributed by Frisch [94] to Novikov [193], Donsker [64] and Furutsu [97].

We begin the proof by defining the following correlation functions:

$$U_{\alpha\beta}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) = \langle u_\alpha(\mathbf{x}_1, t_1) u_\beta(\mathbf{x}_2, t_2)\rangle - \langle u_\alpha(\mathbf{x}_1, t_1)\rangle \langle u_\beta(\mathbf{x}_2, t_2)\rangle , \quad \text{(D.3)}$$

$$B_n^{\alpha_1 \cdots \alpha_n \beta}(\{\mathbf{X}\}_n, \mathbf{Y}, t) = \left\langle (\mathcal{U}_\beta(\mathbf{Y}, t)) \left[\prod_{l=1}^{n} w_{\alpha_l}(\mathbf{X}_l, t) \right] \right\rangle - \langle \mathcal{U}_\beta(\mathbf{Y}, t))\rangle \, F_n^{\alpha_1 \alpha_2 \cdots \alpha_n}(\{\mathbf{X}\}_n, t), \quad \text{(D.4)}$$

$$H_n^{\alpha_1 \alpha_2 \cdots \alpha_n \beta}(\{\mathbf{X}\}_n, t) = \left[\sum_{k=1}^{n} (\partial_{\beta, \mathbf{x}_k} + \partial_{\beta, \mathbf{x}'_k}) F_n^{\alpha_1 \alpha_2 \cdots \alpha_n}(\{\mathbf{X}\}_n, t) \right], \quad \text{(D.5)}$$

and also the following response functions:

$$R_{\alpha\beta} = \left\langle \frac{\delta w_\alpha(\mathbf{X}, t_1)}{\delta u_\beta(\mathbf{y}, t_2)} \right\rangle, \quad \text{(D.6)}$$

$$R_n^{\alpha_1 \cdots \alpha_n \beta}(\{\mathbf{X}\}_n, t, \mathbf{y}, \tau) = \left\langle \frac{\delta}{\delta u_\beta(\mathbf{y}, \tau)} \left[\prod_{l=1}^{n} w_{\alpha_l}(\mathbf{X}_l, t) \right] \right\rangle . \quad \text{(D.7)}$$

Here, we disregard the fact that $u_\alpha(\mathbf{x}, t)$ and $w_\alpha(\mathbf{x}, \mathbf{x}', t)$ are related by definition and assume that the only effect of $u_\alpha(\mathbf{x}, t)$ on $w_\alpha(\mathbf{x}, \mathbf{x}', t)$ is via the sweeping interaction. We also assume that the velocity field $u_\alpha(\mathbf{x}, t)$ is delta-correlated which implies that

$$U_{\alpha\beta}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2) = U_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2)\delta(t_1 - t_2). \quad \text{(D.8)}$$

We begin by splitting $I_n(\{\mathbf{X}\}_n,t)$ into two terms

$$I_n(\{\mathbf{X}\}_n,t) = I_{n,(1)}(\{\mathbf{X}\}_n,t) + I_{n,(2+3)}(\{\mathbf{X}\}_n,t), \tag{D.9}$$

with $I_{n,(1)}(\{\mathbf{X}\}_n,t)$ given by

$$I_{n,(1)}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t) = \sum_{k=1}^{n}(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x}'_k})\left\{\langle \mathcal{U}_\beta(\{\mathbf{X}\}_n,t)\rangle \left\langle\left[\prod_{l=1}^{n} w_{\alpha_l}(\mathbf{X}_l,t)\right]\right\rangle\right\} \tag{D.10}$$

$$= \langle \mathcal{U}_\beta(\{\mathbf{X}\}_n,t)\rangle \left[\sum_{k=1}^{n}(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x}'_k})F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t)\right] \tag{D.11}$$

$$= \langle \mathcal{U}_\beta(\{\mathbf{X}\}_n,t)\rangle\, H_n^{\alpha_1\alpha_2\cdots\alpha_n\beta}(\{\mathbf{X}\}_n,t). \tag{D.12}$$

Here we have used the incompressibility condition.

$$\sum_{k=1}^{n}(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x}'_k})\,\langle \mathcal{U}_\beta(\{\mathbf{X}\}_n,t)\rangle = 0. \tag{D.13}$$

The remaining contribution to $I_n(\{\mathbf{X}\}_n,t)$ reads

$$I_{n,(2+3)}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t) = \frac{1}{2n}\sum_{k=1}^{n}\sum_{l=1}^{n}(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x}'_k})B_n^{\alpha_1\cdots\alpha_n\beta}(\{\mathbf{X}\}_n,\mathbf{X}_l,t). \tag{D.14}$$

Using Gaussian integration by parts we may write

$$B_1^{\alpha\beta}(\mathbf{X},\mathbf{Y},t) = \int d\mathbf{z}d\tau\; R_{\alpha\gamma}(\mathbf{X},t;\mathbf{z},\tau)[U_{\beta\gamma}(\mathbf{y},t;\mathbf{z},\tau)+U_{\beta\gamma}(\mathbf{y}',t;\mathbf{z},\tau)] \tag{D.15}$$

$$= \int d\mathbf{z}\; R_{\alpha\gamma}(\mathbf{X},t;\mathbf{z},t)[U_{\beta\gamma}(\mathbf{y},\mathbf{z})+U_{\beta\gamma}(\mathbf{y}',\mathbf{z})], \tag{D.16}$$

and

$$B_n^{\alpha_1\cdots\alpha_n\beta}(\{\mathbf{X}\}_n,\mathbf{Y},t) = \int d\mathbf{z}d\tau\; R_n^{\alpha_1\cdots\alpha_n\gamma}(\{\mathbf{X}\}_n,t;\mathbf{z},\tau)[U_{\beta\gamma}(\mathbf{y},t;\mathbf{z},\tau)+U_{\beta\gamma}(\mathbf{y}',t;\mathbf{z},\tau)] \tag{D.17}$$

$$= \int d\mathbf{z}\; R_n^{\alpha_1\cdots\alpha_n\gamma}(\{\mathbf{X}\}_n,t;\mathbf{z},t)[U_{\beta\gamma}(\mathbf{y},\mathbf{z})+U_{\beta\gamma}(\mathbf{y}',\mathbf{z})]. \tag{D.18}$$

The key step is to note that

$$R_n^{\alpha_1\cdots\alpha_n\beta}(\{\mathbf{X}\}_n,t,\mathbf{y},t) = \left\langle \frac{\delta}{\delta u_\beta(\mathbf{y},t)}\left[\prod_{l=1}^{n} w_{\alpha_l}(\mathbf{X}_l,t)\right]\right\rangle \tag{D.19}$$

$$= \sum_{k=1}^{n}\left\langle\left[\prod_{l=1,l\neq k}^{n} w_{\alpha_l}(\mathbf{X}_l,t)\right]\frac{\delta w_{\alpha_k}(\mathbf{X}_k,t)}{\delta u_\beta(\mathbf{y},t)}\right\rangle \tag{D.20}$$

$$= \sum_{k=1}^{n} F_{n-1}^{\alpha_1\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n}(\{\mathbf{X}\}_n^k)R_{\alpha_k\beta}(\mathbf{X}_k,t;\mathbf{y},t). \tag{D.21}$$

Here we exploit the fact, first pointed out in Ref. [170], that the variational derivative $(\delta w_{\alpha_k}(\mathbf{X}_k,t))/(\delta u_\beta(\mathbf{y},t))$ is not correlated with the velocity differences $w_{\alpha_l}(\mathbf{X}_l,t)$ because no time is being allowed for the interaction to develop a correlation. This relationship between the response functions in implies a corresponding relationship between $B_n(\{\mathbf{X}\}_n,\mathbf{Y},t)$ and $B_1(\mathbf{X}_k,\mathbf{Y},t)$:

$$B_n^{\alpha_1\cdots\alpha_n\beta}(\{\mathbf{X}\}_n,\mathbf{Y},t)=\int d\mathbf{z}\ \left[\sum_{k=1}^{n}F_{n-1}^{\alpha_1\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n}(\{\mathbf{X}\}_n^k)R_{\alpha_k\beta}(\mathbf{X}_k,t;\mathbf{z},t)\right][U_{\beta\gamma}(\mathbf{y},\mathbf{z})+U_{\beta\gamma}(\mathbf{y}',\mathbf{z})] \tag{D.22}$$

$$=\sum_{k=1}^{n}F_{n-1}^{\alpha_1\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n}(\{\mathbf{X}\}_n^k)\left[\int d\mathbf{z}\ R_{\alpha_k\beta}(\mathbf{X}_k,t;\mathbf{z},t)[U_{\beta\gamma}(\mathbf{y},\mathbf{z})+U_{\beta\gamma}(\mathbf{y}',\mathbf{z})]\right] \tag{D.23}$$

$$=\sum_{k=1}^{n}F_{n-1}^{\alpha_1\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n}(\{\mathbf{X}\}_n^k)B_1^{\alpha_k\beta}(\mathbf{X}_k,\mathbf{Y},t). \tag{D.24}$$

It immediately follows that

$$I_{n,(2+3)}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t)=\frac{1}{2n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x'}_k})F_{n-1}^{\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n}(\{\mathbf{X}\}_n^m)B_1^{\alpha_m\beta}(\mathbf{X}_m,\mathbf{X}_l,t), \tag{D.25}$$

which can be broken down to

$$I_{n,(2)}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t)=\sum_{l=1}^{n}\sum_{m=1}^{n}F_{n-1}^{\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n}(\{\mathbf{X}\}_n^m)I_{\alpha_m}(\mathbf{X}_m,\mathbf{X}_l,t), \tag{D.26}$$

$$I_{n,(3)}^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{X}\}_n,t)=\sum_{l=1}^{n}\sum_{m=1}^{n}B_1^{\alpha_m\beta}(\mathbf{X}_m,\mathbf{X}_l,t)H_{n-1}^{\alpha_1\cdots\alpha_{m-1}\alpha_{m+1}\cdots\alpha_n\beta}(\{\mathbf{X}\}_n^m), \tag{D.27}$$

with

$$I_\alpha(\mathbf{X}_1,\mathbf{X}_2,t)=\sum_{k=1}^{2}(\partial_{\beta,\mathbf{x}_k}+\partial_{\beta,\mathbf{x'}_k})B_1^{\alpha\beta}(\mathbf{X}_1,\mathbf{X}_2,t). \tag{D.28}$$

Appendix E

HÖLDER INEQUALITIES

Let $f(x)$ and $g(x)$ be two functions defined over a domain $x \in \mathcal{A}$, such that

$$f(x) > 0 \text{ and } g(x) > 0, \ \forall x \in \mathcal{A}, \tag{E.1}$$

and let a, b be real numbers such that $(1/a)+(1/b) = 1$. Then the Hölder inequalities, in the integral form, read

$$\int_{\mathcal{A}} f(x)g(x)\, dx \leq \left(\int_{\mathcal{A}} [f(x)]^a \, dx \right)^{1/a} \left(\int_{\mathcal{A}} [g(x)]^b \, dx \right)^{1/b}. \tag{E.2}$$

For the case $a = b = 2$ and $\mathcal{A} = (0, +\infty)$ with $f(x) = k^\alpha \sqrt{E(k)}$ and $g(x) = k^\beta \sqrt{E(k)}$, we have

$$E_{\alpha+\beta} = \int_0^{+\infty} k^{\alpha+\beta} E(k)\, dk \tag{E.3}$$

$$\leq \left(\int_0^{+\infty} (k^\alpha \sqrt{E(k)})^2 \, dk \right)^{1/2} \left(\int_0^{+\infty} (\sqrt{k^\beta E(k)})^2 \, dk \right)^{1/2} = \sqrt{E_{2\alpha} E_{2\beta}}. \tag{E.4}$$

For the cases $(\alpha, \beta) = (0, 1)$ and $(\alpha, \beta) = (0, 2)$ we obtain $E_1^2 \leq E_0 E_2$ and $E_2^2 \leq E_0 E_4$ by raising squares, noting that all the quantities involved are positive. For the case $\alpha \mapsto \kappa$ and $\beta \mapsto \kappa + 2\alpha$ we get the inequality $E_{\kappa+\alpha}^2 \leq E_\kappa E_{\kappa+2\alpha}$ which can be rewritten as

$$\frac{E_{\kappa+\alpha}}{E_\kappa} \leq \frac{E_{\kappa+2\alpha}}{E_{\kappa+\alpha}}. \tag{E.5}$$

This inequality appears to indicate that the function $\mathcal{E}(\kappa, \alpha) \equiv E_{\kappa+\alpha}/E_\kappa$ is an increasing function with respect to κ for $\alpha > 0$ and decreasing for $\alpha < 0$. To prove this, we first note that from Eq.(E.5) we have, for any integer $n > 0$

$$\mathcal{E}(\kappa + \alpha/n, \alpha) = \frac{E_{\kappa+\alpha/n+\alpha}}{E_{\kappa+\alpha/n}} = \prod_{j=1}^{n} \frac{E_{\kappa+\alpha/n+j\alpha/n}}{E_{\kappa+\alpha/n+(j-1)\alpha/n}} \tag{E.6}$$

$$\geq \prod_{j=1}^{n} \frac{E_{\kappa+j\alpha/n}}{E_{\kappa+(j-1)\alpha/n}} = \frac{E_{\kappa+\alpha}}{E_\kappa} = \mathcal{E}(\kappa, \alpha). \tag{E.7}$$

It follows that since $\mathcal{E}(\kappa, \alpha)$ is a differentiable function with respect to κ, that $\partial\mathcal{E}(\kappa, \alpha)/\partial\kappa \geq 0$ for $\alpha > 0$ and $\partial\mathcal{E}(\kappa, \alpha)/\partial\kappa \leq 0$ for $\alpha < 0$, and thus $\mathcal{E}(\kappa, \alpha)$ is an increasing function with respect to κ for $\alpha > 0$ and a decreasing function for $\alpha < 0$. All the inequalities needed for our discussion of Rhines proof can be deduced from this result.

Appendix F

RELATIONSHIP BETWEEN ENERGY SPECTRUM AND F_2

The relationship between the energy spectrum and the second order structure function has been part of the folklore of the statistical theory of turbulence for a long time. The one-dimensional version of this result was discovered independently by Wiener [258], Khinchin [128], and Einstein (for reprint and interesting commentary see [68, 101, 261]), and its relevence to turbulence was first highlighted by Taylor [244]. The three-dimensional version was used by Batchelor [7] in his famous monograph, as well as his papers, and a version for any dimension can be found in Panchev [196], without proof. The one dimensional and three-dimensional version have also been stated without proof by Frisch [94].

The version shown here has been derived independently while referring to Panchev [196], and using the definition of the energy spectrum given by Frisch [94]. The localized energy spectrum is defined as

$$E_{\alpha\beta}(k, \mathbf{x}, t) = \frac{1}{2}\frac{d}{dk}\langle u_\alpha^{<k}(\mathbf{x}, t) u_\beta^{<k}(\mathbf{x}, t)\rangle, \tag{F.1}$$

where $u_\alpha^{<k_0}$ is the filtered velocity field defined as

$$u_\alpha^{<k_0}(\mathbf{x}, t) = \frac{1}{(2\pi)^d}\int d\mathbf{k}\int d\mathbf{l}\; u_\alpha(\mathbf{l}, t)\exp(i\mathbf{k}\cdot(\mathbf{l}-\mathbf{x}))H(k_0 - \|\mathbf{k}\|), \tag{F.2}$$

where H is the Heaviside function

$$H(x) = \begin{cases} 1, & x > 0 \\ 1/2, & x = 0 \\ 0, & x < 0 \end{cases} . \tag{F.3}$$

Integrating over the spatial dependence gives the standard energy spectrum

$$E(k, t) = \int d\mathbf{x} E_{\alpha\alpha}(k, \mathbf{x}, t). \tag{F.4}$$

Likewise, we define a 2nd order structure function $S_2(\rho)$ in terms of F_2 as follows

$$S_2(\rho) \equiv \frac{1}{\gamma_d}\int d\mathbf{x}\int_{SO(d)} d\Omega(A) F_2(\mathbf{x}, \mathbf{x}+\rho A\mathbf{e}, \mathbf{x}, \mathbf{x}+\rho A\mathbf{e}), \tag{F.5}$$

where $\mathbf{e}$ is a fixed unit vector, and $d\Omega(A)$ represents spherical integration defined as

$$\begin{aligned} d\mathbf{x} &= r^{d-1} dr d\Omega(A) \\ d\Omega(A) &= \prod_{\alpha=1}^{d-1} (\sin\phi_\alpha)^{d-1-\alpha} d\phi_\alpha, \end{aligned} \tag{F.6}$$

with $\phi \in [0,\pi]^{d-2} \times [0, 2\pi)$ and

$$\gamma_d \equiv \int_{SO(d)} d\Omega(A) = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{F.7}$$

With these definitions in place, the following relationship between the energy spectrum and the second order structure function can be derived rigorously

$$S_2(\rho, t) = 4 \int_0^{+\infty} dk E(k,t)[1 - \Phi_d(k\rho)], \tag{F.8}$$

where, $\Phi_d(x)$ is the kernel of the spherical Fourier transform given by

$$\begin{aligned} \Phi_d(x) &\equiv \frac{1}{\gamma_d} \int_{SO(d)} d\Omega(A) \exp[ix\mathbf{e} \cdot (A\mathbf{e})] \\ &= \frac{(2\pi)^{d/2}}{\gamma_d} \frac{J_{(d-2)/2}(x)}{x^{(d-2)/2}}. \end{aligned} \tag{F.9}$$

For $d = 2$ we have $\Phi_2(x) = J_0(x)$, where J_0 is the Bessel function.

Using this relationship, it is then possible to prove the relationship between the scaling exponent of F_2 and the scaling exponent of the energy spectrum. In particular, it can be shown that for $1 < m < 3$

$$E(k) = ak^{-m}, \quad \forall k \gg k_0 \Longleftrightarrow S_2(r) = aC(d) r^{m-1}, \quad \forall r \ll k_0^{-1}, \tag{F.10}$$

where the constant $C(d)$ depends on the dimension d and reads

$$C(d) = 4 \int_0^{+\infty} \frac{1 - \Phi_d(x)}{x^m}\, dx. \tag{F.11}$$

Appendix G

3D INTERPRETATION OF SQG TURBULENCE

There has been considerable confusion over the physical interpretation of the surface quasi-geostrophic model. Although its mathematical formulation is in the form of a one-layer model, it represents a three-dimensional system that corresponds to the baroclinic limit of the three-dimensional quasi-geostrophic model. Once that is taken into account, the physical interpretation of the spectra $A(k)$ and $B(k)$ have to be revised.

As derived by Charney [38], 3D QG flow conserves the 3D potential vorticity ξ, which is advected horizontally by the streamfunction ψ. Here, both ψ and ξ are 3D fields. For constant Coriolis parameter f, the governing conservation law for ξ takes the form:

$$\frac{\partial \xi}{\partial t} + J(\psi, \xi) = 0, \tag{G.1}$$

with ξ given by

$$\xi = \Delta\psi + \frac{f^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial \psi}{\partial z}\right) \equiv \mathcal{P}\psi, \tag{G.2}$$

where $\rho_0(z)$ is the ambient air density, and $N^2(z)$ the Brunt-Väisälä frequency. Here we have omitted the forcing and dissipation terms.The streamfunction ψ is also linked with the potential temperature Θ via the hydrostatic relation

$$\Theta = \frac{g}{N}\frac{T}{T_0} = \frac{f}{N}\frac{\partial \psi}{\partial z}. \tag{G.3}$$

The potential temperature is also governed by a 2D-like equation

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) + \frac{T_0 N^2}{g} w = 0, \tag{G.4}$$

where w is the vertical velocity field.

In SQG the potential vorticity ξ is assumed, a priori, to be identically zero for $z > 0$. The streamfunction ψ is solved from $\xi = \mathcal{P}\psi = 0$. With ρ_0 and N^2 taken to be constants, the horizontal Fourier transform of $\psi(x, y, z, t)$ is obtained as

$$\hat{\psi}(\mathbf{k}, z, t) = \hat{\psi}_0(\mathbf{k}, t) e^{-|\mathbf{k}|(N/f)z}, \tag{G.5}$$

using the boundedness boundary condition as $z \to \infty$.

Most of the dynamics in this model are occurring at the surface $z = 0$, where the boundary condition of vanishing vertical velocity w applied to the potential temperature (Θ) equation leads to:

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) = \mathcal{D} + \mathcal{F}. \tag{G.6}$$

Here we have also introduced thermal forcing and dissipation: $\mathcal{D} = \nu \Delta \Theta$ is the thermal diffusion, and

$$\mathcal{F} = Q = \alpha_E(\Theta_0 - \Theta), \tag{G.7}$$

is the thermal heating in the commonly used form of Newtonian cooling (see Ref. [253]) which includes a "forcing" term $\alpha_E \Theta_0$ and the "Ekman damping" term $-\alpha_E \Theta$. This equation is to be solved on a 2D surface $z = 0$. It has the same form as the vorticity equation for 2D turbulence (e.g. (**??**)), except that the spectral relationship between the advected quantity Θ and the advecting field ψ is given instead by

$$\hat{\Theta}(\mathbf{k}, z, t) = \frac{f}{N} \frac{\partial}{\partial z} \left(\hat{\psi}_0(\mathbf{k}, t) e^{-|\mathbf{k}|(N/f)z} \right) \tag{G.8}$$

$$= -|\mathbf{k}| [\hat{\psi}_0(\mathbf{k}, t) e^{-|\mathbf{k}|(N/f)z}] \tag{G.9}$$

$$= -|\mathbf{k}| \hat{\psi}_0(\mathbf{k}, z, t), \tag{G.10}$$

which reduces to $\hat{\Theta}(\mathbf{k}, t) = -|\mathbf{k}| \hat{\psi}(\mathbf{k}, t)$ at $z = 0$. Thus SQG corresponds to $L(k) = k$.

It has been shown by Charney [38], and more generally by Tung and Orlando [253], that the 3D QG energy density

$$\mathcal{E} \equiv \frac{1}{2} \rho_0 \left[|\nabla \psi|^2 + \frac{f^2}{N^2} \left(\frac{\partial \psi}{\partial z} \right)^2 \right], \tag{G.11}$$

is an invariant (i.e. independent of time), *when integrated over the 3D domain.* $\mathcal{E}$ is the sum of the kinetic energy density $\mathcal{E}_K$ and the potential energy density $\mathcal{E}_P$ which are given by

$$\mathcal{E}_K = (1/2)\rho_0 (u^2 + v^2) = (1/2)\rho_0 |\nabla \psi|^2 \tag{G.12}$$

$$\mathcal{E}_P = \frac{1}{2} \rho_0 \left(\frac{f}{N} \right)^2 \left(\frac{\partial \psi}{\partial z} \right)^2 = \frac{1}{2} \rho_0 \Theta^2. \tag{G.13}$$

For SQG, using Parseval's identity, the energies integrated over the horizontal surface are given by

$$E_P = \langle\langle \mathcal{E}_P \rangle\rangle = \frac{1}{2} \rho_0 \langle\langle \Theta^2 \rangle\rangle, \tag{G.14}$$

$$E_K = \langle\langle \mathcal{E}_K \rangle\rangle = \frac{1}{2} \rho_0 \langle\langle |\nabla \psi|^2 \rangle\rangle \tag{G.15}$$

$$= \frac{1}{2} \rho_0 \int \left\langle (i\mathbf{k}\hat{\psi}(\mathbf{k}, t)) \cdot (-i\mathbf{k}\hat{\psi}^*(\mathbf{k}, t)) \right\rangle \, d\mathbf{k} \tag{G.16}$$

$$= \frac{1}{2} \rho_0 \int \left\langle |-k\hat{\psi}(\mathbf{k}, t)|^2 \right\rangle \, d\mathbf{k} = \frac{1}{2} \rho_0 \langle\langle \Theta^2 \rangle\rangle = E_P. \tag{G.17}$$

It is thus seen that the kinetic energy density and the available potential energy density, when integrated horizontally, are equipartioned, and that

$$2B \equiv \langle\langle \Theta^2 \rangle\rangle = (E_P + E_K)/\rho_0 = E/\rho_0, \tag{G.18}$$

is the total energy at the lower surface. The 3D energy is, instead

$$\begin{aligned}
E_{3D} &\equiv \int_0^\infty \langle\langle \mathcal{E} \rangle\rangle dz = \int_0^\infty \rho_0 dz \langle\langle \Theta^2 \rangle\rangle \\
&= \int_0^\infty \rho_0 dz \iint dk_x dk_y \left\langle \left| \hat{\Theta}|_{z=0} \right|^2 \right\rangle e^{-2|\mathbf{k}|(N/f)z} \\
&= \frac{1}{2}\rho_0 \iint dk_x dk_y \frac{f}{N|\mathbf{k}|} \left\langle \hat{\Theta}\hat{\Theta}^* \Big|_{z=0} \right\rangle \\
&= \frac{1}{2}\rho_0 \iint \frac{f}{N} \left\langle -\hat{\psi}\hat{\Theta}^*|_{z=0} \right\rangle dk_x dk_y \\
&= \frac{1}{2}\rho_0 \frac{f}{N} \langle\langle (-\psi\Theta|_{z=0}) \rangle\rangle = \rho_0 (f/N) A,
\end{aligned}$$

with A defined earlier as $A \equiv (1/2)\langle\langle(-\psi\Theta)\rangle\rangle$. Previous authors have made use of the similarity between the form of vorticity equation in 2D turbulence and the temperature equation in SQG turbulence to identify, by analogy, A as the "energy" and B as the "enstrophy" [116, 204]. As pointed out by Tung and Orlando [253] and also here, $2B$ is the total energy integrated over the lower surface, and includes kinetic plus available potential energy. The physical interpretation for A was not given, but can now be seen to be the total energy integrated over the 3D domain. There is no potential enstrophy ($\xi^2/2$) per se in SQG turbulence, because potential vorticity ξ has been taken to be zero identically.

VITA

Eleftherios Gkioulekas was born on 1974 in Thessaloniki, Greece and grew up in Stratoni, Chalkidiki, Greece. He learned to love mathematics from his high-school teacher Alexandros Pistofides, and to love physics from reading *The Feynman Lectures on Physics*. He earned his Bachelor of Science degree in Applied Mathematics in 1997 from the California Institute of Technology. At the University of Washington he earned his Master of Science degree in Applied Mathematics in 2000 and his Doctor of Philosophy degree in Applied Mathematics in 2006.