DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **5**, Number **1**, February **2005**

Website: http://AIMsciences.org

pp. **103–124**

ON THE DOUBLE CASCADES OF ENERGY AND ENSTROPHY IN TWO-DIMENSIONAL TURBULENCE. PART 2. APPROACH TO THE KLB LIMIT AND INTERPRETATION OF EXPERIMENTAL EVIDENCE

Eleftherios Gkioulekas

Department of Applied Mathematics
Box 352420
University of Washington
Seattle, WA, 98195-2420, USA

Ka Kit Tung

Department of Applied Mathematics
Box 352420
University of Washington
Seattle, WA, 98195-2420, USA

Abstract. This paper is concerned with three interrelated issues on our proposal of double cascades intended to serve as a more realistic theory of two-dimensional turbulence. We begin by examining the approach to the KLB limit. We present improved proofs of the result by Fjortoft. We also explain why in that limit the subleading downscale energy cascade and upscale enstrophy cascade are hidden in the energy spectrum. Then we review the experimental evidence from numerical simulations concerning the realizability of the energy and enstrophy cascade. The inverse energy cascade is found to be affected by the presense of a particular solution, and the downscale enstrophy cascade forms only under certain configurations of the dissipative terms. Finally, we amplify the hypothesis that the energy spectrum of the atmosphere reflects a combined downscale cascade of energy and enstrophy. The possibility of the downscale helicity cascade is also considered.

1. **Introduction.** This is the second paper in a series of papers. The goal of this series is to introduce a theoretical framework for the inertial ranges of two-dimensional turbulence with infrared dissipation and finite viscosities. The standard KLB framework [2, 48, 51] is applicable in an unbounded domain without infrared dissipation under the limit $\nu \to 0$. The scenario suggested by KLB is that there is an upscale energy cascade and a downscale enstrophy cascade. Both cascades are pure; there is no downscale energy flux and there is no upscale enstrophy flux. Assuming locality, the purity of the cascades allows the use of dimensional analysis to predict the slope of the energy spectrum for each cascade. The upscale energy cascade is therefore expected to scale as $k^{-5/3}$, and the downscale enstrophy cascade as k^{-3} as a function of the local wavenumber k. In its traditional form, the KLB scenario requires an unbounded domain to allow the upscale energy flux to avoid the need for infrared dissipation by escaping to larger and larger length scales. A

2000 *Mathematics Subject Classification.* 76FXX,60G60.
Key words and phrases. energy, enstrophy, 2d turbulence, QG turbulence.

number of recent theoretical results [83–85] challenge the realizability of cascades as envisaged by Kraichnan for the standard case of Navier-Stokes without an infrared sink but in a bounded domain.

In the more realistic case of a finite domain, a dissipative sink is needed both at large scales and at small scales, in order for cascades to form. Inevitably, the dissipation sink at small scales will dissipate some energy, and the dissipation sink at large scales will dissipate some enstrophy. The presence of both energy and enstrophy flux on either side of the injection scale means that we can no longer predict the shape of the energy spectrum using dimensional analysis.

In the preceding paper [40] we have introduced a statistical theory for this scenario, based on a similar non-perturbative theory introduced by L'vov and Procaccia [56–60] to explain the energy cascade of three-dimensional turbulence. We have shown that as long as the infinite set of balance equations of the generalized structure functions is not truncated by closure approximations, it remains linear and admits two homogeneous solutions, corresponding to the energy and enstrophy cascades, and a particular solution raised by the forcing term and the boundary conditions. The energy spectrum, both downscale and upscale of the injection wavenumber, admits contributions from the energy and enstrophy cascades combined linearly, and a contribution from the particular solution that accounts for the forcing range. They can be written as:

$$\begin{aligned} E(k) &= E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k) + E_{uv}^{(p)}(k), \ \forall k\ell_0 \gg 1 \\ E(k) &= E_{ir}^{(\varepsilon)}(k) + E_{ir}^{(\eta)}(k) + E_{ir}^{(p)}(k), \ \forall k\ell_0 \ll 1, \end{aligned} \tag{1.1}$$

where $E_{uv}^{(\varepsilon)}(k)$ and $E_{ir}^{(\varepsilon)}(k)$ are expected to scale as $k^{-5/3}$, and $E_{uv}^{(\eta)}(k)$ and $E_{ir}^{(\eta)}(k)$ as k^{-3}. These terms are the contribution of the homogeneous solutions to the energy spectrum. The contribution of the particular solution is $E_{uv}^{(p)}(k)$ and $E_{ir}^{(p)}(k)$. The dissipation terms of the Navier-Stokes equations, corresponding to molecular diffusion or hyperdiffusion and Ekman damping or hypodiffusion, modify the linear operator of the statistical theory, and, in so doing, modify the corresponding homogeneous solutions by truncating the power law scaling and replacing part of the inertial range with dissipation ranges governed by exponential scaling. The form of these terms is summarized in the preceding paper [40].

It should be emphasized that the formation of cascades observable in the energy spectrum is by no means guaranteed. There are two prerequisites that need to be satisfied: first, the contribution of the particular solution has to be negligible both downscale and upscale of the injection scale, i.e.

$$\begin{aligned} E_{uv}^{(p)}(k) &\ll E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k), \ \forall k\ell_0 \gg 1 \\ E_{ir}^{(p)}(k) &\ll E_{ir}^{(\varepsilon)}(k) + E_{ir}^{(\eta)}(k), \ \forall k\ell_0 \ll 1. \end{aligned} \tag{1.2}$$

If this condition is satisfied, then we say that the corresponding inertial range is *structurally stable*. Second, the dissipative adjustment of the homogeneous solution has to be such that it does not destroy the power law scaling in the inertial range. Furthermore, the dissipation scales have to be positioned so that the incoming energy and enstrophy can be dissipated.

A careful development of our theory promises to tell us when these prerequisites are satisfied. Meanwhile, it is possible to determine experimentally which of the two prerequisites fails when there are departures from universal scaling. In a numerical simulation where the dissipation operators are artificially localized to act only in the dissipation range, there will be no dissipative adjustment to the homogeneous solution in the spectral region where the dissipation operators are suppressed. If this measure restores universal scaling, then the failure of universality, in the case where dissipation is not localized, should be attributed to dissipative adjustment of the homogeneous solution. If universal scaling is not restored, then it is the contribution of the particular solution that is responsible for changing the slope of the energy spectrum.

The goal of this paper is to examine the following three interrelated issues: First, we show how our theory is reconciled with the KLB model in the limit of large Reynolds number. We refer to this limit as the KLB limit.[1] Second, we review the accumulated theoretical and experimental studies of the inverse energy cascade and the direct enstrophy cascade to highlight the issues that need to be addressed by our theory. Third, we explain why the idea of a downscale double cascade of energy and enstrophy is essential in explaining the energy spectrum of the atmosphere.

The plan of this paper is as follows. In section 2, we consider in detail what happens when the KLB limit is approached. We begin, in sections 2.1, 2.2 with two different proofs that the leading downscale cascade is the enstrophy cascade and the leading upscale cascade is the energy cascade. In section 2.3, we discuss the subleading cascades. We show that for the case of two-dimensional turbulence, the subleading cascades will be hidden in the energy spectrum. However, although the percentage of the energy and enstrophy flux associated with the subleading cascades vanishes very fast with increasing Reynolds numbers, the separation of scales of the subleading cascades will be proportional to the separation of scales of the leading cascades.

In section 3, we review the theoretical and experimental evidence both in support and against the realizability of a direct enstrophy cascade and inverse energy cascade with scaling consistent with the predictions of dimensional analysis. In section 3.1, we discuss in detail the paradox of the inverse energy cascade discovered by Danilov and Gurarie [23]. We show that the experimental evidence suggests that the inverse energy cascade can be dominated by the particular solution, which dominates and hides the $k^{-5/3}$ contribution of the inverse energy cascade due to the finiteness of the boundary. In section 3.2 we highlight that the direct enstrophy cascade can be realized with universal scaling in experiments that use hypodiffusion, but not in experiments that use Ekman damping, where a small dissipative correction to the slope of the energy spectrum is expected. A related finding was previously given by Tran and Shepherd [85] who argued that the k^{-3} scaling of the energy spectrum downscale of injection depends on the use of hypodiffusion as the infrared sink.

In section 4, we assemble the evidence, in light of the theory presented in this paper, in support of the theory by Tung and Orlando [88, 89], that the atmospheric energy spectrum corresponds to a double cascade of energy and enstrophy, both of which are being injected by baroclinic instability at large scales and dissipated at small scales. The bulk of this discussion is given in section 4.1. In section 4.2, we argue that although the nature of the $k^{-5/3}$ spectrum has been clarified, it is the nature of the approximately k^{-3} part that remains controversial. Section 5 concludes the paper.

[1] When we talk about approaching the KLB limit, we mean that there is a very large separation between the injection scale and the dissipation scale in one of the two inertial ranges, either upscale or downscale of injection. In that case, for that particular range, the leading cascade dominates the energy spectrum and the subleading cascade contributes a negligible correction.

2. Approach to the KLB limit.

The misconception that no energy can flow downscale in two-dimensional turbulence is often explained in terms of the celebrated "proof" by Fjortoft [36], that has been copied in many textbooks and review articles, including the recent article by Tabeling [82]. In light of Charney's [12] claim that quasi-geostrophic turbulence is isomorphic to two-dimensional turbulence, this misconception has been carried over to the fundamentally different problem of the energy spectrum of the atmosphere. The goal of this section is to clarify this issue, from the viewpoint of two-dimensional turbulence, and show how our theory is reconciled with the KLB limit.

The fallacy behind Fjortoft's proof has been exposed by Tung and Orlando [89, 90]. Briefly, Fjortoft's argument uses the structure of the quadratic term of the Navier-Stokes equations but is independent of the dissipation terms. The latter play no role in his proof. The problem is that without the dissipation terms, the governing equation is time reversible. For every solution where the energy flows upscale, there exists another solution where it flows downscale, that can be obtained by reversing time's arrow.

The only way to fashion an argument that can select the direction for the fluxes, *without* involving the dissipation terms, is by making intuitively plausible but unproven assumptions. For example, Kraichnan [48] "proves"[2] the direction of fluxes by comparing the slopes of inertial range solutions with the slope of the absolute thermodynamic equilibrium solution. This amounts to choosing the direction of time's arrow by *assuming* a tendency to move towards thermodynamic equilibrium. Another argument by Rhines [74], similarly *assumes* that an amount of energy concentrated around some wave number must have a tendency to spread out. These arguments do not explain *why* it is only these solutions that are more likely to be realized instead of their reversed counterparts.

Despite these criticisms, the claim that *most* of the enstrophy will be dissipated at small scales and *most* of the energy at large scales is correct for the case of two-dimensional turbulence. We present, in this section, two proofs: the first proof is based on a folklore argument by Eyink [28]; the second proof is based on an observation that was communicated to us by Danilov [20]. However, it is also the case that there will exist a small downscale energy flux and a small upscale enstrophy flux accompanying the leading cascades. We show that even though these subleading fluxes decrease rapidly when the length of the corresponding inertial ranges increases, there will always exist a sufficient amount to form subleading cascades with length proportional to the length of the leading cascades.

2.1. Direction of fluxes: First proof.

Let $\varepsilon_{uv}, \eta_{uv}$ be the energy and enstrophy dissipated at small scales and $\varepsilon_{ir}, \eta_{ir}$ the energy and enstrophy dissipated at large scales. In the stationary case, they satisfy

$$\begin{aligned} \varepsilon &= \varepsilon_{ir} + \varepsilon_{uv} \\ \eta &= \eta_{ir} + \eta_{uv}, \end{aligned} \tag{2.1}$$

[2] In his paper, Kraichnan does not intend this argument as a rigorous proof, even though this is how the argument is often misinterpreted. It is intended only as a heuristic explanation. In fact, Kraichnan himself stresses, in the first paragraph of section 3 of his paper, that there is nothing in the conservation laws themselves (derived from the structure of the non-linear interaction term in the Euler equation) that can determine the direction of the fluxes. We are making the same argument: we claim that involving the dissipation terms is a necessary requirement for a correct proof.

where ε is the rate of energy input and η is the rate of enstrophy input. It should emphasized that these variables are defined as dissipation rates, and they are equal to fluxes only when a cascade forms successfully in the corresponding range. We define the forcing scale from the relation $\varepsilon = \eta\ell_0^2$. Note that Eyink also defines[3] *dissipation rate ratios* λ_{ir} and λ_{uv} from the relations $\varepsilon_{ir} = \eta_{ir}\lambda_{ir}^2$ and $\varepsilon_{uv} = \eta_{uv}\lambda_{uv}^2$. These can be interpreted as flux ratios as long as the corresponding double cascade exists. For the remainder of this section we will be calling these quantities "flux ratios".

We obtain the following system of equations

$$\begin{aligned} \eta &= \eta_{ir} + \eta_{uv} \\ \eta\ell_0^2 &= \eta_{ir}\lambda_{ir}^2 + \eta_{uv}\lambda_{uv}^2, \end{aligned} \tag{2.2}$$

and the solution reads

$$\eta_{ir} = \frac{\eta(\ell_0^2 - \lambda_{uv}^2)}{\lambda_{ir}^2 - \lambda_{uv}^2} \quad \varepsilon_{ir} = \frac{\eta\lambda_{ir}^2(\ell_0^2 - \lambda_{uv}^2)}{\lambda_{ir}^2 - \lambda_{uv}^2} \tag{2.3}$$

$$\eta_{uv} = \frac{\eta(\lambda_{ir}^2 - \ell_0^2)}{\lambda_{ir}^2 - \lambda_{uv}^2} \quad \varepsilon_{uv} = \frac{\eta\lambda_{uv}^2(\lambda_{ir}^2 - \ell_0^2)}{\lambda_{ir}^2 - \lambda_{uv}^2}. \tag{2.4}$$

The main difficulty that prevents concluding the argument is that there is no way to calculate the flux ratios λ_{ir} and λ_{uv} independently of their definition.

We propose that the argument can be carried forward in the following way. First, in order for the system to reach a steady state, it is necessary for both large-scale and small-scale dissipation terms to act. It is not possible for either term, by itself, to dissipate together any arbitrary rate of energy and enstrophy injection. We may therefore formulate two alternative theories, only one of which is self-consistent: one in which there is an upscale energy cascade and a downscale enstrophy cascade (what really happens), or one in which there is an upscale enstrophy cascade and a downscale energy cascade. The form of both theories can range from a simple dimensional analysis argument, to a more elaborate theory. The essential point is that both theories, having been formulated, will predict that with increasing Reynolds number the separation of scales in the dual cascade will increase.

According to the argument given later in section 2.3, in the same limit the flux ratios λ_{ir} and λ_{uv} are approximately equal to the corresponding dissipation scales of the dominant cascades. It is true that the argument of section 2.3, makes explicit use of the assumption that the leading downscale cascade is the enstrophy cascade. However, if this assumption is replaced with the assumption that the leading downscale cascade is the energy cascade, and repeat the argument, we will still conclude that the flux ratios still asymptotically coincide with the dissipation scales.

It follows that for large Reynolds numbers we expect the flux ratios to satisfy

$$\lambda_{uv} \ll \ell_0 \ll \lambda_{ir}. \tag{2.5}$$

We stress that we do not need to know in advance the actual direction of the fluxes to establish this inequality. The inequality follows *both* from the realistic theory *and* from the theory which we will reject.

[3]Eyink defines the ratios λ_{uv} and λ_{ir} as dissipation scales, and not as flux ratios or dissipation rate ratios. By definition, these quantities are dissipation rate ratios. In the event of the successful formation of the corresponding double cascade, they will also be flux ratios and, as will be shown later, double as transition scales. As the KLB limit is approached, they will converge to the dissipation scale of the leading cascade. Only then is it valid to interpret them as dissipation scales.

We now show, using this inequality, why the theory of the leading upscale enstrophy cascade and downscale energy cascade is inconsistent. First, note that

$$\begin{aligned}\varepsilon_{uv} \ll \varepsilon_{ir} &\Longleftrightarrow \lambda_{uv}^2(\lambda_{ir}^2 - \ell_0^2) \ll \lambda_{ir}^2(\ell_0^2 - \lambda_{uv}^2) \\ &\Longleftrightarrow 2\lambda_{ir}^2\lambda_{uv}^2 \ll \ell_0^2(\lambda_{ir}^2 + \lambda_{uv}^2) \\ &\Longleftrightarrow 2\lambda_{ir}^2\lambda_{uv}^2 \ll \ell_0^2\lambda_{ir}^2 \Longleftrightarrow \sqrt{2}\lambda_{uv} \ll \ell_0.\end{aligned} \tag{2.6}$$

The physical meaning of this equivalence is that the separation of scales between λ_{uv} and ℓ_0 acts as a "shield" that diverts most of the energy upscale. We write the mathematical steps in detail to stress that the inequality (2.5) is being used to go from the second line to the third line. It is also needed to eliminate the denominators. A similar argument can be provided for the enstrophy as follows:

$$\begin{aligned}\eta_{ir} \ll \eta_{uv} &\Longleftrightarrow \ell_0^2 - \lambda_{uv}^2 \ll \lambda_{ir}^2 - \ell_0^2 \\ &\Longleftrightarrow 2\ell_0^2 \ll \lambda_{ir}^2 + \lambda_{uv}^2 \Longleftrightarrow \sqrt{2}\ell_0 \ll \lambda_{ir}.\end{aligned} \tag{2.7}$$

This shows that the separation of scales between λ_{ir} and ℓ_0 acts as a shield that diverts most of the enstrophy downscale.

The argument that we have outlined in this section is still valid for the case where only one of the two inertial ranges forms successfully. All that is required is that $\lambda_{uv}, \lambda_{ir}$ should satisfy the inequality $\lambda_{uv} \ll \lambda_{ir}$.

2.2. **Direction of fluxes: Second proof.** A different proof of the same claim can be obtained by using the following inequality satisfied by the energy and enstrophy flux both upscale and downscale of injection

$$k^2\varepsilon(k) - \eta(k) \leq 0. \tag{2.8}$$

This inequality was brought to our attention by Danilov [20]. To prove it, note that downscale of injection, in a stationary system, the energy and enstrophy flux are given by

$$\begin{aligned}\varepsilon(k) &= 2\nu \int_k^{+\infty} q^{2\kappa}E(q)dq + 2\beta \int_k^{+\infty} q^{-2m}E(q)dq \\ \eta(k) &= 2\nu \int_k^{+\infty} q^{2\kappa+2}E(q)dq + 2\beta \int_k^{+\infty} q^{-2m+2}E(q)dq,\end{aligned} \tag{2.9}$$

as long as the entire forcing spectrum is localized in the $[0, k]$ interval. These relations are an immediate consequence of the observation that all the energy and enstrophy dissipated at the interval $[k, +\infty)$ has to cross the wave number k to come from the $[0, k]$ interval where it is injected. It follows that

$$k^2\varepsilon(k) - \eta(k) = 2\nu \int_k^{\infty} (k^2 - q^2)q^{2\kappa}E(q)dq + 2\beta \int_k^{\infty} (k^2 - q^2)q^{-2m}E(q)dq \leq 0. \tag{2.10}$$

The same argument can be repeated when the wavenumber k is on the upscale side of injection. In that case, the energy flux and the enstrophy flux satisfy

$$\begin{aligned}\varepsilon(k) &= -2\nu \int_0^k q^{2\kappa}E(q)dq - 2\beta \int_0^k q^{-2m}E(q)dq \\ \eta(k) &= -2\nu \int_0^k q^{2\kappa+2}E(q)dq - 2\beta \int_0^k q^{-2m+2}E(q)dq,\end{aligned} \tag{2.11}$$

and the same inequality follows.

The connection between Danilov's inequality and proving the direction of the leading cascades lies in observing that the potential function $P(k)$, introduced in the preceding paper [40], satisfies the following identity

$$P(k) = \int_0^k 2q\varepsilon(q)dq = k^2\varepsilon(k) - \eta(k). \tag{2.12}$$

Combined with (2.8), it follows that

$$\int_0^k 2q\varepsilon(q)dq \le 0. \tag{2.13}$$

The immediate interpretation of this inequality is that in order for two-dimensional turbulence to reach a steady state, it is necessary that there is a spectral region with negative energy flux. A somewhat similar argument was given previously by Tseskis [86], in terms of the energy transfer rate. This condition requires the presence of an infrared sink. Alternatively, it would be necessary to achieve a state of absolute equilibrium where the energy flux is zero both upscale and downscale of injection.

Consider now the case where there are wavenumber intervals both upscale and downscale of injection where the energy flux is constant:

$$\begin{aligned} \varepsilon(k) &= \varepsilon_{uv}, \quad \forall k \in (k_0, k_{uv}] \\ \varepsilon(k) &= -\varepsilon_{ir}, \quad \forall k \in [k_{ir}, k_0). \end{aligned} \tag{2.14}$$

As discussed in the preceding paper [40], the enstrophy flux is constrained to be constant in the same intervals too. Furthermore, since we can presume that there exists infrared dissipation in the interval $[0, k_{ir}]$, we may safely assume that the energy flux satisfies

$$-\varepsilon_{ir} \le \varepsilon(k) \le 0, \quad \forall k \in [0, k_{ir}]. \tag{2.15}$$

Using these equations combined with Danilov's inequality (2.8), we find

$$\begin{aligned} P(k_{uv}) &= \int_0^{k_{uv}} 2q\varepsilon(q)dq \\ &= -\varepsilon_{ir}(k_0^2 - k_{ir}^2) + \varepsilon_{uv}(k_{uv}^2 - k_0^2) + \int_0^{k_{ir}} 2q\varepsilon(q)dq \le 0, \end{aligned} \tag{2.16}$$

and using (2.15), we have the inequality

$$\varepsilon_{uv}(k_{uv}^2 - k_0^2) - \varepsilon_{ir}k_0^2 \le P(k_{uv}) \le 0, \tag{2.17}$$

that can be rewritten as

$$\frac{\varepsilon_{uv}}{\varepsilon_{ir}} \le \frac{k_0^2}{k_{uv}^2 - k_0^2}. \tag{2.18}$$

For the upscale enstrophy flux η_{ir}, a similar inequality can be established. Choose a wave number k in the upscale range such that $k_{ir} < k < k_0$. The potential function $P(k)$ and its derivative read

$$\begin{aligned} P(k) &= -\varepsilon_{ir}(k^2 - k_{ir}^2) + \int_0^{k_{ir}} 2q\varepsilon(q)dq \\ \frac{\partial P(k)}{\partial k} &= -2\varepsilon_{ir}k. \end{aligned} \tag{2.19}$$

It follows that the upscale enstrophy flux η_{ir} will satisfy

$$\begin{aligned}\eta_{ir} &= -\eta(k) = -\frac{k}{2}\frac{\partial P(k)}{\partial k} + P(k) = \\ &= -\frac{k}{2}\left[-2\varepsilon_{ir}k\right] - \varepsilon_{ir}(k^2 - k_{ir}^2) + \int_0^{k_{ir}} 2q\varepsilon(q)dq \\ &= \varepsilon_{ir}k_{ir}^2 + \int_0^{k_{ir}} 2q\varepsilon(q)dq \leq \varepsilon k_{ir}^2 + \int_0^{k_{ir}} 2q\varepsilon(q)dq.\end{aligned} \tag{2.20}$$

To finalize the proof we employ the same argument as in the preceding section. The two possible theories involve either the case of leading downscale enstrophy cascade and leading upscale energy cascade, or the hypothetical case of leading downscale energy cascade or leading upscale enstrophy cascade. In both cases, it can be shown that when the rate of energy and enstrophy injection is increased, the dissipation wavenumbers go to the limits $k_{uv} \to +\infty$ and $k_{ir} \to 0$. Then we use the inequalities derived above to reject the hypothetical case.

Although both this proof and the preceding proof are mathematically equivalent, one advantage of the present proof is that it shows more clearly why it is necessary to have sinks both upscale and downscale of injection. Furthermore, it does not rely on establishing a connection between the flux ratios and the dissipation scales. It should be noted, however, that the dissipation wavenumbers k_{uv} and k_{ir} belong to the third order structure functions $S_3(r)$, whereas the flux ratios used in the previous proof are claimed to be equal to dissipation length scales that belong to $S_2(r)$. Finally, it is possible to formulate a third proof by making direct use of (2.8), as pointed out by Danilov [21].

2.3. **The subleading cascades.** Is it possible to see the subleading cascades? The answer is that in numerical simulations we can look for the constant energy flux in the downscale cascade, for example. In a recent comment, Smith [78] reported a small energy flux accompanying the enstrophy flux in the downscale range of a numerical simulation of two-dimensional turbulence. Constant downscale energy flux has been observed before in Danilov and Gurarie [22, 23] (see their Figure 1 in [23], and Figure 1,2 in [22]) and Borue [6] (see his Figure 3)). So far as we know, the subleading inverse enstrophy cascade has not been discussed much in the turbulence literature.

In a hypothetical situation where the fluxes are fixed but the inertial ranges are extended indefinitely, the subleading cascades will eventually be exposed after certain "transition" wavenumbers. These can be obtained by comparing the leading and subleading terms in the energy spectrum equation (1.1), and these transition wavenumbers for the downscale and upscale ranges, respectively, are given by

$$k_t^{(uv)} \approx \sqrt{\frac{\eta_{uv}}{\varepsilon_{uv}}} = \frac{1}{\lambda_{uv}} \qquad k_t^{(ir)} \approx \sqrt{\frac{\eta_{it}}{\varepsilon_{ir}}} = \frac{1}{\lambda_{ir}}. \tag{2.21}$$

It follows that the necessary condition for exposing the subleading cascades is: $\ell_{uv} \ll \lambda_{uv}$ and $\ell_{ir} \gg \lambda_{ir}$.

As has been pointed out by Danilov [20], this condition cannot be satisfied for the case of two-dimensional turbulence. In fact, it follows from Danilov's inequality (2.8), that the transition scales will be located in the dissipation range both upscale and downscale of injection. A similar claim was given by Smith [78], however his argument was problematic in some respects, as pointed out by Tung [87]. It should be stressed that the same claim is not applicable in quasi-geostrophic turbulence, where the dynamic is different.

Although the subleading cascades remain effectively hidden, it will be shown that as one approaches the KLB limit the transition scale will converge towards the dissipation scale of the contribution to the energy spectrum by the leading cascade. This convergence is essential in justifying the proof given in section 2.1. Furthermore, it can be shown that although the energy and enstrophy flux associated with the subleading cascades is vanishing rapidly in the KLB limit, there will always be a sufficient amount to form subleading cascades with separation of scales proportional to that of the leading cascades.

We begin with the demonstration that in the KLB limit the transition scales coincide with the dissipation scales of the leading cascade. When the system reaches equilibrium, the downscale enstrophy flux η_{uv} , for example, will be equal to the corresponding dissipation rate at small scales. Since the effect of the sink at large scales can be safely ignored, the dominant contribution to the enstrophy flux is given by the integrals

$$\eta_{uv} \approx 2\nu \int_{1/\ell_0}^{1/\ell_{uv}} k^{2\kappa+2} E(k)dk + 2\nu \int_{1/\ell_{uv}}^{+\infty} k^{2\kappa+2} E(k)dk. \tag{2.22}$$

When we substitute[4]

$$E(k) \approx b_{uv} \eta_{uv}^{2/3} k^{-3} \mathcal{D}_{uv}(k\ell_{uv}), \tag{2.23}$$

from (1.1), we find that the first integral diverges in the limit $\nu \to 0$, whereas the second integral stays finite because it is moderated by dissipative corrections to the energy spectrum. The vanishing viscosity eliminates the second integral and moderates the divergence of the first integral. As a result, the dominant contribution comes from the first integral and it follows that

$$\eta_{uv} \approx b_{uv} \nu \eta_{uv}^{2/3} (1/\ell_{uv})^{2\kappa} + \nu C_1, \tag{2.24}$$

In fact, for the case $\kappa \neq 1$, the relevant coefficient

$$\mathcal{A}(\nu) \equiv \nu (1/\ell_{uv})^{2\kappa}, \tag{2.25}$$

turns out to be constant. For the exceptional case $\kappa = 1$, it will vanish in the limit $\nu \to 0$. However, it does so very slowly. To see this, note that for $\kappa = 1$, $\mathcal{A}(\nu)$ can be evaluated analytically as

$$\mathcal{A}(\nu) = \eta_{uv}^{1/3} \left[\ln \left(\frac{\ell_0}{\ell_{uv}} \right) \right]^{-4/3}. \tag{2.26}$$

This means, for example, that increasing the separation of scales ratio all the way up to 10^{10} will only decrease $\mathcal{A}(\nu)$ by a little more than an order of magnitude. Incidentally, this calculation shows that there is an anomalous sink of enstrophy for hyperviscosity $\kappa > 1$, and practically so for molecular viscosity $\kappa = 1$.

Using a similar argument, the downscale energy flux is given by

$$\varepsilon_{uv} \approx 2\nu \int_{1/\ell_0}^{1/\ell_{uv}} k^{2\kappa} E(k)dk + 2\nu \int_{1/\ell_{uv}}^{+\infty} k^{2\kappa} E(k)dk. \tag{2.27}$$

[4]The asymptotically valid assumption of the almost-pure double cascade, enters the argument at this step.

Again, although the first integral does not balance the viscosity ν, it still is the case that the second integral vanishes more rapidly than the first integral. Using a similar argument, as earlier, the downscale energy flux can be written as:

$$\varepsilon_{uv} \approx b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2} + \nu C_2, \tag{2.28}$$

where again, it is expected that the dominant contribution is the first term.

If the quantities νC_1 and νC_2 are small enough to be negligible, then it follows that

$$\begin{aligned}\lambda_{uv}^2 = \frac{\varepsilon_{uv}}{\eta_{uv}} &\approx \frac{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2} + \nu C_2}{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa} + \nu C_1} \\ &\approx \frac{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2}}{b_{uv}\nu\eta_{uv}^{2/3}(1/\ell_{uv})^{2\kappa}} = \ell_{uv}^2.\end{aligned} \tag{2.29}$$

Using a similar argument, we may show that as $\beta \to 0$, we have $\ell_{ir} \approx \lambda_{ir}$.

The same argument can be repeated for the hypothetical case, which is inconsistent, where the dominant upscale cascade is the enstrophy cascade and the dominant downscale cascade is the energy cascade. The form of the energy spectrum will then be different, but it will be compensated by the laws governing the dominant dissipation scales which will also be different. In particular, using the approximation $E(k) \approx a_{uv}\varepsilon^{2/3}k^{-5/3}\mathcal{D}_{uv}(k\ell_{uv})$ from (1.1), we get

$$\begin{aligned}\eta_{uv} &= a_{uv}\nu\varepsilon_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2/3} + \nu C_1 \\ \varepsilon_{uv} &= a_{uv}\nu\varepsilon_{uv}^{2/3}(1/\ell_{uv})^{2\kappa-2/3-2} + \nu C_2.\end{aligned} \tag{2.30}$$

Except for the case $\kappa = 1$, which is problematic in a number of ways, when the quantities νC_1 and νC_2 are negligible, then we still get $\ell_{uv} \approx \lambda_{uv}$.

A consequence of this coincidence is that it enables the calculation of the separation of scales of the subleading cascades as a function of the separation of scales of the leading cascades. First, we write the downscale energy and enstrophy flux in terms of the corresponding dissipation scales

$$\begin{aligned}\eta_{uv} &= \nu^3[\mathcal{R}_{0,uv}^{(\eta)}]^3[\ell_{uv}^{(\eta)}]^{-6\kappa} \\ \varepsilon_{uv} &= \nu^3[\mathcal{R}_{0,uv}^{(\varepsilon)}]^3[\ell_{uv}^{(\varepsilon)}]^{2-6\kappa}.\end{aligned} \tag{2.31}$$

Then, the flux ratio is given by

$$\lambda_{uv}^2 = \frac{\varepsilon_{uv}}{\eta_{uv}} = \left(\frac{\mathcal{R}_{0,uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right)^3 \left[\frac{\ell_{uv}^{(\eta)}}{\ell_{uv}^{(\varepsilon)}}\right]^{6\kappa-2} [\ell_{uv}^{(\eta)}]^2, \tag{2.32}$$

We have shown that in the KLB limit $\lambda_{uv} \approx \ell_{uv}^{(\eta)}$. It follows that the dissipation scale of the subleading downscale energy cascade is given asymptotically by

$$\ell_{uv}^{(\varepsilon)} \approx \ell_{uv}^{(\eta)} \left(\frac{\mathcal{R}_{0,uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right)^{3/(6\kappa-2)}. \tag{2.33}$$

This equation shows that asymptotically the extent of the subleading downscale energy cascade $\ell_{uv}^{(\varepsilon)}$ is proportional to the extent of the leading downscale enstrophy cascade $\ell_{uv}^{(\eta)}$. For a very large order of hyperdiffusion κ the proportionality constant approaches unity. In that case, $\ell_{uv}^{(\varepsilon)} \approx \ell_{uv}^{(\eta)}$. Nevertheless the ratio of the leading and subleading dissipation scales cannot be taken as 1 in (2.32). A small difference in the dissipation scales ratio $\ell_{uv}^{(\eta)}/\ell_{uv}^{(\varepsilon)}$ can still result in a significant adjustment of the transition scale, because in equation (2.32) it is being raised to very large powers. (see [87])

A similar result can be derived for the upscale range. The upscale energy and enstrophy fluxes are given by

$$\begin{aligned} \varepsilon_{ir} &= \beta^3[\mathcal{R}_{0,ir}^{(\varepsilon)}]^3[\ell_{ir}^{(\varepsilon)}]^{2+6m} \\ \eta_{ir} &= \beta^3[\mathcal{R}_{0,ir}^{(\eta)}]^3[\ell_{ir}^{(\eta)}]^{6m}, \end{aligned} \tag{2.34}$$

and, likewise, the flux ratio reads

$$\lambda_{ir}^2 = \frac{\varepsilon_{ir}}{\eta_{ir}} = \left(\frac{\mathcal{R}_{0,ir}^{(\varepsilon)}}{\mathcal{R}_{0,ir}^{(\eta)}}\right)^3 \left[\frac{\ell_{ir}^{(\varepsilon)}}{\ell_{uir}^{(\eta)}}\right]^{6m} [\ell_{ir}^{(\varepsilon)}]^2, \tag{2.35}$$

and in the KLB limit we find

$$\ell_{ir}^{(\eta)} \approx \ell_{ir}^{(\varepsilon)} \left(\frac{\mathcal{R}_{0,ir}^{(\varepsilon)}}{\mathcal{R}_{0,ir}^{(\eta)}}\right)^{1/2m}. \tag{2.36}$$

The significance of these results is that they highlight that the subleading cascades are not a hypothetical possibility; even far into the KLB limit, there will be sufficient downscale energy flux and upscale enstrophy flux to provide subleading cascades that are proportionally as large as the leading cascades. In fact, if that were not the case it would signal an inconsistency in the overall theory, since the energy flux and the enstrophy flux are constrained to be constant over the same spectral region.

As separation of scales increases, the *percentage* of these counter fluxes decreases rapidly. To see this, use the approximation $\ell_{uv} \approx \lambda_{uv}$ to show that

$$\varepsilon_{uv} = \eta_{uv}\lambda_{uv}^2 \leq \eta\lambda_{uv}^2 = \varepsilon\left(\frac{\lambda_{uv}}{\ell_0}\right)^2 \approx \varepsilon\left(\frac{\ell_{uv}^{(\eta)}}{\ell_0}\right)^2. \tag{2.37}$$

Similarly, for the upscale range we have

$$\eta_{ir} \leq \eta\left(\frac{\ell_0}{\lambda_{ir}}\right)^2 \approx \eta\left(\frac{\ell_0}{\ell_{ir}^{(\varepsilon)}}\right)^2. \tag{2.38}$$

It follows from these inequalities that a separation of scales of one decade is sufficient to reduce the counter fluxes percentagewise to about 1%. However, because the derivation implicitly assumes the KLB limit, it does not follow that a separation of scales of only one decade is sufficient to reach that limit.

3. **Evidence from numerical simulations.** The inertial ranges of two-dimensional turbulence have been studied extensively with numerical simulations and experiments as well as theoretically for many decades. In this section we summarize the accumulated theoretical and experimental insight derived from these studies. Our goal is to give the reader a current account of the extent of theoretical and experimental support for the enstrophy and energy cascade of two-dimensional turbulence. Our conclusion is that there is experimental support for the existence of both cascades under favorable conditions. However, unlike the case of three-dimensional turbulence, neither cascade is completely robust. Furthermore, we note that all present theoretical work implicitly assumes that the cascades have formed successfully and proceeds to explain their statistical behavior. This question of robustness

remains open, and we believe that it can be addressed by our framework. Some initial comments to that effect for the case of inverse energy cascade are given.

3.1. **The inverse energy cascade.** The inverse energy cascade with the $k^{-5/3}$ energy spectrum on the upscale side of injection appeared to be robust. It has been observed in numerous simulations and experiments [37, 42, 67, 77], and we even have experimental indications that there are no intermittency corrections [4, 68]. Yakhot [93] has formulated an interesting theoretical explanation for the lack of intermittency corrections, using a mathematical technique developed by Polyakov [72] for Burger's turbulence. His argument is based on the unproven assumption that the pressure gradients are local.

In the Kolmogorov downscale energy cascade of three-dimensional turbulence, smaller length scales always exist, so we may disregard the presence of boundary conditions and assume that our system is unbounded with impunity. In the two-dimensional inverse energy cascade, on the other hand, the boundary conditions become significant for sufficiently large length scales. If the upscale cascading energy is not dissipated at length scales smaller than the typical length scales of the boundary conditions, then the energy is condensed at large scales leading to steep spectra. This corresponds to the formation of large-scale coherent structures, mentioned by many authors without necessarily elucidating their origin theoretically. The formation of $k^{-5/3}$ scaling, and the subsequent break down due to this condensation effect, have also been observed in numerical simulations [79, 80]. To form a stationary inverse energy cascade it is therefore necessary that large-scale dissipation remove the energy at the length scale much smaller than the typical length scale of the boundary conditions.

Danilov and Gurarie [23] have conducted numerical simulations using $(m, \kappa) = (0, 2)$, and showed that the optimal β yielding an energy spectrum closest to the KLB prediction of $k^{-5/3}$ scaling does not correspond to constant energy flux. Decreasing β improves the energy flux but the slope of the energy spectrum steepens. This behavior is somewhat minimized in simulations using $(m, \kappa) = (0, 8)$, but the reverse relation between optimizing the flux and optimizing the spectrum persists. Sukoriansky *et al* [81] note that using higher order large-scale dissipation ($m > 0$) may produce a constant energy flux, but distorts the spectrum. It has therefore been suggested that the locality of the inverse energy cascade should be called into question.

There are two aspects of this behaviour that call for an explanation. The first is the behavior of the upscale energy flux, which is non-constant. The second is the observed steepening of the energy spectrum. We wish to begin by pointing out that the behaviour of the energy flux, by itself, is not paradoxical but quite reasonable. It makes sense that decreasing β, or using hypodiffusion, improves the upscale energy flux, in the sense of making it more constant over a larger range. It is also reasonable that the upscale energy flux is not constant under Ekman damping. As we have mentioned in the preceding paper [40], the dissipation scale law for the inverse enstrophy cascade breaks down for the case of Ekman damping. It is therefore unlikely that an inverse enstrophy cascade with constant enstrophy flux can be realized. Because the energy flux is linked with the enstrophy flux, the energy flux cannot[5] be constant either.

[5]It is not strange that it is possible to see $k^{-5/3}$ scaling without constant energy flux. This indicates that there is a fortunate cancellation between the corrections to the inertial range scaling

The interesting phenomenon is the steepening of the energy spectrum for small β. Both Danilov and Gurarie [22], and earlier Borue [6], observed that this steepening is caused by coherent structures. These structures cover a relatively small portion of the domain, but they account for most of the energy. When the energy spectrum of the background flow, in which these structures are situated, is evaluated instead, the regular $-5/3$ scaling is restored. This means that the homogeneous solution for the inverse energy cascade still exists, and therefore, the observed steepening of the energy spectrum is caused either by the homogemeous solution associated with an inverse enstrophy cascade, or the particular solution. Danilov and Gurarie [22] examined this first possibility by a very careful analysis of these coherent structures and established that they receive most of their energy from the forcing term. Furthermore, as Danilov [21] has demonstrated, in a paper published in this volume, the inverse enstrophy flux is very small for all cases, and in fact it is larger in the simulations where Ekman damping is used and smaller in the simulation where hypodiffusion is used. It follows that the homogeneous solution associated with the inverse enstrophy cascade is not likely to be responsible for the observed steepening.

The contribution of the particular solution $E_{ir}^{(p)}(k)$ to the energy spectrum plays a crucial role in explaining this paradoxical behavior. We propose that the particular solution dominates and hides the homogeneous solution which continues to exist. Physically, the particular solution is associated with the contribution to the energy spectrum by the coherent structures. The homogeneous solution, on the other hand, is associated with the contribution of the background turbulent flow. The resulting energy spectrum is a linear combination of the two solutions. The corresponding physical interpretation is that the coherent structures coexist side-by-side with the background flow, and each contributes its share to the energy spectrum. The fact that Danilov and Gurarie [22] can separate the two contributions to the energy spectrum is further evidence that the contributions are linearly superimposed. Danilov [21] noted that the steepening of the energy spectrum can be attributed to the non-locality of the triad interactions. The triad interactions are obtained from a Fourier transform that mixes the homogeneous and particular solution together. Nevertheless, the local interactions associated with the homogeneous solution continue to be there, and they remain responsible for a small part of the energy and enstrophy transfer associated with the background flow.

This point is illustrated with the case of the inverse energy cascade in a simulation that uses hypodiffusion. As Danilov [21] reports, the energy spectrum exhibits a strong deviation from $k^{-5/3}$ scaling, which indicates non-locality. However, in the same simulation a very constant energy flux is reported. This apparent discrepancy can be explained as follows: The non-local interactions associated with the coherent structures transfer energy and enstrophy directly from the forcing range to the dissipation range, and they are in fact responsible for the greater part of energy and enstrophy transfer [20]. It follows that the effect of the particular solution to the energy and enstrophy flux in the inertial range is to simply shift it by a constant amount. The energy and enstrophy flux associated with the homogeneous solution is also constant, and as a result the total fluxes are both constant. That the use of hypodiffusion aggravates the departure from universal scaling in the energy spectrum instead of restoring it, is additional evidence that the non-universality of the

---

from the forcing range and the dissipation range. The cancellation is not perfect, and that accounts for the "bulge" observed in some of the numerical simulations [23] where Ekman damping is being used.

inverse energy cascade, and the associated non-locality, result from the particular solution and not from dissipative adjustment of the homogeneous solution.

Eyink [31] recently communicated to us results from recent numerical simulations where it has been shown that it is possible to produce an inverse energy cascade with constant energy flux under hypodiffusion, if the numerical resolution is sufficiently large. The requirement on the resolution probably increases as one increases the order of hypodiffusion. This is significant, because, as we shall argue below, hypodiffusion is required to produce the enstrophy cascade.

Nevertheless, the situation here should be contrasted with the case of the downscale energy cascade in three-dimensional turbulence. In the latter case, the scaling behavior of the energy cascade is robust even under very low numerical resolutions. The inverse energy cascade, on the other hand, is only obtained within carefully defined parameter ranges for the infrared sink and the length scale associated with the size of the domain.

3.2. **The enstrophy cascade.** Numerical simulations do not reproduce the k^{-3} energy spectrum of the downscale range consistently. As a result, this prediction of Kraichnan's is considered by many an unproven conjecture. Alternative theories have been proposed that predict steeper scaling [62, 71, 75]. Kraichnan [48] himself noted that the non-locality of the direct enstrophy cascade makes the application of dimensional analysis inconsistent. However, using a 1-loop closure model [49], he showed that introducing a logarithmic correction to the energy spectrum restores the constant enstrophy flux [50]. The same result can be obtained with other 1-loop models [45].

If higher order closures yield additional higher powers of logarithmic corrections, they may add up to a power law renormalization leading to a steeper spectrum. Eyink has shown recently [30] that such a renormalization does not take place when a downscale enstrophy cascade manifests with constant enstrophy flux, although logarithmic corrections are not excluded. This result is a refinement of an earlier argument [27] that only ruled out energy spectra steeper than $k^{-11/3}$.

Falkovich and Lebedev [33, 34] used a Lagrangian approach [14, 32] to confirm Kraichnan scaling with the logarithmic correction. They also predict that the vorticity structure functions have regular (the scaling exponents form a straight line) logarithmic scaling given by

$$\langle [\zeta(\mathbf{r}_1) - \zeta(\mathbf{r}_2)]^n \rangle \sim [\eta \ln(\ell_0/r_{12})]^{2n/3}. \tag{3.1}$$

Eyink [29] noted that this theory does not follow from first principles and that it rests on an unproven regularity for the velocity field. However, should this regularity condition be proven, it would then follow that the Kraichnan scaling scenario is the only one that is statistically stable [35].

Although Eyink's most recent result [30] shows that intermittency corrections are excluded in the enstrophy cascade, the argument still relies on the successful formation of an enstrophy cascade under given configurations of dissipation at large scales and small scales. An additional argument is then needed to show that the homogeneous solution corresponding to the enstrophy cascade is not hidden by the particular solution and is not destroyed by dissipative adjustments. Given this argument, the results of Eyink [30] combined with the theory by Falkovich and Lebedev [33, 34] give a satisfying theory for the downscale enstrophy cascade.

It is well known that in the presence of coherent structures, the enstrophy range does not follow k^{-3} scaling. Borue [5] showed that using hypodiffusion ($m = 8$ and

$k = 1, 8$) disrupts the coherent structures and with increasing Reynolds number the scaling of the enstrophy range approaches asymptotically Kraichnan scaling. As pointed out by Tran and Shepherd [85], all the successful simulations of the k^{-3} spectral range are done so far with the hypodiffusion device. For example, Lindborg and Alvelius [55] showed that the downscale enstrophy range with a k^{-3} energy spectrum can be created in a high resolution simulation, if one uses hyperdiffusion and hypodiffusion $\kappa = m = 2$ and 4096^2 resolution. According to Pasquero and Falkovich [69], the logarithmic correction can be also observed if the simulation is allowed to run for a very long time. The presence of the logarithmic correction has also been confirmed by Ishihira and Kaneda [44]. Bowman, Shadwick, and Morrison [9] used a reduced statistical description of turbulence, called *spectral reduction* [8], to calculate both the energy spectrum of the enstrophy cascade as well as higher order statistics, and they found agreement with the scaling proposed by Falkovich and Lebedev.

Although these results favor the Kraichnan scaling scenario, no numerical simulation has reproduced clean scaling for the physically relevant case of Ekman damping and molecular diffusion. As a matter of fact, Bernard [3] has given an elementary proof that under Ekman damping it is not possible for the energy spectrum of the downscale cascade to scale as k^{-3} with or without the logarithmic correction. A steeper energy spectrum is predicted instead. From our viewpoint, Ekman damping acts by modifying the operator of the balance equations, consequently it changes the homogeneous solutions responsible for the enstrophy cascade.

Nam *et al.* [64] have derived a law governing the steepening of the enstrophy cascade by Ekman damping, however it cannot be used directly to predict the slope of the energy spectrum from the viscosity parameters without additional experimental input. That the behavior of the spectrum at the downscale range is so dependent on the nature of the energy sink at the largest scales is also one of the surprising and important messages from the work of Tran and Shepherd [85], and later Tran and Bowman [84]. A numerical simulation by Schorghofer [76] using Ekman damping and molecular diffusion showed that the enstrophy range approaches Kraichnan scaling with increasing Reynolds number, but failed to yield the k^{-3} slope with the same precision as simulations employing hyperdiffusion and hypodiffusion. Note that the simulation by Ishihira and Kaneda [44] uses a filtered Ekman damping that acts only on wavenumbers smaller than the injection wavenumber. Such a dissipation filter is effectively a hypodiffusion, and Bernard's argument does not apply in this case.

The accumulated evidence of numerical experiments suggests that the enstrophy cascade can be observed in the energy spectrum, but its existence is fragile and heavily dependent on the dissipation mechanisms. It is quite clear that the enstrophy cascade cannot be realized under Ekman damping, and that hypodiffusion is needed. In light of the numerical results communicated to us by Eyink, it is possible to obtain an inverse energy cascade under hypodiffusion, provided that there is sufficient resolution. It may therefore be possible to obtain the dual cascade with careful tuning of the relevant parameters under even larger numerical resolutions. It remains an open question whether the enstrophy cascade can be realized under hypodiffusion and molecular viscosity.

4. **The atmospheric energy spectrum.** According to Kraichnan [48], the study of two-dimensional turbulence was motivated by the hope that it would prove a useful model for atmospheric turbulence. This idea was later encouraged by Charney [12] who claimed that quasi-geostrophic turbulence is isomorphic to two-dimensional turbulence. Early observations suggested that the energy spectrum of the atmosphere follows a power law behavior (see [89] for review). An analysis of wind and temperature measurements taken during the Global Atmospheric Sampling Program by Nastrom *et al.* [65, 66] showed that there is a robust k^{-3} spectrum extending from approximately 3,000 km to 1,000 km in wavelength and a robust $k^{-5/3}$ spectrum extending from 600 km down to a few kilometers. A theoretical analysis by Gage and Nastrom [39] showed that the observed spectrum indeed represents quasi-two-dimensional turbulence. Recent measurements [18, 19, 61] have confirmed the $k^{-5/3}$ part of the atmospheric energy spectrum, and it has also been reproduced in General Circulation Model simulations [46, 47].

In light of the KLB theory of two-dimensional turbulence, the k^{-3} spectrum has been interpreted as a direct enstrophy cascade driven by enstrophy injection by the baroclinic instability. The small-scale $k^{-5/3}$ spectrum, on the other hand, for a long time has been the source of a bit of a mystery. Because it was widely believed that downscale energy cascade is forbidden in two-dimensional turbulence, one interpretation was that it reflects an inverse energy cascade given by energy injection at small scales [38, 52]. Another explanation was that it represents positive energy flux from large to small scales that results from the breaking of long gravity waves to shorter waves [24, 91].

4.1. **Double cascades in atmospheric turbulence.** Recent observational evidence in the atmosphere shows that there is *both* a downscale flux of enstrophy η_{uv} *and* a downscale flux of energy ε_{uv} over the mesoscales, from a few tens of kilometer to a few thousand kilometers in wavelength [16, 17]. As pointed out by Tung and Orlando [88, 89], under some situations the downscale energy flux, though small compared to its upscale part, can manifest itself in the energy spectrum for wavelumbers k such that $\epsilon_{uv}k^2 > \eta_{uv}$. This then gives rise to the $-5/3$ spectral slope, which is observed in the atmosphere for large wavenumbers.

The present paper has amplified this explanation by demonstrating theoretically that in a double cascade of both energy and enstrophy, the resulting energy spectrum will be the linear combination of contributions from the energy and enstrophy cascade according to (1.1). A different, nonlinear, form of the energy spectrum was predicted by Lilly [53], obtained using the Leith [51] and Pouquet [73] closure approximations, for a double cascade but with a negative energy flux. As discussed in the preceding paper [40] closure approximations break the linearity of the statistical theory, hence the discrepancy in the form of the predicted energy spectrum.

Indeed, Tung and Orlando [88] have also demonstrated that a two-level quasi-geostrophic channel model with thermal forcing, Ekman damping, and hyperdiffusion with $\kappa = 9$ can reproduce the atmospheric energy spectrum. The diagnostic shown in figure 7 of [88], clearly shows both the constant downscale energy and enstrophy fluxes coexisting in the same inertial range. The k^{-3} spectrum of the enstrophy cascade is actually very short, extending from 1,980 km to approximately 800 km. It is also shown that the range corresponding to 8,580 km to 1,980 km, a portion of which was believed to be part of the enstrophy cascade, is actually part of the forcing range.

In their simulation, Tung and Orlando [88] use a very high order hyperviscosity to model dissipation mechanisms that are not included in the original quasi-geostrophic theory or the two-layer model, such as frontogenesis, the generation of gravity waves, etc. As Tung [87] has pointed out, the hyperviscosity coefficient has to be adjusted as a function of the resolution to control the rate of downscale energy flux. It may be objected that in a simulation where the quasi-geostrophic theory is well resolved, the rate of downscale energy flux should be independent of the resolution, and it shouldn't be necessary to adjust the hyperviscosity efficient. In the real atmosphere, the quasi-geostrophic theory is valid in the inertial range but not in the dissipation range, because it does not account for all the dissipation mechanisms at work. The renormalization of the hyperviscosity coefficient is needed to account for these additional dissipation mechanisms. As long as the correct fluxes are provided, it does not matter whether the dissipation range is governed by our renormalized hyperdiffusion or by resolving the actual physical mechanisms in a more realistic model.

In a recent comment, Smith [78] argued that in a simulation where the Kolmogorov dissipation scale is resolved, the transition scale will have to be in the dissipation range. Smith's argument suffers from a number of problems discussed by Tung [87]. One of these problems, locating the dissipation scales for the leading and subleading downscale cascades, can only be addressed within the framework proposed in the preceding paper [40]. Furthermore, the more convincing argument communicated to us by Danilov [20], presented in section 2.3 of this paper, is applicable for two-dimensional turbulence, but cannot be extended to the two-layer model of quasi-geostrophic turbulence.

Contrary to Charney's claim [12], quasi-geostrophic turbulence is fundamentally different from two-dimensional turbulence, especially in the dissipative spectral region [89, 90]. Therefore, these results from two-dimensional turbulence theory do not contradict the Tung and Orlando model. The principle of linear superposition of the energy and enstrophy cascade, on the other hand, is a deeper mathematical result and it is shared by both dynamical systems. It should be stressed that this simulation has indeed established the crucial fact: that the quasi-geostrophic theory, valid in the inertial range, does admit a double cascade of energy and enstrophy and does yield the observed energy spectrum with the transition scale located at in agreement with observations.

4.2. **The possibility of a helicity cascade.**

Quasi-two-dimensional turbulence differs from two-dimensional turbulence in that it also satisfies the conservation of helicity law. This leaves open the possibility of downscale helicity cascades. Numerical models using a small number of vertical layers axiomatically exclude helicity cascades.

The idea that the conservation of helicity law in three-dimensional turbulence may lead to helicity cascades was first proposed by Brissaud *et al.*[11]. By dimensional analysis, the energy spectrum associated with the helicity cascade is expected to have $k^{-7/3}$ scaling. It is also possible to show that in the helicity cascade we have $\zeta_3 = 2$ [15, 41]. A theoretical treatment of the double cascade of helicity and energy in stratified and compressible non-entropic gases was given by Moiseev and Chkhetiani [63], using the framework of the Hopf formalism [43]. A transition scale from $k^{-7/3}$ to $k^{-5/3}$ scaling, similar to the transition scale for the double downscale cascade of energy and enstrophy, was also derived.

Helicity cascades have been observed in quasi-two-dimensional turbulence experiments [1]. It has been established that three-dimensional instabilities are responsible for injecting helicity into the system. When these instabilities are suppressed by constraints such as stratification, rotation, a magnetic field, etc. then the helicity cascade is replaced by an enstrophy cascade. A typical example are the energy spectra reported for mercury flows constrained by a magnetic field with variable intensity [70]. There is also some recent interest in the double cascade of energy and helicity in three-dimensional turbulence, as opposed to quasi-two-dimensional turbulence [7, 13, 25, 26, 92].

So far as we know, it is unclear whether the inertial range observed in the subsynoptic scales is an enstrophy cascade or a helicity cascade. It has been suggested that in the spectrum reported by Nastrom and Gage [65] the $-7/3$ slope gives a better fit than the -3 slope (see discussion in section 6.5 of [10], and figure 3 of [1]). The question can be decided by analyzing third order structure functions of velocity differences, as pioneered by Lindborg [54]. The presence of a constant enstrophy flux has been clearly established for scales between 300 km and 1,500 km [16]. However, a remarkably clean r^2 ($\zeta_3 = 2$) dependency governs the off diagonal third order functions in the stratosphere from 10 km to 1,000 km in scale. It is possible that this dependency is the footprint of an extensive helicity cascade. More work in this direction would go a long way to clarify this issue.

5. **Conclusions.** The direction of the energy and enstrophy fluxes cannot be determined without considering the effect of dissipation. Previous arguments that rely only on the structure of the quadratic term in the Navier-Stokes equations are not correct. We provide a more careful argument to show that, in the limit of large separation of scales, most of the energy will be dissipated at large scales, and most of the enstrophy at small scales. The only self-consistent possibility, as long as universality is not broken, is therefore a leading downscale enstrophy cascade and a leading upscale energy cascade. It is also shown, however, that even in the limiting case where the separation of scales in the double cascades is very large, the subleading downscale energy cascade and the subleading upscale enstrophy cascade continue to exist. Although they are shown to be hidden, their extent is asymptotically proportional to the extent of the corresponding leading cascades.

We have shown that the experimental evidence from numerical simulations shows that the inverse energy cascade is not structurally stable. The direct enstrophy cascade can be reproduced in numerical simulations that use hypodiffusion and hyperviscosity. Hypodiffusion is known to be necessary, but it has not been determined whether hyperviscosity is required.

We also clarified, in light of the theoretical framework presented in this paper, the numerical issues with the simulation by Tung and Orlando [88]. We have explained that the simulation does settle the crucial question of whether a downscale energy and enstrophy cascade can coexist at length scales governed by quasi-geostrophic theory. We also raised the possibility that the spectrum of the real atmospheric turbulence could be a triple downscale cascade of energy, enstrophy, and helicity. The presence of a helicity cascade cannot be predicted by numerical simulations that do not resolve the vertical direction. There are however some peculiaritities in the observational measurements of third-order structure functions that suggest that a helicity cascade may indeed be present in atmospheric turbulence.

Acknowledgement. It is a pleasure to thank Dr. Sergey Danilov for his meticulous comments on our manuscript that helped us improve our paper. We thank Dr. Gregory Eyink for sharing with us the unpublished results from recent numerical simulations by his research group, and for explaining to us relevant aspects of his theoretical work. We also thank Dr. Chuong Van Tran and Dr. John Bowman for comments and discussions. This work is supported in part by the National Science Foundation, under grants ATM 98-13770, ATM 01-32727, and DMS 03-27650.

REFERENCES

[1] A.Bershadskii, E.Kit, and A. Tsinober, *Spontaneous breaking of reflexional symmetry in real quasi-two-dimensional turbulence: stochastic travelling waves and helical solitons in atmosphere and laboratory*, Proc. R. Soc. Lond. A **441** (1993), 147–155.

[2] G.K. Batchelor, *Computation of the energy spectrum in homogeneous, two dimensional turbulence*, Phys. Fluids Suppl. II **12** (1969), 233–239.

[3] D. Bernard, *Influence of friction on the direct cascade of the two dimensional forced turbulence*, Europhys. Lett. **50** (2000), 333–339.

[4] G. Boffetta, A. Celani, and M. Vergassola, *Inverse energy cascade in two dimensional turbulence: deviations from gaussian behavior*, Phys. Rev. E **61** (2000), 29–32.

[5] V. Borue, *Spectrum exponents of the enstrophy cascade in stationary two dimensional homogeneous turbulence*, Phys. Rev. Letters **71** (1993), 3967–3970.

[6] ———, *Inverse energy cascade in stationary two dimensional homogeneous turbulence*, Phys. Rev. Letters **72** (1994), 1475–1478.

[7] V. Borue and S.A.Orzag, *Spectra in helical three-dimensional homogeneous isotropic turbulence*, Phys. Rev. E **55** (1997), 7005–7009.

[8] J. Bowman, *A wavenumber partitioning scheme for two-dimensional statistical closures*, J. Sci. Comp. **11** (1996), 343–367.

[9] J.C. Bowman, B.A. Shadwick, and P.J. Morrison, *Spectral reduction: a statistical description of turbulence*, Phys. Rev. Letters **83** (1999), 5491–5494.

[10] H. Branover, A. Eidelman, E. Golbraikh, and S. Moiseev, *Turbulence and structures: chaos, fluctuations, and helical self organization in nature and the laboratory*, Academic Press, San Diego, 1999.

[11] A. Brissaud, U. Frisch, J. Leorat, M. Lesieur, and A. Mazure, *Helicity cascades in fully developed isotropic turbulence*, Phys. Fluids **16** (1973), 1366–1368.

[12] J.G. Charney, *Geostrophic turbulence*, J. Atmos. Sci. **28** (1971), 1087–1095.

[13] Q. Chen, S. Chen, and G.L. Eyink, *The joint cascade of energy and helicity in three-dimensional turbulence*, Phys. Fluids **15** (2003), 361–374.

[14] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, *Statistics of a passive scalar advected by a large-scale two dimensional velocity field: analytic solution*, Phys. Rev. E **51** (1995), 5609–5627.

[15] O.G. Chkhetiani, *On the third moments in helical turbulence*, JETP Letters **63** (1996), 808–812.

[16] J.Y.N. Cho and E. Lindborg, *Horizontal velocity structure functions in the upper troposphere and lower stratosphere. 1. Observations*, J. Geophys. Res. **106 D10** (2001), 10,223–10,232.

[17] ———, *Horizontal velocity structure functions in the upper troposphere and lower stratosphere. 2. Theoretical considerations*, J. Geophys. Res. **106 D10** (2001), 10,223–10,241.

[18] J.Y.N. Cho, R.E. Newell, and J.D. Barrick, *Horizontal wavenumber spectra of winds, temperature, and trace gases during the Pacific Exploratory Missions: 2. Gravity waves, quasi-two-dimensional turbulence, and vortical modes*, J. Geophys. Res. **104 D13** (1990), 16297–16308.

[19] J.Y.N. Cho, Y. Zhu, R.E. Newell, B.E. Anderson, J.D. Barrick, G.L. Gregory, G.W. Sachse, M.A. Carroll, and G.M. Albercook, *Horizontal wavenumber spectra of winds, temperature, and trace gases during the Pacific Exploratory Missions: 1. Climatology*, J. Geophys. Res. **104 D5** (1999), 5697–5716.

[20] S. Danilov, *personal communication*.

[21] ———, *Non-universal features of forced 2d turbulence in the energy and enstrophy ranges*, Discrete and Continuous Dynamical Systems Series B **5** (2005), 67–78.

[22] S. Danilov and D. Gurarie, *Forced two-dimensional turbulence in spectral and physical space*, Phys. Rev. E **63** (2001), 061208.

[23] ———, *Non-universal features of forced two dimensional turbulence in the energy range*, Phys. Rev. E **63** (2001), 020203.
[24] E.M. Dewan, *Stratospheric spectra resembling turbulence*, Science **204** (1979), 832–835.
[25] P.D. Divletsen and P. Giuliani, *Cascades in helical turbulence*, Phys. Rev. E **63** (2001), 036304.
[26] ———, *Dissipation in helical turbulence*, Phys. Fluids **13** (2001), 3508–3509.
[27] G.L Eyink, *Exact results on scaling exponents in the two dimensional enstrophy cascade*, Phys. Rev. E **74** (1995), 3800–3803.
[28] ———, *Exact results on stationary turbulence in two dimensions: Consequences of vorticity conservation*, Physica D **91** (1996), 97–142.
[29] ———, *Eyink replies*, Phys. Rev. Letters **76** (1996), 1975.
[30] ———, *Dissipation in turbulent solutions of 2d Euler equations*, Nonlinearity **14** (2001), 787-802
[31] ———, *personal communication.*
[32] G. Falkovich, K. Gawedzki, and M. Vergassola, *Particles and fields in fluid turbulence*, Rev. Mod. Phys., **73** (2001), 913–975.
[33] G. Falkovich and V. Lebedev, *Non-local vorticity cascade in two dimensions*, Phys. Rev. E, **423** (1994), 1800–1803.
[34] ———, *Universal direct cascade in two dimensional turbulence*, Phys. Rev. E, **50** (1994), 3883–3899.
[35] ———, *Comment on: Exact results on scaling exponents in the two dimensional enstrophy cascade*, Phys. Rev. Letters, **76** (1996), 1974.
[36] R. Fjortoft, *On the changes in the spectral distribution of kinetic energy for two dimensional non-divergent flow*, Tellus, **5** (1953), 225–230.
[37] U. Frisch and P.L. Sulem, *Numerical simulation of the inverse cascade in two dimensional turbulence*, Phys. Fluids, **27** (1984), 1921–1923.
[38] K.S. Gage, *Evidence for a $k^{-5/3}$ law inertial range in mesoscale two-dimensional turbulence*, J. Atmos. Sci., **36** (1979), 1950–1954.
[39] K.S. Gage and G.D. Nastrom, *Theoretical interpretation of atmospheric wavenumber spectra of wind and temperature observed by commercial aircraft during GASP*, J. Atmos. Sci., **43** (1986), 729–740.
[40] E. Gkioulekas and K.K. Tung, *On the double cascades of energy and enstrophy in two dimensional turbulence. Part 1. Theoretical formulation*, Discrete and Continuous Dynamical Systems Series B, **5** (2005), 79–102.
[41] T. Gomez, H. Politano, and A. Pouquet, *Exact relationship for third-order structure functions in helical flows*, Phys. Rev. E, **61** (2000), 5321–5325.
[42] J.R. Herring and J.C. McWilliams, *Comparison of direct numerical simulation of two dimensional turbulence with two point closure: the effects of intermittency*, J. Fluid. Mech., **153** (1985), 229–242.
[43] E. Hopf, *Statistical hydromechanics and functionals calculus*, J. Ratl. Mech. Anal., **1** (1952), 87–123.
[44] T. Ishihira and Y. Kaneda, *Energy spectrum in the enstrophy transfer range of two-dimensional forced turbulence*, Phys. Fluids, **13** (2001), 544–547.
[45] Y. Kaneda and T. Ishihira, *Nonuniversal k^{-3} energy spectrum in stationary two-dimensional homogeneous turbulence*, Phys. Fluids, **13** (2001), 1431–1439.
[46] J.N. Koshyk and K. Hamilton, *The horizontal kinetic energy spectrum and spectral budget simulated by a high-resolution troposphere-stratosphere mesosphere GCM*, J. Atmos. Sci., **58** (2001), 329–348.
[47] J.N. Koshyk, K. Hamilton, and J.D. Mahlman, *Simulation of the $k^{-5/3}$ mesoscale spectral regime in the GFDL SKYHI general circulation model*, Geophys. Res. Lett., **26** (1999), 843–846.
[48] R.H. Kraichnan, *Inertial ranges in two dimensional turbulence*, Phys. Fluids, **10** (1967), 1417–1423.
[49] ———, *An almost Markovian Galilean invariant turbulence model*, J. Fluid. Mech., **47** (1971), 513–524.
[50] ———, *Inertial range transfer in two and three dimensional turbulence*, J. Fluid. Mech., **47** (1971), 525–535.
[51] C.E. Leith, *Diffusion approximation for two dimensional turbulence*, Phys. Fluids, **11** (1968), 671–673.

[52] D.K. Lilly, *Stratified turbulence and the mesoscale variability of the atmosphere*, J. Atmos. Sci., **40** (1983), 749–761.
[53] ______, *Two dimensional turbulence generated by energy sources at two scales*, J. Atmos. Sci., **46** (1989), 2026–2030.
[54] E. Lindborg, *Can the atmospheric kinetic energy spectrum be explained by two dimensional turbulence?*, J. Fluid. Mech.,**388** (1999), 259–288.
[55] E. Lindborg and K. Alvelius, *The kinetic energy spectrum of two dimensional enstrophy turbulence cascade*, Phys. Fluids, **12** (2000), 945–947.
[56] V.S. L'vov and I. Procaccia, *Cornerstones of a theory of anomalous scaling in turbulence*, Physica Scripta **T64** (1996), 131–135.
[57] ______, *Exact resummations in the theory of hydrodynamic turbulence. III. Scenarios for anomalous scaling and intermittency*, Phys. Rev. E, **53** (1996), 3468–3490.
[58] ______, *Fusion rules in turbulent systems with flux equilibrium*, Phys. Rev. Letters, **76** (1996), 2898–2901.
[59] ______, *Towards a nonperturbative theory of hydrodynamic turbulence: fusion rules, exact bridge relations, and anomalous viscous scaling functions*, Phys. Rev. E, **54** (1996), 6268–6284.
[60] ______, *Viscous lengths in hydrodynamic turbulence are anomalous scaling functions*, Phys. Rev. Letters, **77** (1996), 3541–3544.
[61] A. Marenco, V. Thouret, P. Nedelec, H. Smit, M. Helten, D. Kley, F. Karcher, P. Simon, K. Law, J. Pyle, G. Poschmann, R. Wrede, C. Hume, and T. Cook, *Measurement of ozone and water vapor by airbus in-service aircraft: The MOZAIC airborne program an overview*, J. Geophys. Res., **103 D19** (1998), 25631–25642.
[62] H.K. Moffatt, *Advances in Turbulence* (G.Comte-Bellot and J. Mathieu, eds.), Springer-Verlag, Berlin, 1986, p. 284.
[63] S.S. Moiseev and O.G. Chkhetiani, *Helical scaling in turbulence*, JETP, **83** (1996), 192–198.
[64] K. Nam, T.M. Antonsen, P.N. Guzdar, and E. Ott, *k spectrum of finite lifetime passive scalars in lagrangian chaotic fluid flow*, Phys. Rev. Letters, **83** (1999), 3426–3429.
[65] G.D. Nastrom and K.S. Gage, *A climatology of atmospheric wave number spectra of wind and temperature observed by commercial aircraft*, J. Atmos. Sci., **42** (1984), 950–960.
[66] G.D. Nastrom, K.S. Gage, and W.H. Jasperson, *The atmospheric kinetic energy spectrum,* $10^0 - 10^4$ *km*, Nature, **310** (1984), 36–38.
[67] J. Paret and P. Tabeling, *Experimental observation of two dimensional inverse energy cascade*, Phys. Rev. Letters, **79** (1997), 4162–4165.
[68] ______, *Intermittency in the two dimensional inverse cascade of energy: experimental observations*, Phys. Fluids, **10** (1998), 3126–3136.
[69] C. Pasquero and G. Falkovich, *Stationary spectrum of vorticity cascade in two dimensional turbulence*, Phys. Rev. E, **65** (2002), 056305.
[70] I. Platnieks and S.F. Seluto, *The effect of initial boundary conditions, upon the formation and development of turbulent structure.*, Liquid metal magnetohydrodynamics (J. Lielpeteris and R. Morue, eds.), Kluwer, Dordecht, 1989, pp. 433–439.
[71] A.M. Polyakov, *The theory of turbulence in two dimensions*, Nuclear Physics B, **396** (1993), 367–385.
[72] ______, *Turbulence without pressure*, Phys. Rev. E, **52** (1995), 6183–6188.
[73] A. Pouquet, M. Lesieur, J.C. Andre, and C. Basdevant, *Evolution of high reynolds number two-dimensional turbulence*, J. Fluid. Mech., **75** (1975), 305–319.
[74] P.B. Rhines, *Geostrophic turbulence*, Ann. Rev. Fluid Mech., **11** (1979), 401–441.
[75] P.G. Saffman, *On the spectrum and decay of random two dimensional vorticity distributions at large reynolds number*, Stud. Appl. Math., **50** (1971), 377.
[76] N. Schorghofer, *Energy spectra of steady two dimensional turbulence flows*, Phys. Rev. E, **61** (1999), 6572–6577.
[77] E.D. Siggia and H. Aref, *Point vortex simulation of the inverse energy cascade in two dimensional turbulence*, Phys. Fluids, **24** (1981), 171–173.
[78] K.S. Smith, *Comment on: The* k^{-3} *and* $k^{-5/3}$ *energy spectrum of the atmospheric turbulence: quasi-geostrophic two level model simulation*, J. Atmos. Sci., **61** (2004), 937-942
[79] L.M. Smith and V. Yakhot, *Bose condensation and small-scale structure generation in a random force driven 2d turbulence*, Phys. Rev. Letters, **71** (1993), x352–355.
[80] ______, *Finite size effects in forced two dimensional turbulence*, J. Fluid. Mech., **274** (1994), 115–138.

[81] S. Sukoriansky, B. Galperin, and A. Chekhlov, *Large-scale drag representation in simulations of two dimensional turbulence*, Phys. Fluids, **11** (1999), 3043–3053.
[82] P. Tabeling, *Two dimensional turbulence: a physicist approach*, Phys. Rep., **362** (2002), 1–62.
[83] C.V. Tran and J.C. Bowman, *Robustness of the inverse cascade in two dimensional turbulence*, Phys. Rev. E, **69** (2004), 036303
[84] ______, *On the dual cascade in two dimensional turbulence*, Physica D, **176** (2003), 242–255.
[85] C.V. Tran and T.G. Shepherd, *Constraints on the spectral distribution of energy and enstrophy dissipation in forced two dimensional turbulence*, Physica D, **165** (2002), 199–212.
[86] A.L. Tseskis, *The hydrodynamics of two-dimensional turbulence: Relation between self-similar ranges and anomaly of third moments*, JETP, **95** (2002), 49–56.
[87] K.K. Tung, *Reply to comments by K. Shafer Smith*, J. Atmos. Sci., **61** (2004), 937-948
[88] K.K. Tung and W.W. Orlando, *The k^{-3} and $k^{-5/3}$ energy spectrum of the atmospheric turbulence: quasi-geostrophic two level model simulation*, J. Atmos. Sci., **60** (2003), 824–835.
[89] ______, *On the differences between 2d and qg turbulence*, Discrete and Continuous Dynamical Systems Series B, **3** (2003), 145–162.
[90] K.K. Tung and W.T. Welch, *Remarks on note on geostrophic turbulence*, J. Atmos. Sci., **58** (2001), 2009–2012.
[91] T.E. VanZadt, *A universal spectrum of buoyancy waves in the atmosphere*, Geophys. Res. Lett., **9** (1982), 575–578.
[92] F. Waleffe, *The nature of triad interactions in homogeneous turbulence*, Phys. Fluids A, **4** (1992), 350–363.
[93] V. Yakhot, *Two dimensional turbulence in the inverse cascade range*, Phys. Rev. E, **60** (1999), 5544–5551.

Received October 2003; revised January 2004; final version July 2004

E-mail address: lf@amath.washington.edu
E-mail address: tung@amath.washington.edu