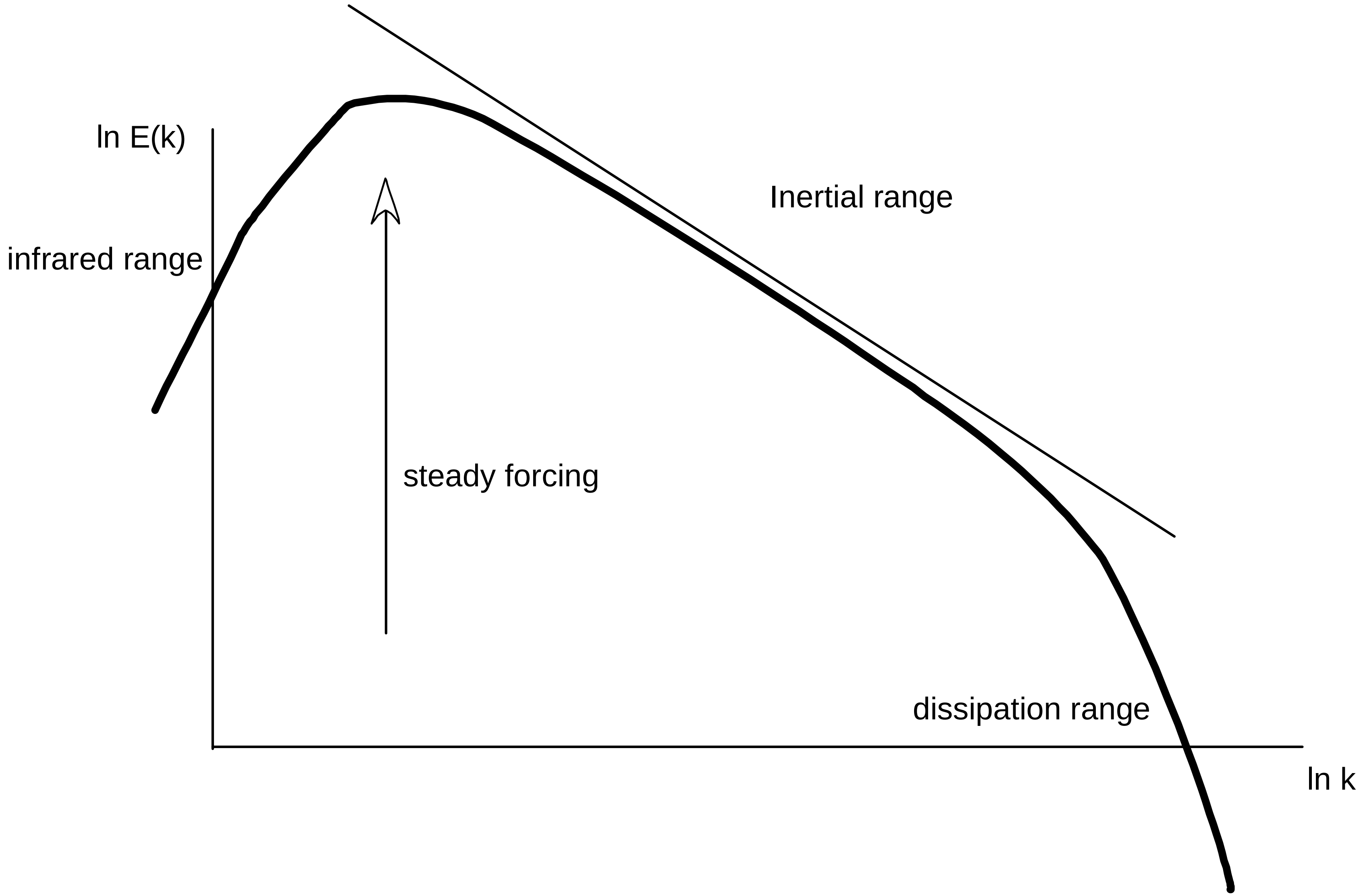
The question of universal scaling coefficients for inertial range structure functions.

Mogens V Melander
Department of Mathematics
SMU, Dallas TX

3D Homogeneous Isotropic incompressible turbulence in equilibrium



$$S_p(\ell) \equiv \left\langle X^p(\ell) \right\rangle = C_p \ell^{\zeta_p}$$

$$S_p / X_o^p = K_p (\ell / \ell_o)^{\zeta_p}$$

Universality if K_p is independent of the forcing for properly chosen ℓ_o and X_o.

Argument 1 against universality

The large scales are determined by the forcing which can be specified arbitrarily.

Thus, $S_p\left(\ell_o\right)$ is also arbitrary and can therefore not be given by a universal expression

Counter example

$$\phi(x,\ell)dx = \Pr\{x < X(\ell) < x + dx\}$$

$$\frac{\partial\phi}{\partial(1/\ell)} = \frac{\partial^2\phi}{\partial x^2} \text{ for } \ell \le \ell_o \text{ with } \phi(x,\ell_o) = f(x).$$

$$\phi(x,\ell) = \frac{1}{\sqrt{4\pi(\ell^{-1} - \ell_o^{-1})}} \int_{-\infty}^{\infty} f(\eta) e^{-(x-\eta)^2/(4(\ell^{-1}-\ell_o^{-1}))} d\eta.$$

$$S_p(\ell) \equiv \left\langle X^p(\ell) \right\rangle = \int_{-\infty}^{\infty} x^p \phi(x,\ell)dx = \frac{1}{\sqrt{4\pi(\ell^{-1} - \ell_o^{-1})}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p f(\eta) e^{-(x-\eta)^2/(4(\ell^{-1}-\ell_o^{-1}))} d\eta dx$$

$$S_p(\ell) = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{p} \binom{p}{j} \int_{-\infty}^{\infty} \eta^j f(\eta) d\eta \int_{-\infty}^{\infty} \xi^{p-j} e^{-\xi^2} d\xi \left(4(\ell^{-1} - \ell_o^{-1})\right)^{\frac{p-j}{2}} = \sum_{j=0}^{p} S_j(\ell_o) Q_{p-j} \left((\ell^{-1} - \ell_o^{-1})\right)^{\frac{p-j}{2}}$$

$$Q_{p-j} = \frac{2^{p-j}}{\sqrt{\pi}} \binom{p}{j} \int_{-\infty}^{\infty} \xi^{p-j} e^{-\xi^2} d\xi$$

$$S_p(\ell) = \left((\ell^{-1} - \ell_o^{-1})\right)^{\frac{p}{2}} \sum_{j=0}^{p} S_j(\ell_o) Q_{p-j} \left((\ell^{-1} - \ell_o^{-1})\right)^{\frac{-j}{2}} = \ell^{-\frac{p}{2}} \{S_o(\ell_o) Q_p + \mathrm{O}(\ell^{-\frac{1}{2}})\} = \ell^{-\frac{p}{2}} \{Q_p + \mathrm{O}(\ell^{-\frac{1}{2}})\}$$

Argument 2 against universality

The averaging of power laws with common scaling exponents does not allow universal scaling coefficients.

Assuming that homogenous isotropic turbulence can have patches with different structure functions the non-universality follows.

Averaging power laws

$$S_p^{(a)} = \left(X_o^{(a)} \right)^p K_p (\ell / \ell_o^{(a)})^{\zeta_p}$$

$$S_p^{(b)} = \left(X_o^{(b)} \right)^p K_p (\ell / \ell_o^{(b)})^{\zeta_p}$$

$$\left(S_p^{(a)} + S_p^{(b)} \right) / 2 = \frac{1}{2} \left(\left(X_o^{(a)} / \ell_o^{(a)\zeta_p / p} \right)^p + \left(X_o^{(b)} / \ell_o^{(b)\zeta_p / p} \right)^p \right) K_p \ell^{\zeta_p}$$

in general

$$\left(X_o^{(c)} \right)^p = \frac{1}{2} \left(\left(X_o^{(a)} / \ell_o^{(a)\zeta_p / p} \right)^p + \left(X_o^{(b)} / \ell_o^{(b)\zeta_p / p} \right)^p \right)$$

cannot be satisfied for all p

Intrinsic scales collapse data in the dissipation range

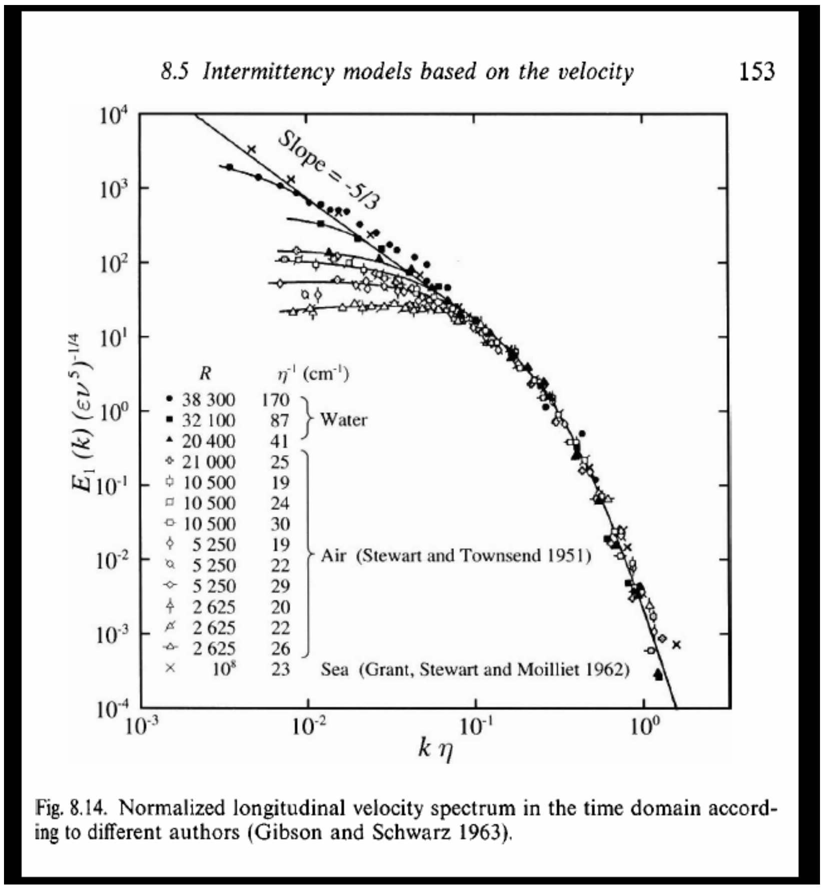
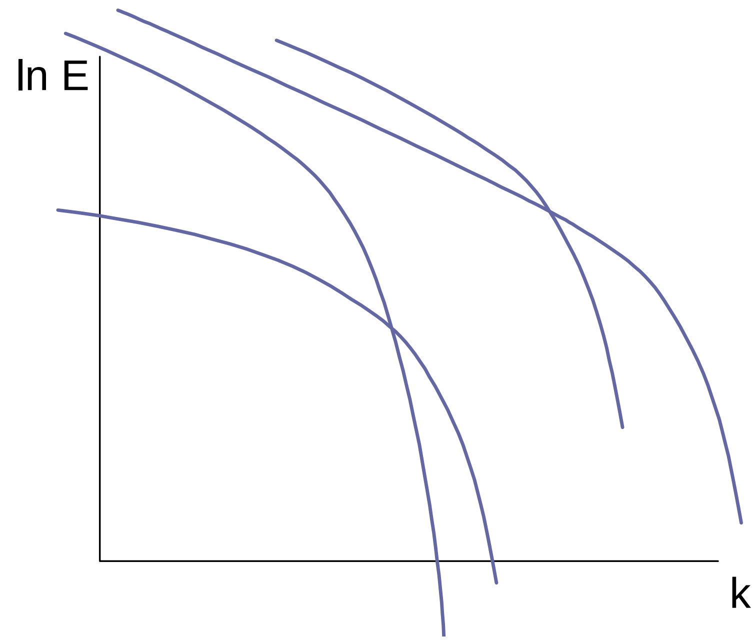


Fig. 8.14. Normalized longitudinal velocity spectrum in the time domain according to different authors (Gibson and Schwarz 1963).



Without intrinsic scales

Intrinsic inertial range units

Measured power laws:

$$S_p / X_{mes}^p = K_{p,mes} (\ell / \ell_{mes})^{\zeta_p}$$

Intrinsic power laws

$$S_p / X_o^p = \left(\left(\frac{X_{mes}}{X_o} \right)^p K_{p,mes} (\ell_o / \ell_{mes})^{\zeta_p} \right) (\ell / \ell_o)^{\zeta_p}$$

intrinsic coefficient K_p

Intrinsic units ℓ_o and X_o must be obtained solely from the measured powers laws.

The integral scale, micro scale, and dissipation scale do not qualify!

Inertial range length scale

$$0 \le \left\langle \left(X^2 - \left\langle X^2 \right\rangle \right)^2 \right\rangle = \left\langle X^4 \right\rangle - \left\langle X^2 \right\rangle^2$$

$$= S_4 - S_2^2 = C_4 \ell^{\zeta_4} - \left(C_2 \ell^{\zeta_2} \right)^2$$

$$= C_4 \ell^{\zeta_4} \left(1 - \frac{C_2^2}{C_4} \ell^{2\zeta_2 - \zeta_4} \right)$$

$$\ell_{\max} = \left(C_4 / C_2^2 \right)^{\left(2\zeta_2 - \zeta_4 \right)^{-1}}$$

anomalous scaling means $2\zeta_2 > \zeta_4$

The inequality shows that S_2 and S_4 are not moments of a pdf when $\ell > \ell_{\max}$.

$$S_p = X_o^p K_p (\ell / \ell_o)^{\zeta_p}$$

Redundancy in the choice of X_o and K_p

$$S_p = \left(\frac{X_o^p}{y^p} \right) (y^p K_p) (\ell / \ell_o)^{\zeta_p} \qquad y \text{ arbitrary}$$

We are free to choose, say $K_2 = 1$.

Then, with $\ell_{\mathrm{o}} = \ell_{\max}$ we have $X_o = \sqrt{S_2(\ell_{\max})}$

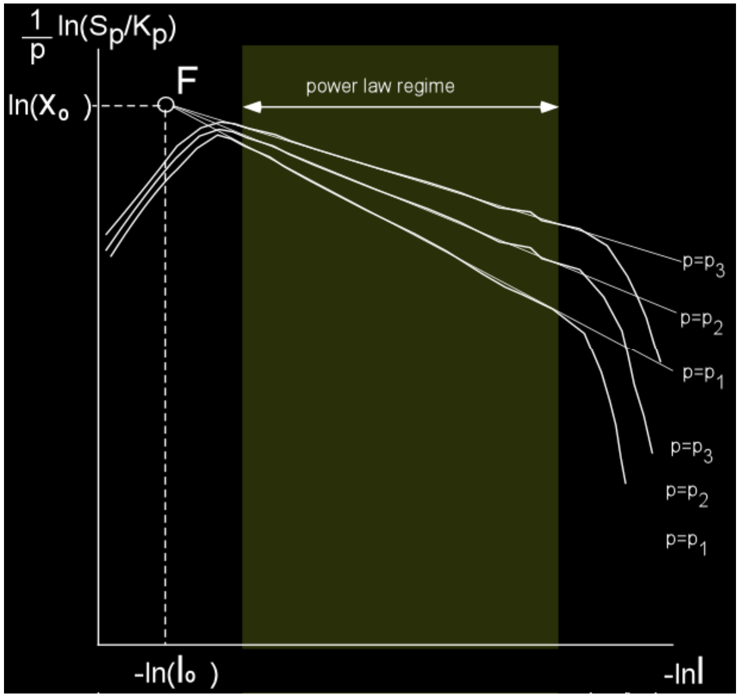
All other K_p are now uniquely determined:

$$K_p = S_p(\ell_o) / S_2^{p/2}(\ell_o)$$

$$S_p / X_o^p = K_p(\ell / \ell_o)^{\zeta_p}$$

With K_p known we can rewrite for log-log plotting:

$$\frac{1}{p}\ln\left(\frac{S_p}{K_p}\right) = \left(\ln X_o - \frac{\zeta_p}{p}\ln \ell_o\right) + \frac{\zeta_p}{p}\ln \ell$$

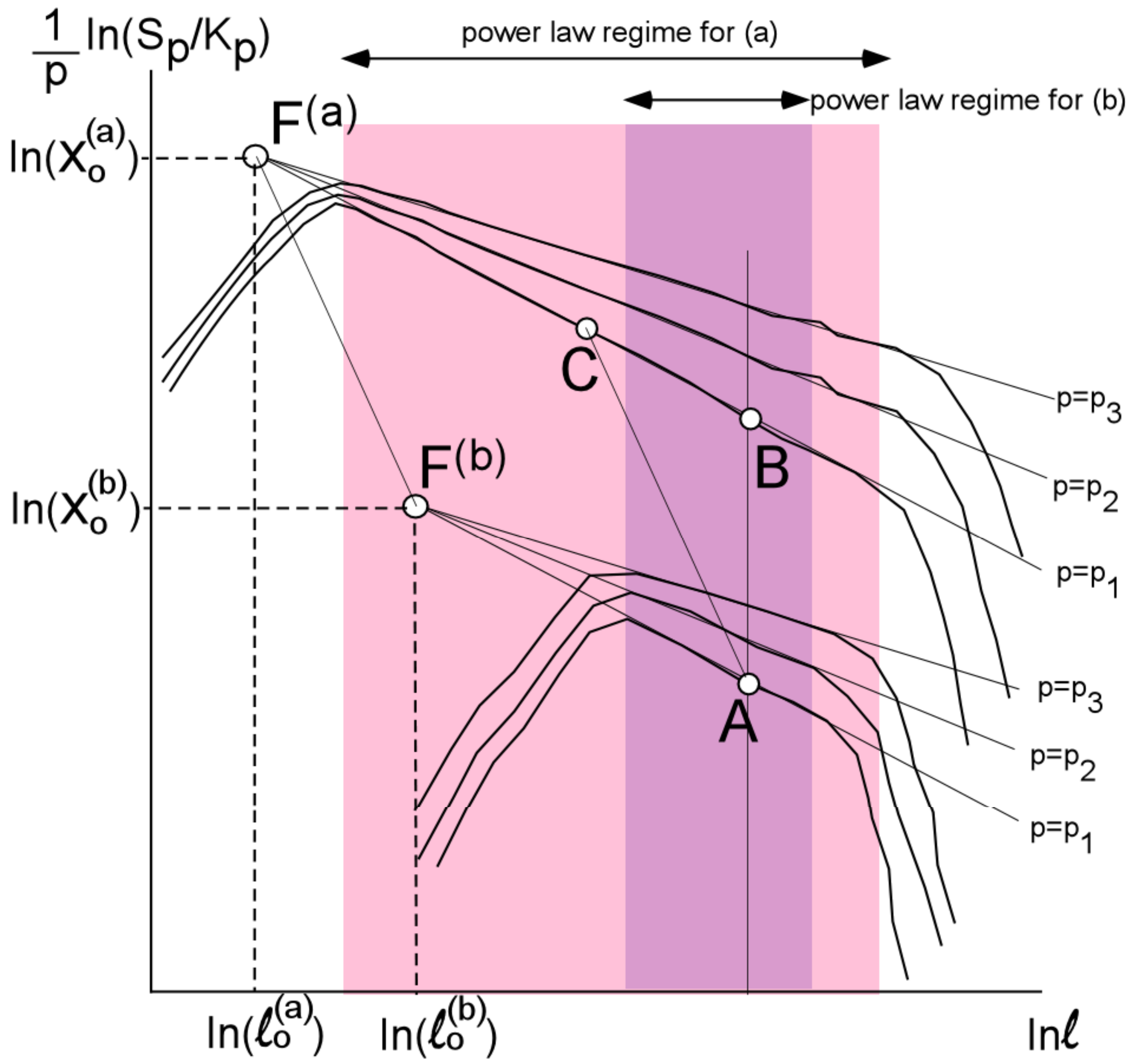


Suppose we have two datasets with the same K_p ,
but different values for ℓ_o and X_o
and qualitatively different large scales.

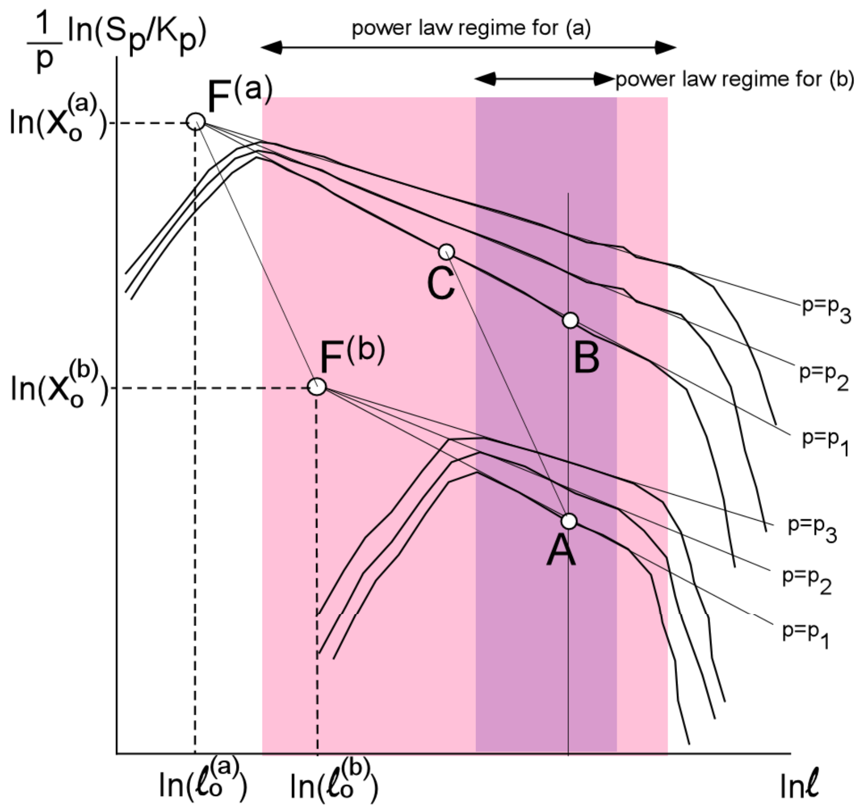
$$S_p^{(a)} = \left(X_o^{(a)} \right)^p K_p (\ell / \ell_o^{(a)})^{\zeta_p}$$

$$S_p^{(b)} = \left(X_o^{(b)} \right)^p K_p (\ell / \ell_o^{(b)})^{\zeta_p}$$

How should we average the two datasets?

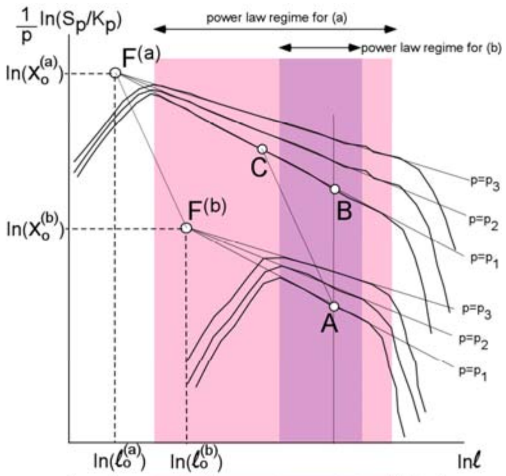


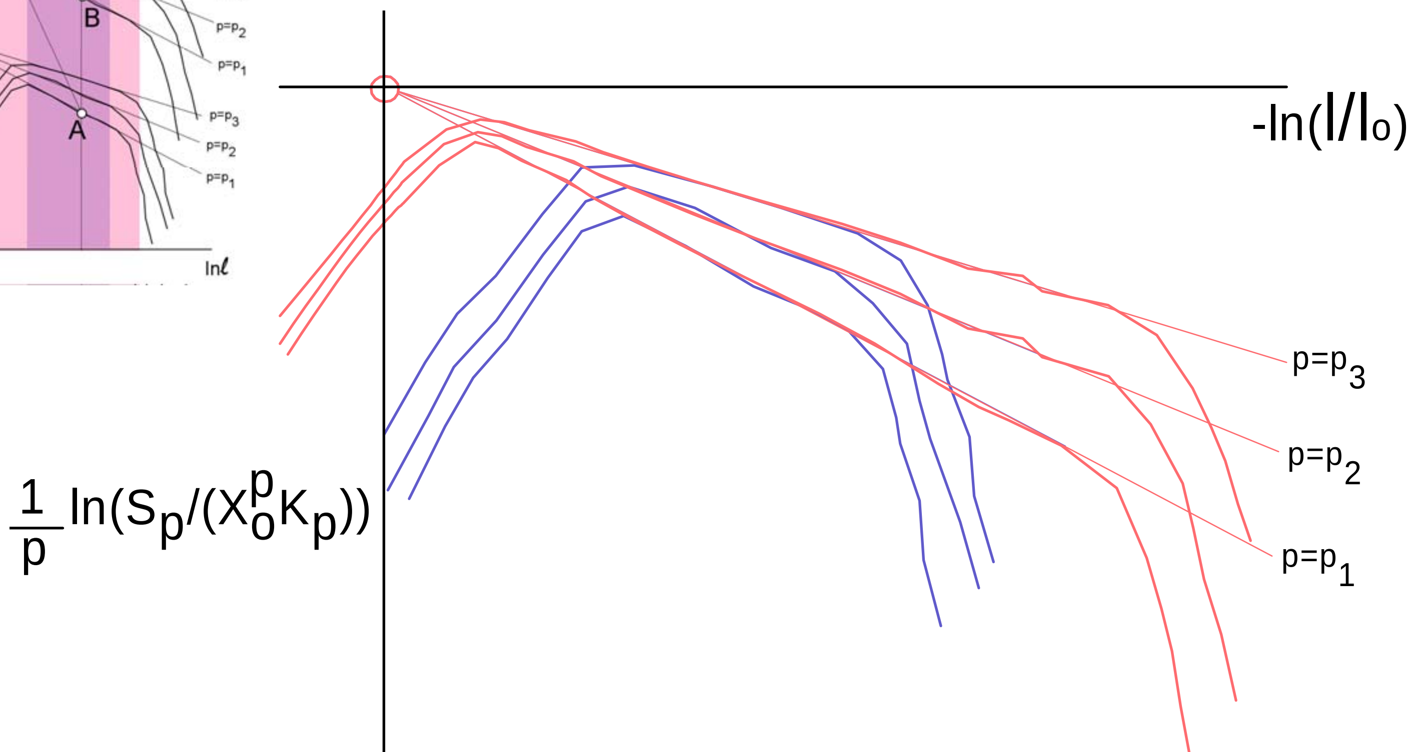
$\frac{1}{p}\ln(S_p/K_p)$
power law regime for (a)
power law regime for (b)
$F^{(a)}$
$\ln(X_o^{(a)})$
$F^{(b)}$
$\ln(X_o^{(b)})$
C
B
A
$p=p_3$
$p=p_2$
$p=p_1$
$p=p_3$
$p=p_2$
$p=p_1$
$\ln(\ell_o^{(a)})$
$\ln(\ell_o^{(b)})$
$\ln\ell$



$$\left(S_p^{(a)}+S_p^{(b)}\right)/2=\frac{1}{2}\left(\left(X_o^{(a)}/\ell_o^{(a)\zeta_p/p}\right)^p+\left(X_o^{(b)}/\ell_o^{(b)\zeta_p/p}\right)^p\right)K_p\ell^{\zeta_p}$$

Averaging data corresponding to the same ℓ leads to a non-unversal coefficient.





Averaging according to

$$\frac{1}{2}\left(S_p^{(a)}(\ell/\ell_o^{(a)})/X_o^{(a)} + S_p^{(b)}(\ell/\ell_o^{(b)})/X_o^{(b)}\right)$$

preserves the universal coefficient K_p

Conclusion so far:

The use of an intrinsic length scale for the inertial range is critical.

- There are many ways to introduce an intrinsic length scale.

- Some are computationally practical, e.g. the previous l(max).
- Others are theoretically natural.

- All are equivalent.

Example 1: Log Poisson model

$$\zeta_p = \frac{p}{9} + 2 - 2\left(\frac{2}{3}\right)^{p/3}$$

Suppose we have a pdf with these scaling exponents

$\phi(x,\ell), \quad x \geq 0.$

The moments are then given by

$S_p(\ell) = C_p \ell^{\zeta_p}$

where the coefficients can be expressed in terms of $f(x) = \phi(x,1)$

via a Mellin transform :

$$\mathrm{C_p} = \int_0^\infty x^{p-1}(xf(x))dx = M[xf(x), p]$$

Among the many operational rules for the Mellin transform, we find the scaling law

$$M\left[g(x), z\right] = G(z)$$

$$\Updownarrow$$

$$M^{-1}\left[a^z G(z), x\right] = g\left(\frac{x}{a}\right),$$ where a is a positive constant.

It is possible to express $\phi(\mathrm{x},\ell)$ in terms of $f(x)$.

$$x\phi(x,\ell) = M^{-1}[C_p \ell^{\zeta_p}, x] = M^{-1}[C_p \ell^{p/9+2-2(2/3)^{p/3}}, x] =$$

$$= \ell^2 M^{-1}[(\ell^{1/9})^p C_p \exp(\ln(\ell^{-2})(2/3)^{p/3}), x]$$

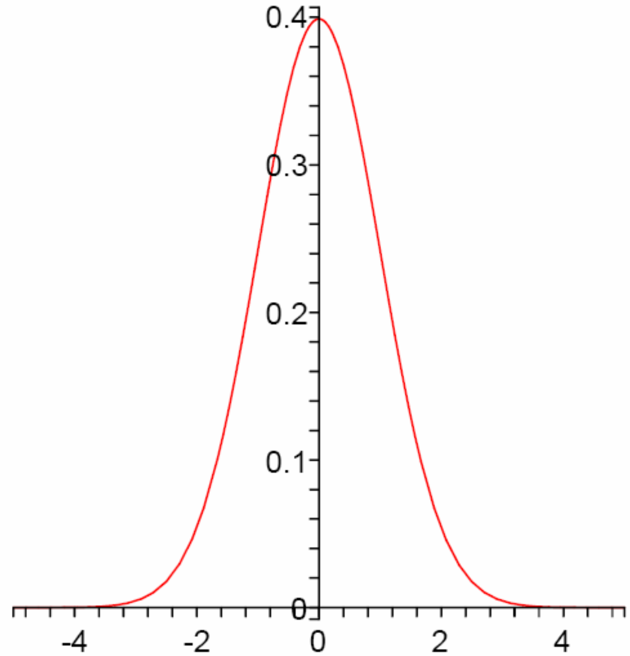
$$= \ell^2 M^{-1}[C_p \exp(\ln(\ell^{-2})(2/3)^{p/3}), x/\ell^{1/9}]$$

$$= \ell^2 M^{-1}[C_p \sum_{m=0}^{\infty} \frac{(\ln(\ell^{-2})(2/3)^{p/3})^m}{m!}, x/\ell^{1/9}]$$

$$= \ell^2 \sum_{m=0}^{\infty} \frac{(\ln(\ell^{-2}))^m}{m!} M^{-1}[(2/3)^{pm/3}\ C_p, x/\ell^{1/9}]$$

$$= \ell^2 \sum_{m=0}^{\infty} \frac{(\ln(\ell^{-2}))^m}{m!} M^{-1}[\ C_p, \left(x/\ell^{1/9}\right)/(2/3)^{m/3}]$$

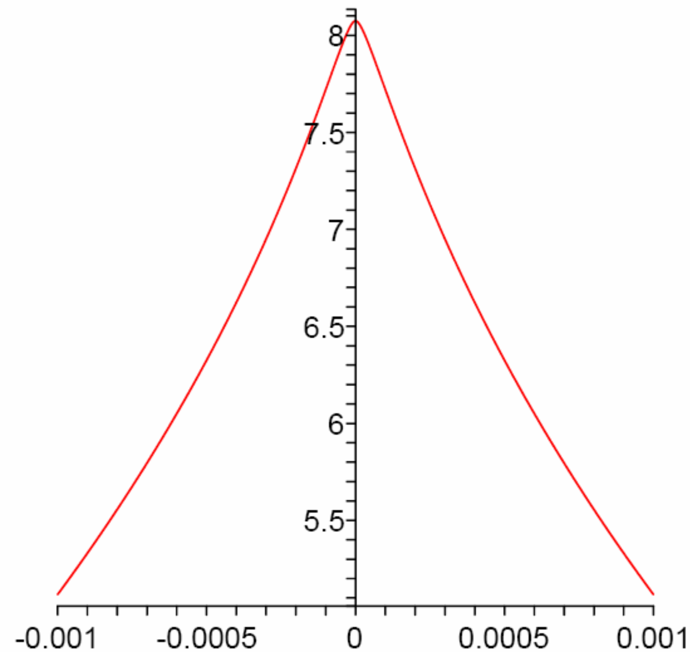
$$\phi(x,\ell) = \ell^{17/9} \sum_{m=0}^{\infty} \left(\frac{3}{2}\right)^{m/3} \frac{(\ln(\ell^{-2}))^m}{m\,!} f\left(\left(\frac{3}{2}\right)^{m/3} \frac{x}{\ell^{1/9}}\right)$$



$\ell = 0$

$\phi(x,\ell)$ versus x

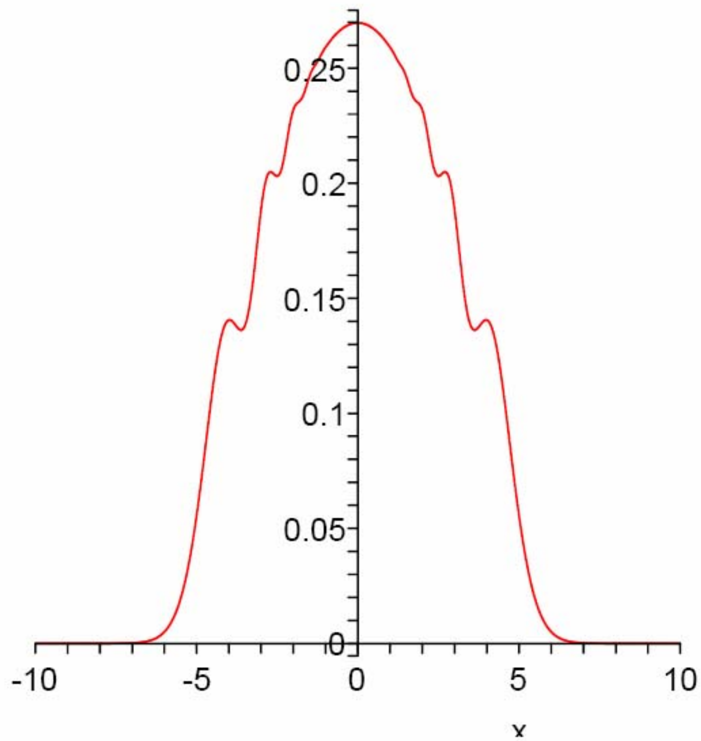
with $f(x)$ as a Gaussian



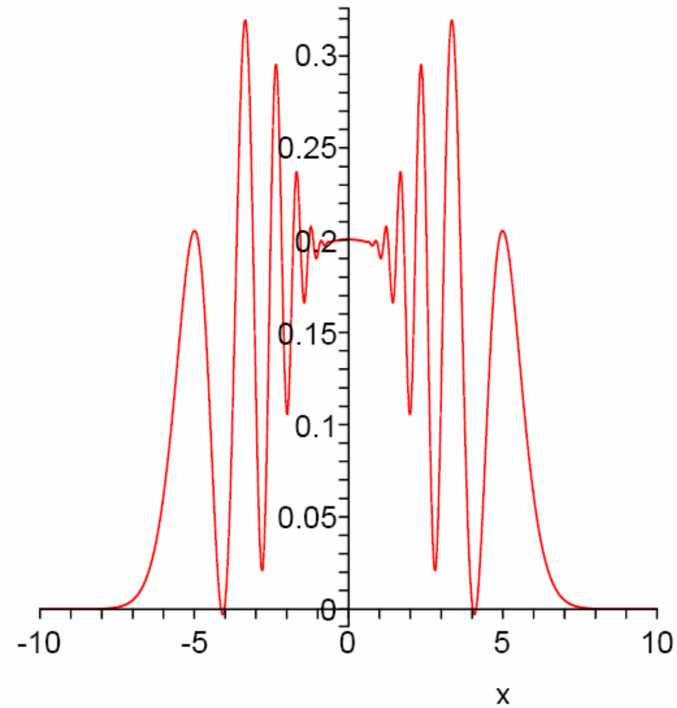
$\ell = 0.1$

$\ln \phi(x,\ell)$ versus x

$$\phi(x,\ell) = \ell^{17/9} \sum_{m=0}^{\infty} \left(\frac{3}{2}\right)^{m/3} \frac{(\ln(\ell^{-2}))^m}{m!} f\left(\left(\frac{3}{2}\right)^{m/3} \frac{x}{\ell^{1/9}}\right)$$



$\ell = 15$



$\ell = 31.5$

$\phi(x,\ell)$ versus x

with $f(x)$ as a Gaussian

$$\phi(x,\ell) = \ell^{17/9}\sum_{m=0}^{\infty}\left(\frac{3}{2}\right)^{m/3}\frac{(\ln(\ell^{-2}))^m}{m!}f\left(\left(\frac{3}{2}\right)^{m/3}\frac{x}{\ell^{1/9}}\right)$$

The natural intrinsic length scale (for this example)

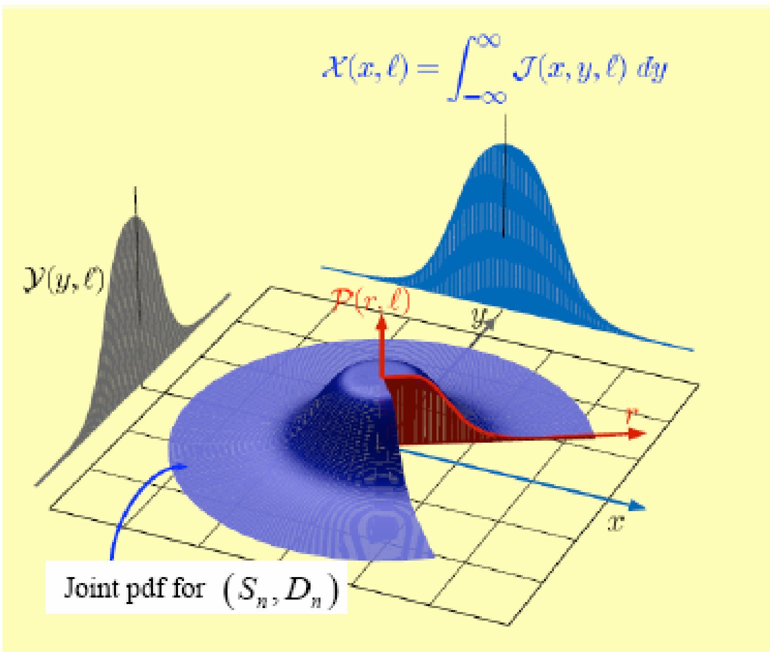
is then $\ell_{nat} \approx 31.2$,

because $\phi(x,\ell)$ takes on negative values precise when $\ell > \ell_{nat}$

Example 2: Inertial range similarity

Let $J(z,\ell)$ be the pdf for a complex variable $z = x + iy$ and depending parametrically on ℓ.

Consider, the axisymmetric compoment $J_o(z,\ell)$ of J.



The object of interest is the collection of moments of $A = |z|$, i.e., $S_p(\ell) = \langle A^p \rangle$

The radial profile $P(r,\ell)$ provides all moments:

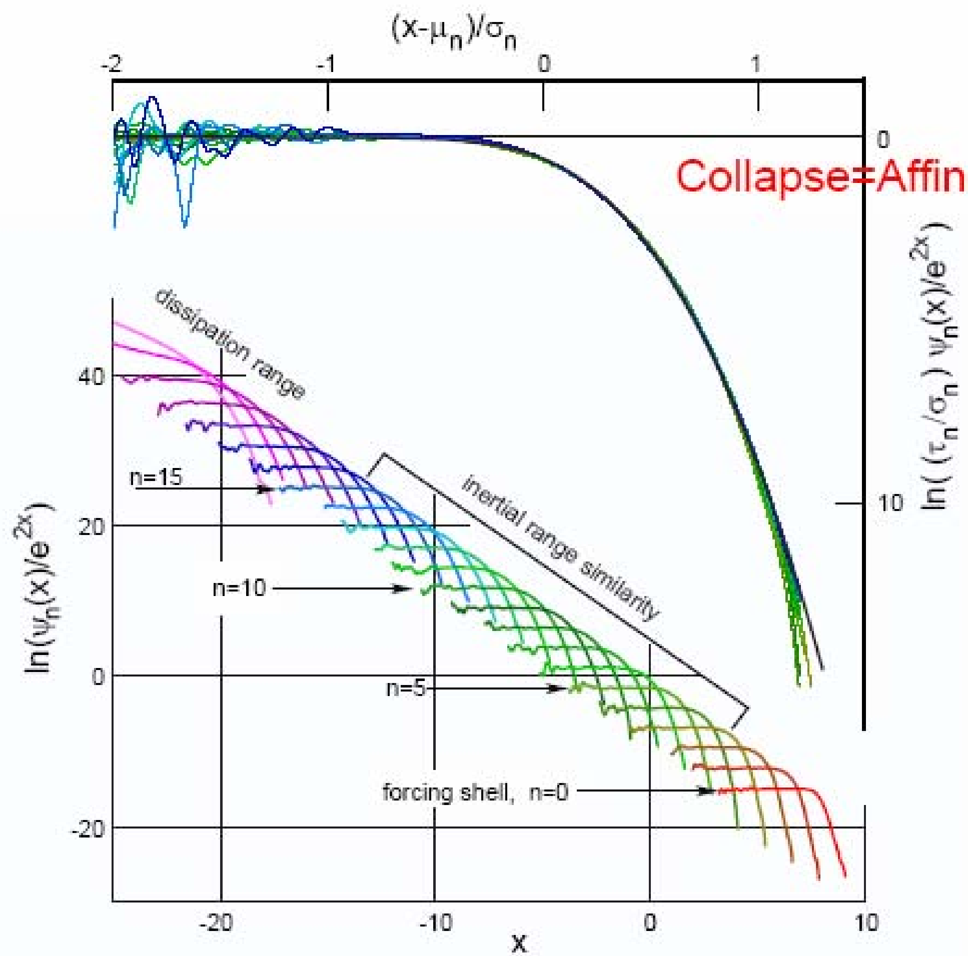
$$S_p(\ell) = 2\pi \int_0^\infty r^{p+1} P(r,\ell) dr = 2\pi M[P(r,\ell), p+2]$$

Computational evidence from shell models and Navier Stokes suggest the similarity

$$P(r,\ell) = C(\ell) f\left[\frac{\ln r - \mu(\ell)}{\sigma(\ell)}\right]$$

with some unknown functions f, μ, σ, C.

Radial profile log-log plot

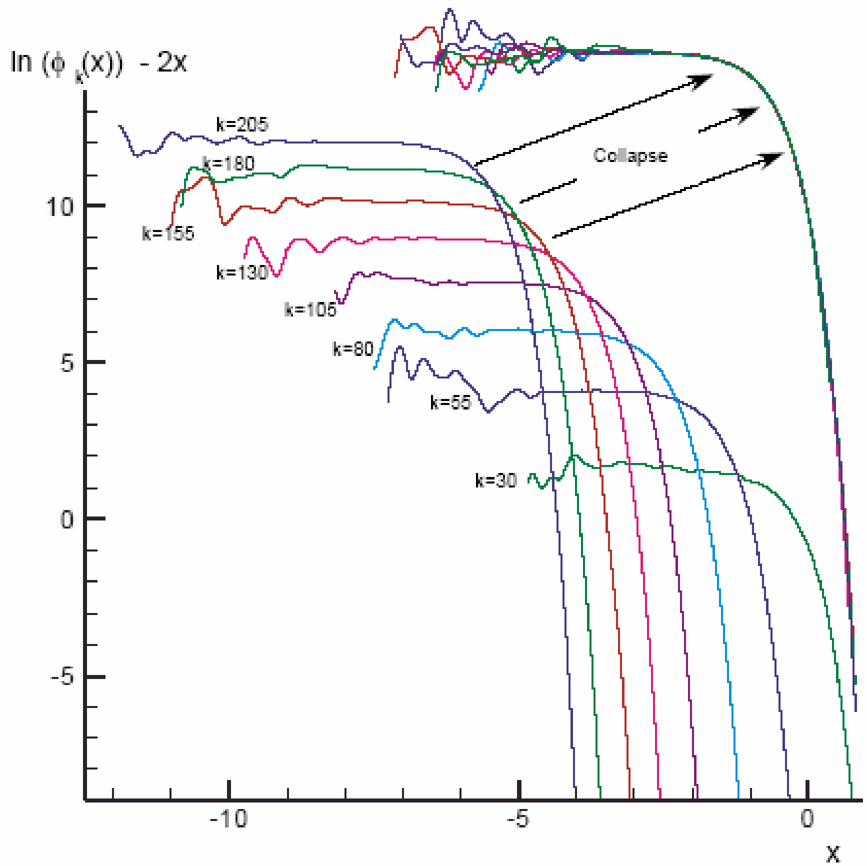


Collapse=Affine Symmetry

$$C_n F\left(\frac{x-\mu_n}{\sigma_n}\right)$$

DNS data

Radial profile log-log plot



DNS by LANL group: Holm, Kurien & Taylor

Inertial range similarity theory

Assumptions:

(1) The moments are power laws:

$$S_p(\ell) = C_p \ell^{\zeta_p}$$

(2) The radial profile is self-similar

$$P(r,\ell) = C(\ell) f\left[\frac{\ln r - \mu(\ell)}{\sigma(\ell)}\right]$$

- arXiv:physics/0702073

Results

3D Scale definition

$$\ell \propto 1/\left|\vec{k}\right|$$

$$S_p(\ell) = C_p \left(\frac{\ell}{\ell_o}\right)^{\zeta_p}$$

$$\zeta_p - \frac{p}{3} = \frac{a}{3(\beta - 1)\beta}\left(3(p+2)^{\beta} + \left(2^{\beta} - 5^{\beta}\right)(p+2) + 2\times 5^{\beta} - 5\times 2^{\beta}\right)$$

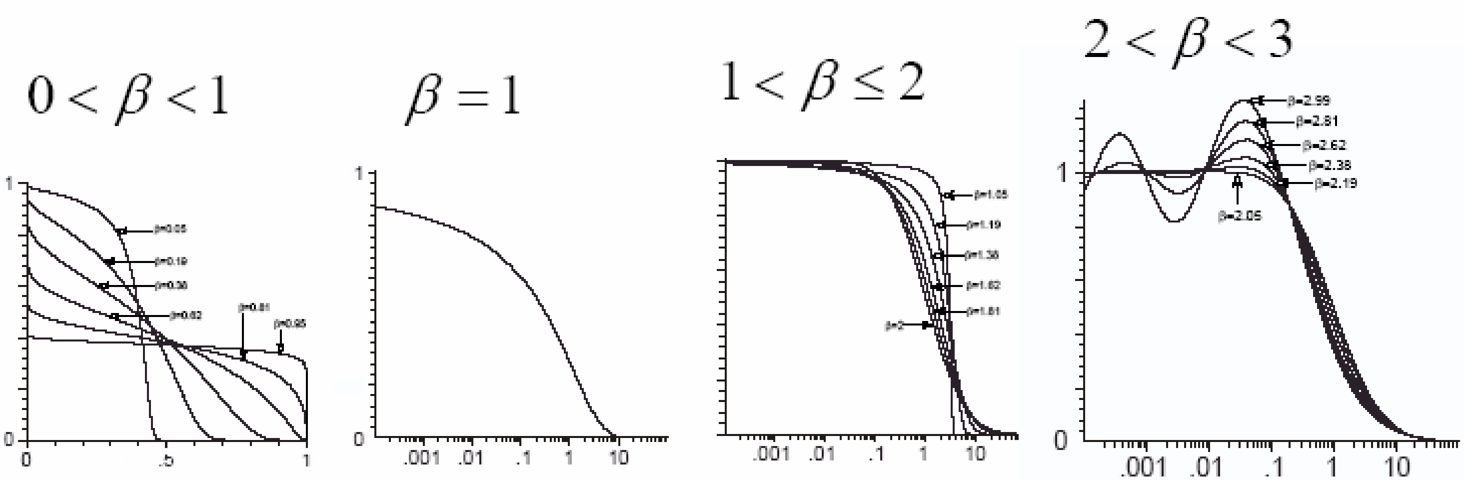
$$\beta \neq 0,1$$

$$a \neq 0 \qquad C_p = \frac{2}{p+2}\left(\frac{5C_3}{2}\right)^{p/3} \qquad p > -2$$

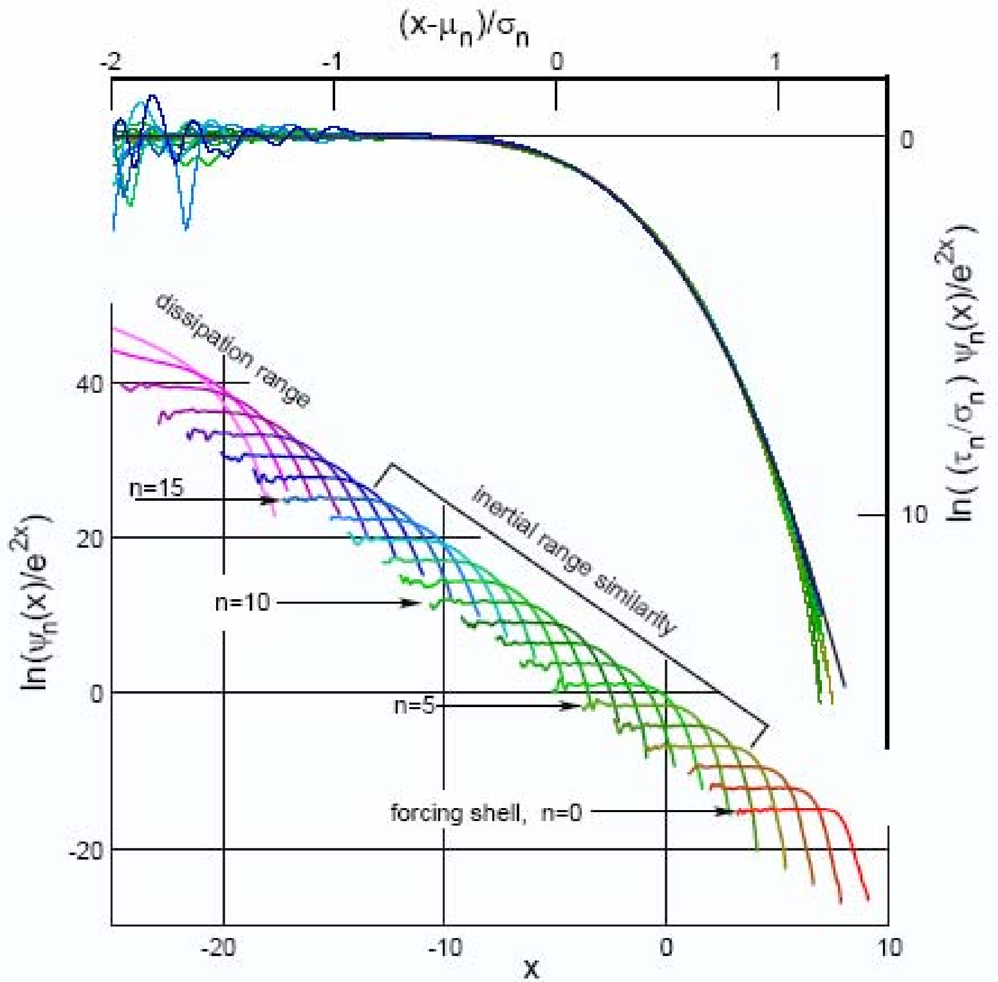
parameters: "intermittency parameter" a , "super exponent" β , intrinsic length scale ℓ_o C_3 a.k.a. dissipation

Radial profile in standard form

$$b(r)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-1} e^{z^{\beta}\,\mathrm{sgn}(\beta-1)-z\ln x}\,dz \qquad c>0 \quad \beta\neq 0,1$$

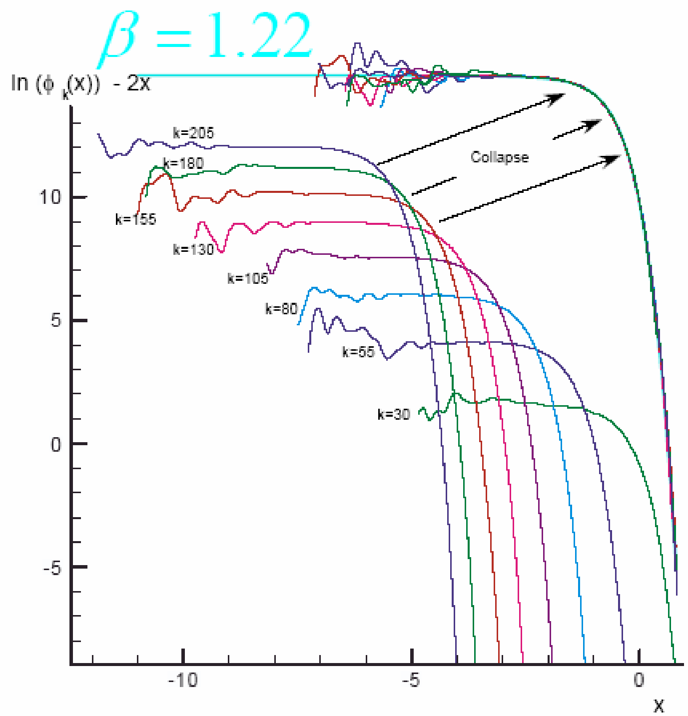


Collapse of shell model data



$\beta = 1.83$

DNS collapse onto theoretical pdf



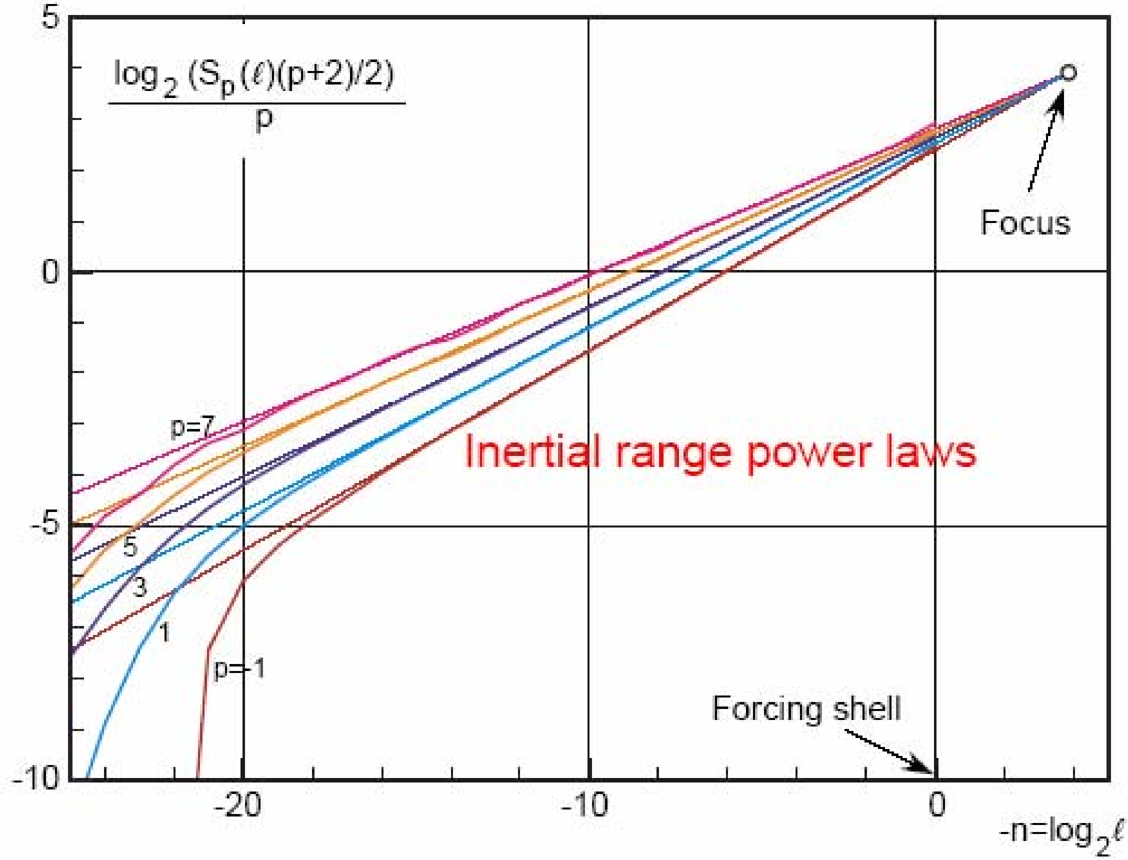
The intrinsic length scale

$$\sigma = \left(\ln \ell_o - \ln \ell\right)^{1/\beta}$$

$$P_0(r;\ell) = C(\ell) f\left(\frac{\ln r - \mu(\ell)}{\sigma(\ell)}\right)$$

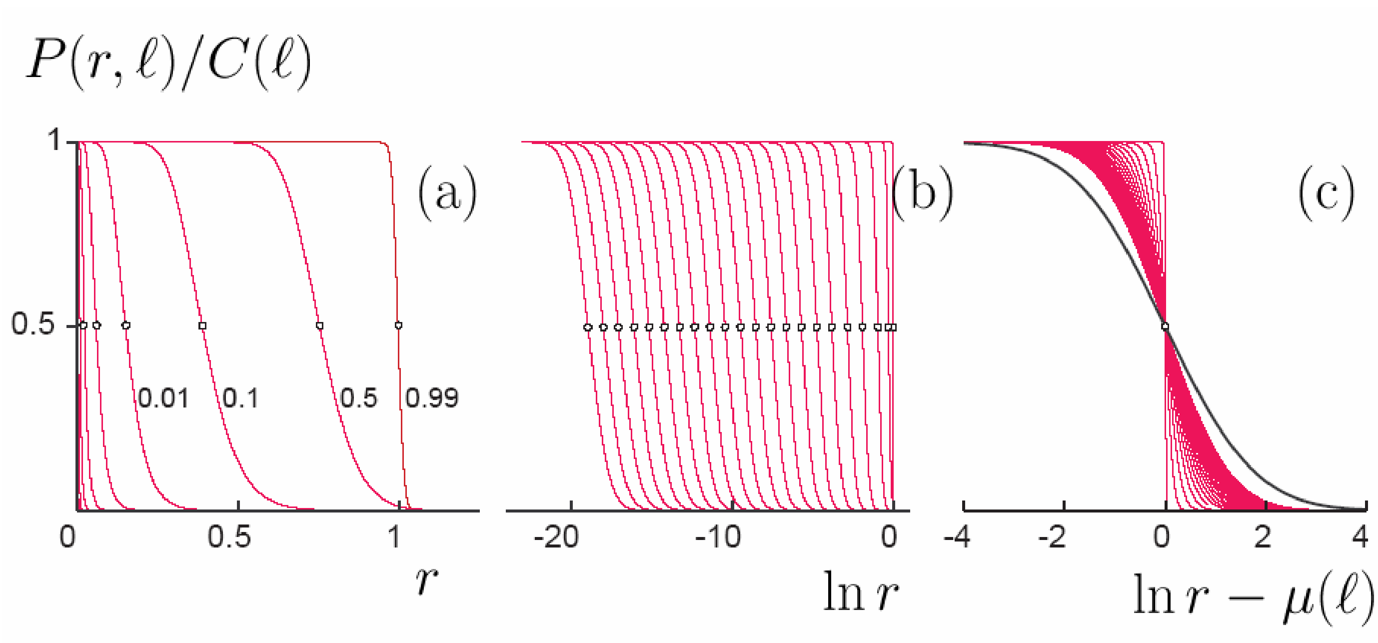
Intrinsic length scale

small scale



large scale

The limit $\ell \to \ell_o -$



Conclusion

- Without the intrinsic inertial range length scale universal scaling coefficients can not be identified.
- The matter of universality does not depend on how the intrinsic scale is chosen.
- The natural intrinsic scale is the largest l for which the scaling laws corresponds to a pdf.