2-D Turbulence for Forcing in all Scales

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1. Dissipation cut-off estimates via finite time averages

2. Cascades of pseudo fluxes

3. Dissipation law

4. Inertial range

Navier-Stokes eqns w/ per BCs on $\Omega = [0, L]^2$

$$\frac{du}{dt} + \nu A u + B(u,u) = f(t), \quad u = u(t) \in H\ ,\ t \geq t_0\ ,\ u(t_0) = u_0\ .$$

$A = -\mathcal{P}\Delta$, $\quad B(u,v) = \mathcal{P}\left((u\cdot\nabla)v\right)$, $\quad \mathcal{P} =$ Helmholtz-Leray proj.

$$u = \sum_{k\in\mathbb{Z}^2} \hat{u}_k(t) e^{i\kappa_0 k\cdot x}\ , \qquad u_{\kappa,\kappa'} = \sum_{\kappa \leq \kappa_0|k| < \kappa'} \hat{u}_k e^{i\kappa_0 k\cdot x}$$

$$\kappa_0 = \frac{2\pi}{L}\ , \quad \hat{u}_0 = 0\ , \quad \hat{u}_k^* = \hat{u}_{-k}\ .$$

$$p_\kappa = u_{\kappa_0,\kappa}\ , \qquad q_\kappa = u_{\kappa,\infty}\ ,$$

Pseudo-flux of Enstrophy

$$(\mathrm{NSE}, Aq_k), \quad |\cdot| = |\cdot|_{L^2}, \quad \|\cdot\| = |A^{1/2}\cdot| \ldots$$

$$\frac{1}{2}\frac{d}{dt}\|q_\kappa\|^2 + \nu|Aq_\kappa|^2 = \frac{1}{\kappa_0^2}\mathfrak{F}_\kappa = -(B(u,u), Aq_\kappa) + (f, Aq_\kappa)$$

net rate of exchange of enstrophy from low to high modes

$$\mathfrak{F}_\kappa = \mathfrak{E}_\kappa^{\rightarrow} - \mathfrak{E}_\kappa^{\leftarrow} + \kappa_0^2(f, Aq_\kappa)$$

$$\mathfrak{E}_\kappa^{\rightarrow}(u) = -\kappa_0^2(B(p_\kappa, p_\kappa), Aq_\kappa) \qquad \mathfrak{E}_\kappa^{\leftarrow}(u) = -\kappa_0^2(B(q_\kappa, q_\kappa), Ap_\kappa)$$

enstrophy cascade: $\langle \mathfrak{F}_\kappa \rangle \approx$ const for $\underline{\kappa}_i \le \kappa \le \overline{\kappa}_i$

Original dissipation cut-off estimate for time indep. force

$$\eta = \nu \kappa_0^2 \langle |Au|^2 \rangle_\infty \,, \qquad \kappa_\eta = \left(\frac{\eta}{\nu^3} \right)^{1/6} \,, \quad G = \frac{|f|}{\nu^2 \kappa_0^2} \,,$$

$\langle \cdot \rangle_\infty =$ "infinite time" average (via generalized, H-B limit)

$=$ ensemble average over global attractor

Theorem. [Foias-Manley-Temam '93]

$$G^{1/6} \lesssim \frac{\kappa_\eta}{\kappa_0} \leq G^{1/3} \qquad \textit{(both acheived)}$$

Framework for <u>time dep force</u>

$$\overline{|f|} = \sup_{t\in\mathbb{R}} |f(t)| < \infty \qquad \overline{G} = \frac{\overline{|f|}}{\nu^2\kappa_0^2}$$

finite time ave $\quad \langle\cdot\rangle = \langle\cdot\rangle_{t_1}^{t_2} = \dfrac{1}{t_2 - t_1}\displaystyle\int_{t_1}^{t_2} \cdot \, dt$

$$t_2 - t_1 \ge T_{\min} \qquad \overline{G} \ge \overline{G}_{\min}$$

$f \in L^\infty(t_0,\infty; D(A^{j/2}))$ and either $\begin{cases} \dot{f} \in L^\infty(t_0,\infty;H) \\ |\langle f\rangle| > 0 \end{cases}$

Two forms of diss cut-off estimates:

$$\left(\frac{\overline{G}}{C_*}\right)^{1/6} \le \frac{\kappa_\eta}{\kappa_0} \le (C^*\overline{G})^{1/3}$$

$$\left[\frac{\kappa_\tau \kappa_\sigma}{\kappa_0^2}\right]^{1/3} \left(\frac{\overline{G}}{K_*}\right)^{1/6} \le \frac{\kappa_\eta}{\kappa_0} \le \left[\frac{\kappa_0^2}{\kappa_\tau \kappa_\sigma}\right]^{1/3} (C^*\overline{G})^{1/3}$$

where

$$\kappa_\sigma = \left(\frac{\langle |Au|^2\rangle}{\langle \|u\|^2\rangle}\right)^{1/2} \qquad \kappa_\tau = \left(\frac{\langle \|u\|^2\rangle}{\langle |u|^2\rangle}\right)^{1/2}$$

Ave enstrophy lower bound:

$$\langle \|u\|^2\rangle \ge C(\nu\kappa_0)^2\overline{G}$$

As in [Foias-J.-Manley-Rosa '02] for $\mathfrak{E}_\kappa = \mathfrak{E}_\kappa^{\rightarrow} - \mathfrak{E}_\kappa^{\leftarrow}$

Theorem. *Under conditions on f such that*

$$\left(\frac{\overline{G}}{C_*}\right)^{1/6} \leq \frac{\kappa_\eta}{\kappa_0} \, ,$$

$$\textit{we have for} \quad t_2 - t_1 \geq \frac{2C_*\overline{G}}{\delta\nu\kappa_0^2} \quad \textit{and any} \quad \delta > 0 \, ,$$

$$1 - \left(\frac{\kappa}{\kappa_\sigma}\right)^2 - \delta \leq \frac{\langle \mathfrak{F}_\kappa \rangle}{\eta} \leq 1 + \delta \, .$$

Suff cond for direct enstr cascade: $\kappa_0 \ll \kappa_\sigma \implies$

$$\langle \mathfrak{F}_\kappa \rangle \approx \eta \quad \text{for} \quad \kappa_0 \leq \kappa \ll \kappa_\sigma$$

Pseudo-flux of Energy

net rate of exchange of energy from low to high modes

$$\mathfrak{f}_\kappa = -(B(u,u),q_\kappa) + \kappa_0^2(f,q_\kappa) = \mathfrak{e}_\kappa^{\rightarrow} - \mathfrak{e}_\kappa^{\leftarrow} + \kappa_0^2(f,q_\kappa)$$

net rate of exchange of energy from high to low modes

$$\mathfrak{g}_\kappa = -(B(u,u),p_\kappa) + \kappa_0^2(f,p_\kappa) = -[\mathfrak{e}_\kappa^{\rightarrow} - \mathfrak{e}_\kappa^{\leftarrow}] + \kappa_0^2(f,p_\kappa)$$

where

$$\mathfrak{e}_\kappa^{\rightarrow}(u) = -\kappa_0^2(B(p_\kappa,p_\kappa),q_\kappa)\ , \quad \text{and} \quad \mathfrak{e}_\kappa^{\leftarrow}(u) = -\kappa_0^2(B(q_\kappa,q_\kappa),p_\kappa)\ .$$

Theorem. *Under the conditions on f such that* $\langle \|u\|^2 \rangle \geq C(\nu\kappa_0)^2\overline{G}$

we have for $t_2 - t_1 \geq \dfrac{2\overline{G}}{C\delta\nu\kappa_0^2}$ *and any* $\delta > 0$,

$$1 - \left(\frac{\kappa}{\kappa_\tau}\right)^2 - \delta \leq \frac{\langle \mathfrak{f}_\kappa \rangle}{\epsilon} \leq 1 + \delta \ ,$$

and

$$1 - \left(\frac{\kappa_\sigma}{\kappa}\right)^2 - \delta \leq \frac{\langle \mathfrak{g}_\kappa \rangle}{\epsilon} \leq 1 + \delta \ .$$

direct cascade of pseudo energy flux: $\langle \mathfrak{f}_\kappa \rangle \approx \epsilon$ for $\kappa_0 \leq \kappa \ll \kappa_\tau$

inverse cascade of pseudo energy flux: $\langle \mathfrak{g}_\kappa \rangle \approx \epsilon$ for $\kappa_\sigma \ll \kappa$

Effect of forcing terms on the pseudo fluxes

For time-indep $f = f_{\underline{\kappa},\overline{\kappa}}$, nonlin fluxes $\mathfrak{E}_\kappa = \mathfrak{E}_\kappa^{\rightarrow} - \mathfrak{E}_\kappa^{\leftarrow}$, $\mathfrak{e}_\kappa = \mathfrak{e}_\kappa^{\rightarrow} - \mathfrak{e}_\kappa^{\leftarrow}$

$$\text{satisfy} \qquad \langle \mathfrak{E}_\kappa \rangle_\infty, \langle \mathfrak{e}_\kappa \rangle_\infty \begin{cases} \geq 0\,, & \text{if } \kappa > \overline{\kappa} \\ \leq 0\,, & \text{if } \kappa \leq \underline{\kappa} \end{cases}$$

Yet $\mathfrak{F}_\kappa = \mathfrak{E}_\kappa + (f, Aq_\kappa)$, $\mathfrak{f}_\kappa = \mathfrak{e}_\kappa + (f, q_\kappa)$ and $\mathfrak{g}_\kappa = -\mathfrak{e}_\kappa + (f, p_\kappa)$

$$\text{satisfy} \quad \langle \mathfrak{F}_\kappa \rangle > 0 \text{ for } \kappa < \kappa_\sigma \quad \text{and} \quad \langle \mathfrak{f}_\kappa \rangle > 0 \text{ for } \kappa < \kappa_\tau$$

$$\langle \mathfrak{g}_\kappa \rangle > 0 \text{ for } \kappa > \kappa_\sigma$$

For general force conclude: $|\langle \mathfrak{e}_\kappa \rangle| < \langle (f, p_\kappa) \rangle$ for $\kappa > \kappa_\sigma$

Rationale for forcing terms in the pseudo-fluxes

net rate of energy exchange into low modes : $\mathfrak{g}_\kappa = -\mathfrak{e}_\kappa + (f, p_\kappa)$

injection of energy into p_κ from sources <u>external</u> to p_κ

minus loss of energy from p_κ to q_κ

$$
\begin{aligned}
\frac{\mathfrak{e}_\kappa}{\kappa_0^2} &= -(B(u,u), q_\kappa) = (B(u,u), p_\kappa) \\
&= (B(q_\kappa, q_\kappa), p_\kappa) + (B(p_\kappa, q_\kappa), p_\kappa) + \underbrace{(B(q_\kappa, p_\kappa), p_\kappa)}_{0} + \underbrace{(B(p_\kappa, p_\kappa), p_\kappa)}_{0} \\
&= (B(q_\kappa, q_\kappa), p_\kappa) - (B(p_\kappa, p_\kappa), q_\kappa)
\end{aligned}
$$

Rationale for forcing terms in the psedo-fluxes

$$\underbrace{\frac{d}{dt}|p_\kappa|^2}_{\text{net change}} + \underbrace{\nu\|p_\kappa\|^2}_{\text{internal source}} = \underbrace{-(B(u,u)),p_\kappa) + (f,p_\kappa)}_{\text{external source}} = \frac{\mathfrak{g}_\kappa}{\kappa_0^2}$$

- viscous term has same sign for all κ
- f may have mixing effect similar to nonlinear term

2-D dissipation law $\eta \sim \kappa_0^3 U^3$ where $U = \kappa_0 \langle |u|^2 \rangle^{1/2}$

Theorem. [Foias-J.-Manley-Rosa '02] $f = f_{\underline{\kappa},\overline{\kappa}}$, *time indep*

$$\eta \lesssim \left(\frac{\overline{\kappa}}{\kappa_0}\right)^4 \kappa_0^3 U^3 \,, \quad \textit{where} \quad \langle \cdot \rangle = \langle \cdot \rangle_\infty$$

Theorem. *If*

$$\left[\frac{\kappa_\tau \kappa_\sigma}{\kappa_0^2}\right]^{1/3} \left(\frac{\overline{G}}{K_*}\right)^{1/6} \le \frac{\kappa_\eta}{\kappa_0} \le \left[\frac{\kappa_0^2}{\kappa_\tau \kappa_\sigma}\right]^{1/3} (C^* \overline{G})^{1/3} \,,$$

then

$$\eta \le C^* K_* \kappa_0^3 U^3 \,, \quad \textit{where} \quad \langle \cdot \rangle = \langle \cdot \rangle_{t_1}^{t_2}$$

Proof:

$$\left[\frac{\kappa_0^2}{\kappa_\tau\kappa_\sigma}\right] C^*\overline{G} \geq \left(\frac{\kappa_\eta}{\kappa_0}\right)^3 = \frac{1}{\nu\kappa_0^2}\langle|Au|^2\rangle^{1/2} = \frac{1}{\nu}\left[\frac{\kappa_\tau\kappa_\sigma}{\kappa_0^2}\right]\langle|u|^2\rangle^{1/2}$$

$$\implies \quad C^*\overline{G} \geq \frac{1}{\nu}\left[\frac{\kappa_\tau\kappa_\sigma}{\kappa_0^2}\right]^2 \langle|u|^2\rangle^{1/2}$$

$$C^*\frac{\langle|Au|^2\rangle}{\nu^2\kappa_0^4} = C^*\left(\frac{\kappa_\eta}{\kappa_0}\right)^6 \geq \left[\frac{\kappa_\tau\kappa_\sigma}{\kappa_0^2}\right]^2 \frac{C^*\overline{G}}{K_*} \geq \frac{1}{\nu K_*}\left[\frac{\kappa_\tau\kappa_\sigma}{\kappa_0^2}\right]^4 \langle|u|^2\rangle^{1/2}$$

$$= \frac{1}{\nu\kappa_0^8 K_*}\frac{\langle|Au|^2\rangle^2}{\langle|u|^2\rangle^{3/2}}$$

$$\nu\langle|Au|^2\rangle \leq C^* K_* \kappa_0^4 \langle|u|^2\rangle^{3/2}$$

2-D dissipation law equiv to $\kappa_\tau \kappa_\sigma \sim \kappa_0 \kappa_\eta$

Theorem. *If*

$$\eta \leq C^* K_* \kappa_0^3 U^3 \ ,$$

then

$$\kappa_\tau \kappa_\sigma \leq (C^* K_*)^{1/3} \kappa_0 \kappa_\eta \ .$$

Proof: The (one-sided) dissipation law can be rewritten as

$$\nu\langle |Au|^2\rangle \le C^* K_* \kappa_0^4 \langle |u|^2\rangle^{3/2} = C^* K_* \kappa_0^4 \kappa_\tau^{-3} \langle \|u\|^2\rangle^{3/2} ,$$

which is equivalent to

$$\langle |Au|^2\rangle^{2/3} \le (C^* K_*)^{2/3} \nu^{-2/3} \kappa_0^{8/3} \kappa_\tau^{-2} \langle \|u\|^2\rangle ,$$

which is equivalent to

$$\kappa_\tau^2 \kappa_\sigma^2 = \kappa_\tau^2 \frac{\langle |Au|^2\rangle}{\langle \|u\|^2\rangle} \le (C^* K_*)^{2/3} \kappa_0^2 \left(\frac{\kappa_0^2 \langle |Au|^2\rangle}{\nu^2} \right)^{1/3} = (C^* K_*)^{2/3} \kappa_0^2 \kappa_\eta^2 .$$

Inertial range

1. A significant amount of enstrophy should be in the inertial range.

2. This range should be wide, in particular $\underline{\kappa}_i \ll \overline{\kappa}_i \sim \kappa_\eta$.

3. The enstrophy cascade should hold over this range.

4. The power law

$$\mathsf{e}_{\kappa,2\kappa} \sim \frac{\eta^{2/3}}{\kappa^2}$$

   should hold for all $\kappa \in [\underline{\kappa}_i, \overline{\kappa}_i]$.

Logarithmic correction

As in [Rose-Sulem'78], [Ohkitani'89]

$$\frac{\eta^{2/3}}{\kappa^2} \prec \mathsf{e}_{\kappa,2\kappa} \prec \frac{\eta^{2/3}}{\kappa^2} \ ,$$

where

$a \prec b$ when $a \leq C(\log(s\overline{G}))^\alpha b$ for some $\alpha \in \mathbb{R}$, and large enough $\overline{G}$

and where C and s are shape factors, with a similar convention for $\succ$.

Theorem. *For any* κ *such that*

$$\frac{\eta^{2/3}}{\kappa^2} \prec \mathsf{e}_{\kappa,2\kappa}$$

we have $\kappa \prec \kappa_\eta$*, and consequently*

$$\overline{\kappa}_i \prec \kappa_\eta \ .$$

Theorem. *If*

$$\kappa_\eta \prec \kappa_\sigma \ ,$$

then

$$\mathsf{e}_{\kappa,2\kappa} \prec \frac{\eta^{2/3}}{\kappa^2} \ , \quad \textit{for all } \kappa \geq \kappa_0 \ .$$

Moreover, for any κ such that

$$\frac{\eta^{2/3}}{\kappa^2} \prec \mathsf{e}_{\kappa,2\kappa} \prec \frac{\eta^{2/3}}{\kappa^2} \ ,$$

we have $\kappa_\tau \prec \kappa$, and consequently

$$\kappa_\tau \prec \underline{\kappa}_i \ , \qquad \kappa_\tau \prec \kappa_0$$

Theorem. *If*

$$\mathsf{e}_{\kappa,2\kappa} \prec \frac{\eta^{2/3}}{\kappa^2}$$

holds for

$$\kappa_1 = \left(1 - \frac{C}{\log \overline{G}}\right)^{-1/2} \kappa_\tau \leq \kappa \leq \kappa_2 = \left(1 - \frac{C}{\log \overline{G}}\right)^{1/2} \kappa_\eta$$

then for all $\overline{G}$ large enough,

$$\kappa_\eta \prec \kappa_\sigma$$

and

$$\frac{\langle \|q_{\kappa_1}\|^2 \rangle}{\langle \|p_{\kappa_1}\|^2 \rangle} \geq \frac{1}{\log \overline{G}} \; .$$

Theorem. *If* $t_2 - t_1 \geq T_{\min}$ *and* $\overline{G} \geq \overline{G}_{\min}$, *then*

$$\left(\frac{\overline{G}}{C_*}\right)^{1/6} \leq \frac{\kappa_\eta}{\kappa_0} \leq (C^*\overline{G})^{1/3}$$

force	$T_{\min}$	$\overline{G}_{\min}$	C_*	C^*
$f, \dot{f} \in L^\infty(t_0, \infty; H)$	$\frac{c}{\nu\kappa_0^2\Gamma^2}$	$c(\Gamma_1^2 + 1)$	$\frac{c}{\Gamma^2}$	c
$f \in L^\infty(t_0, \infty; H)$, $\lvert\langle f\rangle\rvert > 0$	$\frac{c}{\nu\kappa_0^2\Gamma_0^2}$	$\frac{c}{\Gamma_0^2}$	$\frac{c}{\Gamma_0^2}$	c

$$\Gamma_1 = \sup\left(\frac{\langle|\dot{f}|^2\rangle}{\nu^2\kappa_0^4\langle|f|^2\rangle}\right)^{1/2} \quad \Gamma = \inf\frac{\langle|f|^2\rangle^{1/2}}{\overline{|f|}} \quad \Gamma_0 = \inf\frac{|\langle f\rangle|}{\overline{|f|}}$$

sup, inf are over all t_1,t_2 s.t. $t_2 - t_1 \geq 1/(\nu\kappa_0^2)$.

Theorem. *If* $f \in L^\infty(t_0,\infty;X)$, $t_2 - t_1 \ge T_{\min}$ *and* $\overline{G} \ge \overline{G}_{\min}$, *then*

$$\left[\frac{\kappa_\tau\kappa_\sigma}{\kappa_0^2}\right]^{1/3}\left(\frac{\overline{G}}{K_*}\right)^{1/6} \le \frac{\kappa_\eta}{\kappa_0} \le \left[\frac{\kappa_0^2}{\kappa_\tau\kappa_\sigma}\right]^{1/3}(C^*\overline{G})^{1/3}$$

force	$T_{\min}$	$\overline{G}_{\min}$	K_*	C^*
$X = D(A),\ \dot{f} \in L^\infty(t_0,\infty;H)$	$\frac{\overline{G}^{1/2}\log^{1/4}\overline{G}}{\nu\kappa_0^2}$	$\frac{c(\psi_2^2+\Gamma_1^2)}{\varphi_2\log^{1/2}\overline{G}}$	$\frac{c\psi_2\log^{1/2}\overline{G}}{\Gamma}$	$c\psi_2$
$X = D(A^{3/2}),\ \lvert\langle f\rangle\rvert > 0$	$\frac{c\overline{G}^{1/2}}{\nu\kappa_0^2}$	$\frac{c\theta_2}{\Gamma_0\left[\log\frac{e\theta_3}{\theta_2}\right]^{1/2}}$	$\frac{c\theta_2\left[\log\frac{e\theta_3}{\theta_2}\right]^{1/2}}{\Gamma_0}$	$c\psi_2$

$$\psi_j^2 = \sup\frac{\langle|A^{j/2}f|^2\rangle}{\kappa_0^{2j}\langle|f|^2\rangle} \qquad \varphi_j = \frac{\overline{|A^{j/2}f|}}{\kappa_0^{2j}\overline{|f|}} \qquad \theta_j = \sup\frac{|A^{j/2}\langle f\rangle|}{\kappa_0^{j}|\langle f\rangle|}$$

Sharpening in diss cut-off est that results if $\kappa_\tau \kappa_\sigma \sim \kappa_0 \kappa_\eta$

Corollary. *If* $t_2 - t_1 \geq T_{\min}$, $\overline{G} \geq \overline{G}_{\min}$, *and*

$$\frac{\kappa_\tau \kappa_\sigma}{\kappa_0^2} \sim \frac{\kappa_\eta}{\kappa_0} \,,$$

then

$$\left(\frac{\overline{G}}{K_*}\right)^{1/4} \leq \frac{\kappa_\eta}{\kappa_0} \leq (C^* \overline{G})^{1/4} \,.$$

Summary

- $\left(\frac{\kappa_\tau \kappa_\sigma}{\kappa_0^2}\right)^{1/3} \overline{G}^{1/6} \lesssim \kappa_\eta/\kappa_0 \lesssim \left(\frac{\kappa_0^2}{\kappa_\tau \kappa_\sigma}\right)^{1/3} \overline{G}^{1/3}$
- Tme dep forcing in all scales. Shape factors fixed as $\overline{G} \to \infty$
- Lower bound on $\kappa_\eta/\kappa_0 \implies$ direct enstrophy cascade if $\kappa_\sigma \gg \kappa_0$
- Energy power law $\implies \kappa_\sigma \sim \kappa_\eta$ (up to log)
- 2-D diss law $\eta \sim \kappa_0^3 U^3$ equiv to $\kappa_\tau \kappa_\sigma \sim \kappa_0 \kappa_\eta$
- $\kappa_\tau \kappa_\sigma \sim \kappa_0 \kappa_\eta \implies$ sharp bound $\overline{G}^{1/4} \sim \kappa_\eta/\kappa_0$