Inviscid Regularization of Hydrodynamics Equations: Global Regularity, Numerical Analysis & Statistical Behavior

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The Navier-Stokes

The Navier-Stokes equations

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla} \vec{v}) - \nu \Delta \vec{v} = -\vec{\nabla} p + \vec{f} \qquad in\ Q$$

Incompressibilitycondition

$$\vec{\nabla} \cdot \vec{v} = 0 \qquad in\ Q$$

initial condition

$$\vec{v}(0, x) = v_o \qquad in\ \Omega$$

boundary condition

$$\vec{v} = 0 \qquad on\ [0, T] \times \partial\Omega$$

or periodic boundary condition

cylinder

$$Q = (0, T) \times \Omega$$

unknowns:

$\vec{v} : [0, T] \times \Omega \to \mathbb{R}^n$ – is the velocity field

$p : [0, T] \times \Omega \to \mathbb{R}$ – is the pressure

forcing

$$\vec{f} : [0, T] \times \Omega \to \mathbb{R}^n$$

ν – is the viscosity

The Navier-Stokes Equations

$$\frac{\partial}{\partial t}\vec{u} - \upsilon \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \frac{1}{\rho_0}\nabla p = \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

Plus Boundary conditions, say periodic in the box

$$\Omega = [0, L]^3$$

- We will assume that $\rho_0 = 1$
- Denote by $\langle \varphi \rangle = \int_\Omega \varphi(x)\,dx$
- Observe that if $\langle \vec{u}_0 \rangle = \langle \vec{f} \rangle = 0$ then $\langle \vec{u} \rangle = 0$.
- Poncare' Inequality

For every $\varphi \in H^1$ with $\langle \varphi \rangle = 0$ we have

$$\|\varphi\|_{L^2} \leq cL \|\nabla \varphi\|_{L^2}$$

Sobolev Spaces

$$H^s(\Omega) = \{\varphi = \sum_{\vec{k}\in \mathsf{Z}^d} \hat{\varphi}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}\frac{2\pi}{L}}$$

such that

$$\sum_{\vec{k}\in \mathsf{Z}^{\mathsf{d}}} \left|\hat{\varphi}_{\vec{k}}\right|^2 (1+\left|\vec{k}\right|^2)^s < \infty\}$$

Navier-Stokes Equations Estimates

- Formal Energy estimate

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 + \int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{u} + \int\nabla p\cdot\vec{u} = \left(\vec{f},\vec{u}\right)$$

- Observe that since $\nabla\cdot\vec{u} = 0$ we have

$$\int(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{u}\,dx = \int\nabla p\cdot\vec{u}\,dx = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 = \left(\vec{f},\vec{u}\right)$$

By the Cauchy-Schwarz and Poincare' inequalities

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 \le \left\|\vec{f}\right\|_{L^2}^2 \|\vec{u}\|_{L^2}^2 \le cL\left\|\vec{f}\right\|_{L^2} \|\nabla\vec{u}\|_{L^2}$$

By the Young's inequality

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \upsilon\|\nabla\vec{u}\|_{L^2}^2 \le \frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2 + \frac{\upsilon}{2}\|\nabla\vec{u}\|_{L^2}^2$$

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}\|_{L^2}^2 + \frac{\upsilon}{2}\|\nabla\vec{u}\|_{L^2}^2 \le \frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2$$

By Poincare' inequality

$$\frac{d}{dt}\left\|\vec{u}\right\|_{L^2}^2 + c\,\frac{\upsilon}{L^2}\left\|\vec{u}\right\|_{L^2}^2 \le \frac{cL^2}{\upsilon}\left\|\vec{f}\right\|_{L^2}^2$$

By Gronwall's inequality

$$\left\|\vec{u}(t)\right\|_{L^2}^2 \le e^{-c\upsilon L^{-2}t}\left\|\vec{u}(0)\right\|_{L^2}^2 + \frac{cL^4}{\upsilon^2}\left(1-e^{-c\upsilon L^{-2}t}\right)\left\|\vec{f}\right\|_{L^2}^2 \qquad \forall t\in[0,\mathrm{T}]$$

and

$$\upsilon\int_0^{\mathrm{T}}\left\|\nabla\vec{u}(\tau)\right\|_{L^2}^2 d\tau \le K\left(L,\left\|\vec{u}_0\right\|_{L^2},\left\|\vec{f}\right\|_{L^2},\upsilon,\mathrm{T}\right)$$

Theorem (Leray 1932-34)

For every $\mathrm{T} > 0$ there exists a weak solution (in the sense of distribution) of the Navier-stokes equations, which also satisfies

$$\vec{u} \in C_w([0,\mathrm{T}], L^2(\Omega)) \cap L^2([0,\mathrm{T}], H^1(\Omega))$$

The uniqueness of weak solutions in the three dimensional Navier-Stokes equations case is still an open question.

Strong Solutions of Navier-Stokes

$$\vec{u} \in C([0,\mathrm{T}], H^1(\Omega)) \cap L^2([0,\mathrm{T}], H^2(\Omega))$$

Enstrophy

$$\left\|\nabla \times \vec{u}\right\|_{L^2}^2 = \left\|\vec{\omega}\right\|_{L^2}^2 = \left\|\nabla \vec{u}\right\|_{L^2}^2$$

Formal Enstrophy Estimates

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\vec{u}\right\|_{L^2}^2+\upsilon\left\|\Delta\vec{u}\right\|_{L^2}^2+\int(\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})+\int\nabla p(-\Delta\vec{u})=\int\vec{f}\cdot(-\Delta\vec{u})$$

Observe that $\int\nabla p\cdot(-\Delta\vec{u})dx=0$

By Cauchy-Schwarz $\left|\int\vec{f}\cdot(-\Delta\vec{u})\right|\leq\frac{\left\|\vec{f}\right\|_{L^2}^2}{\upsilon}+\frac{\upsilon}{4}\left\|\Delta\vec{u}\right\|_{L^2}^2$

By Hőlder inequality

$$\left|\int(\vec{u}\cdot\nabla)\vec{u}\cdot(-\Delta\vec{u})\right|\leq\left\|\vec{u}\right\|_{L^4}\left\|\nabla\vec{u}\right\|_{L^4}\left\|\Delta\vec{u}\right\|_{L^2}$$

Calculus/Interpolation (Ladyzhenskaya) Inequatlities

$$\|\varphi\|_{L^4} \leq \begin{cases} c\,\|\varphi\|_{L^2}^{1/2}\;\;\|\nabla\varphi\|_{L^2}^{1/2} & 2-D \\ c\,\|\varphi\|_{L^2}^{1/4}\;\;\|\nabla\varphi\|_{L^2}^{3/4} & 3-D \end{cases}$$

Denote by $\quad y = e_0 + \|\nabla\vec{u}\|_{L^2}^2$

The Two-dimensional Case

$$\dot{y} \le c\, y^2 \qquad \& \qquad \int_0^{T} y(\tau)\,d\tau \le K(T)$$

$$\Rightarrow y(t) \le \widetilde{K}(T)$$

Global regularity of strong solutions to the two-dimensional Navier-Stokes equations.

Navier-Stokes Equations

- Two-dimensional Case

  * Global Existence and Uniqueness
    of weak and strong solutions

  * Finite dimension global attractor

One can instead use the following Sobolev inequality

$$\|\vec{u}\|_{L^6} \le c\|\nabla \vec{u}\|_{L^2}$$

Which leads to $\dot{y} \le c y^3 \quad \& \quad \int_0^{T} y(\tau)d\tau \le K$

Theorem (Leray 1932-1934)

There exists $T_*(\|\vec{u}_0\|_{L^2}, \|\vec{f}\|_{L^2}, \upsilon, L)$ such that

$y(t) < \infty$ for every $t \in [0, T_*)$.

Navier-Stokes Equations

- The Three-dimensional Case

  * Global existence of the weak solutions

  * Short time existence of the strong solutions

  * Uniqueness of the strong solutions

- Open Problems:

  * Uniqueness of the weak solution

  * Global existence of the strong solution.

Vorticity Formulation

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla)\vec{\omega} - \underline{\underline{(\vec{\omega} \cdot \nabla)\vec{u}}} = \nabla \times \vec{f}$$

Vorticity Stretching Term $(\vec{\omega} \cdot \nabla)\vec{u}$

Two dimensional case $(\vec{\omega} \cdot \nabla)\vec{u} \equiv \vec{0}$

$$\frac{\partial \vec{\omega}}{\partial t} - \nu \Delta \vec{\omega} + (\vec{u} \cdot \nabla)\vec{\omega} = \nabla \times \vec{f}$$

$\left|\vec{\omega}(x,t)\right|^2$ Satisfies a maximum principle.

The Three-dimensional Case

$$(\vec{\omega} \cdot \nabla)\vec{u} \not\equiv 0$$

$$\vec{\omega} \sim \mathsf{z}$$

$$(\vec{\omega} \cdot \nabla)\vec{u} \sim \mathsf{z}^2$$

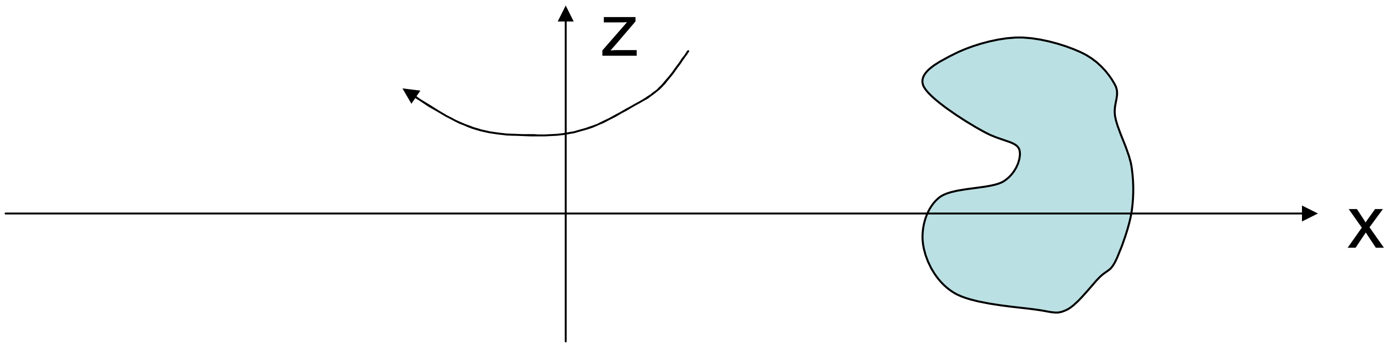
For large initial data $\vec{\omega}_0$ the vorticity balance takes the form

$$\dot{\mathsf{z}} \sim \mathsf{z}^2 \Rightarrow$$ **Potential "Blow Up"!!**

Special Results of Global Existence for the three-dimensional Navier-Stokes

Theorem (Fujita and Kato)

Let $\|u_0\|_{H^{1/2}}$ be small enough. Then the 3D Navier-Stokes equations are globally well-posed for all time with such initial data. The same result holds if the initial data is small in $L^3(\Omega)$ (Kato, Giga & Miyakawa)



• Ω – Revolution Domain around the z - axis [away from z - axis]

• Let us move to Cylindrical coordinates

Theorem (Ladyzhenskaya) Let

$$\vec{u}_0(x,y,z) = (\varphi_r^0(r,z), \varphi_\theta^0(r,z), \varphi_z^0(r,z))$$

be axi-symmetric initial data. Then the three-dimensional Navier-Stokes equations have globally (in time) strong solution corresponding to such initial data. Moreover, such strong solution remains axi-symmetric.

Theorem (Leiboviz, Mahalov and E.S.T.)

Consider the three-dimensional Navier-Stokes equations in an infinite pipe. Let

$$\vec{u}_0 = (\varphi_r^0(r, n\theta + \alpha z), \varphi_\theta^0(r, n\theta + \alpha z), \varphi_z^0(r, n\theta + \alpha z))$$

(Helical symmetry). For such initial data we have global existence and uniqueness. Moreover, such a solution remains helically symmetric.

Remarks

- For axi-symmetric and helical flows the vorticity stretching term is nontrivial, and the velocity field is three-dimensional.

- In the inviscid case, i.e. $\upsilon = 0$, the question of global regularity of the three-dimensional helical or axi-symmetrical Euler equations is still open. Except the invariant sub-spaces where the vorticity stretching term is trivial.

- **Theorem [Cannone, Meyer & Planchon] [Bondarevsky] 1996**

Let M be given, as large as we want, and let $\|u_0\|_{H^1} \leq M$
Then there exists K(M) such that for every initial data of the form

$$\vec{u}_0 = \sum_{|\vec{k}| \geq K(M)} \hat{\vec{u}}^0_{\vec{k}} e^{i\vec{k}\cdot\vec{x}\frac{2\pi}{L}}$$

[VERY OSCILLATORY]

the three-dimensional Navier-Stokes equations have global existence of strong solutions.

Remark Such initial data satisfies $\|u_0\|_{H^{1/2}} << 1.$

So, this is a particular case of Kato's Theorem.

The Effect of Rotation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p + \vec{\Omega} \times \vec{u} = 0$$

$$\nabla \cdot \vec{u} = 0$$

- There is $\Omega_0(\mathrm{T}, \vec{u}_0)$ such that if $|\Omega| > \Omega_0$ the solution exists on $[0, \mathrm{T})$.
- That is there exists $\mathrm{T}_0\ (\vec{u}_{0,} |\vec{\Omega}|)$ such that the solution

exists on $[0, \mathrm{T}_0)$. Observe that

$$\mathrm{T}_0 \to \infty \text{ as } |\vec{\Omega}| \to \infty$$

- Babin - Mahalov - Nicolaenko.
- Embid - Majda.
- Chemin, Ghalagher, Granier, Masmoudi,...
- Liu and Tadmor.

An Illustrative Example

Inviscid Burgers Equation

$$u_t + u u_x = 0 \quad \text{in } \mathrm{R}$$

$$u(x,0) = u_0(\mathrm{x})$$

- If $u_0(x)$ is decreasing function on some subinterval of R then the solution of the above equation develops a singularity (Shock) in finite time.

The solution is given implicitly by the relation:

$$u(x,t) = u_0(x - t u(x,t))$$

The Effect of the Rotation

$$u \in \mathrm{C} \qquad z \in \mathrm{C}$$

$$u_t + uu_z + i\Omega u = 0$$

$$u_0(z) = u(z,0)$$

$$v(z,t) = e^{i\Omega t} u(z,t)$$

$$v_t + e^{-i\Omega t} v v_z = 0$$

$$v(z,t) = v_0\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)$$

$$\frac{\partial}{\partial z} v = \frac{v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}{1 + \frac{e^{-i\Omega t} - 1}{-i\Omega} v_0'\left(z - \frac{e^{-i\Omega t} - 1}{-i\Omega} v(z,t)\right)}$$

If $\Omega >> 1$, (i.e. $\Omega > \Omega_0(u_0)$)

$\frac{\partial}{\partial z} v$ remains finite and the solution remains regular for all $t \geq 0$.

The above complex system is equivalent to 2D Rotating Burgers:

$$u = u_1 + iu_2, \qquad z = x + iy$$

$$\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \Omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{u} = 0$$

Direct Numerical Simulation (DNS)

$$Re = \frac{\text{Nonlinear Intensity}}{\text{Viscous Strength}}$$

$$= \frac{UL}{\nu}$$

Re = 96000

$N^3 = 7.9 \times 10^{11}$, $M = 2.1 \times 10^5$

Simulations are performed at 1 Gigaflop

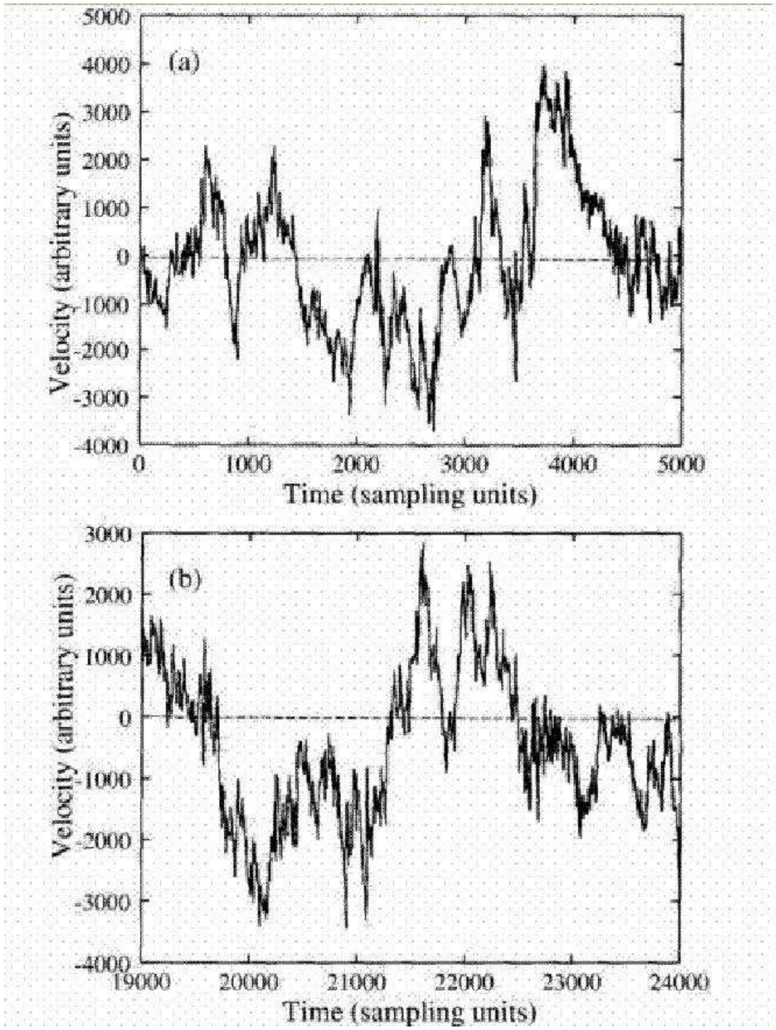
(Assuming 1000 operations per mode per step)

Do we have the time?

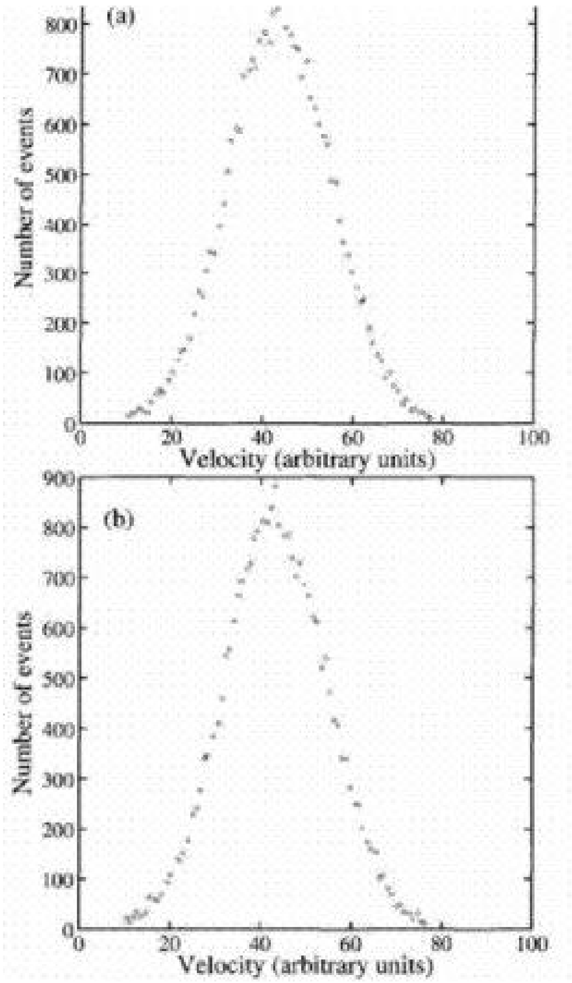
How Long Does it Take to Finish The Job?

5000 Years

Experimental data from Wind Tunnel

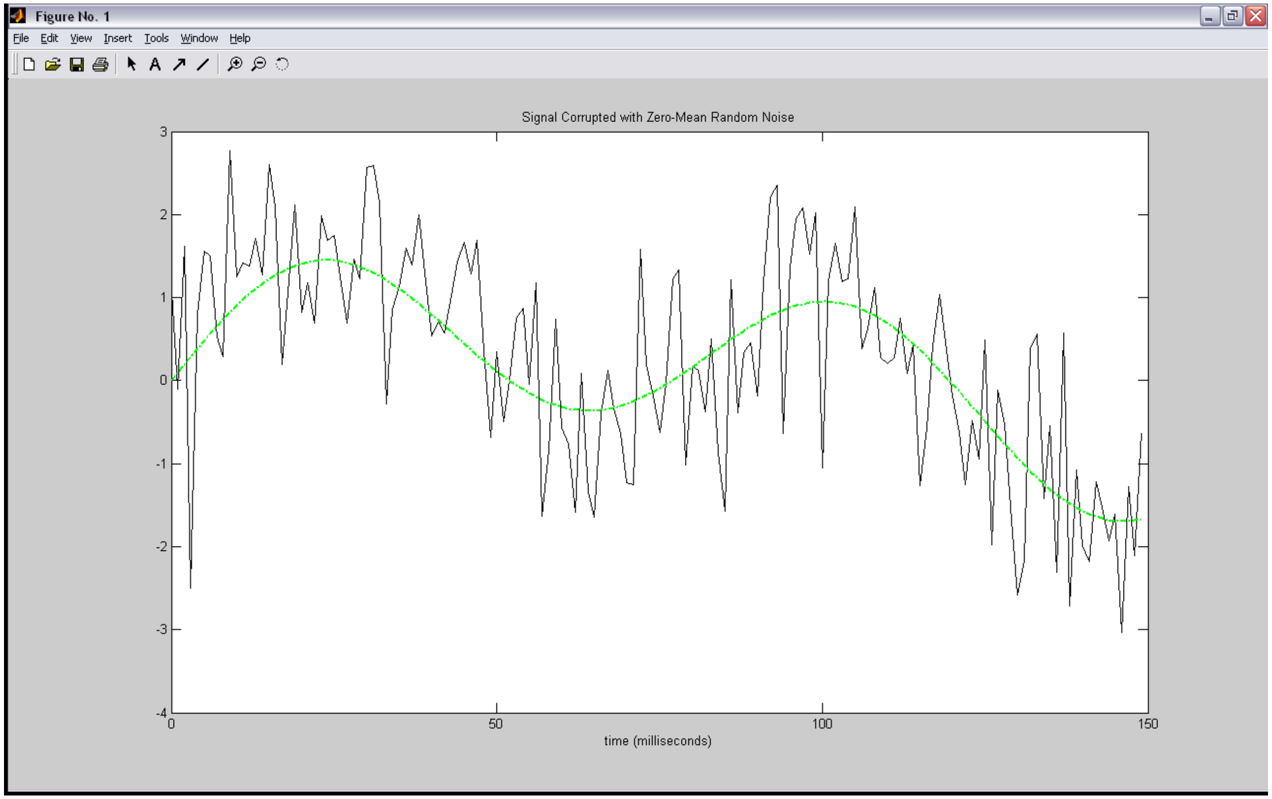


Histogram for the experimental data



Reynolds Equations

$$\phi = \bar{\phi} + \phi'$$



$\bar{\phi}$ – mean

$$\langle \phi \rangle (x) = \bar{\phi}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(x,t)\ dt$$

ϕ' – fluctuations around the mean

Averaged Equations of Motion

$$v = \bar{v} + v'$$ 1

$$p = \bar{p} + p'$$ 2

$$\overline{v'} = \overline{p'} = 0$$ 3

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla} \vec{v}) - \nu \Delta \vec{v} = -\vec{\nabla} p$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

NSE

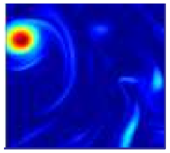
$$\overline{v'} = 0 \qquad \nabla \cdot v = 0 \qquad \nabla \cdot \bar{v} = \nabla \cdot v' = 0$$ 4

Reynolds averaged Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} + \overline{(v' \cdot \nabla) v'} = \nu \Delta \bar{v} - \nabla \bar{p}$$

$$\nabla \cdot \bar{v} = 0$$

Incompressibility condition



Closure Problem

Reynolds averaged Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} + \overline{\nabla \cdot (v' \otimes v')} = \nu \Delta \bar{v} - \nabla \bar{p}$$

$$\nabla \cdot \bar{v} = 0$$

Incompressibility condition

Fundamental Problem in Turbulence – The Closure Problem

(equations are not closed: more unknowns than equations)

Turbulence Modeling

Reynolds averaged Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} + \boxed{\overline{\nabla \cdot (v' \otimes v')}} = \nu \Delta \bar{v} - \nabla \bar{p}$$

$$\nabla \cdot \bar{v} = 0$$

Incompressibility condition

How to model this in terms of $\bar{v}$?

How to close the Reynolds averaged system?

$$\tau_{ij}^{R} = ((v - \bar{v}) \otimes (v - \bar{v}))_{ij}$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

Large Eddy Simulations

- Spatial Filtering
- Large Eddy Simulations
- Sub-grid Scale Model

Let ϕ be nice/smooth spatial filter/kernel

$$\bar{v} = \int \phi(x-y)v(y)$$

$$\frac{\partial \bar{v}}{\partial_t} - \nu \Delta \bar{v} + (\bar{v} \cdot \nabla)\bar{v} = -\nabla \cdot (\tau^R + \bar{p}I)$$

$$\nabla \cdot \bar{v} = 0$$

Here again the problem is to model:

$$div \;\; \tau^R$$

and close the system in terms of $\bar{v}$

$$\tau^R_{ij} = ((v-\bar{v}) \otimes (v-\bar{v}))_{ij}$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

Smogarinsky Model

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i}\right)$$

$$|\bar{S}|^2 = 2\sum_{i,j}(\bar{S}_{ij})^2$$

$$\tau_{ij}^R \approx -2\nu_T \bar{S}_{ij}$$

$$\nu_T = l_S^2 |\bar{S}|$$

$$\tau_{ij}^R = ((v - \bar{v}) \otimes (v - \bar{v}))_{ij}$$

$$= \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

$$\partial_t \bar{v} - \nu \Delta \bar{v} + (\bar{v} \cdot \nabla)\bar{v} = -\nabla \bar{p} + \nu_1 \nabla \cdot (|S| S(\bar{v})) + \bar{f}$$

This and a more general model was also introduced and studied by Ladyzhenskaya.

She proved global existence and uniqueness of the three-dimensional Smagorinsky model.

Navier-Stokes-α

- S. Chen
- C. Foias
- D. Holm
- E. Olson
- S. Wynne
- E. S. Titi

Camassa-Holm Water Wave Equation

Camassa-Holm Water Wave Equation

Hamiltonian

$$\int \left(|u|^2 + \alpha^2 |u_x|^2\right)\, dx$$

Inviscid Equations

Euler equations

Hamiltonian

$$\int |u(x,t)|^2 \; dx$$

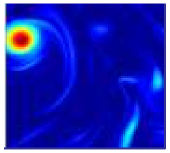
$$\nabla \cdot u = 0 \text{ constraint}$$

Euler-α equations (Holm-Marsden-Ratiu)

Hamiltonian

$$\int |u|^2 + \alpha^2 |\nabla u|^2 \; dx$$

$$\nabla \cdot u = 0 \text{ constraint}$$



Euler-α (Inviscid Second-Grade Fluid)

$$\frac{\partial v}{\partial t} + (u \cdot \nabla)v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta)u$$

Or Equivalently

$$\frac{\partial v}{\partial t} - u \times (\nabla \times v) + \nabla p = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta)u$$

Euler-α (inviscid second grade fluid)

$$-\nu\Delta u$$

$$\frac{\partial v}{\partial t} + (u \cdot \nabla) v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

Navier-Stokes-α (The viscous Camassa-Holm equations)

$$\frac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla) v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

$$\frac{\partial v}{\partial t} - \nu \Delta v - u \times (\nabla \times v) + \nabla p = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

Vorticity Formulation

NSE $\quad \omega = \nabla \times u$

$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = \nabla \times f$$

$$\nabla \cdot u = 0$$

VCHE $\quad q = \nabla \times v \qquad v = u - \alpha^2 \Delta u$

$$\frac{\partial q}{\partial t} - \nu \Delta q + (u \cdot \nabla)q - (q \cdot \nabla)u = \nabla \times f$$

$$\nabla \cdot u = 0$$

$$u \cdot \nabla q - q \cdot \nabla u \quad \Longleftrightarrow \quad v \cdot \nabla q - q \cdot \nabla v$$

Dimension of Global Attractor (NS-α)

$$d(\mathcal{A}) \leq c\left(\frac{L}{\alpha}\right)^{3/2}\left(\frac{L}{l_d}\right)^{3}$$

The Navier-Stokes-α as Sub-grid Scale Model

$$\tau_\alpha = 2\nu(1 - \alpha^2\Delta)D - pI + \alpha^2\dot{D}$$

$$D = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\Omega = \frac{1}{2}(\nabla u - \nabla u^T)$$

$$\dot{D} = u \cdot \nabla D + D\Omega - \Omega D$$

The Navier-Stokes-α as a Closure Model

The Time Averaged Reynolds Navier-Stokes Equations

$$(\bar{v} \cdot \nabla)\bar{v} = \nabla \cdot \tau$$

$$\tau = \nu(\nabla \bar{v} + \nabla \bar{v}^T) - \bar{p}I - \overline{v' \otimes v'}$$

The Navier-Stokes-α as a Closure Model

$$(u \cdot \nabla)u = \nabla \cdot \tau_\alpha$$

$$\tau_\alpha = 2\nu(1 - \alpha^2\Delta)D - pI + \alpha^2\dot{D}$$

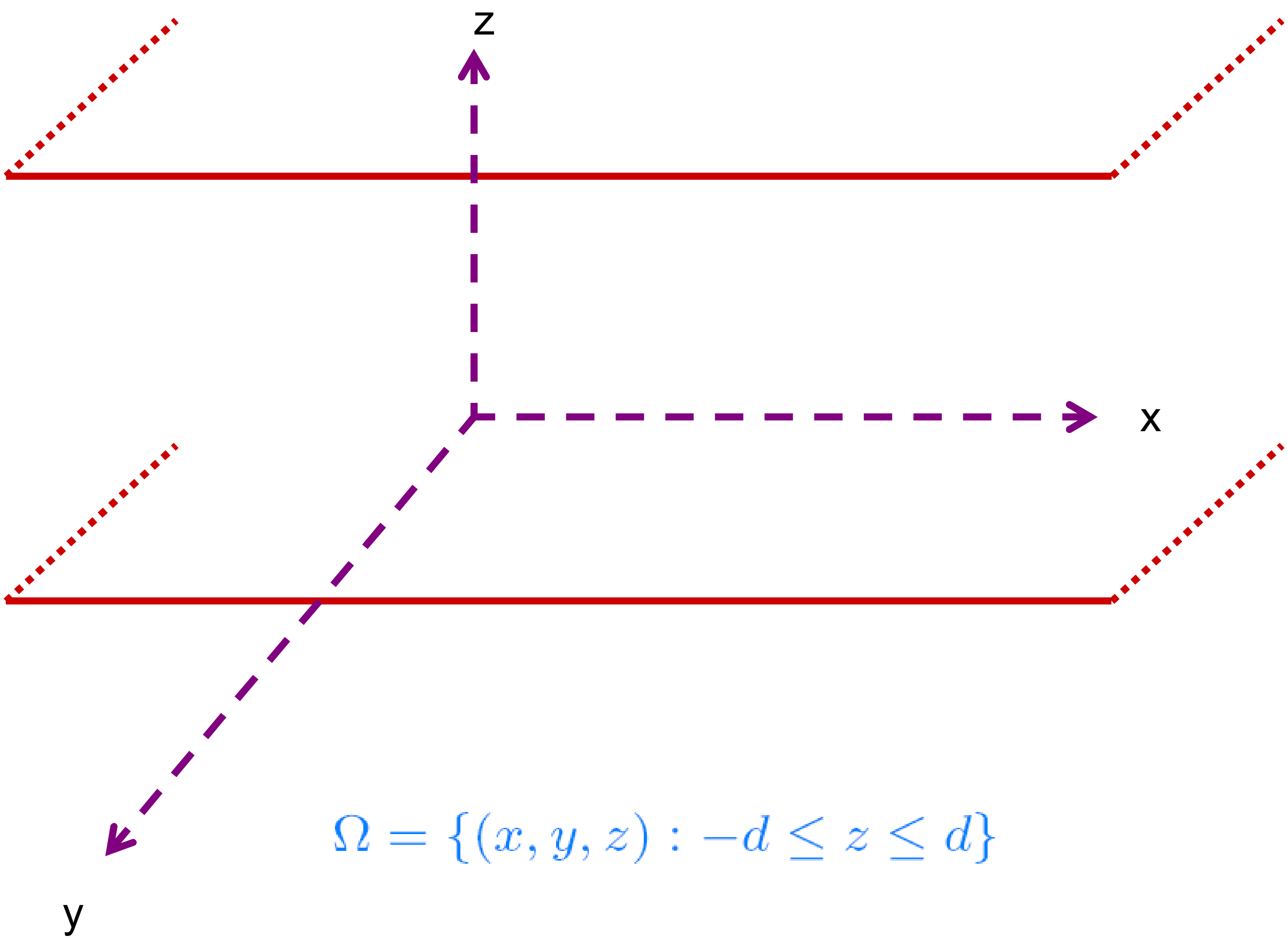
where

$$D = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\Omega = \frac{1}{2}(\nabla u - \nabla u^T)$$

$$\dot{D} = u \cdot \nabla D + D\Omega - \Omega D$$

Turbulent Channel Flow



$$\Omega = \{(x, y, z) : -d \leq z \leq d\}$$

Reynolds Averaged Equations

$$-\nu\Delta \langle u\rangle = \langle (u\cdot\nabla u)\rangle + \nabla\langle p\rangle = 0$$

$$\nabla\cdot\langle u\rangle = 0$$

Facts:

(i)

$$\langle u\rangle = \begin{pmatrix} \bar{U}(z) \\ 0 \\ 0 \end{pmatrix}$$

(ii)

$$\bar{U}(z) = \bar{U}(-z)$$

(iii)

$$u = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \langle u\rangle = \begin{pmatrix} u+\bar{U} \\ v \\ w \end{pmatrix}$$

Reynolds Stresses

The Reynolds stresses

$$\left\langle u^2 \right\rangle, \left\langle uv \right\rangle, \left\langle uw \right\rangle, \left\langle v^2 \right\rangle, \left\langle vw \right\rangle, \left\langle w^2 \right\rangle$$

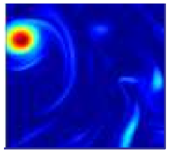
are functions of z alone.

Reynolds Equations

$$-\nu \bar{U}'' + \partial_z \left\langle wu \right\rangle = -\partial_x \bar{P}$$

$$\partial_z \left\langle wv \right\rangle = -\partial_y \bar{P}$$

$$\partial_z \left\langle w^2 \right\rangle = -\partial_z \bar{P}$$



Steady Navier-Stokes-α

ansatz $$u = \begin{pmatrix} U(z) \\ 0 \\ 0 \end{pmatrix}$$

Steady NS-α	Reynolds equations
$-\nu U'' + \nu\alpha^2 U'''' = -\partial_x p$	$-\nu\bar{U}'' + \partial_z \langle wu \rangle = -\partial_x \bar{P}$
$0 = -\partial_y p$	$\partial_z \langle wv \rangle = -\partial_y \bar{P}$
$0 = -\partial_z (p - \alpha^2 (U')^2)$	$\partial_z \langle w^2 \rangle = -\partial_z \bar{P}$

Identifying Terms in VCHE & Reynolds equations

$$\text{(i)} \quad \bar{U} = U$$

$$\text{(ii)} \quad \partial_z \langle wu \rangle = \nu\alpha^2 U'''' + p_0$$

$$\text{(iii)} \quad \partial_z \langle wv \rangle = 0$$

$$\text{(iv)} \quad \nabla(\bar{P} + \langle w^2 \rangle) = \nabla(p - p_0 x - \alpha^2 (U')^2)$$

The General Solution of VCHE

$$U(z) = a\left(1 - \frac{\cosh(z/\alpha)}{\cosh(d/\alpha)}\right) + b\left(1 - \frac{z^2}{d^2}\right)$$

a, b constants

Physical Parameters

- Boundary Stress

$$\pm\tau_0 = -\langle \tau_{13} \rangle |_{z=\pm d} = \nu \bar{U}'(z) + \langle wu \rangle |_{z=\pm d}$$

$$\tau_0 = -\nu \bar{U}'(z=-d)$$

- Averaged Streamwise Velocity Across the Channel

$$\bar{u} = \frac{1}{2d}\int_{-d}^{d} U(z)dz$$

- Reynolds Numbers

$$R = \frac{\bar{u}d}{\nu} \qquad R_0 = \frac{\tau_0^{1/2} d}{\nu}$$

- Length Scales

$$d, \qquad \alpha, \qquad l_* = \frac{\nu}{\tau_0^{1/2}}$$ wall unit

Normalized quantities

Let $$\eta = \frac{z + d}{l_*}$$ normalized distance from the wall

$$\phi(\eta) = \frac{U(\eta l_* - d)}{\tau_0^{1/2}}$$

normalized velocity profile

Drag Law

The drag law for the wall friction

$$D = \frac{2\tau_0}{\bar{u}^2} = \frac{2R_0^2}{R^2}$$

$$\sqrt{\frac{2}{D}} = \frac{R}{R_0}$$

Profile

The Profile ϕ depends on:

$$(\text{i}) \quad c, R, R_0 \text{ or}$$

$$(\text{ii}) \quad c, R, D$$

Blasius drag law

$$D = \lambda R^{-1/4} \qquad \lambda = \text{constant}$$

Blasius Law

Having Blasius law as an input into our theory we obtain:

$$\text{(i)}\quad R_0 = \sqrt{\frac{\lambda}{2}}\, R^{7/8}$$

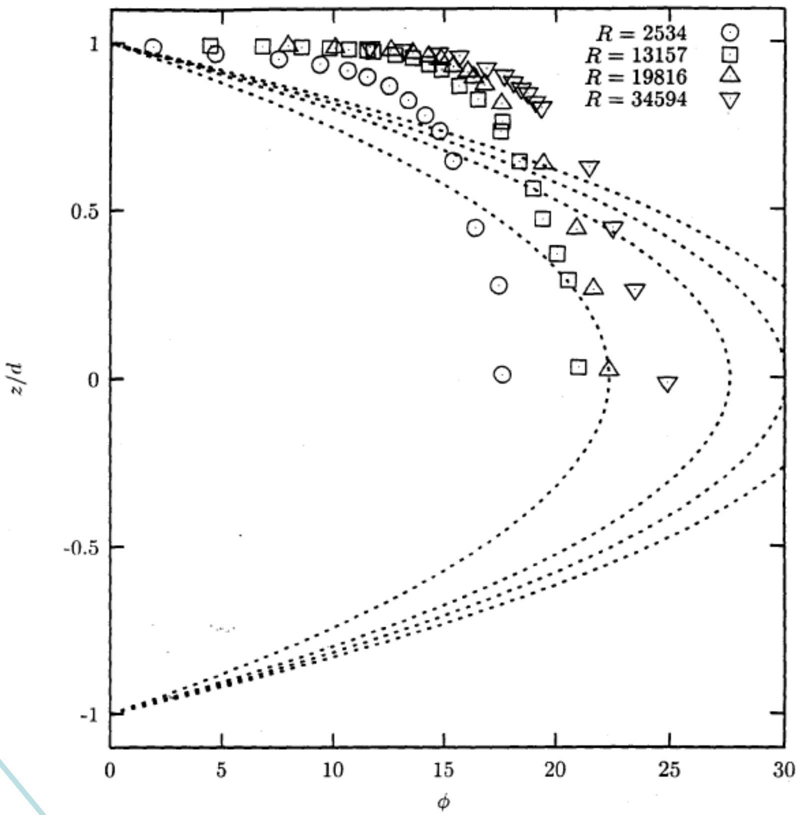
$$\text{(ii)}\quad \frac{d}{\alpha} = \frac{\lambda}{2c} R^{3/4}$$

(iii) Let l_d be the Kolmogorov Fluctuation Dissipation Length

$$\text{(iv)}\quad \frac{d}{l_d} \sim R^{3/4}$$ Classical Theory of Turbulence

$$\Longrightarrow \alpha \sim l_d$$

Figure 1 and Figure 2



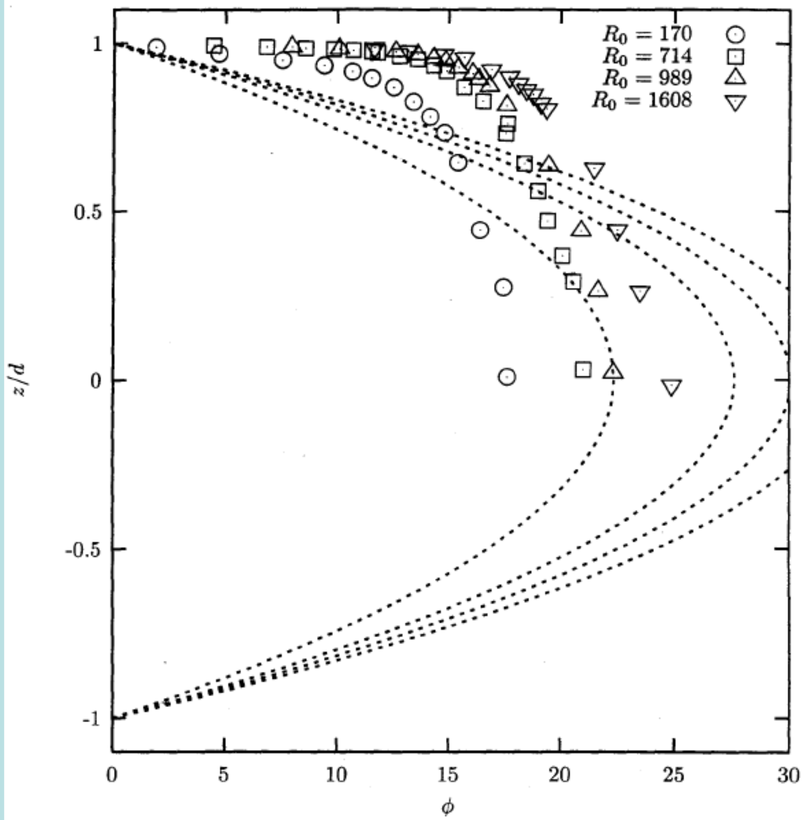
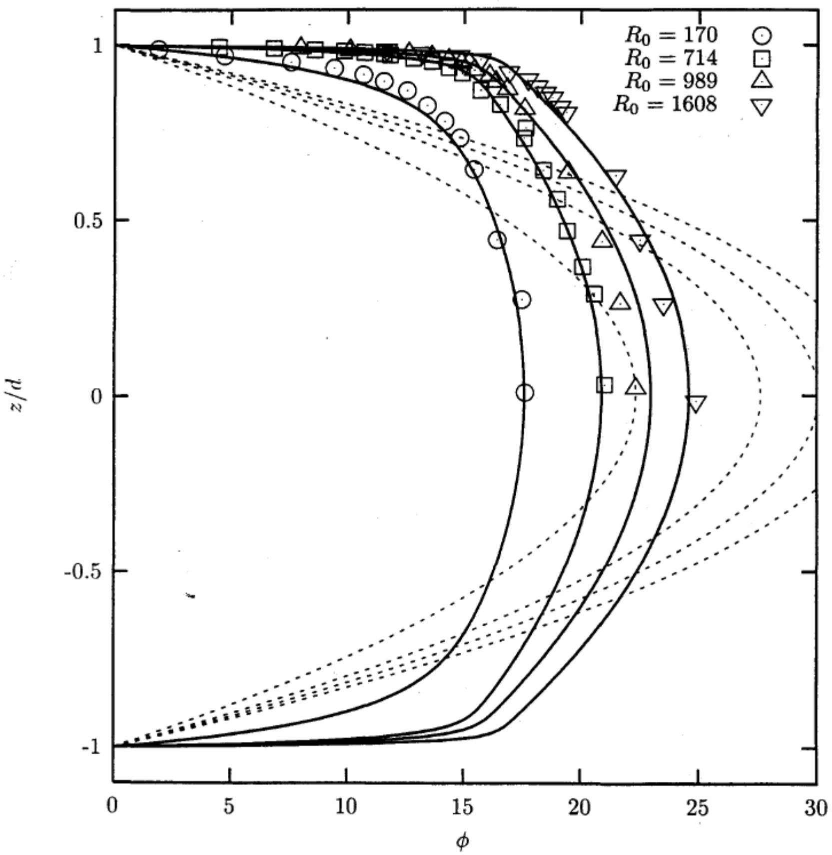


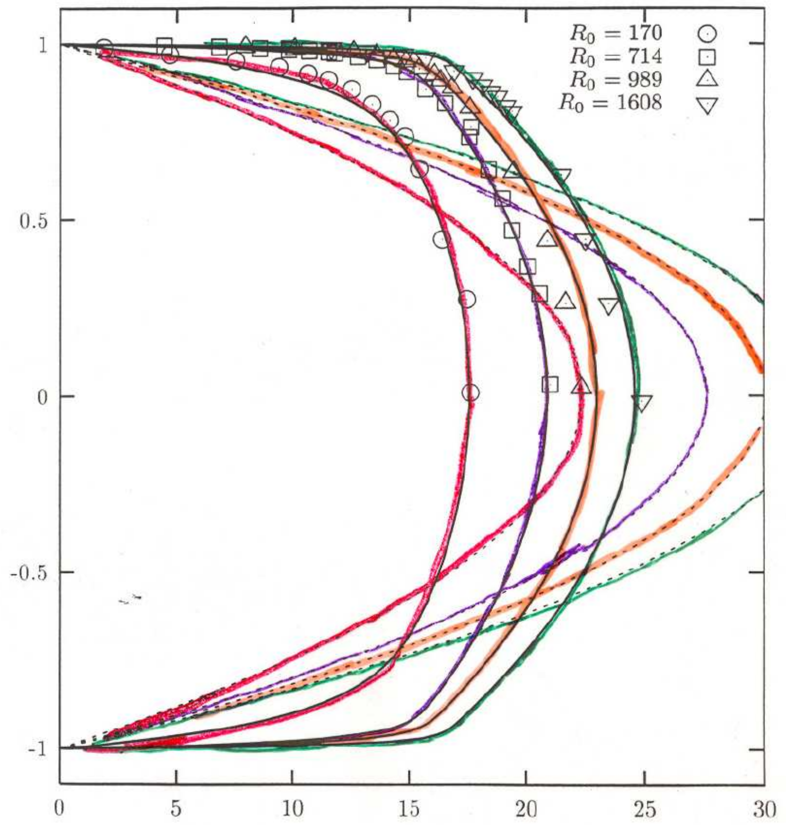
Figure 3



R_c	R_0	ϕ_m	$b/2a$	d/α
2970	170	17.6	1.2	12.850378
14914	714	20.9	1.1	48.782079
22776	989	23	.9	65.777777
39582	1608	24.6	.9	100.569105

$$\phi = \frac{a}{u_*}\left(1 - \frac{\cosh(z/\alpha)}{\cosh(d/\alpha)}\right) + \frac{b}{u_*}\left(1 - \frac{z^2}{d^2}\right)$$

Figure 4



R_c	R_0	ϕ_m	$b/2a$	d/α
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14914	714	20.9	1.1	48.782079
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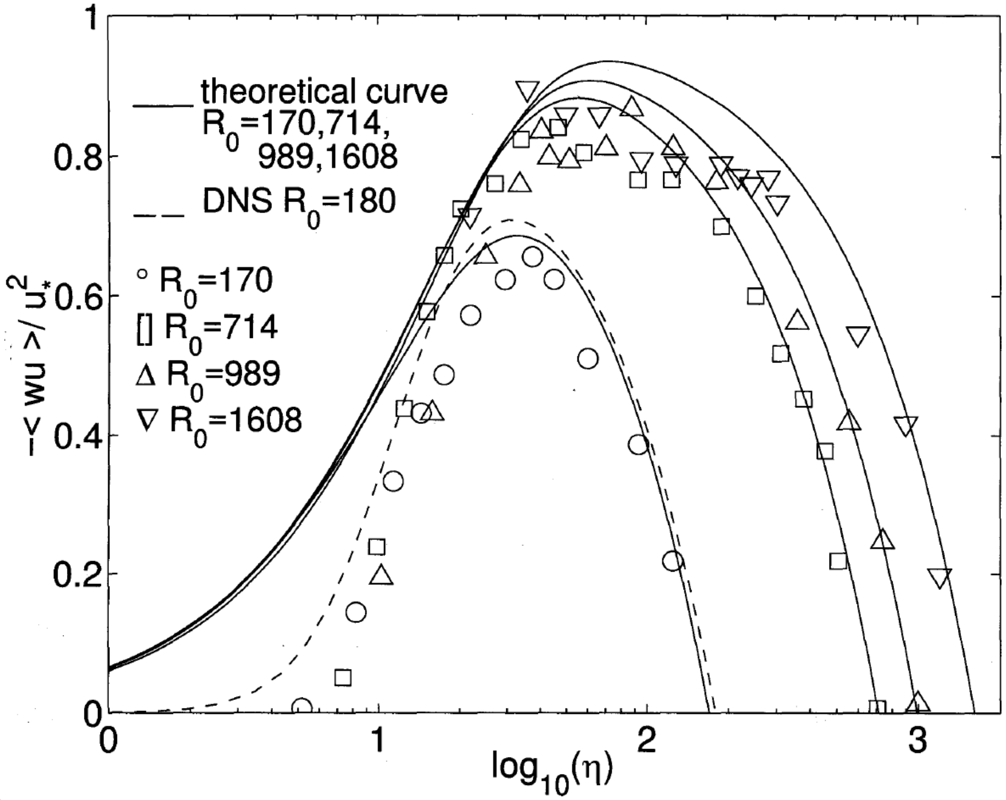
$$\phi = \frac{a}{u_*}\left(1 - \frac{\cosh(z/\alpha)}{\cosh(d/\alpha)}\right) + \frac{b}{u_*}\left(1 - \frac{z^2}{d^2}\right)$$

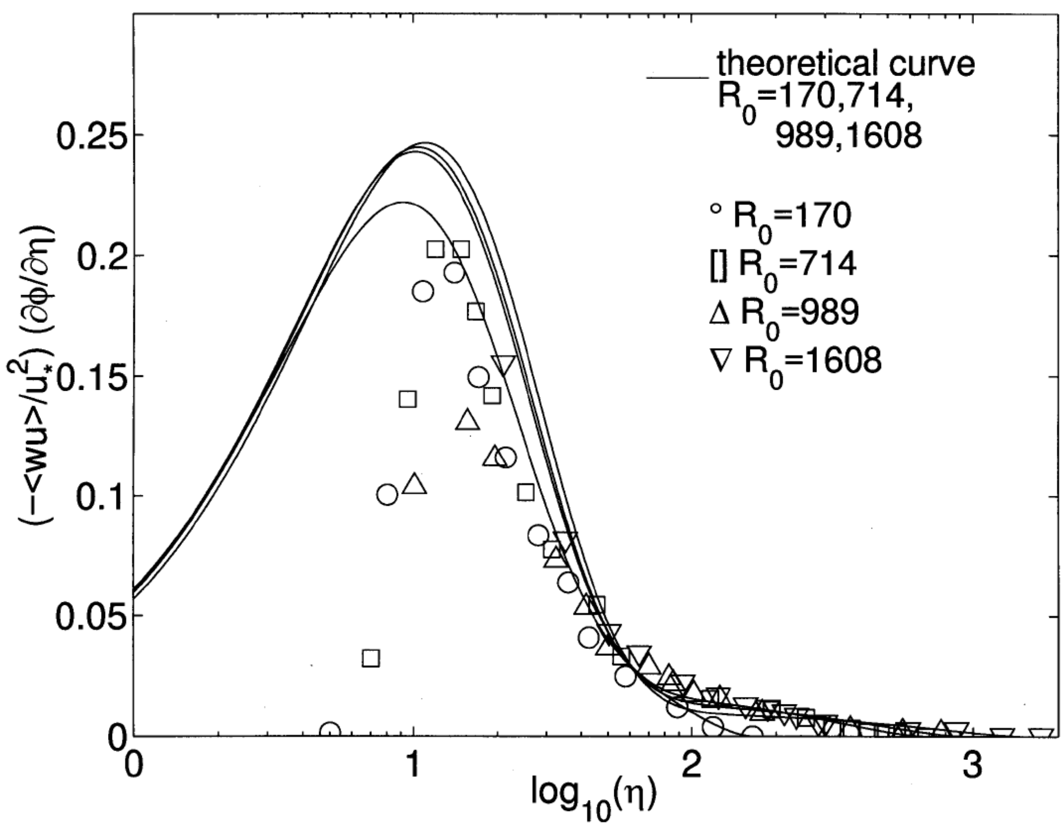
- Experimental data from:
  T. Wei and W.W. Willmarth
- Having blasius drag law
  λ = 0.06

Figure 5



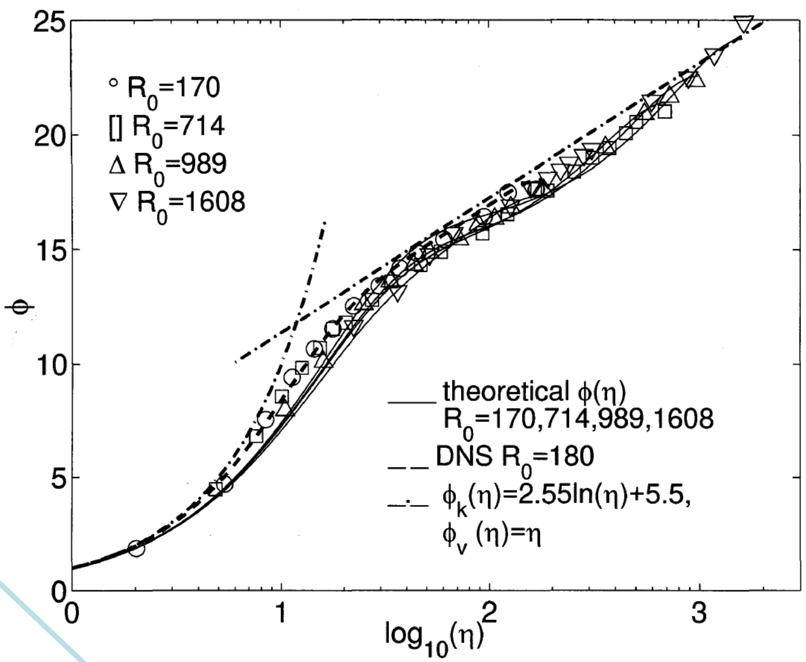
- Experimental data from:
  T. Wei and W.W. Willmarth
- DNS Kim, Moin & Moser
- Having Blasius drag law
  $\lambda = 0.06$

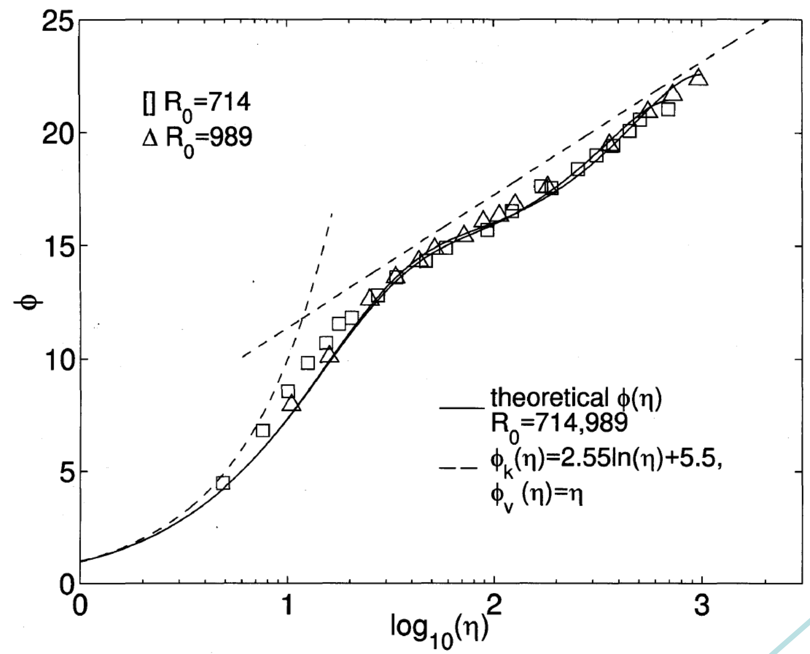
Figure 6



- Experimental data from:
  T. Wei and W.W. Willmarth
- Having blasius drag law
  $\lambda = 0.06$

Figure 7 and Figure 8





- Experimental data from:
  T. Wei and W.W. Willmarth
- DNS Kim, Moin & Moser
- Having blasius drag law
  $\lambda = 0.06$

Room for Improvement

α -- constant away from the boundary

Near the boundary

α -- is a function of the distance from the boundary

First Attempt:

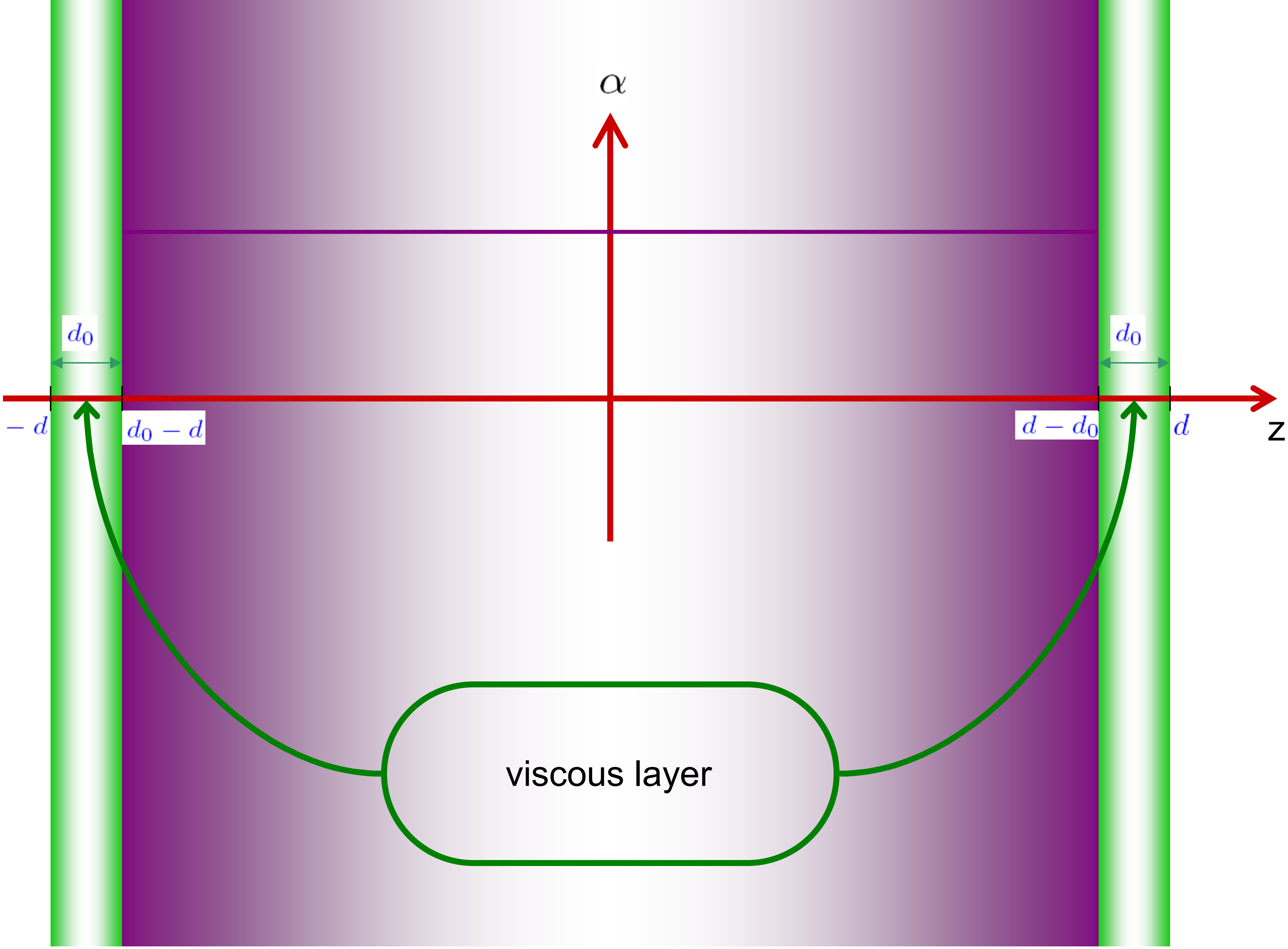
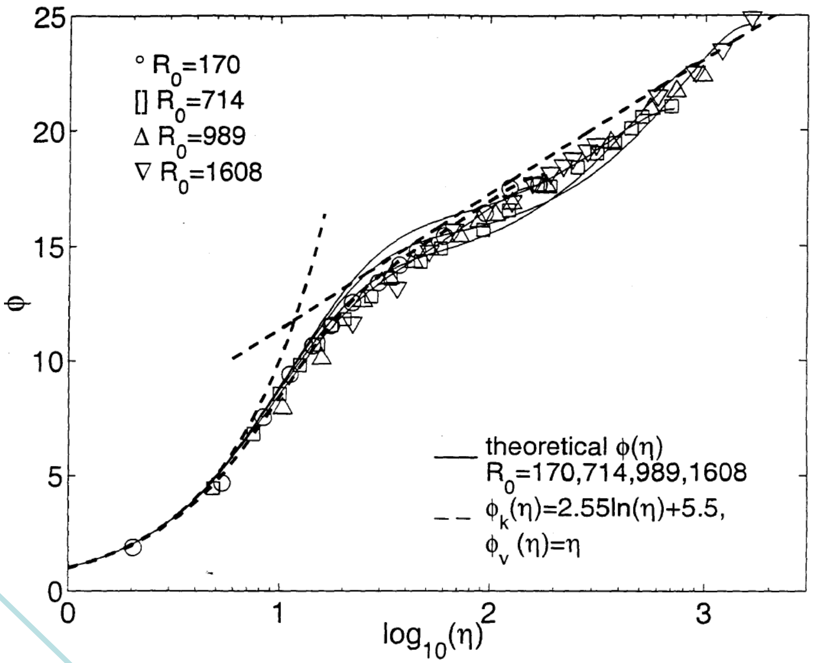
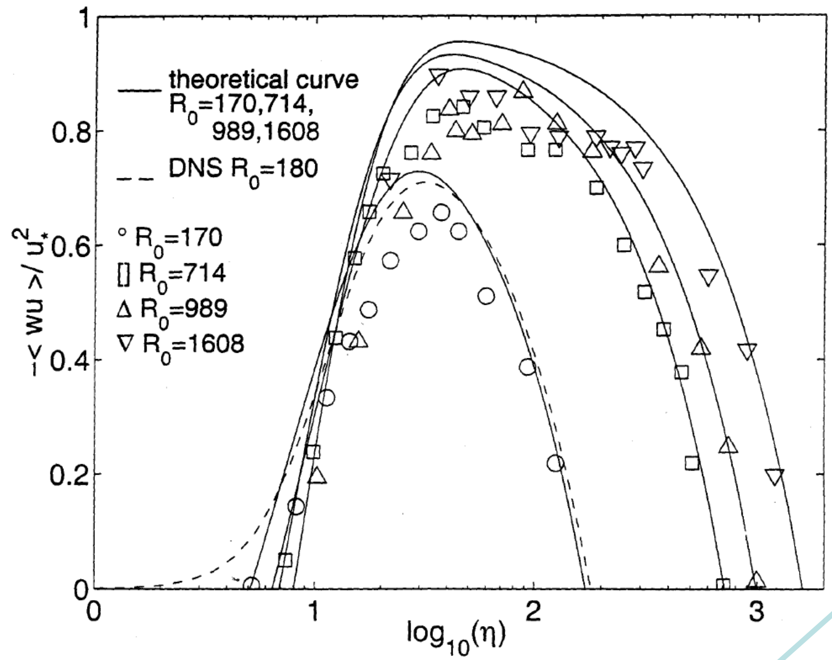


Figure 9 and Figure 10



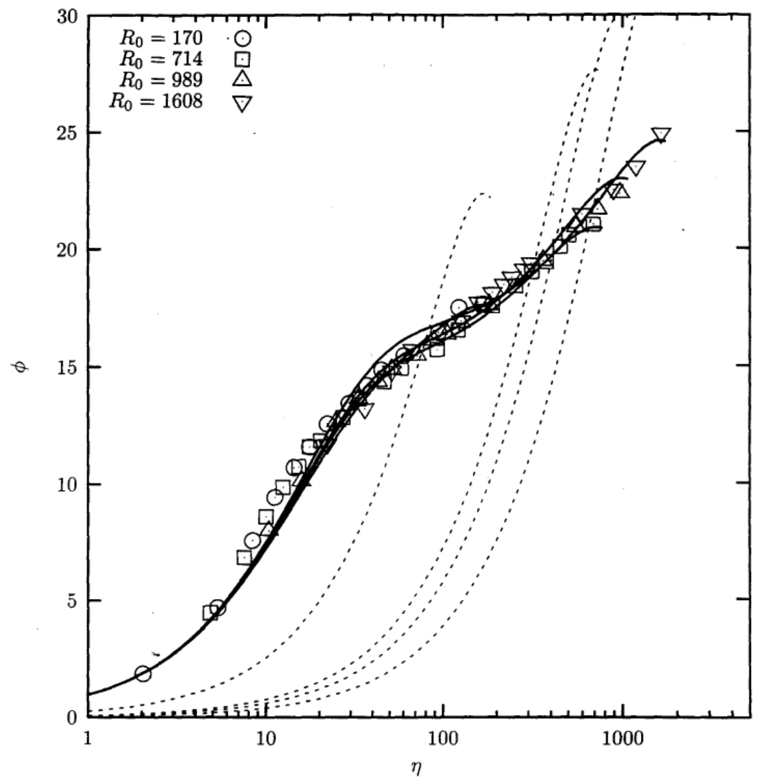


$$\frac{d-d_0}{d} = [.97, .991, .993, .997]$$

Blasius Drag Law $D = .06R^{-1/4}$

Using $\phi(R_0) = \phi_{max}$ as input

Figure 11 and Figure 12



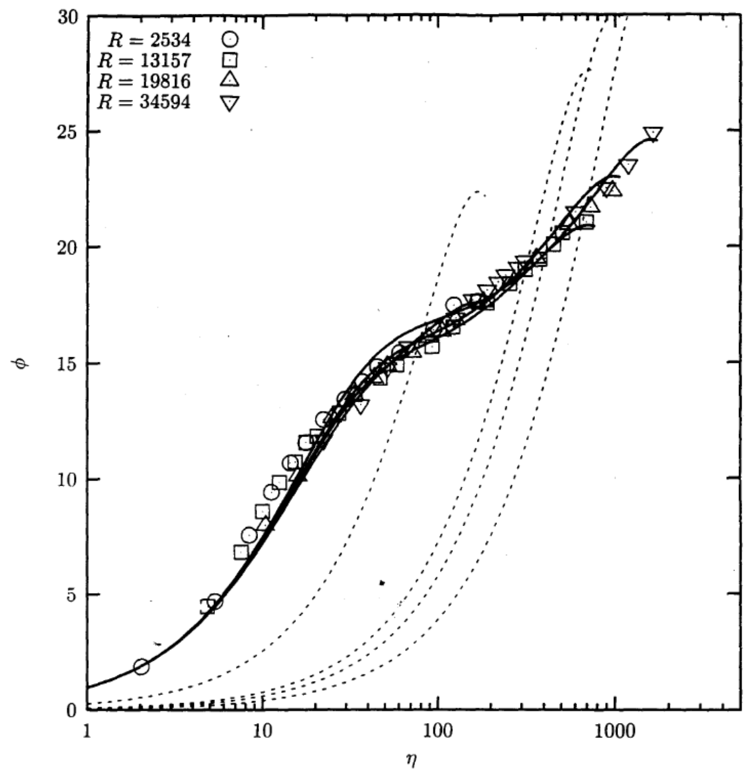
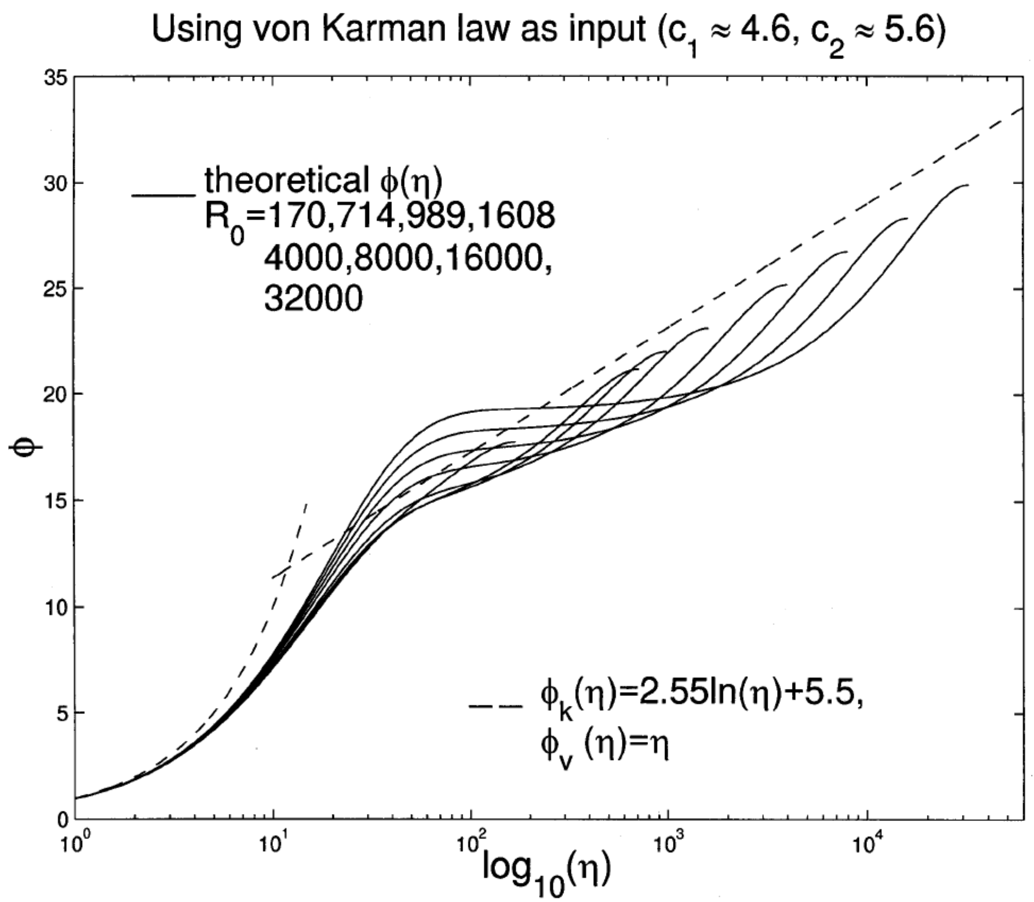
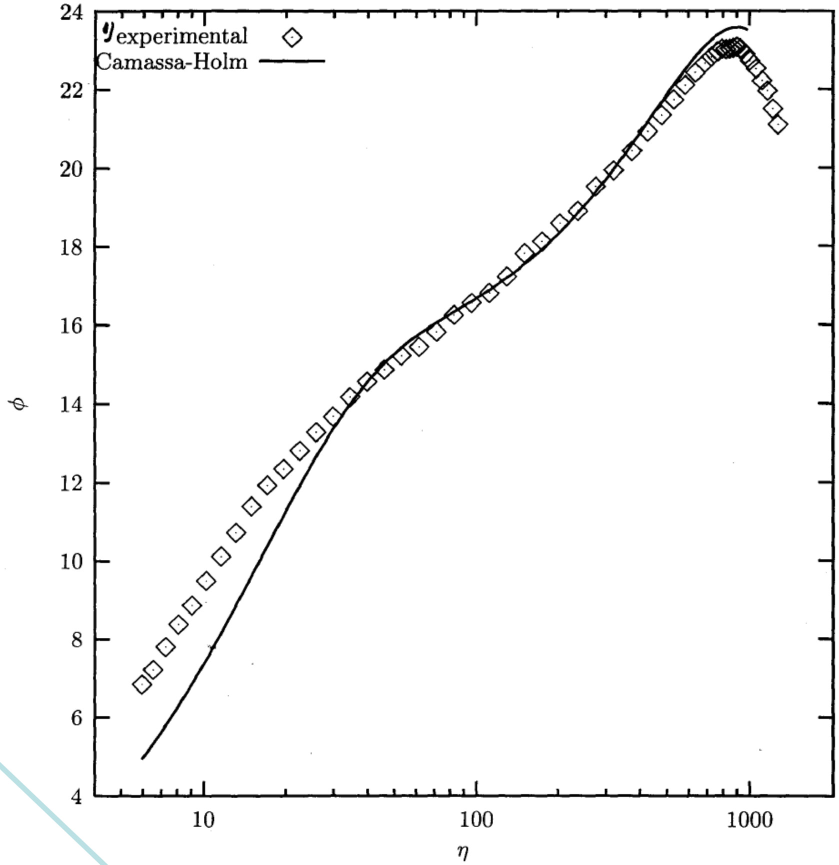


Figure 13



Von Karman drag law

An illustration (Pipe flow)

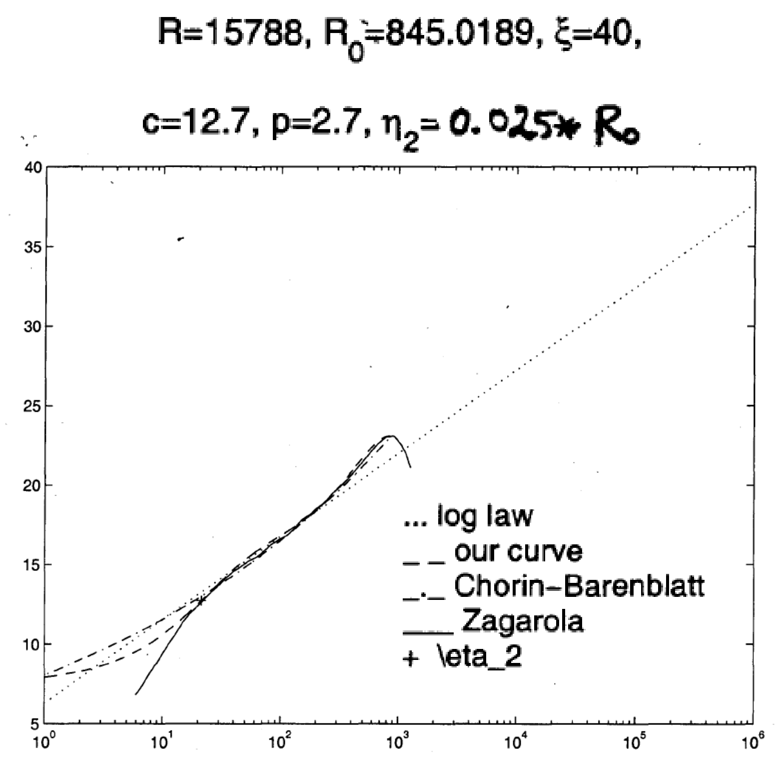


R	R_0	ϕ_m	q_1	ξ
31577	899.98	23.586	.370349	59.926629

Zagarola's thesis,
Princeton University

$$\phi = \phi_m \left\{ \frac{2q_1\eta}{R_0}\left(1 - \frac{\eta}{2R_0}\right) + (1-q_1)\left(1 - \frac{I_0\big(\xi(1-\eta/R_0)\big)}{I_0(\xi)}\right) \right\}$$

Figure 15 and Figure 16



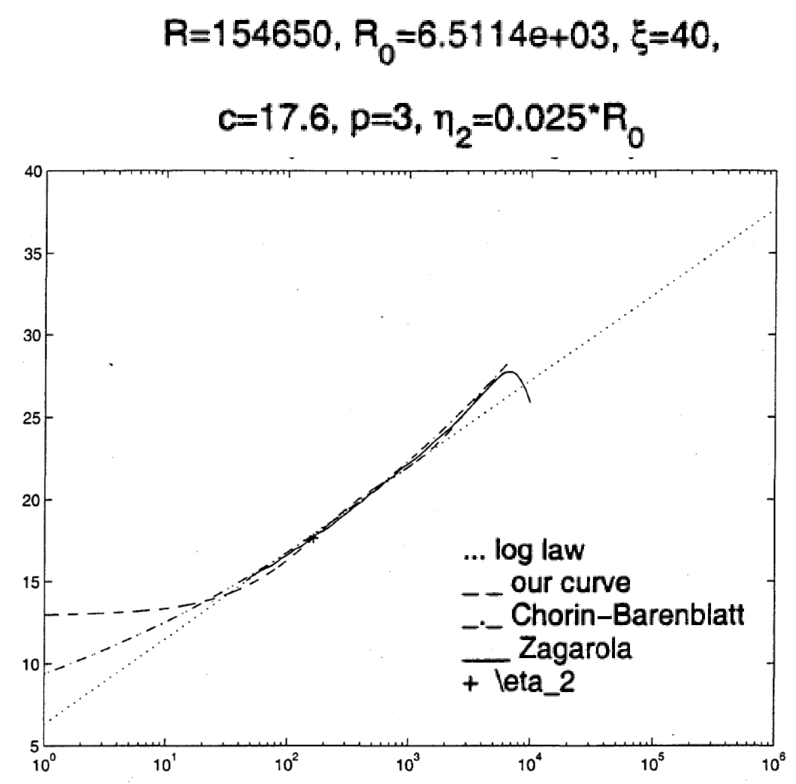
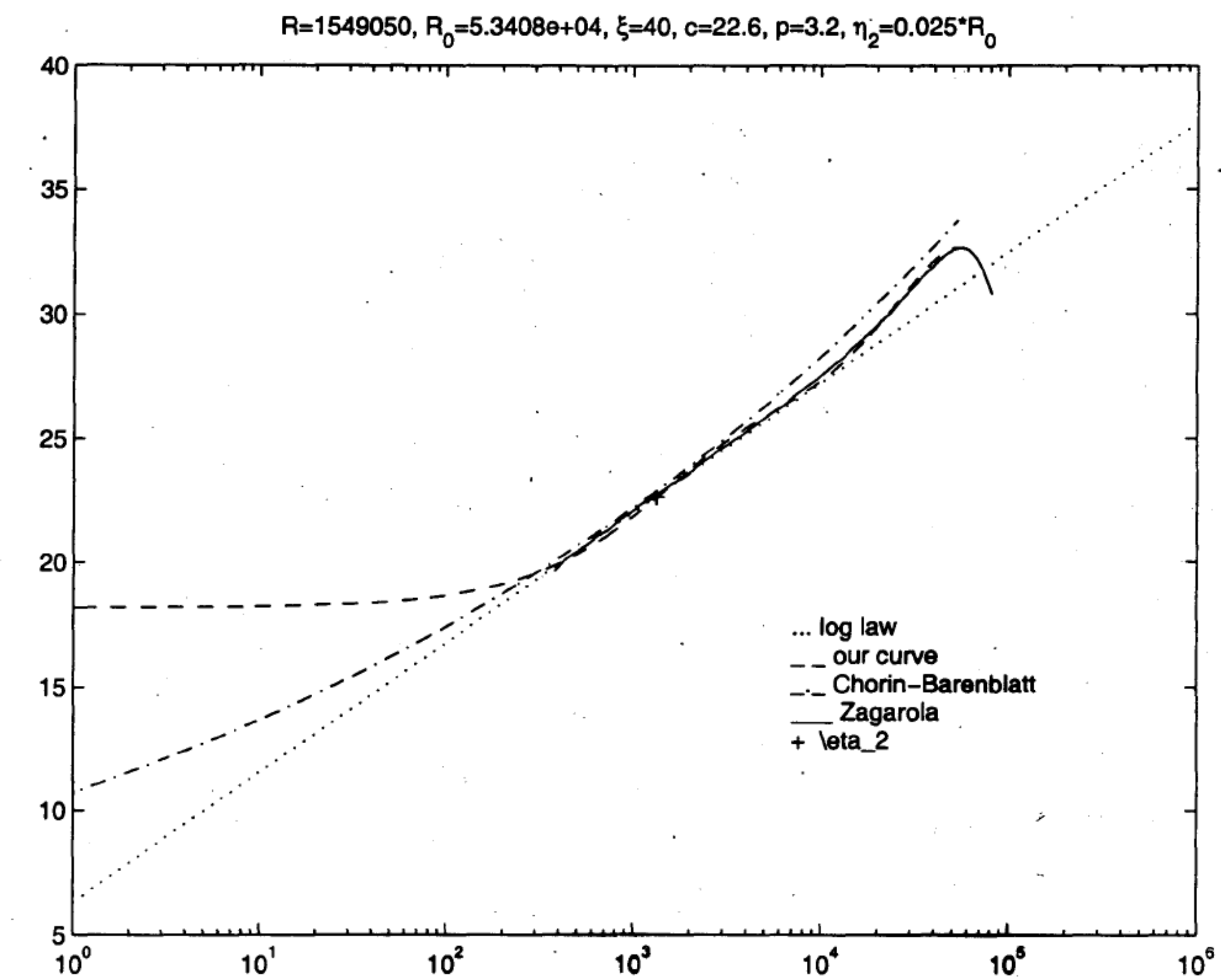
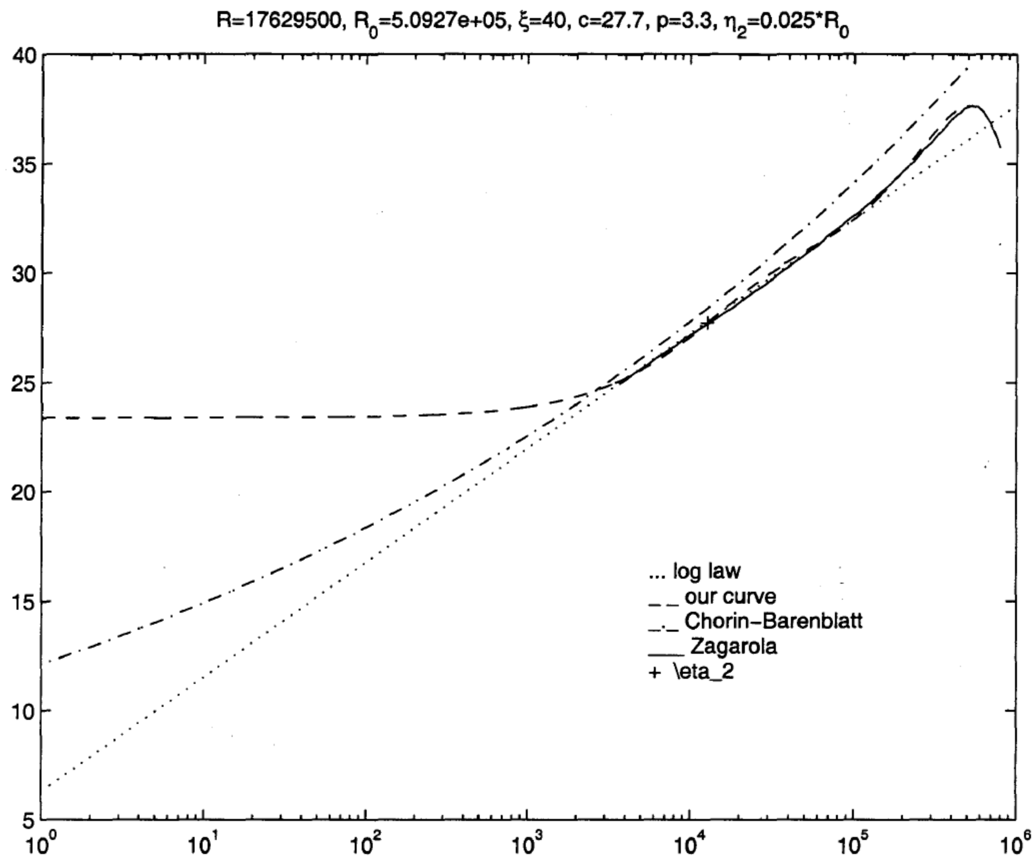


Figure 17



R=1549050, R_0=5.3408e+04, ξ=40, c=22.6, p=3.2, η_2=0.025*R_0
40
35
30
25
20
15
10
5
10^0
10^1
10^2
10^3
10^4
10^5
10^6
... log law
_ _ our curve
_._ Chorin-Barenblatt
____ Zagarola
+ \eta_2

Figure 18



R=17629500, R_0=5.0927e+05, ξ=40, c=27.7, p=3.3, η_2=0.025*R_0
40
35
30
25
20
15
10
5
10^0
10^1
10^2
10^3
10^4
10^5
10^6
... log law
_ _ our curve
_._ Chorin-Barenblatt
____ Zagarola
+ \eta_2

Energy Spectrum

C. Foias et al. / Physica D 152–153 (2001) 505–519

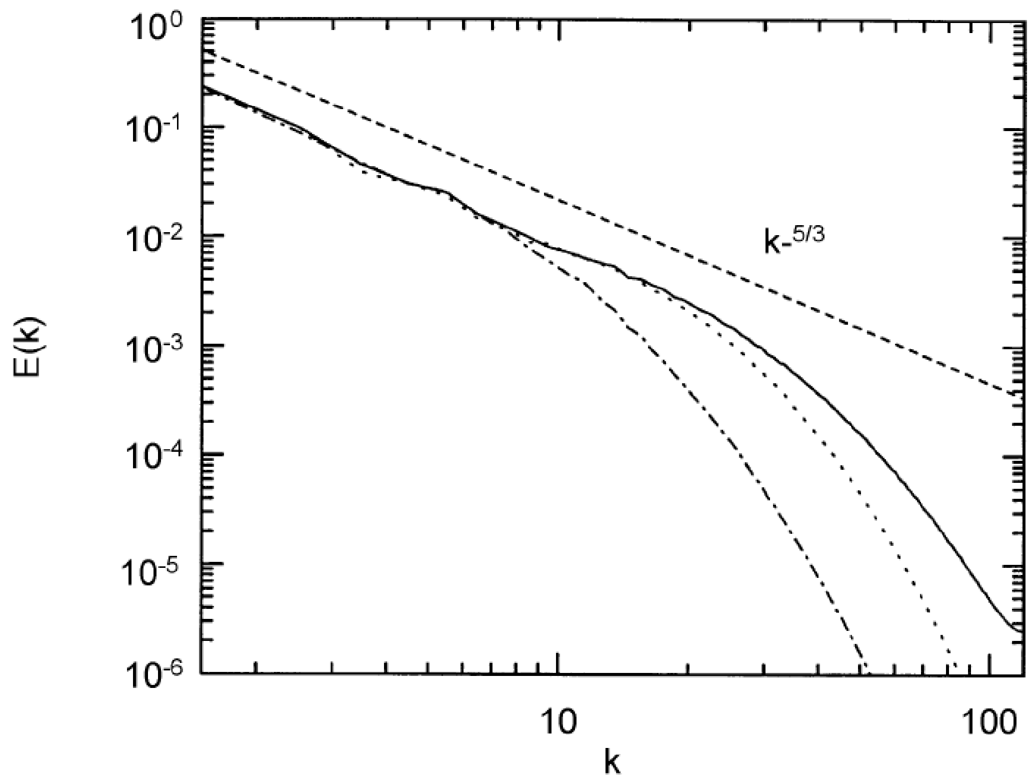
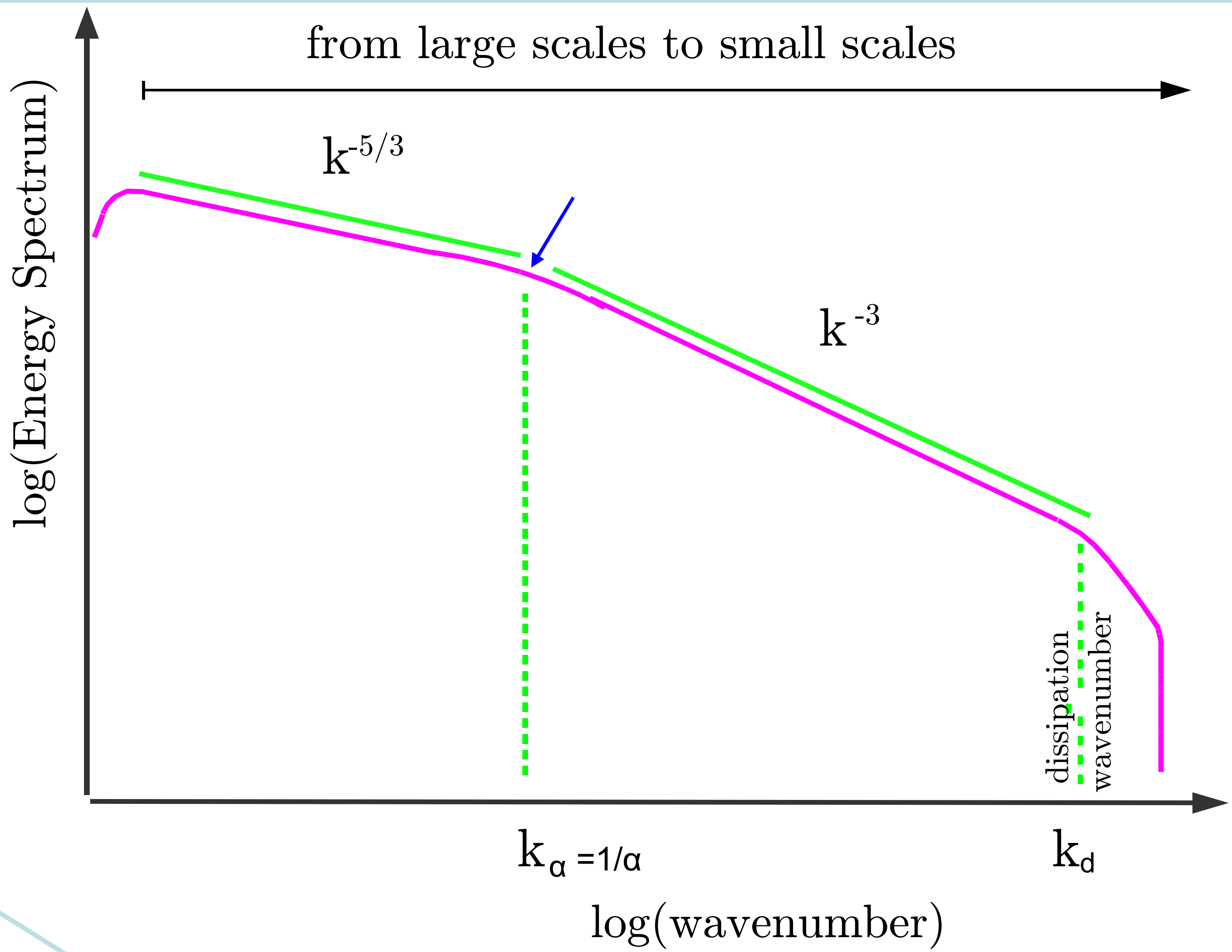


Fig. 1. The DNS energy spectrum, $E(k) = E_\alpha(k)$, versus the wavenumber k for three cases with the same viscosities, same forcings and mesh sizes of 256^3 for $\alpha = 0$ (solid line), $\frac{1}{32}$ (dotted line) and $\frac{1}{8}$ (dotted-dash line). In the inertial range ($k < 20$), a power spectrum with $k^{-5/3}$ can be identified. For finite α, this behavior is seen to roll off to a steeper spectrum for $k \geq 1/\alpha$.

Energy Spectrum (NS-α)



The Navier-Stokes-α subgrid scale model of turbulence

$$\frac{\partial}{\partial_t} v - \nu \Delta v + (u \cdot \nabla) v + \sum_{j=1}^{3} v_j \nabla u_j + \nabla p = f$$

$$\nabla \cdot u = \nabla \cdot v = 0,$$

$$v = u - \alpha^2 \Delta u$$

original velocity

Making the nonlinearity milder

Turns into a complete gradient under the channel and pipe symmetry

The smallest eddy scale still participating actively in the evolution of the flow

Inviscid equation – introduced by Holm, Marsden and Ratiu (Phys. Rev. Let. 1998), called Lagrangian-Averaged Euler - α (No global well-posedness.)

Lagrangian-**A**veraged **N**avier-**S**tokes-α model (LANS-α) or viscous **C**amassa-**H**olm equations (VCHE)

For α =0 we recover NSE

Leray-α Model

NS-α

$$
\begin{cases}
\frac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla) v - \sum_{j=1}^{3} v_j \nabla u_j + \nabla \pi = f \\
\nabla \cdot u = 0 \\
v = (I - \alpha^2 \Delta) u
\end{cases}
$$

Cheskidov, Holm, Olson, Titi (Royal Soc. A, MPES 2005)

The Leray-α analytic subgrid scale model of turbulence

$$
\frac{\partial}{\partial_t} v - \nu \Delta v + (u \cdot \nabla) v + \nabla p = f
$$

$$
\nabla \cdot u = \nabla \cdot v = 0,
$$

$$
v = u - \alpha^2 \Delta u
$$

Aside: Leray Acta Math. 1934 – Regularized NSE

$u = \phi_\alpha * v$

ϕ_α - the Green's function associated with $(1 - \alpha^2 \Delta)$

Navier-Stokes

The Navier-Stokes equations can also be written as:

$$\frac{\partial v}{\partial t} - \nu \Delta v - v \times (\nabla \times v) + \nabla q = 0$$

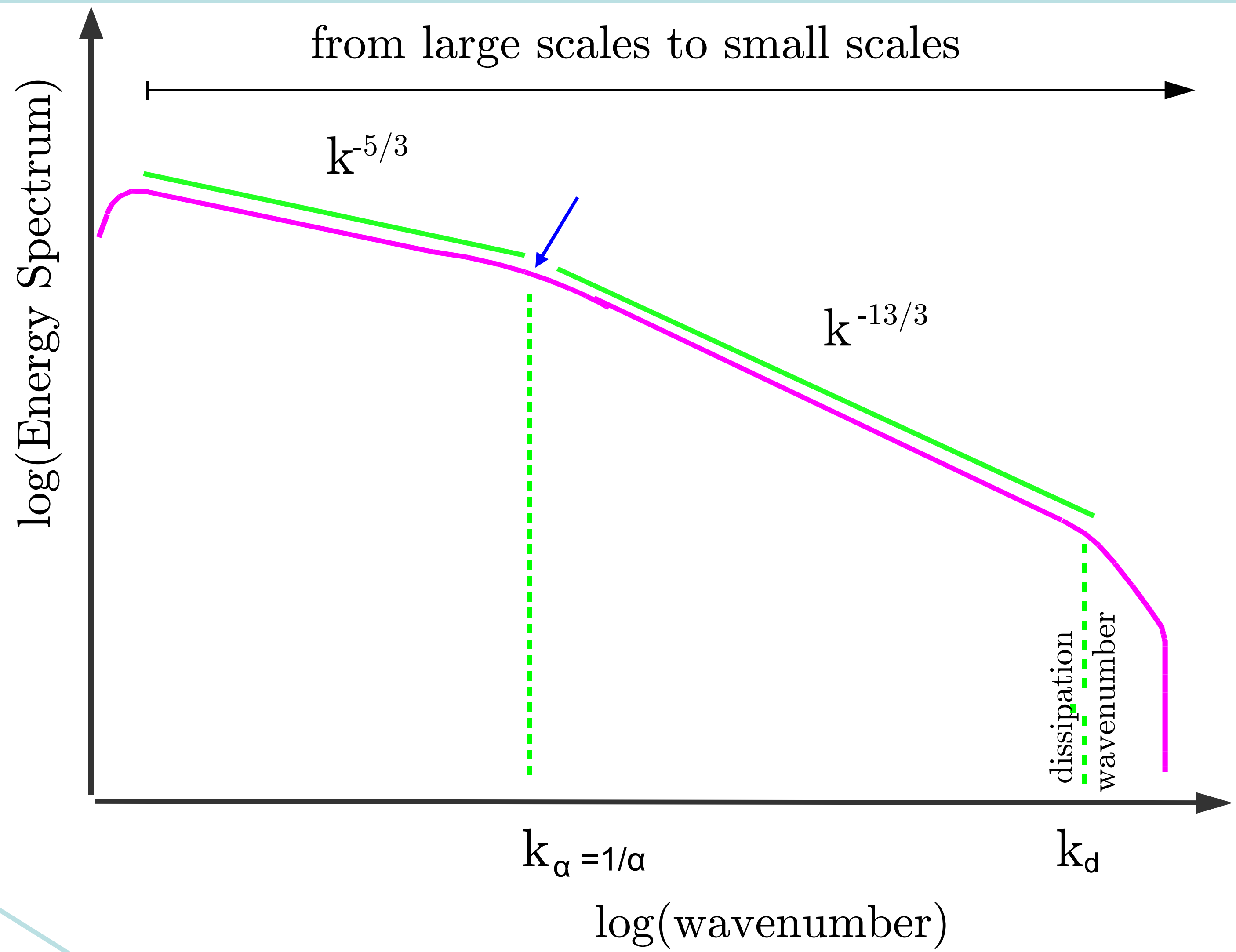
If we mollify the nonlinear term in the previous formulation in the spirit of Leray regularization we obtain the NS-α Model:

$$\frac{\partial v}{\partial t} - \nu \Delta v - u \times (\nabla \times v) + \nabla p = f$$

$$\nabla \cdot u = 0$$

$$v = (I - \alpha^2 \Delta) u$$

Energy Spectrum (Leray-α)



Clark-α Model

C. Cao, D. Holm and E.S.T., *Jour. Of Turbulence,* ***6*** (2005)

The Clark-alpha subgrid scale model of turbulence

$$\partial_t v - \nu\Delta v + (u\cdot\nabla)v + (v\cdot\nabla)u - (u\cdot\nabla)u - \alpha^2\nabla\cdot(\nabla u\cdot\nabla u^T) + \nabla q = g,$$

$$\nabla\cdot u = \nabla\cdot v = 0$$

Global Existence and Uniqueness

Attractors dimension and Energy Sepctrum like Navier-Stokes-alpha

ML-α Model

A. Ilyin, E. Lunasin and E.S.T., *Journ. Nonlinear Science,* **19,** (2006)

Modified Leray-α sub-grid scale Model

$$\frac{\partial}{\partial_t} v - \nu \Delta v + (v \cdot \nabla) u + \nabla p = f$$

$$\nabla \cdot v = 0$$

$$v = u - \alpha^2 \Delta u$$

Global Existence and Uniqueness

Attractor’s dimension and Energy Spectrum like Navier-Stokes-alpha

Simplified Bardina Model

Y. Cao, E. Lunasin, and E.S.T, *Comm. Math Sci.* ***4,*** *(2006)*

Simplified Bardina turbulence model

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) u = -\nabla p + f,$$

$$\nabla \cdot u = \nabla \cdot v = 0,$$

$$v = u - \alpha^2 \Delta u,$$

Y. Cao, E. Lunasin, E.S. Titi (CMS 2006)

Simplified Bardina turbulence model

$$\partial_t v - \nu \Delta v + (u\cdot\nabla)u = -\nabla p + f,$$
$$\nabla\cdot u = \nabla\cdot v = 0,$$
$$v = u - \alpha^2 \Delta u,$$

1980 Bardina $\quad \mathcal{R}(v,v) \approx \overline{\bar{v}\otimes\bar{v}} - \bar{\bar{v}}\otimes\bar{\bar{v}}$

2003 Layton, Lewandowski $\quad \mathcal{R}(v,v) \approx \overline{\bar{v}\otimes\bar{v}} - \bar{v}\otimes\bar{v}$

The Navier-Stokes equations

$$\partial_t v - \nu \Delta v + \nabla\cdot(v\otimes v) = -\nabla p + f,$$
$$\nabla\cdot v = 0,$$
$$v(x,0) = v^{in}(x),$$

Reynolds Average Navier-Stokes

$$\partial_t \bar{v} - \nu \Delta \bar{v} + \nabla\cdot(\overline{v\otimes v}) = -\nabla\bar{p} + \bar{f},$$
$$\nabla\cdot\bar{v} = 0,$$

$$\nabla\cdot(\overline{v\otimes v}) = \nabla\cdot(\bar{v}\otimes\bar{v}) + \nabla\cdot\mathcal{R}(v,v),$$
$$\mathcal{R}(v,v) = \overline{v\otimes v} - \bar{v}\otimes\bar{v}$$

Simplified Bardina Model

Improvement from Layton and Lewandowski (2003)

initial data: $f \in L^2,\ u(0) = u^{in} \in H^1$

weak solution: $u \in C([0,T]; H^1) \cap L^2([0,T]; H^2)$

$$\frac{du}{dt} \in L^2\left([0,T); L^2\right)$$

$$d_H(\mathcal{A}) \le d_F(\mathcal{A}) \le c \left(\frac{L}{\alpha}\right)^{12/5} \left(\frac{L}{l_d}\right)^{12/5}$$

Y. Cao, E. Lunasin, E.S. Titi (CMS 2006)

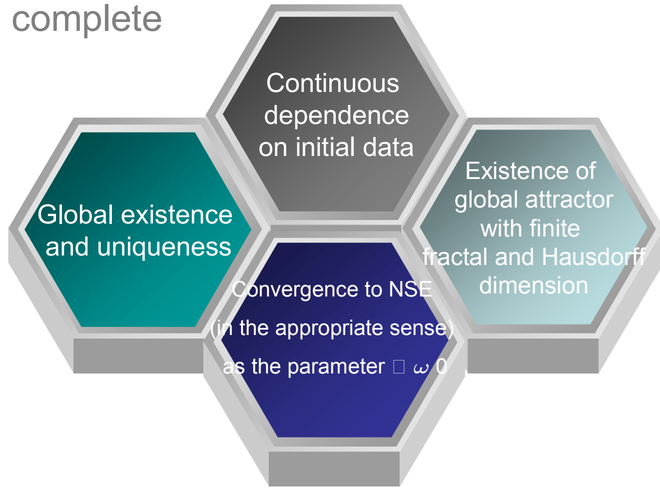
Simplified Bardina turbulence model

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) u = -\nabla p + f,$$

$$\nabla \cdot u = \nabla \cdot v = 0,$$

$$v = u - \alpha^2 \Delta u,$$

The mathematical theory of simplified Bardina is complete



Excellent match with experimental data

Energy spectra

Inviscid Simplified Bardina Model

Y. Cao, E. Lunasin, E.S.T., *Communications in Math. Sciences,* **4** (2006)

$$\begin{aligned} \partial_t v - \cancel{\nu\Delta v} + (u\cdot\nabla)u &= -\nabla p + f, \\ \nabla\cdot u = \nabla\cdot v &= 0, \\ v &= u - \alpha^2\Delta u, \end{aligned}$$

$$\begin{aligned} -\alpha^2\Delta\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p &= f, \\ \nabla\cdot u &= 0, \\ u(x,0) &= u^{in} \end{aligned}$$

This result has important application in computational fluid dynamics when the inviscid model is considered as a regularizing model of the 3D Euler equations.

Also note that the inviscid simplified Bardina model is a globally well-posed model approximating the Euler equations without adding hyperviscous regularizing term.

Euler-Voigt Model

A. Larios and E.S.T. (2009)

$$
\begin{cases}
-\alpha^2 \Delta \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = f, \\
\nabla \cdot u = 0, \\
u(x,0) = u^{in}
\end{cases}
$$

- High order regularity: If $u^{in} \in H^m$, for $m \geq 1$, then $u(t) \in H^m$ for all $t \in (-\infty, \infty)$.
- Similar result are also valid for Gevery regularity.

Blow-up Critertion for Euler Equations

Assume $u^{in} \in H^s$, for some $s \geq 3$. Suppose there exists a finite time $T^{**} > 0$ such that the solutions u_α of the Euler-Voight model with $u_\alpha^{in} = u^{in}$, for each $\alpha > 0$, satisfy

$$\sup_{t \in [0,T^{**})} \limsup_{\alpha \to 0} \alpha^2 \|\nabla u_\alpha(t)\|^2_{L^2([0,1]^3)} > 0.$$

Then the three-dimensioal Euler equations with initial data u^{in} develop a singularity in the interval $[0, T^{**}]$.

The Navier-Stokes-Voigt Model

$$-\alpha^2 \Delta \partial_t u + \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f$$

$$\nabla \cdot u = 0$$

This is a global regularization of the three-dimensional Navier-Stokes.

This regularization works also in the case of no-slip Dirichlet Boundary conditions.

Navier-Stokes-Voigt equations

$$-\alpha^2\Delta\frac{\partial u}{\partial t}+\frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f,$$
$$\nabla\cdot u=0,$$
$$u(x,0)=u^{in}$$

This model was introduced by Oskolkov (1973) as a model of motion of linear, viscoelastic fluids. Models dynamics of Kelvin-Voight viscoelastic incompressible fluids.

Global attractors, estimates of the number of determining modes by
V. Kalantarov and E.S.Titi (2009) Chinese Anals of Math.

Surface Quasi-Geostrophic

In two-dimensions:

$$\theta_t + u \cdot \nabla \theta = 0$$

$$u = \nabla^{\perp} (-\Delta)^{-1/2} \theta$$

.

$\nabla\theta$ **Satisfies:**

$$\frac{\partial}{\partial t}(\nabla\theta) + (u \cdot \nabla)(\nabla\theta) + (\nabla u)^T(\nabla\theta) = 0$$

But morally speaking:

$$|\nabla\theta|^2 \sim |(\nabla u)^T(\nabla\theta)|$$

Thus it is like $$\frac{dz}{dt} \sim z^2$$

Inviscid Regularization of the Surface Quasi-Geostrophic

B. Khouider and E.S. Titi, *Communications Pure Applied Math.* (2007)

$$-\alpha^2 \Delta \theta_t + \theta_t + u \cdot \nabla \theta = 0$$

$$u = \nabla^{\perp} \left(-\Delta \right)^{-1/2} \theta$$

.

Blow-up criterion for SQG

Theorem: [Khouider-Titi, 2007]

If for some $T^* > 0$ we have

$$\sup_{t\in[0,T^*)} \limsup_{\alpha\to 0} \alpha^2 \|\nabla\theta(t)\|_{L^2}^2 > 0.$$

Then the SQG has a singularity $[0, T^*)$.

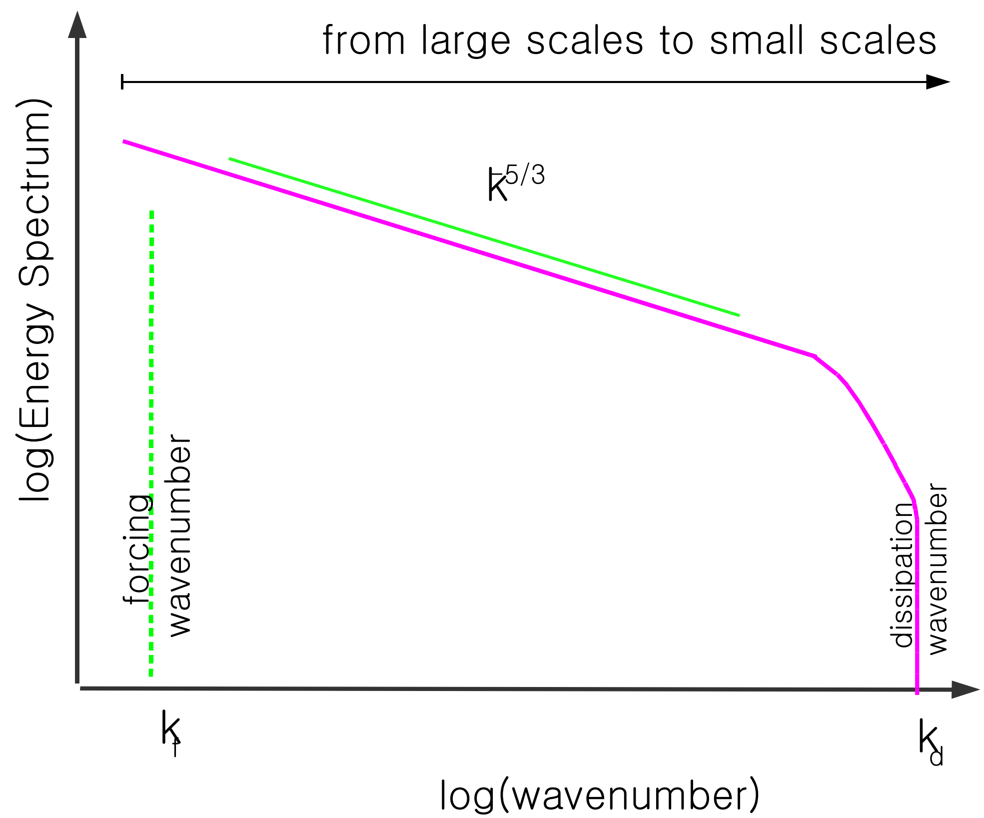
Energy Spectra for Navier-Stokes

S. Kurien
E. M. Lunasin
M. Taylor
E. S. Titi

What has been done in 3D NS-α? Recall: 3D NSE Energy Spectrum

$$E \sim E(k, \varepsilon) = Ck^{-5/3}\varepsilon^{2/3}$$

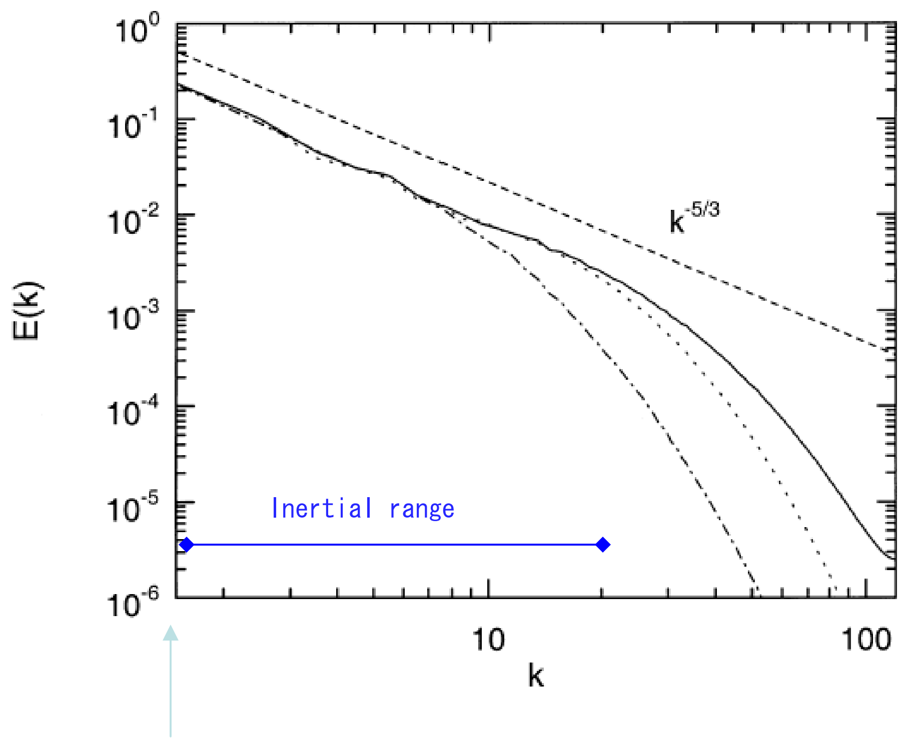
$$k_d(\nu, \varepsilon) = C\nu^{-3/4}\varepsilon^{1/4} = \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$$



What has been done in 3D NS-α?

Large scale dynamics of the flow is captured by the NS-α eqautions.

S. Chen et al. / Physica D 133 (1999) 66–83



256^3

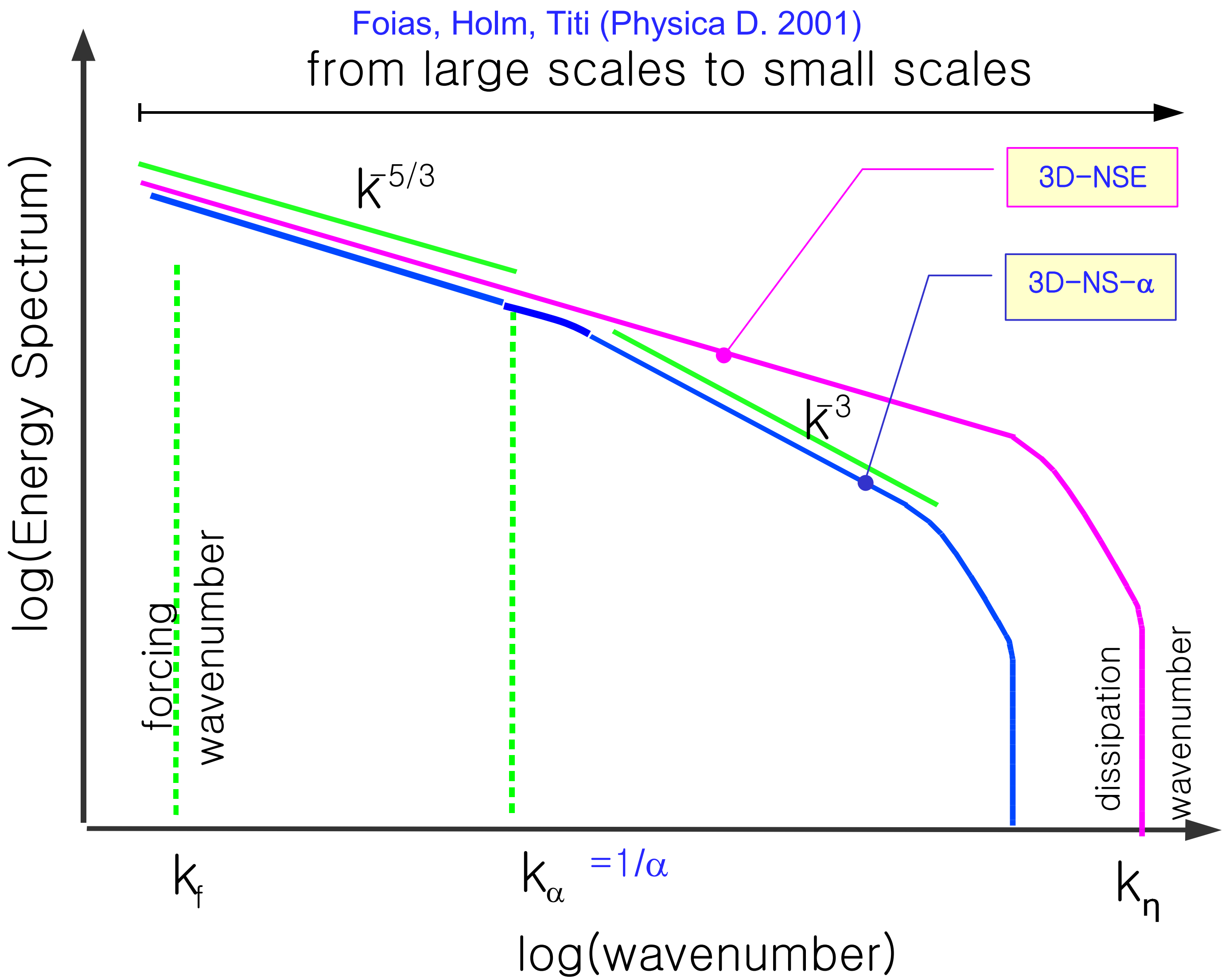
α= 0

α = 1/8

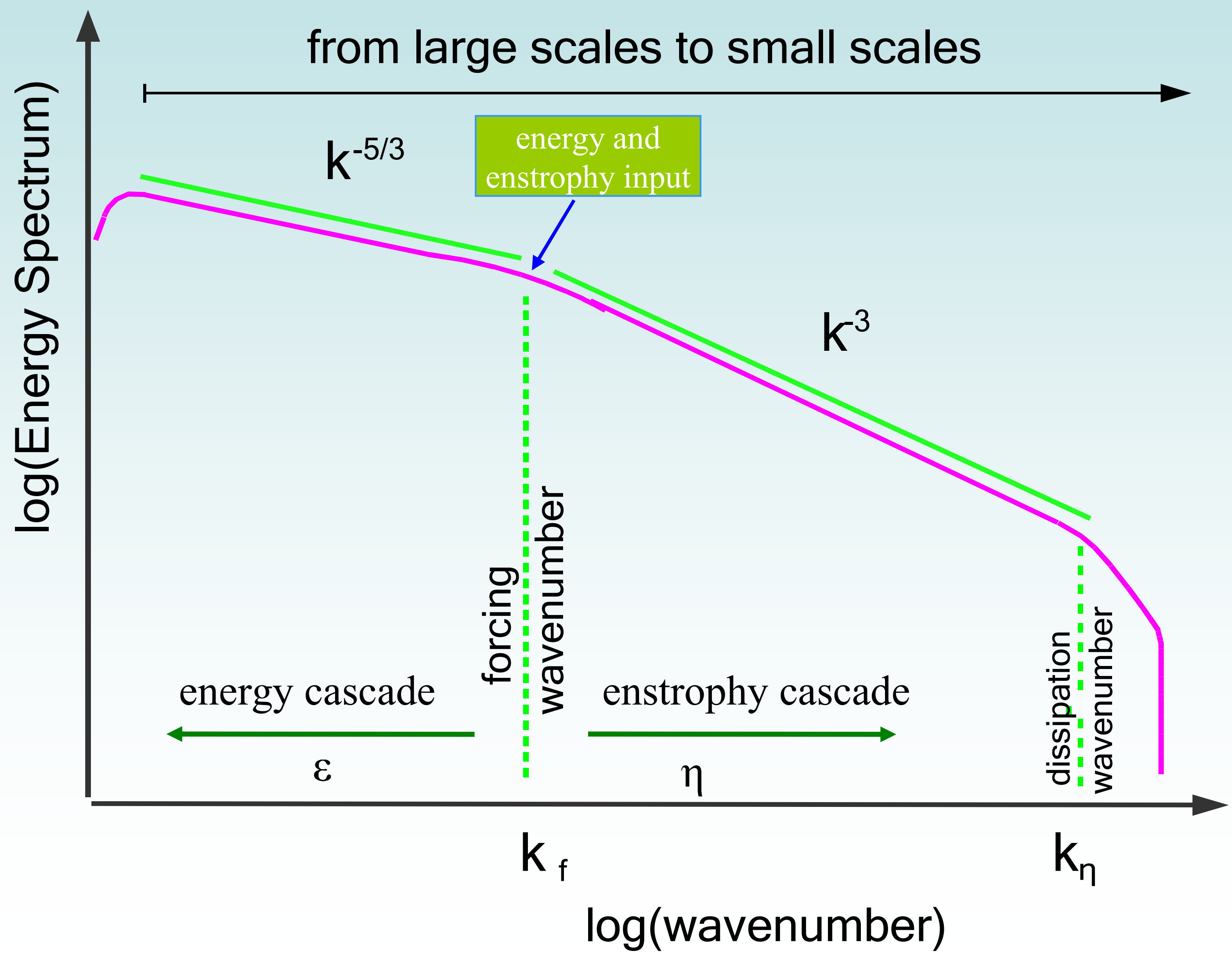
α = 1/32

Also by Mohseni, Kosovic, Shkoller and J. Marsden (2003 Phys. Fluids)

Energy Spectrum of the 3D NS-□



Energy Spectrum of Two-Dimensional Navier-Stokes equations



Observation:

In 2d NS-α the conserved $(\nu = 0 \text{ and } f = 0)$
"energy" and "enstrophy" are as follows

Recall that we have two kinds of velocity

NS-α

$$\partial_t v - u \times (\nabla \times v) + \nabla \tilde{p} = \nu \Delta v + f$$

$$v = (1 - \alpha^2 \Delta) u$$

Don't forget

v un-smoothed velocity field
u smoothed velocity field

$$\frac{1}{2}\frac{d}{dt}\langle v, u\rangle = -\nu(|\nabla u|^2 + \alpha^2 |\Delta u|^2) + \langle f, u\rangle$$

$$\langle u \times \nabla \times v, u\rangle = 0$$

energy conserved $:= \frac{1}{2}(|u|^2 + \alpha^2 |\nabla u|^2)$

NS-α vorticity formulation

$$\partial_t q + (u \cdot \nabla) q = \nu \Delta q + \nabla \times f$$

vorticity $q = \nabla \times v$

$$(q \cdot \nabla) u = \vec{0}$$

$$\frac{1}{2}\frac{d}{dt}|q|^2 = -\nu |\nabla q|^2 + \langle \nabla \times f, q\rangle$$

$$\langle u \cdot \nabla q, q\rangle = \vec{0}$$

enstrophy conserved $:= \frac{1}{2}|q|^2$

Analytical Result 1: The transfer and cascade for the 2d NS-α:

- the energy and enstrophy transfers are as follows:

  below the injection of energy, the energy and enstrophy go from high modes to low modes;

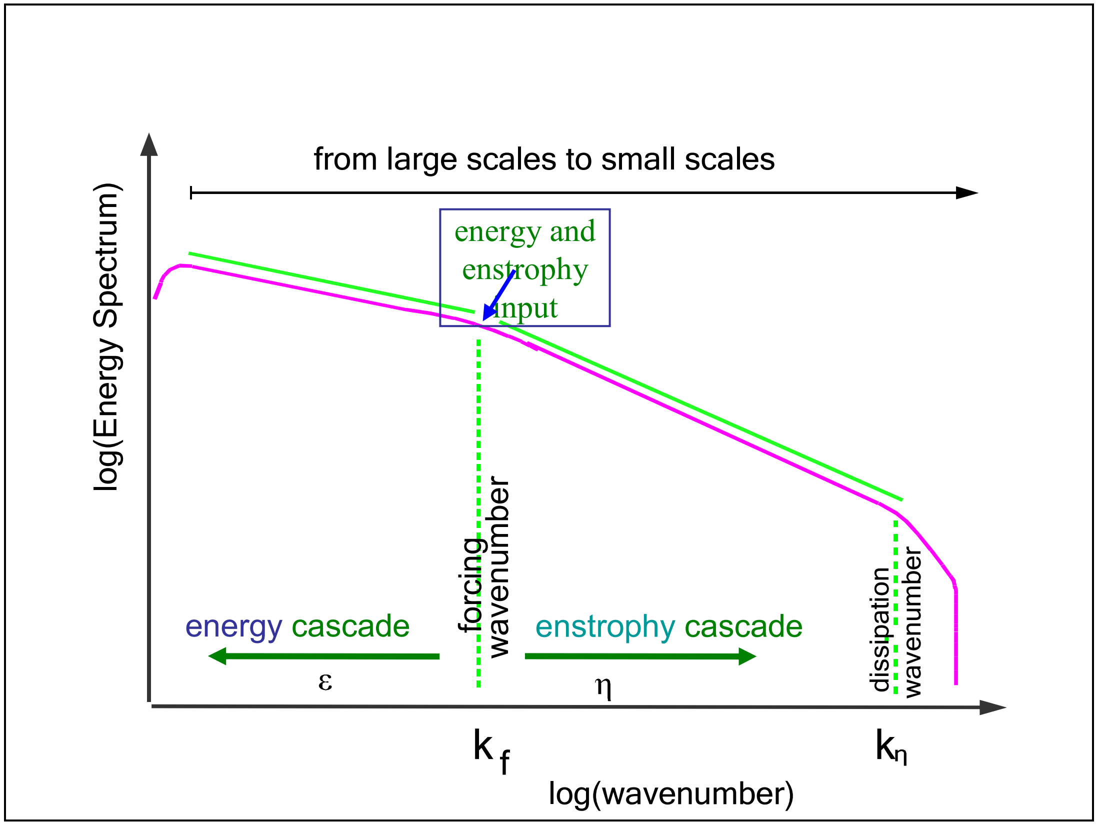
  above the injection of energy, the energy and enstrophy go from low to high modes.

b. The energy and enstrophy cascades are as follows:

  below the injection of energy, we have **inverse energy cascade** ;

  above the injection of energy, we have **direct enstrophy cascade**.

Proof: LKTT (2007, JOT)



Analytical Result 2: Power laws for the 2D NS-α

Proof: LKTT (2007, JOT):

a. Split the flow into 3 wavenumber ranges : [1,k), [k,2k), [2k,∞)
Assume $k_f < k$

b. Define the energy of an eddy of size 1/k as:

c. Enstrophy balance for eddy of size 1/k: where Z_k represents the net amount of enstrophy per unit time transferred into wavenumbers larger than k.

d. Candidates for averaged velocity:

$$u = u_k^< + u_k + u_k^>$$
$$v = v_k^< + v_k + v_k^>$$
$$q = q_k^< + q_k + q_k^> .$$

$$E_\alpha(k) = (1+\alpha^2|k|^2) \sum_{|j|=k} |\hat{u}_j|^2$$

$$\frac{1}{2}\frac{d}{dt}(q_k, q_k) + \nu(-\Delta q_k, q_k) = Z_k - Z_{2k}$$

$$Z_k := -b(u_k^<, q_k^<, q_k + q_k^>) + b(u_k + u_k^>, q_k + q_k^>, q_k^<)$$

Don't forget

$$U_k^0 = \left\langle \frac{1}{L^3} \int_\Omega |v_k|^2 dx \right\rangle^{1/2} \sim$$

$$U_k^1 = \left\langle \frac{1}{L^3} \int_\Omega u_k \cdot v_k dx \right\rangle^{1/2}$$

$$U_k^2 = \left\langle \frac{1}{L^3} \int_\Omega |u_k|^2 dx \right\rangle^{1/2} \sim$$

Therefore we get the following 3 characteristic timescales:

$$\tau_k^n := \frac{1}{kU_k^n} = \frac{(1+\alpha^2k^2)^{(n-1)/2}}{k^{3/2}(E_\alpha(k))^{1/2}} \quad (n=0,1,2)$$

Dissipation rate:

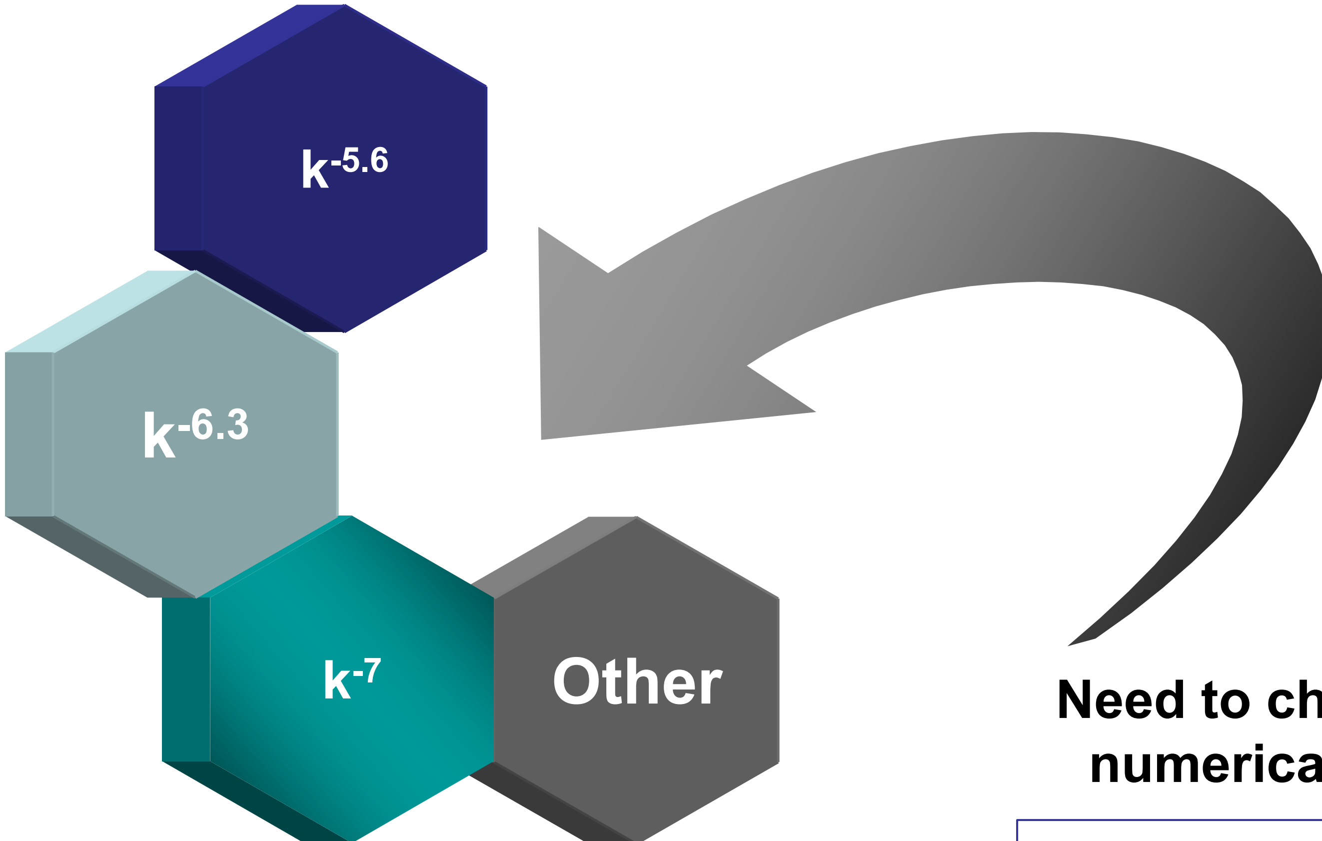
$$\eta \sim \frac{1}{\tau_k^n}\int_k^{2k}(1+\alpha^2k^2)k^2E_\alpha(k)dk \sim \frac{k^{9/2}\left(E_\alpha(k)\right)^{3/2}}{(1+\alpha^2k^2)^{(n-3)/2}}$$

Hence,

$$E_\alpha(k) \sim \frac{\eta^{2/3}(1+\alpha^2k^2)^{(n-3)/3}}{k^3}$$

Main Result: The kinetic energy spectrum for the variable u is:

$$E^u(k) \equiv \frac{E_\alpha(k)}{1+\alpha^2k^2} \sim \begin{cases} \dfrac{\eta_\alpha^{2/3}}{k^3}, & \text{when } k\alpha \ll 1\,, \\ \dfrac{\eta_\alpha^{2/3}}{\alpha^{2(6-n)/3}k^{(21-2n)/3}}, & \text{when } k\alpha \gg 1\,. \end{cases}$$

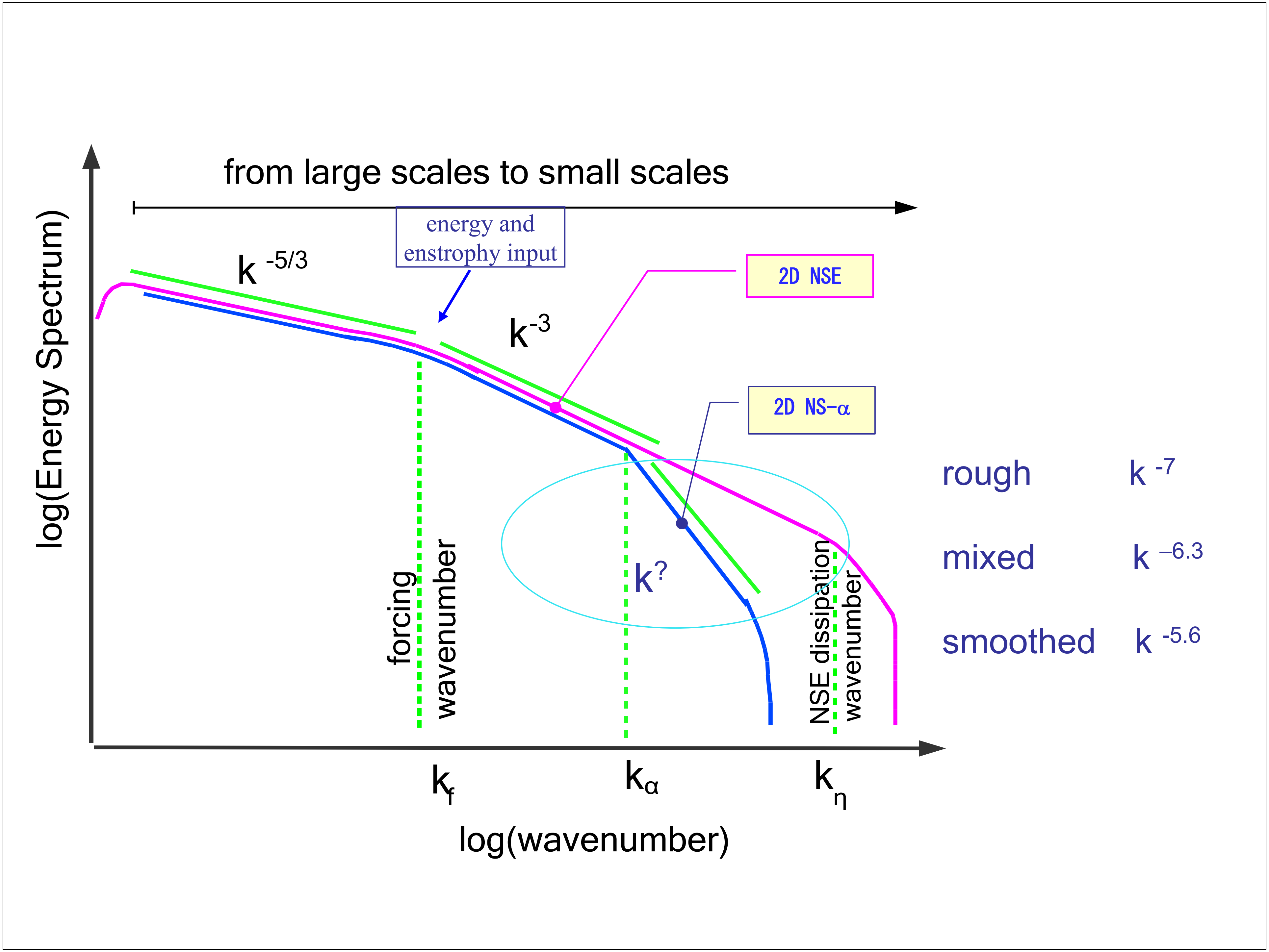


Need to check numerically

$$U_k^0 = \left\langle \frac{1}{L^3} \int_\Omega |v_k|^2 dx \right\rangle^{1/2} \sim$$

$$U_k^1 = \left\langle \frac{1}{L^3} \int_\Omega u_k \cdot v_k dx \right\rangle^{1/2}$$

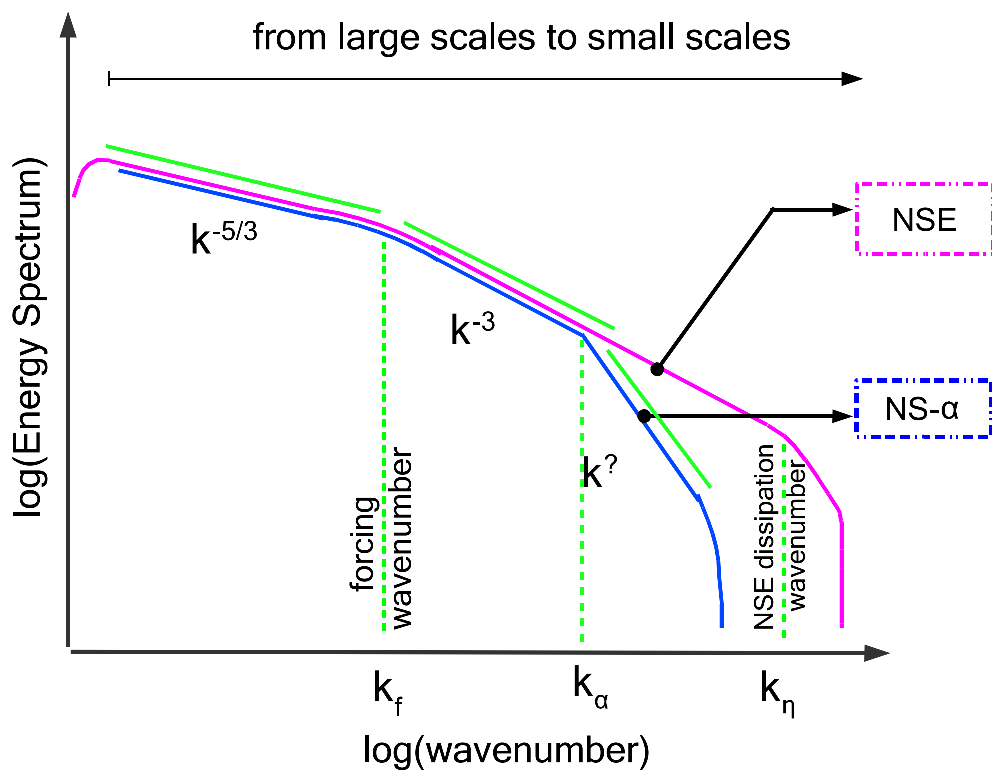
$$U_k^2 = \left\langle \frac{1}{L^3} \int_\Omega |u_k|^2 dx \right\rangle^{1/2} \sim$$



from large scales to small scales
log(Energy Spectrum)
energy and
enstrophy input
k -5/3
k-3
2D NSE
2D NS-α
k?
forcing
wavenumber
NSE dissipation
wavenumber
rough k -7
mixed k −6.3
smoothed k -5.6
k_f
k_α
k_η
log(wavenumber)

Establish two power laws in the enstrophy inertial subrange range numerically.

Verify the semi-rigorous arguments.



What has been done in 2D NS-α?

B. Nadiga and S. Shkoller (2001 Phys. Fluids) –
inverse energy inertial range.

Power law prediction for the energy spectrum for $k > k_\alpha$ in the forward enstrophy cascade regime is $k^{-5.6}$ (not enough resolution to verify).

Figure 1. Energy spectra for a 256^2 simulation with fixed viscosity and varying hypoviscosity coefficient □.

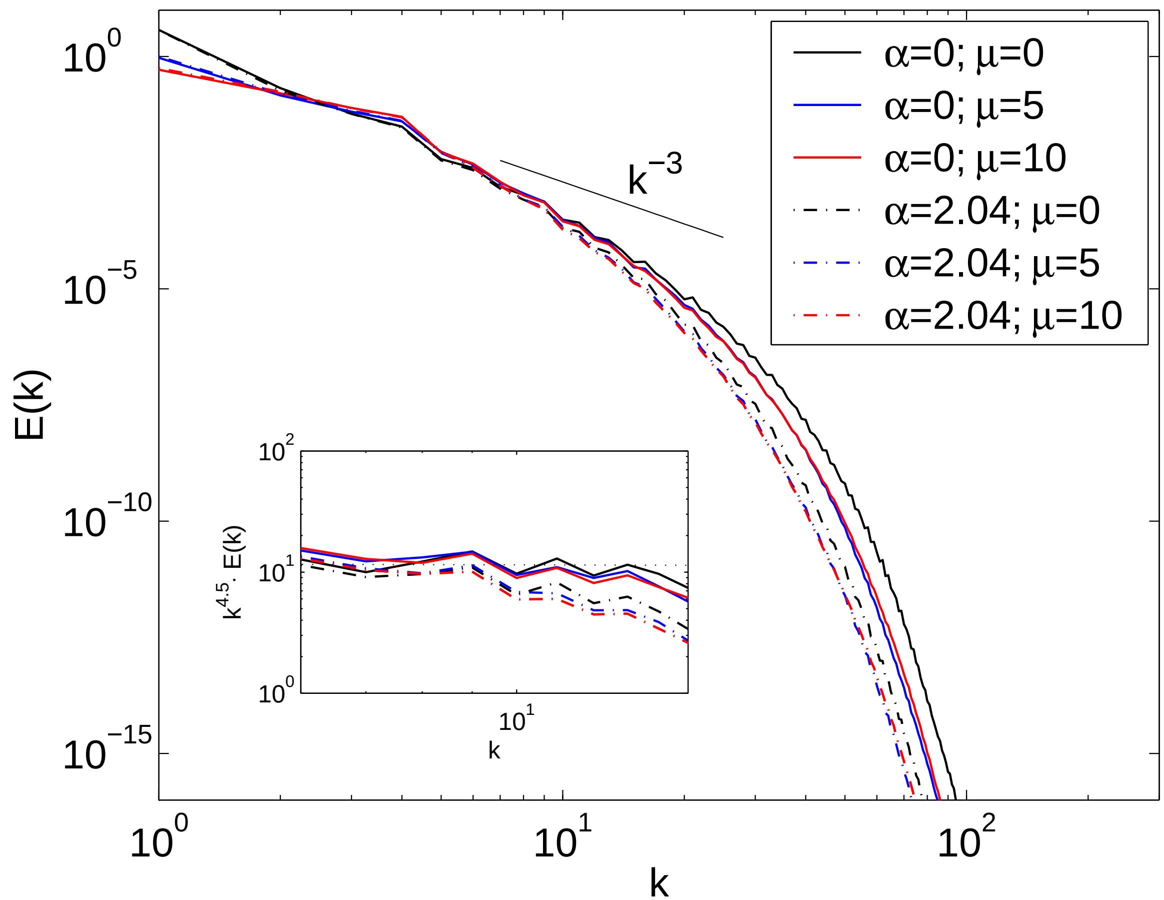
The wavenumber k is in multiples of 2 □. The solid lines are the DNS □=0 calculations of E(k).

The dotted lines are the NS-□ model calculations of $E^u(k)$ for small □.

The behaviour of the spectra is largely independent of the magnitude of the hypoviscosity in the enstrophy cascade subrange (6 < k < 15).

The inset shows the spectra compensated by $k^{4.5}$.

The resolution of this simulation is far to small to observe the expected scaling exponent.



Scale (to prevent trivial dynamics)

$$\partial_t v - \nu \Delta v - u \times \nabla \times v = -\left(\alpha/L\right)^2 \nabla p + \left(\alpha/L\right)^2 f$$
$$\nabla \cdot u = \nabla \cdot v = 0$$
$$v = u - \alpha^2 \Delta u$$

Take the limit $\alpha \to \infty$

$$\partial_t v - \nu \Delta v - u \times \nabla \times v = -\nabla p + f$$
$$\nabla \cdot u = \nabla \cdot v = 0$$
$$v = -L^2 \Delta u$$

1024^2 simulation: Why NS-α equations?

Figure 2. Energy spectra for 1024^2 simulation.

The black curve is the DNS ($\alpha = 0$) which shows close to k^{-3} scaling in the enstrophy cascade range $6 < k < 20$.

The solid red curve is the $E^u(k)$ spectrum as $\alpha=\infty$ which scales close to k^{-7} in the enstrophy cascade range $6< k < 25$.

The energy spectra for intermediate values of α tend to the $\alpha = \infty$ limit as α increases.

The inset shows the DNS energy spectrum (black) compensated by $k^{3.7}$ and the $\alpha=\infty$ energy spectrum (red) compensated by $k^{7.4}$

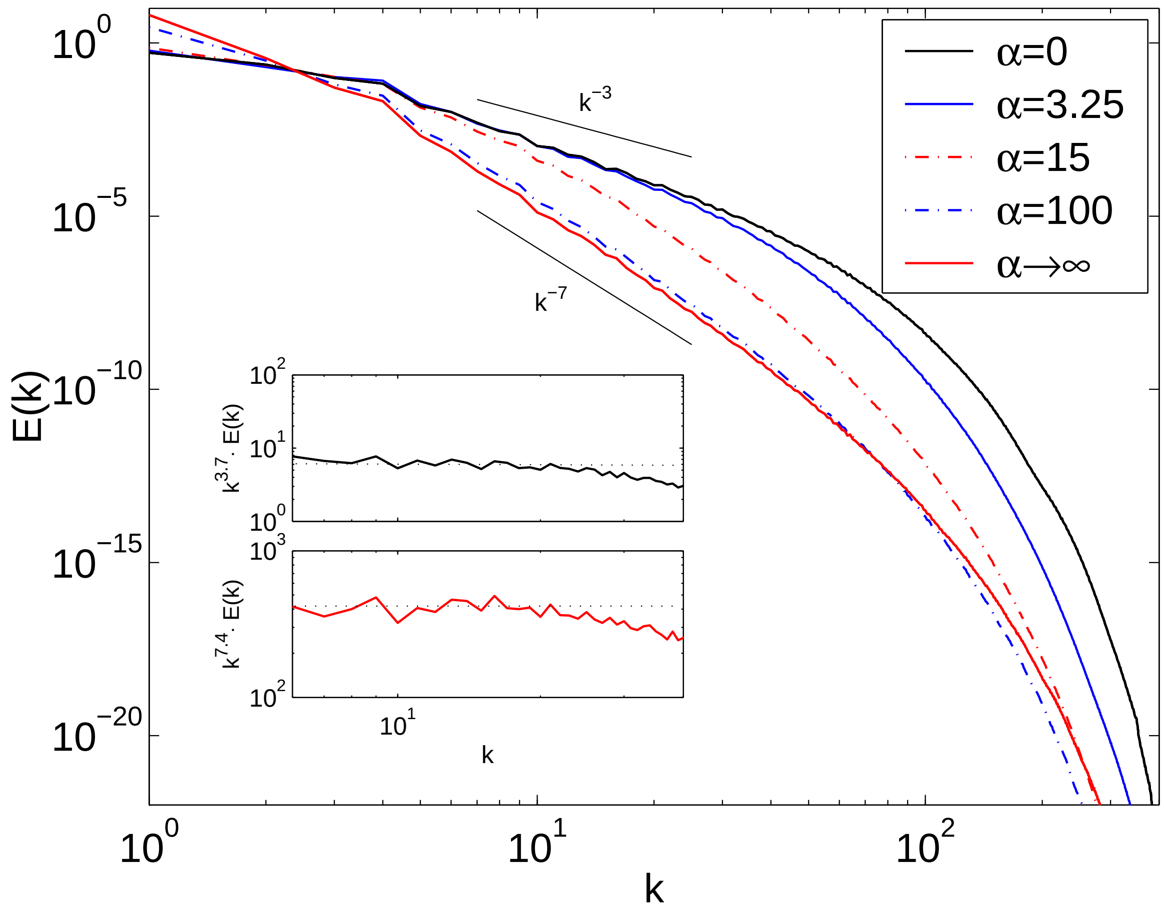


Figure 3. Energy spectra for 2048^2 simulation.

The wavenumber is in multiples of 2^n.

The black curve is the energy spectrum of the DNS which shows close to k^{-3} scaling in the enstrophy cascade range 6<k<35.

The solid red curve is the $E^u(k)$ spectrum of the case $\alpha = \infty$ which scales approximately as $k^{-7.1}$ in the wavenumber region 6<k<25.

The inset shows the DNS energy spectrum (black) compensated by $k^{3.5}$ and the $\alpha=\infty$ energy spectrum (red) compensated by $k^{7.1}$

2048^2

Comparing energy spectra for different values of α

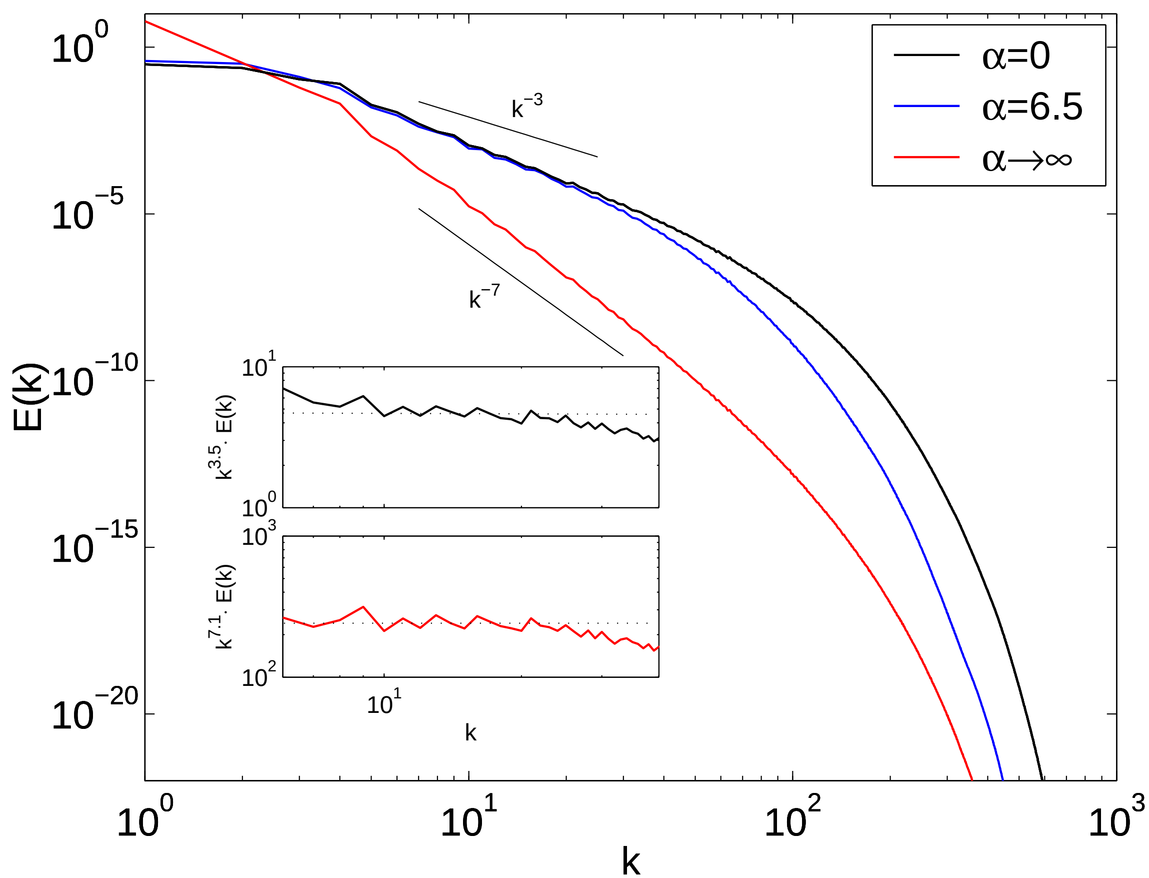


Figure 4. Energy spectra for 4096^2 simulation.

The black curve is the spectrum for the DNS, the red curve is the spectrum for $\alpha \to \infty$.

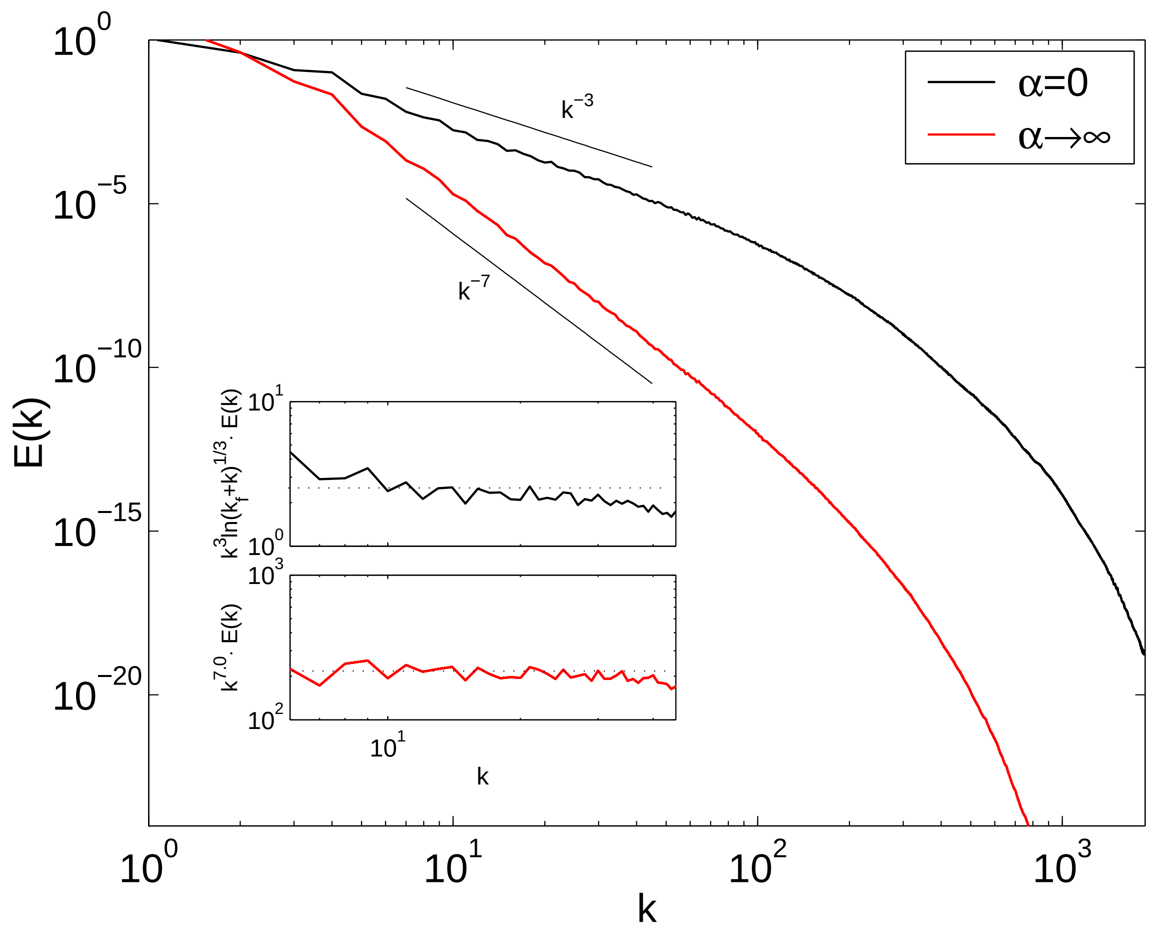
The black curve in the inset corresponds to the NSE energy spectrum compensated by $k^3 \ln(k_f+k)^{1/3}$.

The red curve in the inset is the energy spectrum $E^u(k)$ for NS-α compensated by k^7.

The region $6 < k < 40$ is flat indicating the nominal range over which the k^{-7} scaling holds.

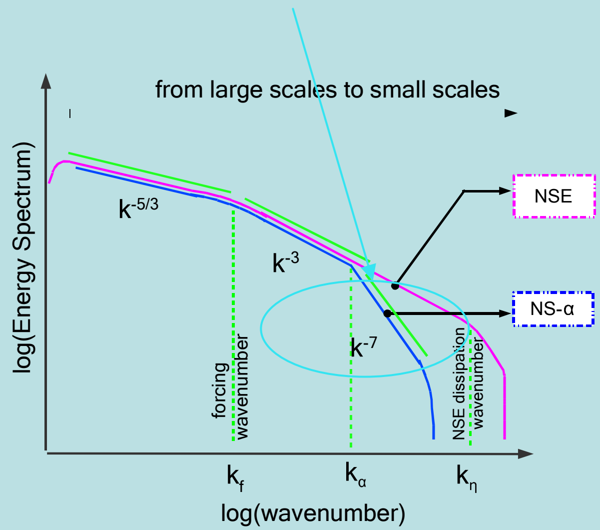
4096^2

Power law for NS-α



Conclusion:

k^{-7} power law



4096^2

Power law for NS-α

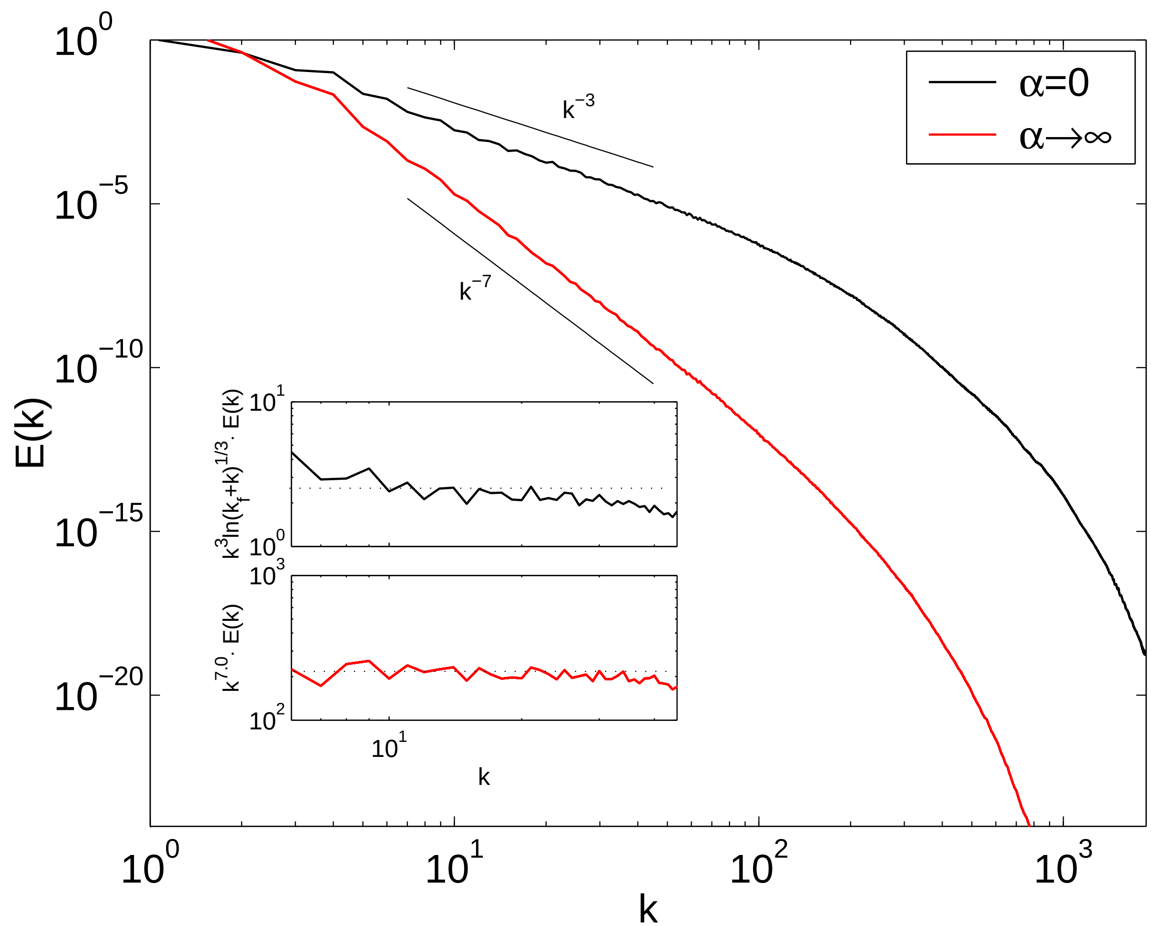
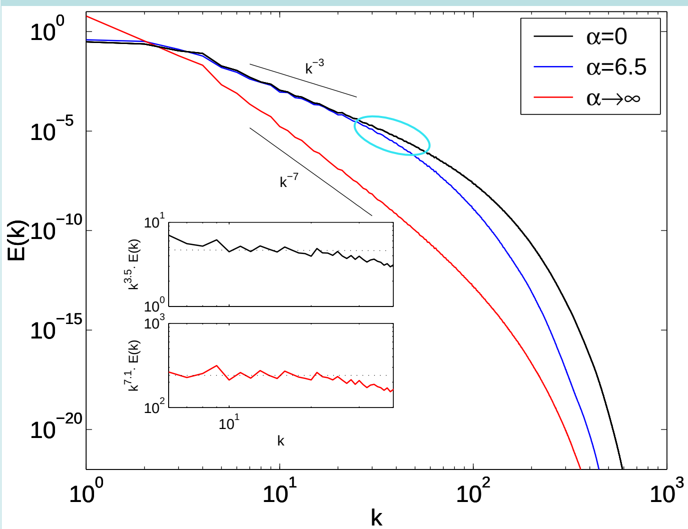


Figure 5. Compensated energy spectra for 2048² simulation for □ =6.5 ($k_□$=39.75; vertical dashed line).

The energy spectrum is compensated by k^7, $k^{19/3}$, and, $k^{17/3}$ respectively.

The region 39 < k < 70 in the first subplot follows a flat regime which indicates the nominal range over which the k^{-7} scaling holds.



2048²

Power law for finite $□ = 6.5$

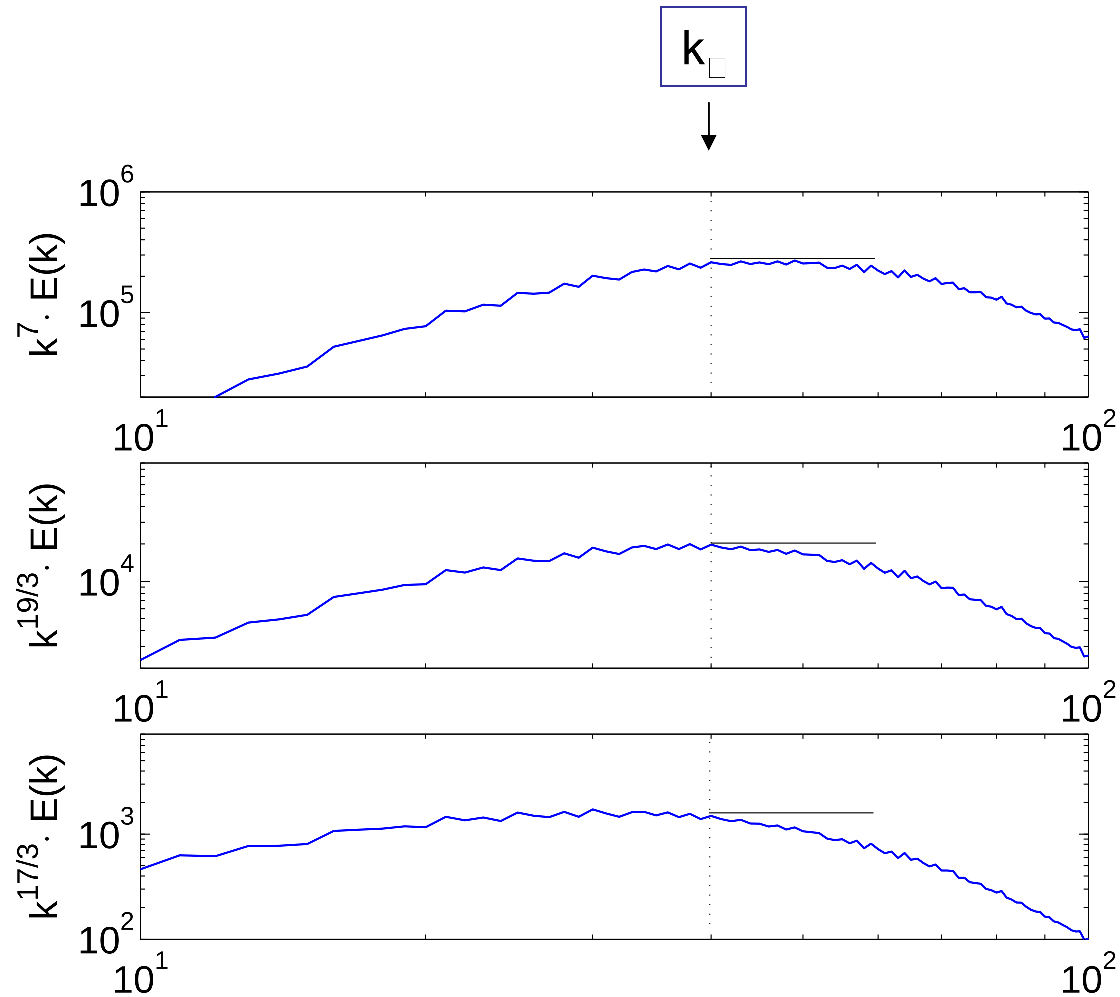


Figure 6. Isosurfaces of vorticity $r\Theta$ v for the 1024^2 simulation. □ = 0, 3.25, 15, 100, ı, reading each row of figures from left to right. The vorticity field exhibits increasingly fine structures as □ is increased.

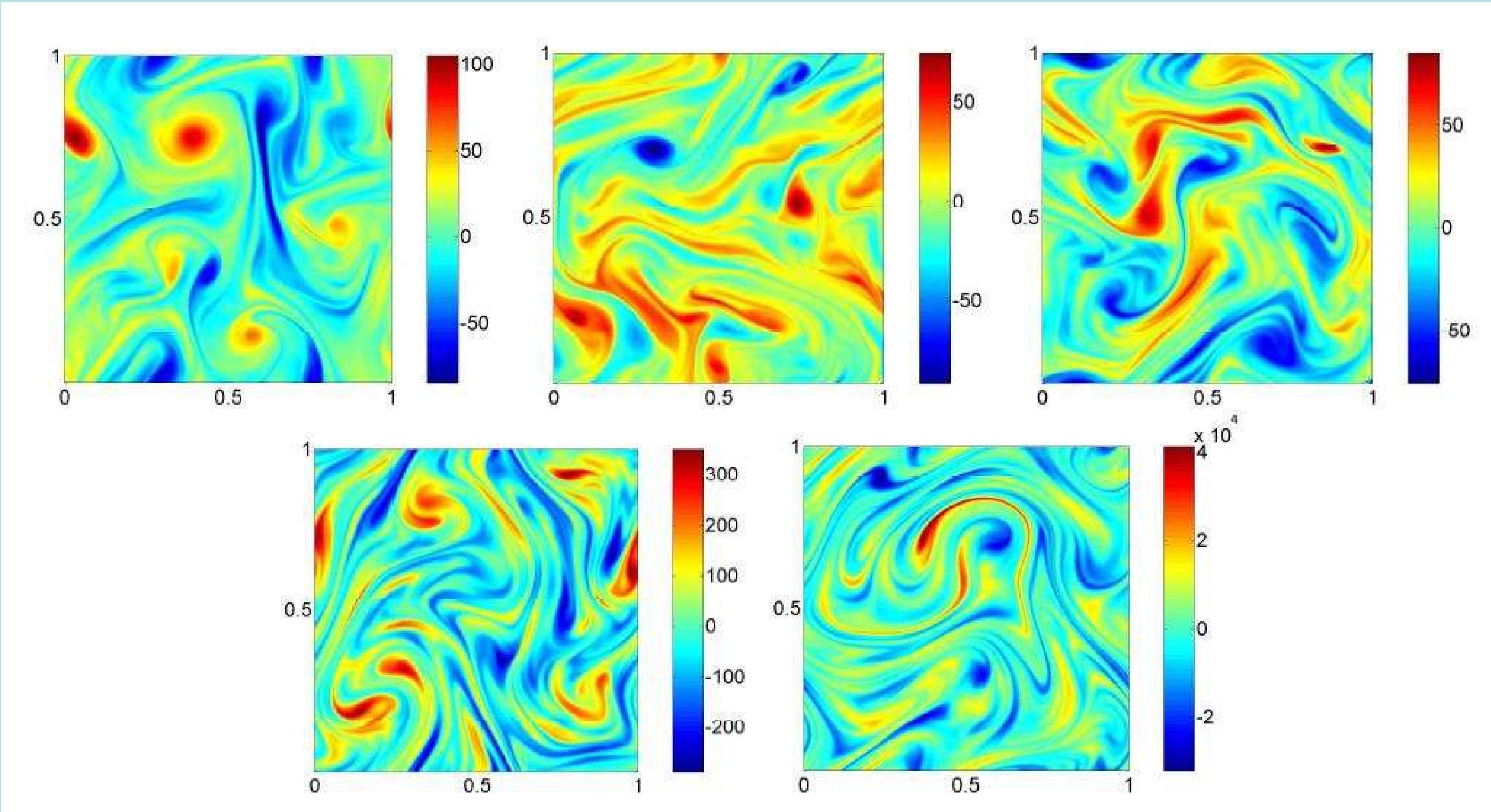
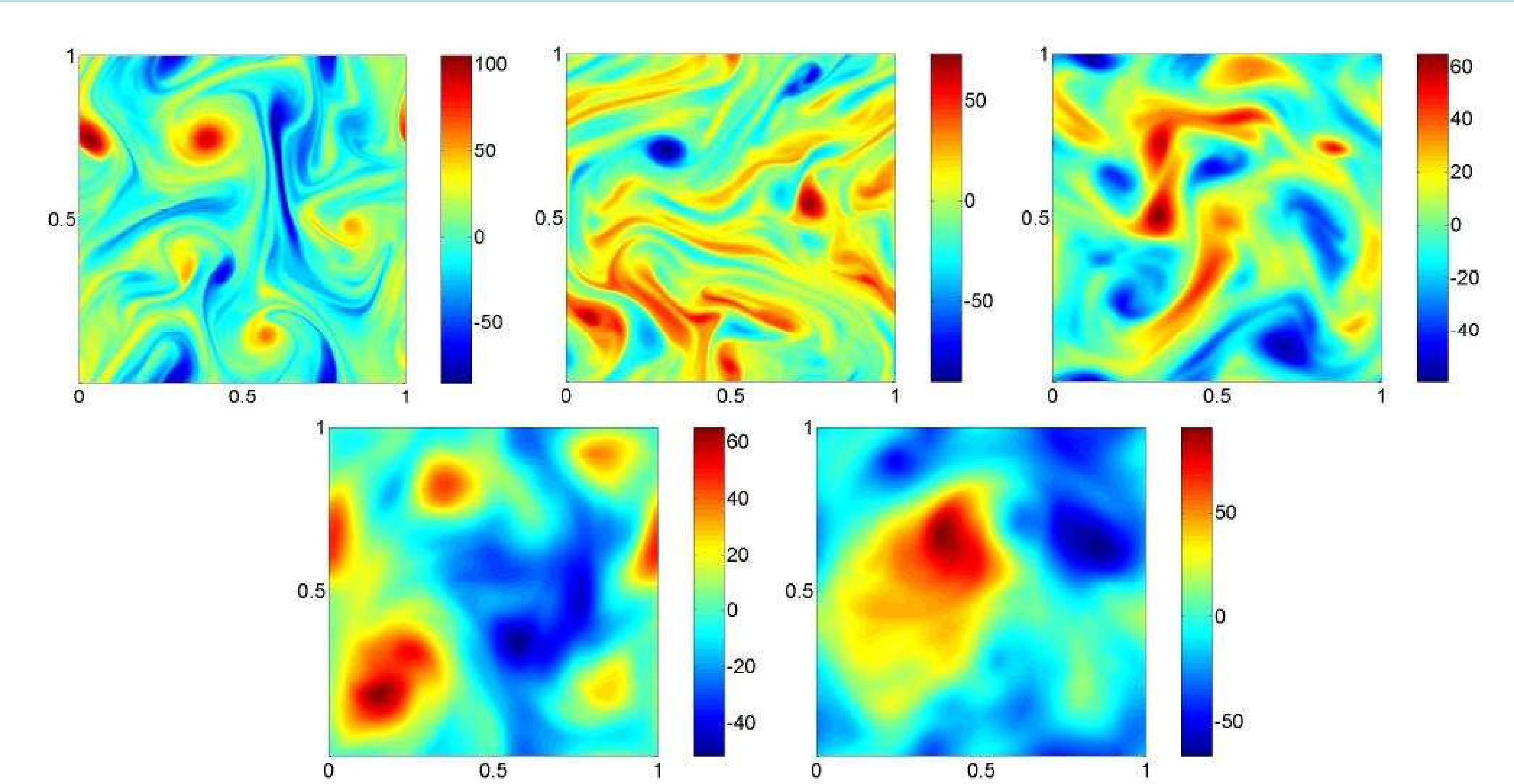


Figure 7. Isosurfaces of vorticity *rΘ* u for the 1024^2 simulation. □ = 0, 3,25, 15, 100, ı, reading each row of figures from left to right. The structures become smoother with increasing □.



Numerical Results for the Two-dimensional Leray-α Model

2D NS-α

Energy conserved* = .5 (u,v)
Enstrophy conserved* = .5 ($\nabla\times$ v, $\nabla\times$ v)

3 Power Laws

vv: $k^{-21/3} = k^{-7}$

uv: $k^{-19/3} \sim k^{-6.3}$

uu: $k^{-17/3} \sim k^{-5.6}$

Numerical Results: ($\alpha \to \infty$)

1024^2: $k^{-7.4}$
2048^2: $k^{-7.1}$
4096^2: k^{-7}

* -- in the absence of viscosity and forcing

2D Leray-α

That is, the characteristic time scale for eddies of size less than α is given by the timescale which depends on the average velocity given by the first one below because of the form of the [illegible] ading enstrophy

Energy conserved* = .5 (u,v)
Enstrophy conserved* = .5 ($\nabla\times$ u, $\nabla\times$ v)

3 Power Laws

vv: $k^{-21/3} = k^{-5.6}$

uv: $k^{-19/3} \sim k^{-5}$

uu: $k^{-17/3} \sim k^{-4.3}$

$$U_k^0 = \left\langle \frac{1}{L^3}\int_\Omega |v_k|^2 dx \right\rangle^{1/2}$$

$$U_k^1 = \left\langle \frac{1}{L^3}\int_\Omega u_k \cdot v_k dx \right\rangle^{1/2}$$

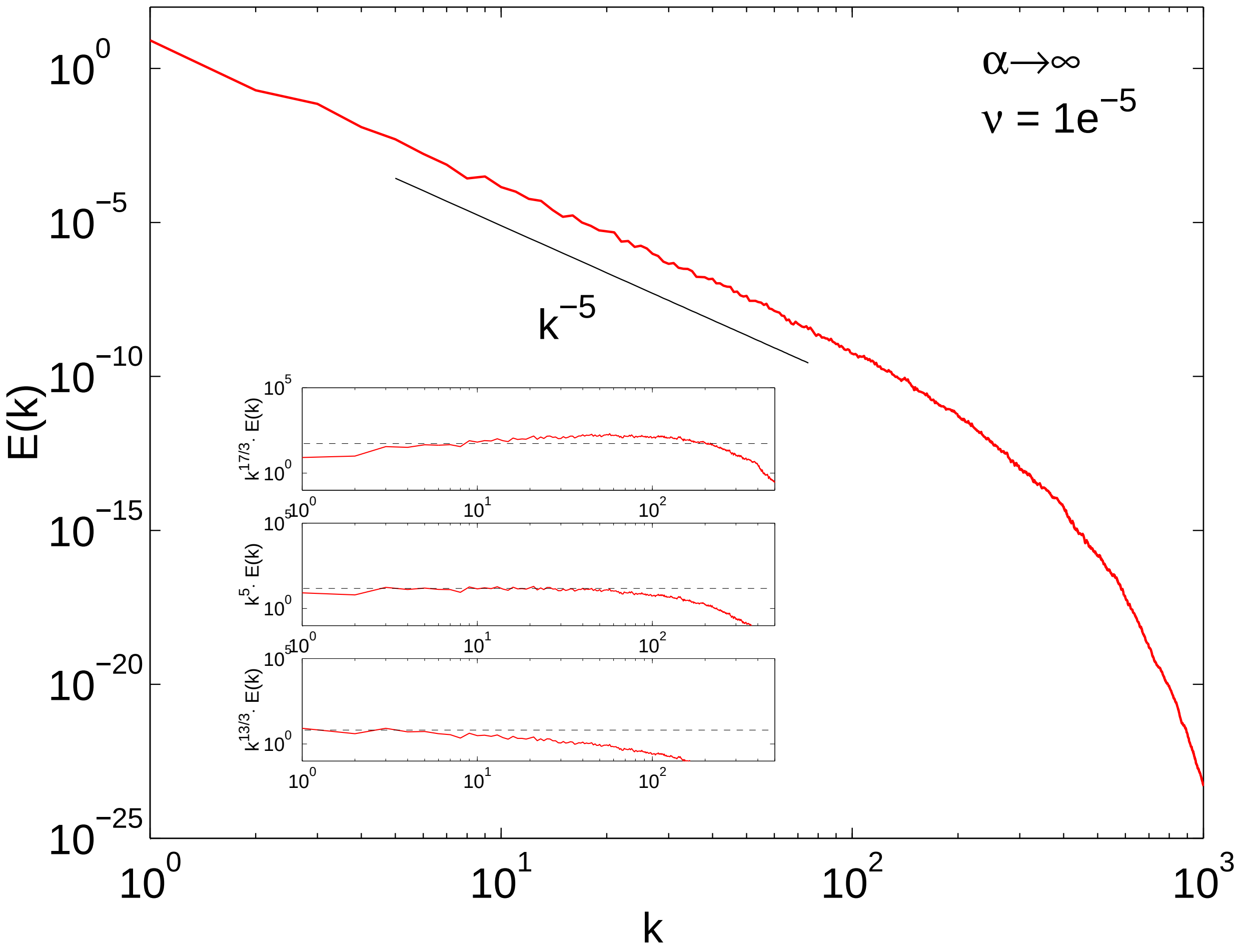
$$U_k^2 = \left\langle \frac{1}{L^3}\int_\Omega |u_k|^2 dx \right\rangle^{1/2}$$

Numerical Results: ($\alpha \to \infty$)

4096^2: k^{-5}

* -- in the absence of viscosity and forcing

4096² simulation of the 2D NS-∞ equation



The red curve is the 2D Leray-α spectrum for $\alpha \to \infty$. The red curves in the inset are the energy spectrum compensated by $k^{17/3}$, $k^{15/3}$, $k^{13/3}$, respectively.

The region $7 < k < 70$ in the *second* subplot follows a flat regime which indicates the nominal range over which the k^{-5} scaling holds.

Back to the Navier-Stokes-Voigt Model

$$-\alpha^2 \Delta \partial_t u + \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f$$

$$\nabla \cdot u = 0$$

- **They are globally regular (even in the inviscid case).**
- **They have finite dimensional attractor.**
- **Unlike the Navier-Stokes equations they are NOT PARABOLIC. But they have a regular attractor. [V. Kalantarov, B. Levant and E. S. Titi, *Journal of Nonlinear Science, 2008*]**
- **They have the same steady state like the Navier-Stokes equations.**
- **They have the same infinite-time averaged Reynolds Equations.**

- **Question: Do they have the same statistics as the Navier-Stokes equations?**

In the 3-d case:

Stationay Statistical Solutions of the Navier-Stokes-Voigt model converge to a Stationary Statistical Solution of the Navier-Stokes Equations.

[Ramos, Titi, *Discr. and Cont. Dyn. Systems*, 2009].

Computational Study with Sabra Shell Model:

Structure functions of the Navier-Stokes-Voigt regularization are investigated in comparison to the those of the Navier-Stokes in the context of Sabra Shell Model. [Levant, Ramos, Titi, *Comm. Math. Sci*., 2009].

Sabra shell model of turbulence

- The equation describe the evolution of complex Fourier-like components $u_n,\ n = 1, 2, \cdots$ of the velocity field $\mathbf{u} = (u_1, u_2, u_3, \ldots)$.

$$\frac{du_n}{dt} = ik_n\left(2\, u_{n+2}\, u^*_{n+1} - \varepsilon\, u_{n+1}\, u^*_{n-1} - \frac{1-\varepsilon}{2} u_{n-1}\, u_{n-2}\right) - \nu\, k_n^2\, u_n + f_n,$$

with the boundary conditions $u_{-1} = u_0 = 0$.

- The scalar wave numbers satisfy $k_n = k_0\, 2^n$.

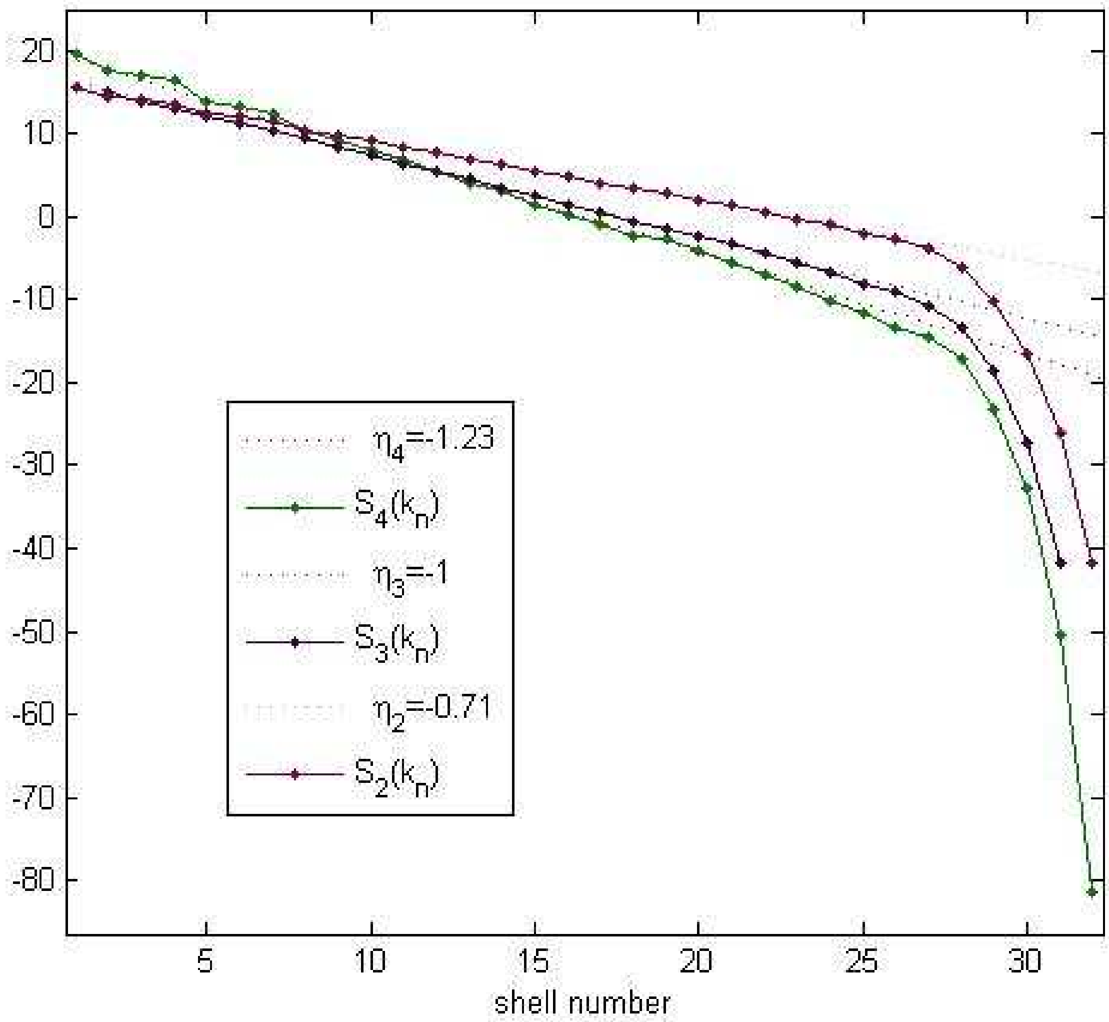
Voigt Regularization of the Sabra shell model of turbulence

We consider the following regularization of the Sabra shell model inspired by the Navier-Stokes-Voigt regularization where

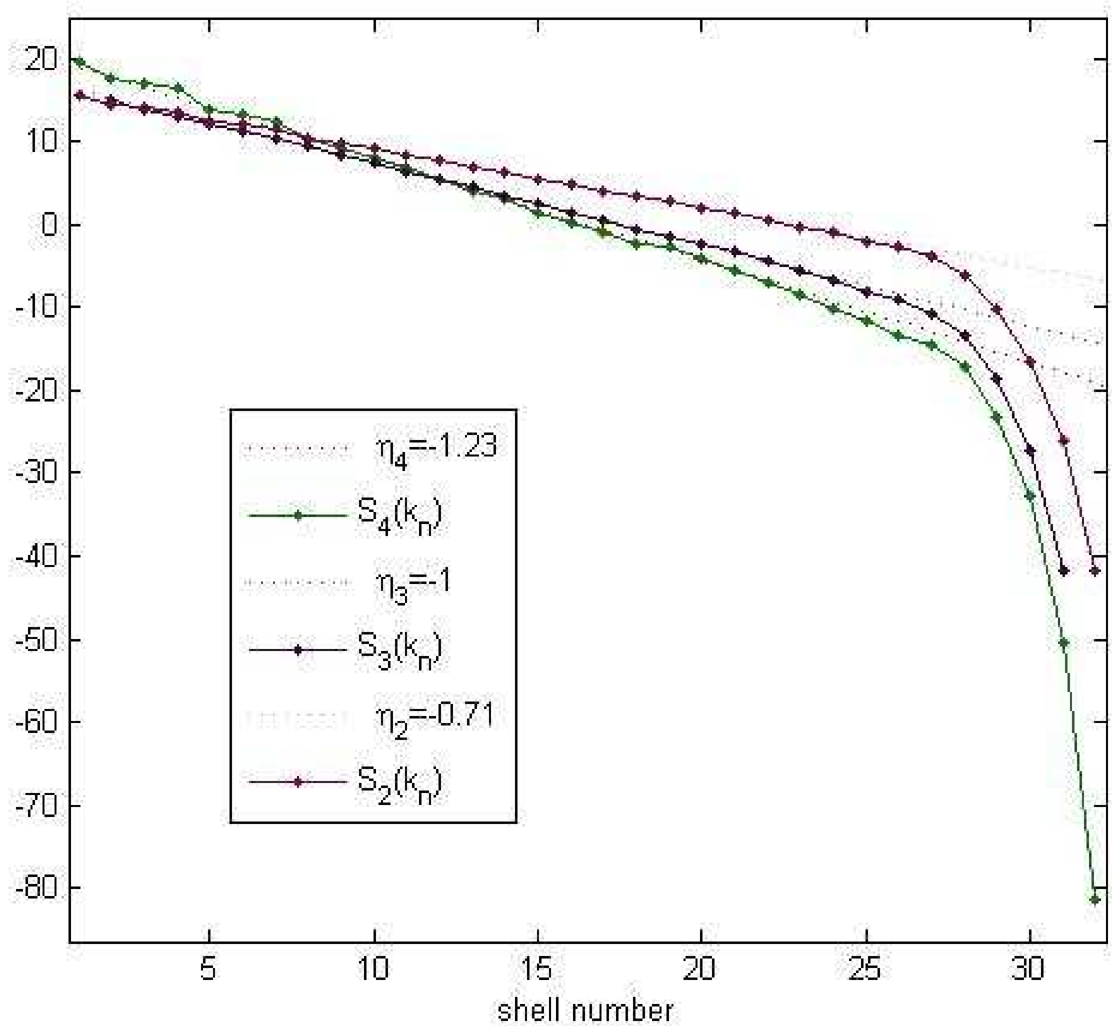
$$\frac{du_n}{dt} \quad \text{is replaced by} \quad (1+\alpha^2 k_n^2)\frac{du_n}{dt}.$$

In Levant, Ramos,& Titi, *Comm. Math. Sci.* (2009), **we investigate the effect of the regularization parameter α on the statisical properties of the solutions.**

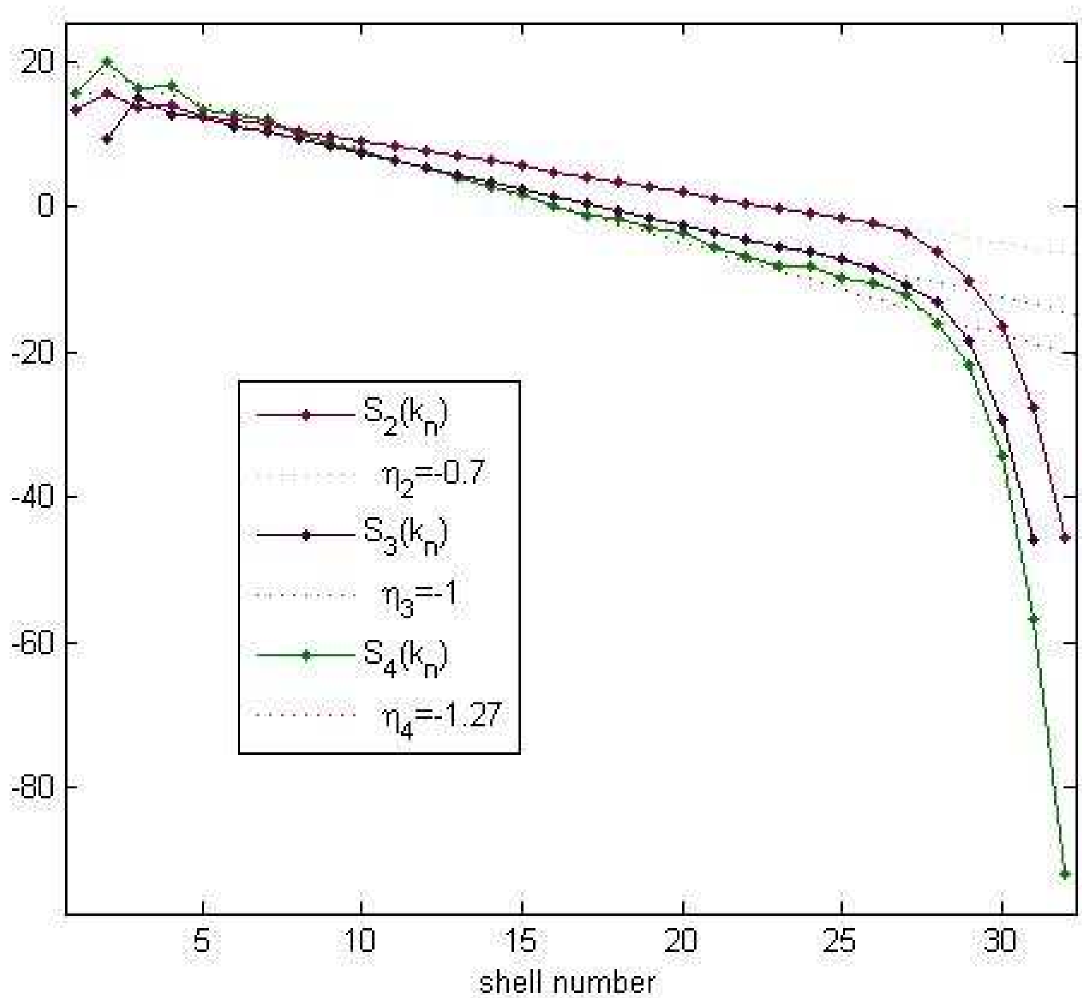
$\nu = 10^{-9}$ and $\alpha = 0$



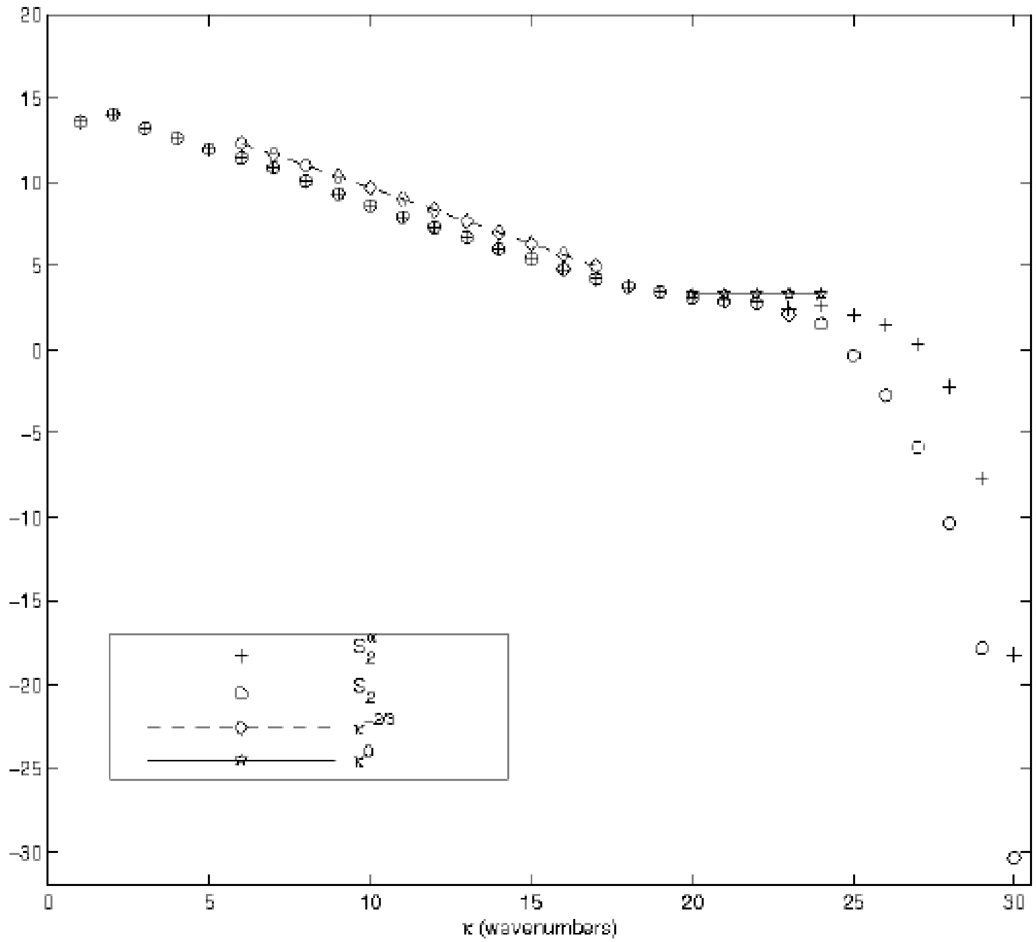
$\nu = 10^{-9}$ and $\alpha = 10^{-9}$



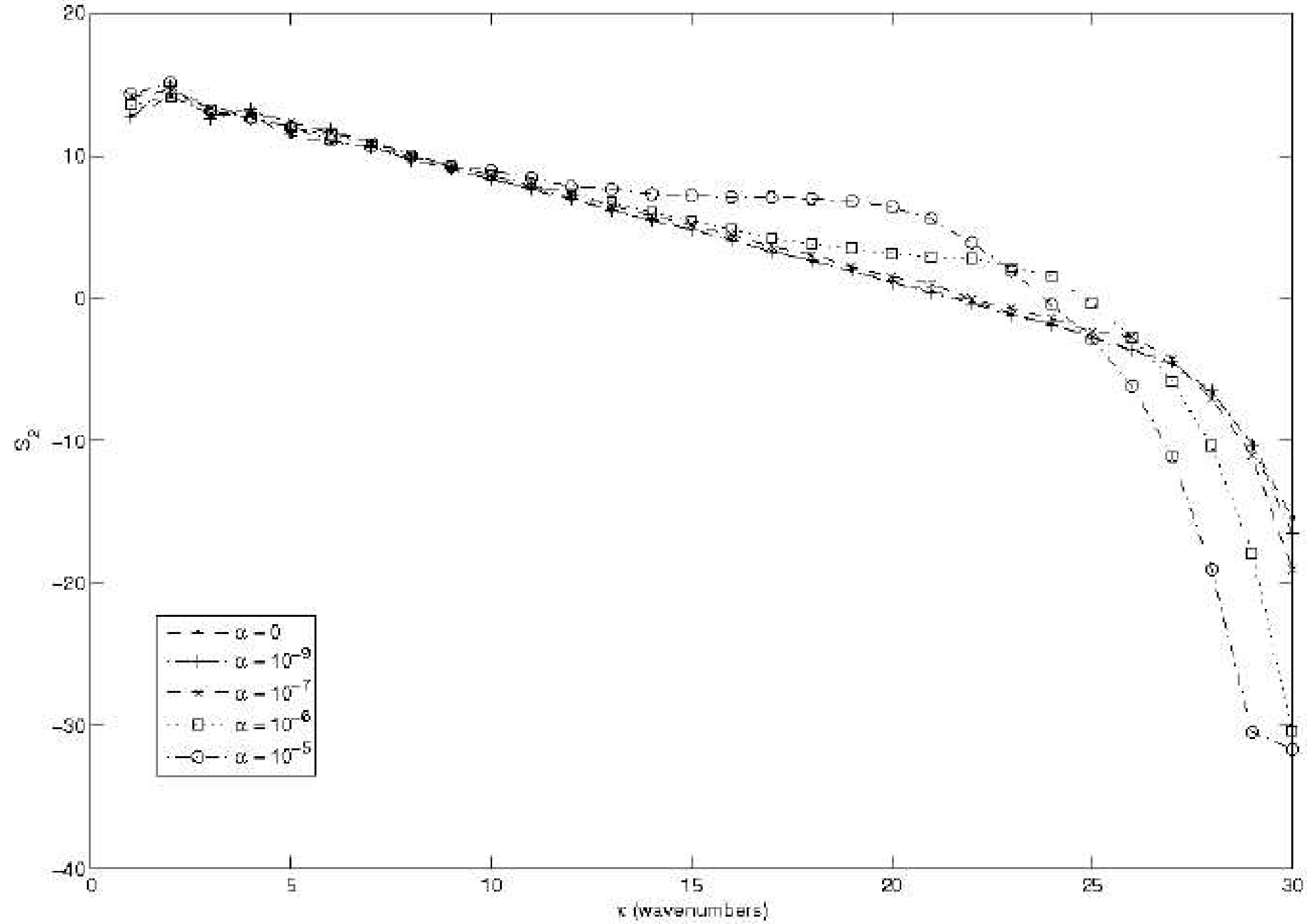
$\nu = 10^{-9}$ and $\alpha = 10^{-7}$



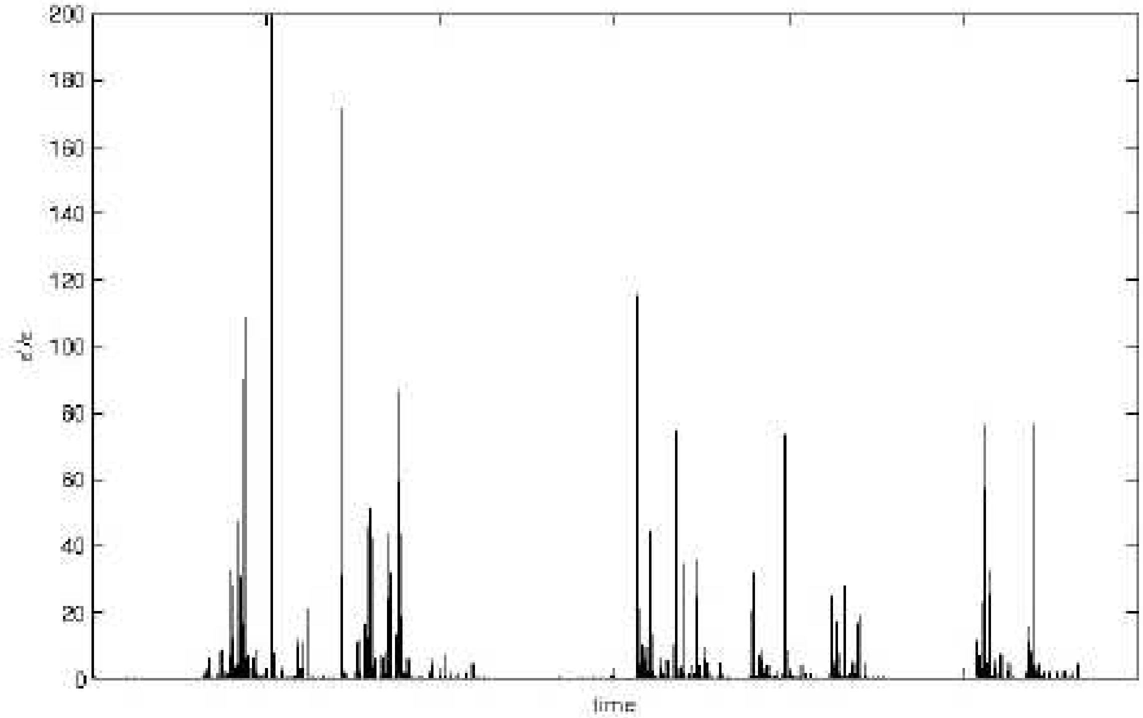
$\nu = 10^{-9}$, $\alpha = 10^{-6}$



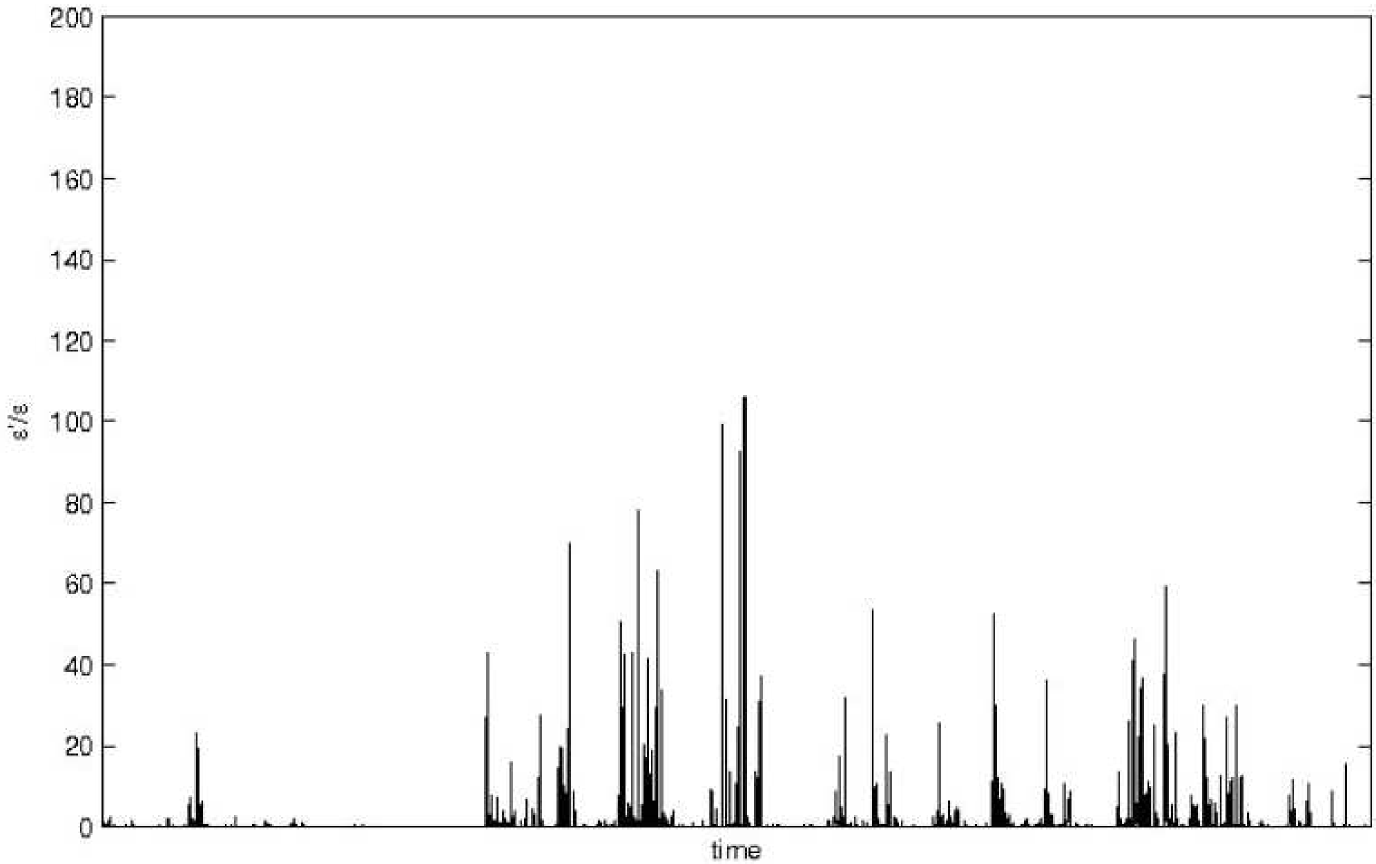
$\nu = 10^{-9}, \alpha = 10^{-5} - 10^{-8}$



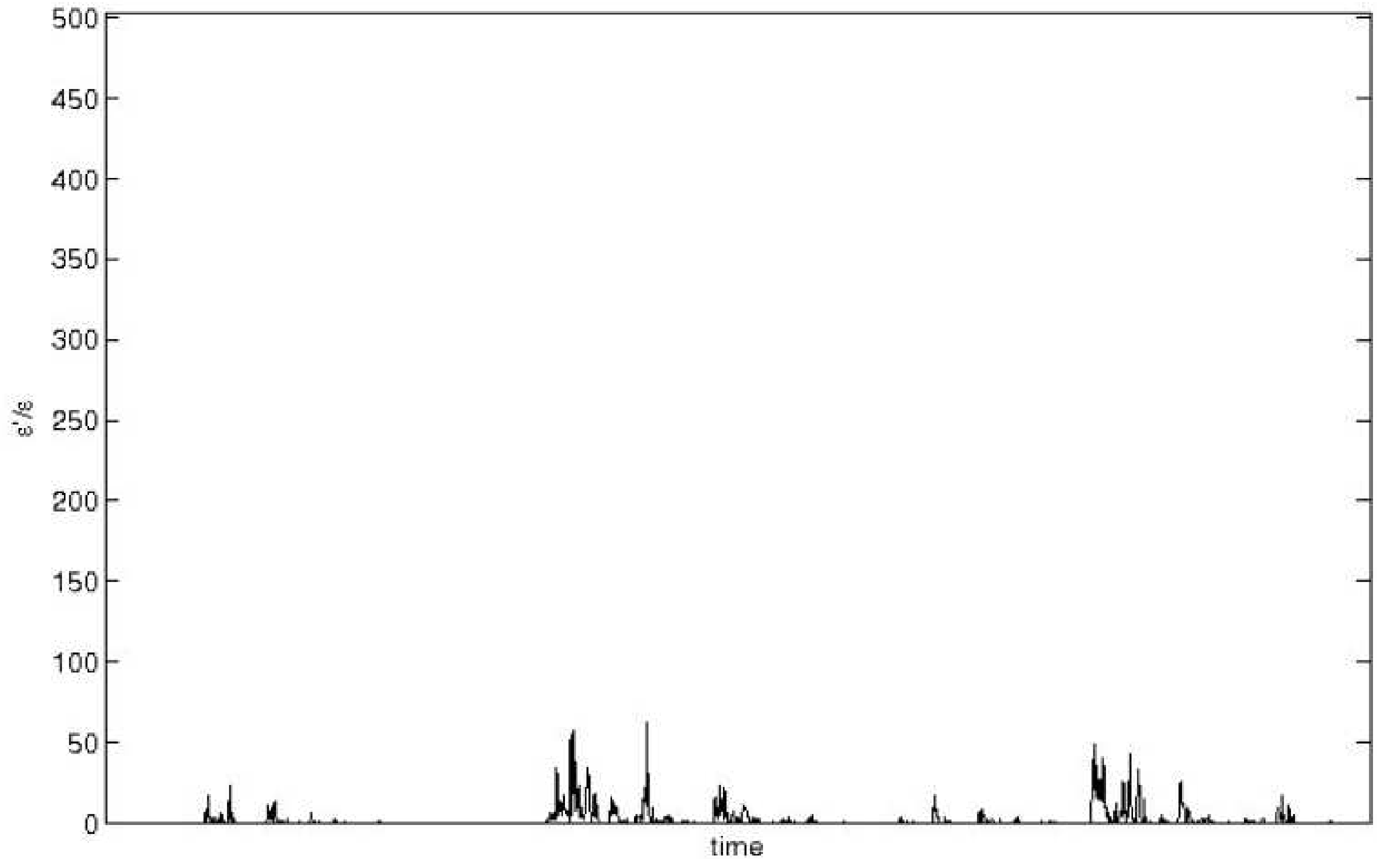
Intermittency - $\nu = 10^{-9}$



Intermittency - $\nu = 10^{-9}$, $\alpha = 10^{-7}$



Intermittency -$\nu = 10^{-9}$, $\alpha = 10^{-6}$



Magneto-Hydrodynamics-α

J. Linshiz and E.S. Titi, *Journal of Math. Physics*, (2007).

Magnetohydrodynamic (MHD) equations

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v - \nu \Delta v + \nabla \pi + \frac{1}{2} \nabla |B|^2 = (B \cdot \nabla)\, B$$
$$\frac{\partial B}{\partial t} + (v \cdot \nabla)\, B - (B \cdot \nabla)\, v - \eta \Delta B = 0$$
$$\nabla \cdot v = \nabla \cdot B = 0$$

v - fluid velocity field, B -magnetic field, π - pressure

$\nu > 0$ - kinematic viscosity, $\eta > 0$ - magnetic diffusivity

- Lagrangian of the ideal MHD:

$$\mathcal{L}[u, D, B] = \int \left(\frac{1}{2} D|u|^2 - \pi(D-1) - \frac{1}{2}|B|^2 \right) dx$$

- Averaged Lagrangian:

$$\bar{\mathcal{L}} = \int \left(\frac{1}{2} D \left(|u|^2 + \alpha^2 |\nabla u|^2 \right) - \pi(D-1) - \frac{1}{2} \left(|B|^2 + \alpha_M^2 |\nabla B|^2 \right) \right) dx$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + (u \cdot \nabla)\, v + \sum_j v_j \nabla u_j - \nu \Delta v + \nabla p + \sum_j (B_s)_j \nabla B_j = (B_s \cdot \nabla)\, B \\
&\frac{\partial B_s}{\partial t} + (u \cdot \nabla)\, B_s - (B_s \cdot \nabla)\, u - \eta \Delta B = 0 \\
&v = \left(1 - \alpha^2 \Delta\right) u, \qquad B = \left(1 - \alpha_M^2 \Delta\right) B_s \\
&\nabla \cdot u = \nabla \cdot v = \nabla \cdot B_s = \nabla \cdot B = 0
\end{aligned}
$$

u, B_s, p - ‘filtered’ fluid velocity, magnetic field and pressure,
$\alpha > 0$, $\alpha_M > 0$ - length-scale parameters, represent the width of the filter

- We have shown the global well-posedness and regularity of solutions of a 3D MHD-α model, which is a particular case of the LAMHD-α model without enhancing the dissipation for the magnetic field, i.e. $\alpha_M = 0$.

Inviscid Regularization of the 3D MHD Equations.

$$-\alpha_F^2 \frac{\Delta \partial v}{\partial t} + \frac{\partial v}{\partial t} + (v \cdot \nabla)\, v - \nu \Delta v + \nabla \pi + \frac{1}{2} \nabla |B|^2 = (B \cdot \nabla)\, B$$

$$-\alpha_M^2 \frac{\Delta \partial B}{\partial t} + (v \cdot \nabla)\, B - (B \cdot \nabla)\, v - \eta \Delta B = 0$$

$$\nabla \cdot v = \nabla \cdot B = 0$$

where $\nu \geq 0 \quad \text{and} \quad \eta \geq 0$

Global existence and uniqueness Larios, E.S.T. (2009)

Nonlinear Schrödinger Equation

$$iv_t + \Delta v + |v|^{2\sigma} v = 0, \qquad x \in \mathbb{R}^N \quad t \in \mathbb{R},$$
$$v(0) = v_0,$$

One has global existence and uniqueness for $0<\sigma<2/N$.

Nonlinear Schroedinger-Helmholtz Equation

Y. Cao, Z. Musslimani, *Nonlinearity*, **21** (2008)

$$iv_t + \Delta v + u|v|^{\sigma-1}v = 0, \tag{1}$$

$$u - \alpha^2 \Delta u = |v|^{\sigma+1}, \tag{2}$$

$$v(0) = v_0, \tag{3}$$

One can show global existence and uniqueness for $1 \leq \sigma < \frac{4}{N}$.

Two-dimensional Euler-α

$$\partial_t q + (u \cdot \nabla) q = 0,$$

$$q = \nabla \times v, \qquad u - \alpha^2 \Delta u = v.$$

Global Existence for Radon Measures.
[Oliver-Shkoller].

α-Regularization of two-dimensional Vortex Sheet

- Bardos-Linshiz-Titi, *Comm. Pure Applied Mathematics,* (2009).

- Convergence of 2d Euler-α to Euler.
- Convergence of Radon measure Solutions, with distinguished sign, to a Delort weak solution of Euler.
- Global well-posedness of the α Kelvin-Helmholtz Problem, for Lipschitz curves.

Axi-symmetric 3d Euler Without Swirl

- Global existence of axi-symmetric 3d Euler without swirl [Yudovich, (1963)]
- No results are know concerning axi-symmetric vortex sheets, even without swirl, for the 3d Euler equations.
- Global Regularty of the 3d Euler-α without swirl was established by Busuioc and Ratiu (2004)

Axi-symmetric 3d Euler-α Without Swirl - Classical Solutions

- Global Regularty of the 3d Euler-α without swirl was established by Busuioc and Ratiu (2004)

Theorem *Let* $u_0 \in H^3(I\!R^3)$, $\mathrm{curl} v_0/r \in L^2(I\!R^3)$ *and* $\mathrm{curl} v_0 \in L^p(I\!R^3)$, *for some* $p \in [1, 2]$. *Then the 3d axi-symmetric Euler*$-\alpha$ *equations without swirl have global solution.*

Axi-symmetric 3d Euler-α without swirl

- Q. Jiu, Z. Niu, Titi and Z. Xin (2009)

Theorem *Assume that the initial velocity is divergence free, axisymmetric without swirl and* $\mathrm{curl} v_0/r \in L^p_c$ *with* $p > \frac{3}{2}$. *Then for any* $T > 0$, *there exists a unique solution of* 3d *Euler*$-\alpha$ *over the interval* $[0, T]$.

Theorem *Assume that the initial velocity is divergence free, axisymmetric without swirl and* $\mathrm{curl} v_0/r \in M_c(\mathbb{R}^3)$. *Then for any* $T > 0$, *there exists a global weak solution* $u \in L^\infty([0,T] \times \mathbb{R}^3)$ *of the* 3d *Euler*$-\alpha$. *Moreover, we have that* $\nabla u \in L^\infty((0,T); L^a + L^\infty)$ *with* $1 \leq a < 3$ *and* $D^2 u \in L^\infty((0,T); L^b + L^\infty)$ *with* $1 \leq b < \frac{3}{2}$.

Thank You!