P2.18 ON THE DOUBLE CASCADES OF ENERGY AND ENSTROPHY IN QG TURBULENCE [1]

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1. INTRODUCTION

According to Kraichnan (1967), the study of two-dimensional turbulence was motivated by the hope that it would prove a useful model for atmospheric turbulence. This idea was later encouraged by Charney (1971) who claimed that quasi-geostrophic turbulence is isomorphic to two-dimensional turbulence. The question that was then posed was whether the energy spectrum of the atmosphere at length scales that are orders of magnitude larger than the depth of the atmosphere can be explained in terms of the theory of two-dimensional turbulence. This question continues to be debated today.

Early observations suggested that the energy spectrum of the atmosphere follows a k^{-3} power law behavior (see Tung & Orlando (2003*b*) for review) consistent with an enstrophy cascade. An analysis of wind and temperature measurements taken during the Global Atmospheric Sampling Program by Nastrom & Gage (1984) showed that there is a robust k^{-3} spectrum extending from approximately 3,000 km to 1,000 km in wavelength (the "synoptic scales") and a robust $k^{-5/3}$ spectrum extending from 600 km down to a few kilometers (the "mesoscales"). A theoretical analysis by Gage & Nastrom (1986) showed that the observed spectrum indeed represents quasi-two-dimensional turbulence. Recent measurements (Cho, Newell & Barrick, 1999*a*; Cho, Zhu, Newell, Anderson, Barrick, Gregory, Sachse, Carroll & Albercook, 1999*b*; Marenco, Thouret, Nedelec, Smit, Helten, Kley, Karcher, Simon, Law, Pyle, Poschmann, Wrede, Hume & Cook, 1998) have confirmed the $k^{-5/3}$ part of the atmospheric energy spectrum, and it has also been reproduced in General Circulation Model simulations (Koshyk & Hamilton, 2001; Koshyk, Hamilton & Mahlman, 1999).

2. THE DOUBLE CASCADE THEORY

It was conjectured by Tung & Orlando (2003*a*) that the observed atmospheric energy spectrum results from the downscale cascade of enstrophy and energy injected at the large scales by baroclinic instability and dissipated at the smallest length scales. If η_{uv} is the downscale enstrophy flux and ε_{uv} is the downscale energy flux, it was suggested that the transition from -3 slope to -5/3 slope occurs at the transition wavenumber k_t with order of magnitude estimated by $k_t = \sqrt{\eta_{uv}/\varepsilon_{uv}}$. Tung & Orlando (2003*a*) have also demonstrated numerically that a two-layer quasi-geostrophic channel model with thermal forcing, Ekman damping, and hyperdiffusion can reproduce the atmospheric energy spectrum. The diagnostic shown in figure 7 of Tung & Orlando (2003*a*), clearly shows both the constant downscale energy and enstrophy fluxes coexisting in the same inertial range. Furthermore, recent measurements and data analysis by Cho & Lindborg (2001) have confirmed the existence of a downscale energy flux and estimate $\eta_{uv} \approx 2 \times 10^{-15}\mathrm{s}^{-3}$ and $\varepsilon_{uv} \approx 6 \times 10^{-11}\mathrm{km}^2\mathrm{s}^{-3}$. From these estimates we find the mean value of the transition scale $k_t = \sqrt{\eta_{uv}/\varepsilon_{uv}} \approx 0.57 \times 10^{-2}\mathrm{km}^{-1}$ and $\lambda_t = 2\pi/k_t \approx 1 \times 10^3\mathrm{km}$ which has the correct order of magnitude.

This theory is contrary to the widely accepted misconception that the argument by Fjørtøft (1953) forbids a downscale energy flux in two-dimensional turbulence, and through the isomorphism theorem of Charney (1971) also in quasi-geostrophic turbulence. This misconception has been clarified by Merilees & Warn (1975) and Tung & Welch (2001). In fact, as has been pointed out by previous authors (Borue, 1994; Eyink, 1996), as long as the dissipation terms at large-scale and small scales have finite viscosity coefficients and the inertial ranges exist, the downscale enstrophy flux will be accompanied by a small downscale energy flux, and the upscale energy flux will be accompanied by a small upscale enstrophy flux.

3. SUPERPOSITION PRINCIPLE

Gkioulekas & Tung (2005*a*) have shown that these small fluxes are associated with a subleading downscale energy cascade and a subleading inverse enstrophy cascade that contribute linearly to the total energy spectrum in addition to the contributions from the dominant cascades. As a result, in the downscale inertial range, the total energy spectrum $E(k)$ has the following three contributions:

$$E(k) = E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k) + E_{uv}^{(p)}(k),\ \forall k\ell_0 \gg 1, \quad (1)$$

where $E_{uv}^{(\varepsilon)}(k)$, $E_{uv}^{(\eta)}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E_{uv}^{(\varepsilon)}(k) &= a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3}\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)}) \\ E_{uv}^{(\eta)}(k) &= b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}), \end{aligned} \quad (2)$$

with $\mathcal{D}_{uv}^{(\varepsilon)}$ and $\mathcal{D}_{uv}^{(\eta)}$ describing the dissipative corrections. The scales $\ell_{uv}^{(\varepsilon)}$,$\ell_{uv}^{(\eta)}$ are the dissipation length scales for the downscale energy and enstrophy cascade. Finally, $E_{uv}^{(p)}(k)$ is the contribution from the effect of forcing and the sweeping interactions. The latter can become significant via the violation of statistical homogeneity caused by the boundary conditions (see Gkioulekas & Tung (2005*a*) for details).

[1] Extended abstract presented at the 15th Conference on Atmospheric and Oceanic Fluid Dynamics of the American Meteorological Society, Cambridge Massachussets, June 2005

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Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrum will take the simple form

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3} + b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}. \quad (3)$$

It should be emphasized that the formation of cascades observable in the energy spectrum is by no means guaranteed. There are two prerequisites that need to be satisfied: first, the contribution of the particular solution $E_{uv}^{(p)}(k)$ has to be negligible both downscale and upscale of the injection scale, i.e.

$$\begin{aligned} E_{uv}^{(p)}(k) &\ll E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k),\ \forall k\ell_0 \gg 1 \\ E_{ir}^{(p)}(k) &\ll E_{ir}^{(\varepsilon)}(k) + E_{ir}^{(\eta)}(k),\ \forall k\ell_0 \ll 1. \end{aligned} \quad (4)$$

Second, the dissipative adjustment $\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)})$ and $\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)})$ of the homogeneous solutions has to be such that it does not destroy the power law scaling in the inertial range. Furthermore, the dissipation scales $\ell_{uv}^{(\eta)}$ and $\ell_{uv}^{(\varepsilon)}$ have to be positioned so that the incoming energy and enstrophy can be dissipated.

This principle of linear superposition of the enstrophy cascade and the energy cascade is similar to the superposition of isotropic and anisotropic contributions to the generalized structure functions (Arad, L'vov & Procaccia, 1999). Furthermore, a similar principle of superposition has been proposed to explain the mechanism behind intermittency corrections in the direct energy cascade of three-dimensional turbulence (Belinicher, L'vov, Pomyalov & Procaccia, 1998*a*; Belinicher, L'vov & Procaccia, 1998*b*; L'vov & Procaccia, 1998); the same idea is implicit in the multi-fractal model of Frisch (1995)

4. DANILOV INEQUALITY

In two-dimensional turbulence, the energy flux $\Pi_E(k)$ and the enstrophy flux $\Pi_G(k)$ are constrained by

$$k^2\Pi_E(k) - \Pi_G(k) < 0, \quad (5)$$

for all wavenumbers outside of the forcing range. This inequality was communicated to us by Danilov (Gkioulekas & Tung, 2005*b*) and it implies that the contribution of the downscale energy cascade to the energy spectrum is overwhelmed by the contribution of the downscale enstrophy cascade and cannot be seen visually on a plot. This result was conjectured earlier by Smith (2004) who debated the theory of Tung & Orlando (2003*a*) by arguing that the downscale energy cascade can never have enough flux to move the transition wavenumber k_t into the inertial range. The obvious counterargument is that the two-layer model is a different dynamical system than the two-dimensional Navier-Stokes equations, and although the superposition principle is a deep mathematical result that is valid in both cases, it is not obvious that the Danilov inequality cannot be violated in the two-layer model (Gkioulekas & Tung, 2005*b*; Tung, 2004).

In the two-layer model forcing is due to thermal heating, which injects energy directly into the baroclinic part of the total energy. The two-layer fluid sits atop of an Ekman boundary layer near the ground, which introduces Ekman damping in the lower layer (Holton, 1979) but *not* in the upper layer. Following Salmon (1998), one may then derive the governing equations for the model, which read:

$$\frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1) = d_1 + f_1 \quad (6)$$

$$\frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2) = d_2 + f_2, \quad (7)$$

where ζ_1 is the potential vorticity of the top layer and ζ_2 the potential vorticity of the bottom layer. The relationship between the vorticities ζ_1 and ζ_2 and the streamfunctions ψ_1 and ψ_2 reads:

$$\zeta_1 = \Delta\psi_1 - \frac{k_R^2}{2}(\psi_1 - \psi_2), \quad (8)$$

$$\zeta_2 = \Delta\psi_2 + \frac{k_R^2}{2}(\psi_1 - \psi_2), \quad (9)$$

Here, k_R is the Rossby radius of deformation wavenumber and it is taken as a given constant. The dissipation terms d_1 and d_2 include momentum dissipation of relative vorticity, $\Delta\psi_i$, in each layer, and Ekman damping from the lower boundary layer, and they read:

$$d_1 = \nu(-\Delta)^{p+1}\psi_1, \quad (10)$$

$$d_2 = \nu(-\Delta)^{p+1}\psi_2 - \nu_E\Delta\psi_2. \quad (11)$$

The two inviscid quadratic invariants are the total energy E and potential enstrophy G, defined as

$$E \equiv \frac{1}{2}\iint -(\psi_1\zeta_1 + \psi_2\zeta_2)\, dxdy \quad (12)$$

$$G \equiv \frac{1}{2}\iint (\zeta_1^2 + \zeta_2^2)\, dxdy \quad (13)$$

Tung & Gkioulekas (2005) have shown that it is the asymmetric presence of Ekman damping on the bottom layer but not the top layer which causes the violation of the Danilov inequality in the two-layer model. As a result, the top layer has more energy than the bottom layer, as is realistic in the atmosphere, and provided that the difference in energy between the two layers is large enough, the downscale energy cascade will be observable in the energy spectrum. It should be noted that the simulation of Tung & Orlando (2003*a*) has already shown quite convincingly that it is possible to have an observable downscale energy cascade. The only issue that required clarification was to understand why it happens in the two-layer model but not in two-dimensional turbulence.

5. CONCLUSION

We have shown that for the case of finite Reynolds number, the enstrophy cascade of two-dimensional turbulence is accompanied with a hidden downscale energy cascade. Both cascades provide contributions to the energy spectrum that combine linearly, despite the nonlinearity of the governing Navier-Stokes equations. A mathematical constraint prevents the contribution of the downscale energy cascade from becoming dominant in the inertial range. However, in a two-layer model, this mathematical constraint is violated. The violation is caused by the baroclinicity induced by the presence of Ekman damping on the bottom layer but not the top layer.

References

Arad, I., L'vov, V. & Procaccia, I. 1999 Correlation functions in isotropic and anisotropic turbulence: The role of the symmetry group. *Phys. Rev. E* **59**, 6753–6765.

Belinicher, V., L'vov, V., Pomyalov, A. & Procaccia, I. 1998*a* Computing the scaling exponents in fluid turbulence from first principles: Demonstration of multiscaling. *J. Stat. Phys.* **93**, 797–832.

Belinicher, V., L'vov, V. & Procaccia, I. 1998*b* A new approach to computing the scaling exponents in fluid turbulence from first principles. *Physica A* **254**, 215–230.

Borue, V. 1994 Inverse energy cascade in stationary two dimensional homogeneous turbulence. *Phys. Rev. Lett.* **72**, 1475–1478.

Charney, J. 1971 Geostrophic turbulence. *J. Atmos. Sci.* **28**, 1087–1095.

Cho, J. & Lindborg, E. 2001 Horizontal velocity structure functions in the upper troposphere and lower stratosphere. 1. Observations. *J. Geophys. Res.* **106 D10**, 10,223–10,232.

Cho, J., Newell, R. & Barrick, J. 1999*a* Horizontal wavenumber spectra of winds, temperature, and trace gases during the Pacific Exploratory Missions: 2. Gravity waves, quasi-two-dimensional turbulence, and vortical modes. *J. Geophys. Res.* **104 D13**, 16297–16308.

Cho, J., Zhu, Y., Newell, R., Anderson, B., Barrick, J., Gregory, G., Sachse, G., Carroll, M. & Albercook, G. 1999*b* Horizontal wavenumber spectra of winds, temperature, and trace gases during the Pacific Exploratory Missions: 1. Climatology. *J. Geophys. Res.* **104 D5**, 5697–5716.

Eyink, G. 1996 Exact results on stationary turbulence in two dimensions: Consequences of vorticity conservation. *Physica D* **91**, 97–142.

Fjørtøft, R. 1953 On the changes in the spectral distribution of kinetic energy for two dimensional non-divergent flow. *Tellus* **5**, 225–230.

Frisch, U. 1995 *Turbulence: The legacy of A.N. Kolmogorov*. Cambridge: Cambridge University Press.

Gage, K. & Nastrom, G. 1986 Theoretical interpretation of atmospheric wavenumber spectra of wind and temperature observed by commercial aircraft during GASP. *J. Atmos. Sci.* **43**, 729–740.

Gkioulekas, E. & Tung, K. 2005*a* On the double cascades of energy and enstrophy in two dimensional turbulence. Part 1. Theoretical formulation. *Discrete and Continuous Dynamical Systems B* **5**, 79–102.

Gkioulekas, E. & Tung, K. 2005*b* On the double cascades of energy and enstrophy in two dimensional turbulence. Part 2. Approach to the KLB limit and interpretation of experimental evidence. *Discrete and Continuous Dynamical Systems B* **5**, 103–124.

Holton, J. R. 1979 *An introduction to dynamic meteorology*. San Diego: Academic Press.

Koshyk, J. & Hamilton, K. 2001 The horizontal kinetic energy spectrum and spectral budget simulated by a high-resolution troposphere-stratosphere mesosphere GCM. *J. Atmos. Sci.* **58**, 329–348.

Koshyk, J., Hamilton, K. & Mahlman, J. 1999 Simulation of the $k^{-5/3}$ mesoscale spectral regime in the GFDL SKYHI general circulation model. *Geophys. Res. Lett.* **26**, 843–846.

Kraichnan, R. 1967 Inertial ranges in two dimensional turbulence. *Phys. Fluids* **10**, 1417–1423.

L'vov, V. & Procaccia, I. 1998 Computing the scaling exponents in fluid turbulence from first principles: The formal setup. *Physica A* **257**, 165–196.

Marenco, A., Thouret, V., Nedelec, P., Smit, H., Helten, M., Kley, D., Karcher, F., Simon, P., Law, K., Pyle, J., Poschmann, G., Wrede, R., Hume, C. & Cook, T. 1998 Measurement of ozone and water vapor by airbus in-service aircraft: The MOZAIC airborne program an overview. *J. Geophys. Res.* **103 D19**, 25631–25642.

Merilees, P. & Warn, T. 1975 On energy and enstrophy exchanges in two-dimensional non-divergent flow. *J. Fluid. Mech.* **69**, 625–630.

Nastrom, G. & Gage, K. 1984 A climatology of atmospheric wave number spectra of wind and temperature observed by commercial aircraft. *J. Atmos. Sci.* **42**, 950–960.

Salmon, R. 1998 *Lectures on Geophysical Fluid Dynamics*. New York: Oxford University Press.

Smith, K. 2004 Comment on: The k^{-3} and $k^{-5/3}$ energy spectrum of the atmospheric turbulence: quasi-geostrophic two level model simulation. *J. Atmos. Sci.* **61**, 937–942.

Tung, K. 2004 Reply to comments by K. Shafer Smith. *J. Atmos. Sci.* **61**, 943–948.

Tung, K. & Gkioulekas, E. 2005 An inequality between fluxes of energy and enstrophy in 2d and qg turbulence. Submitted to J. Fluid Mech.

Tung, K. & Orlando, W. 2003*a* The k^{-3} and $k^{-5/3}$ energy spectrum of the atmospheric turbulence: quasi-geostrophic two level model simulation. *J. Atmos. Sci.* **60**, 824–835.

Tung, K. & Orlando, W. 2003*b* On the differences between 2d and qg turbulence. *Discrete and Continuous Dynamical Systems B* **3**, 145–162.

Tung, K. & Welch, W. 2001 Remarks on note on geostrophic turbulence. *J. Atmos. Sci.* **58**, 2009–2012.