Dissipation range and anomalous sinks in steady two-dimensional turbulence

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Publications

- This presentation is based on
  1. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 79-102
  2. E. Gkioulekas and K.K. Tung (2005), *Discrete and Continuous Dynamical Systems B*, **5**, 103-124.
  3. E. Gkioulekas (2007), *Physica D*, **226**, 151-172
  4. E. Gkioulekas (2008), *Phys. Rev. E* **78**, 066302
  5. E. Gkioulekas (2009), *Phys. Rev. E*, submitted [arXiv:0903.2863]
- Other relevant papers include:
  1. U. Frisch, *Proc. R. Soc. Lond. A* **434** (1991), 89–99.
  2. U. Frisch, *Turbulence: The legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge, 1995.
  3. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **52** (1995), 3840–3857.
  4. V.S. L'vov and I. Procaccia, *Phys. Rev. E* **54** (1996), 6268–6284.

K41 theory. I.

- In three-dimensional turbulence there is an energy cascade from large scales to small scales driven by the nonlinear term of the Navier-Stokes equations
- Kolmogorov (1941) predicts that the structure functions $S_n(\mathbf{x}, r\mathbf{e})$ of longitudinal velocity differences, defined as

$$S_n(\mathbf{x}, r\mathbf{e}) = \langle \{[\mathbf{u}(\mathbf{x} + r\mathbf{e}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \mathbf{e}\}^n \rangle \tag{1}$$

  are governed by self-similar scaling $S_n(\mathbf{x}, \lambda r\mathbf{e}) = \lambda^{\zeta_n} \lambda S_n(\mathbf{x}, r\mathbf{e})$ for scales r in the inertial range $\eta \ll r \ll \ell_0$ (intermediate asymptotics) with
  - $\ell_0 =$ forcing length scale
  - $\eta = (\nu^3/\varepsilon)^{1/4} =$ dissipation scale. (Kolmogorov microscale)
  - $\varepsilon =$ rate of energy injection
- Kolmogorov (1941) predicts that $\zeta_n = n/3$ and thus $S_n(\mathbf{x}, r\mathbf{e}) \sim C_n(\varepsilon r)^{n/3}$ in the inertial range.

K41 theory. II.

- Oboukhov (1941) argued that the energy spectrum $E(k)$ will scale as $E(k) \sim k^{-1-\zeta_2}$, and will thus be given by

$$E(k) \sim C\varepsilon^{2/3}k^{-5/3} \tag{2}$$

- 1962: First experimental confirmation of the Kolmogorov-Oboukhov prediction by measurement of oceanic currents.
- 1962: Kolmogorov predicts **intermittency corrections** to ζ_n:

$$\zeta_n = \frac{n}{3} - \frac{\mu n(n-3)}{18} \tag{3}$$

- Not self-consistent statistically, because ζ_n should not decrease.
- The existence of intermittency corrections confirmed by experimental measurements
- The problem of calculating ζ_n rigorously is still open.

Governing equations for 2D

- In 2D turbulence, the scalar vorticity $\zeta(x, y, t)$ is governed by

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = -[\nu(-\Delta)^{\kappa} + \beta(-\Delta)^{-m}]\zeta + F, \tag{4}$$

  where $\psi(x, y, t)$ is the streamfunction and $\zeta(x, y, t) = -\nabla^2 \psi(x, y, t)$.

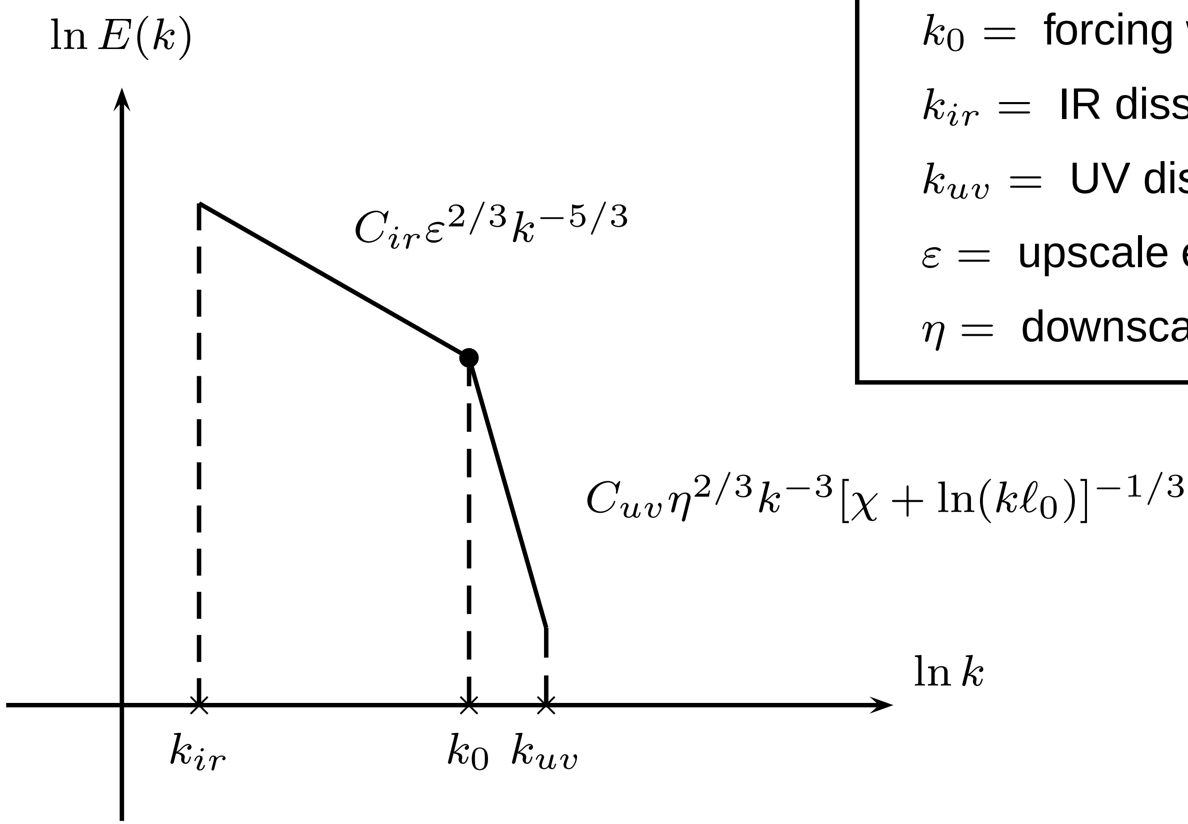
- The Jacobian term $J(\psi, \zeta)$ describes the advection of ζ by ψ, and is defined as

$$J(\psi, \zeta) = \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} - \frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial y}. \tag{5}$$

- Two conserved quadratic invariants: energy E and enstrophy G defined as

$$E(t) = -\frac{1}{2}\int \psi(x, y, t)\zeta(x, y, t)\, dxdy \quad G(t) = \frac{1}{2}\int \zeta^2(x, y, t)\, dxdy. \tag{6}$$

KLB theory



$k_0 =$ forcing wavenumber

$k_{ir} =$ IR dissipation wavenumber

$k_{uv} =$ UV dissipation wavenumber

$\varepsilon =$ upscale energy flux

$\eta =$ downscale enstrophy flux

The generalized balance equations. I

- We employ the balance equations introduced by L'vov and Procaccia (1996).
- Define the fully unfused correlation tensors for velocity u_α and vorticity ζ:

$$F_n^{\alpha_1\alpha_2\dots\alpha_n}(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left\langle \prod_{k=1}^n \left[u_{\alpha_k}(\mathbf{x}_k, t) - u_{\alpha_k}(\mathbf{x'}_k, t)\right] \right\rangle, \quad (7)$$

$$V_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left\langle \prod_{k=1}^n \left[\zeta(\mathbf{x}_k, t) - \zeta(\mathbf{x'}_k, t)\right] \right\rangle \quad (8)$$

- The relation between F_n and V_n is $V_n = \mathcal{T}_n F_n$ or:

$$V_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \prod_{k=1}^n [\varepsilon_{\alpha_k\beta_k}(\partial_{\alpha_k,\mathbf{x}_k} + \partial_{\alpha_k,\mathbf{x'}_k})] F_n^{\alpha_1\cdots\alpha_n}(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) \quad (9)$$

The generalized balance equations. II

- F_n and V_n satisfy the balance equations:

$$\frac{\partial F_n}{\partial t} + \mathcal{O}_n F_{n+1} + I_n = \mathcal{D}_n F_n + Q_n \tag{10}$$

$$\frac{\partial V_n}{\partial t} + \mathcal{T}_n \mathcal{O}_n F_{n+1} + \mathcal{I}_n = \mathcal{D}_n V_n + \mathcal{Q}_n \tag{11}$$

Here $Q_n, \mathcal{Q}_n$ are forcing terms and $I_n, \mathcal{I}_n$ are sweeping terms, $\mathcal{O}_n$ local interactions, and $\mathcal{D}_n$ the dissipation operator.

- Belinicher, L'vov, Pomyalov and Procaccia (1998) argue that in 3D turbulence, the scaling of the downscale energy cascade originates from the solvability condition on the homogeneous equation

$$\mathcal{O}_n F_{n+1} = 0 \tag{12}$$

This argument leads to a scheme for computing the scaling exponents ζ_n of F_n.

The generalized balance equations. III

- In two-dimensional turbulence, homogeneous solutions originate from

$$\mathcal{O}_n F_{n+1} = 0 \Longrightarrow \text{1 solution: energy cascade} \quad (13)$$

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} = 0 \Longrightarrow \text{2 solutions: energy and enstrophy cascade} \quad (14)$$

- We conjecture that the balance equations essentially have two homogeneous solutions (energy/enstrophy cascade) from $\mathcal{T}_n \mathcal{O}_n F_{n+1} = 0$, and a particular solution (coherent structures) which is caused by Q_n and I_n.
- The realistic solutions for each cascade include a dissipation range. These solutions originate from the modified equation

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} - \mathcal{T}_n \mathcal{D}_n F_n = 0. \quad (15)$$

- The dissipative terms modify the linear operator $\mathcal{O}_n$ thus truncating the inertial range with the dissipation range.

Requirements for universal inertial ranges. I

- The existence of an inverse energy cascade or an enstrophy cascade requires:
  - A region $\mathcal{A}_n \subseteq \mathbb{R}^{2n}$ where the corresponding leading homogeneous solution dominates the particular solution.
  - A region $\mathcal{B}_n \subseteq \mathbb{R}^{2n}$ where dissipative effects on the leading homogeneous solution are negligible.
  - An overlap $\mathcal{I}_n = \mathcal{A}_n \cap \mathcal{B}_n$ with non-zero measure.
- The region $\mathcal{I}_n$ is thus a multidimensional representation of the extent of the inertial range associated with the generalized structure function F_n.
- Within the region $\mathcal{I}_n$ we expect that F_n will be self-similar according to the following scaling law:

$$F_n(\lambda\{\mathbf{X}\}_n, t) = \lambda^{\zeta_n} F_n(\{\mathbf{X}\}_n, t). \tag{16}$$

when $\{\mathbf{X}\}_n \in \mathcal{I}_n$ and $\lambda \in (1-\varepsilon, 1+\varepsilon)$ with ε small.

Argument outline. I

- Recall that within the inertial range $E(k) \sim k^{-1-\zeta_2}$
- If we require the cascades to have universal scaling exponents ζ_n, can the region $\mathcal{J}_n$ have a non-zero measure?
- **Step 1:** Universality $\Longrightarrow$ Fusion rules hypothesis
  - Define $F_n^{(p)}(r,R) = F_n(r\{\mathbf{X}_k\}_{k=1}^p, R\{\mathbf{X}_k\}_{k=p+1}^n)$.
  - The fusion rules give the scaling properties of $F_n^{(p)}$ in terms of the following general form:
$$F_n^{(p)}(\lambda_1 r, \lambda_2 R) = \lambda_1^{\xi_{np}} \lambda_2^{\zeta_n - \xi_{np}} F_n^{(p)}(r,R) \tag{17}$$
  - $\xi_{np} = \zeta_p$ for the direct enstrophy cascade ($1 < p < n-1$)
  - $\xi_{np} = \zeta_n - \zeta_{n-p}$ for the inverse energy cascade ($1 < p < n-1$)

Argument outline. II

- **Step 2:** Fusion rules hypothesis $\Longrightarrow$ Locality
  - The integrals in the nonlinear interactions term $\mathcal{O}_n F_{n+1}$ are local.
  - Thus, the scaling exponent of $\mathcal{O}_n F_{n+1}$ is $\zeta_{n+1} - 1$.
- **Step 3:** Locality $\Longrightarrow$ Stability
  - Assume random gaussian forcing.
  - The scaling exponent of Q_n is $q_n = q_2 + \zeta_{n-2}$
  - Compare Q_n with $\mathcal{O}_n F_{n+1}$.
  - Enstrophy cascade marginally stable.
  - Inverse energy cascade stable.
  - Details in
    - E. Gkioulekas (2008), *Phys. Rev. E* **78**, 066302
- **Step 4:** Fusion rules $\wedge$ Locality $\Longrightarrow$ Anomalous sinks
  - Locate dissipation scales
  - Establish anomalous sinks

Enstrophy cascade sink

- Assume $F_n(R) \sim (\eta_{uv}^{1/3} R)^n [\ln(\ell_0/R)]^{a_n}$.
- Falkovich and Lebedev theory: $a_n = 2n/3$.
- Consider $F_n^{(1)}(r, R)$ with $r \ll R \ll \ell_0$.
- Calculate dissipative length scale function $r = \ell_{uv}^{(n)}(R)$ whose graph traces out the dissipative boundary of the enstrophy inertial range in the (r, R) plane.
- Observable dissipative length scale: $\ell_{uv}^{(n)}(\lambda_{uv}^{(n)}) = \lambda_{uv}^{(n)}$
- Admissibility condition $a\lambda_{uv}^{(n)} > \ell_{uv}^{(n)}(a\lambda_{uv}^{(n)})$, $\forall a \in (1, \ell_0/\lambda_{uv}^{(n)})$ is satisfied.
- This gives the enstrophy dissipation rate η_{uv} as:

$$\eta_{uv} \sim \nu^{1-(\zeta_2-2(\kappa+1))/(\xi_{2,1}-2(\kappa+1))} [\ln(\ell_0/\lambda_{uv})]^{a_3-1}. \tag{18}$$

- Anomalous enstrophy sink when
  - $\xi_{2,1} = \zeta_2$ (Fusion rules hypothesis)
  - $a_3 = 1$ (Falkovich and Lebedev theory)

Inverse energy cascade sink

- Assume $F_n(r) \sim (\varepsilon_{ir} r)^{n/3} (r/\ell_0)^{\zeta_n - n/3}$ and $F_3(r) \sim \varepsilon_{ir} r$.
- Consider $F_n^{(1)}(r, R)$ with $\ell_0 \ll R \ll r \ll$.
- Calculate dissipative length scale function $R = \ell_{ir}^{(n)}(r)$ whose graph traces out the dissipative boundary of the inverse energy cascade range in the (r, R) plane.
- Observable dissipative length scale: $\ell_{ir}^{(n)}(\lambda_{ir}^{(n)}) = \lambda_{ir}^{(n)}$
- Admissibility condition $a\lambda_{ir}^{(n)} < \ell_{ir}^{(n)}(a\lambda_{ir}^{(n)})$, $\forall a \in (\ell_0/\lambda_{ir}^{(n)}, 1)$ requires $\zeta_{n+1} - \zeta_n < 2m + 1, \; \forall n > 2.$
- This gives the energy dissipation rate ε_{ir} as:

$$\varepsilon_{ir} \sim \beta^{1-(\zeta_2+2m)/(\zeta_2-\xi_{2,1}+2m)}. \tag{19}$$

- Anomalous energy sink when
  - $\xi_{2,1} = 0$ (Fusion rules hypothesis)

Conclusions

- The hypothesis that there may be an anomalous enstrophy sink at small scales and an anomalous energy sink at large scales emerges as a consequence of the fusion rules hypothesis.
- The logarithmic correction of Kraichnan to the enstrophy cascade energy spectrum plays an essential role in ensuring that the inertial range of the enstrophy cascade is not entirely contaminated by dissipation, when $\kappa = 1$.
- If there are intermittency corrections to the scaling exponents ζ_n , then the scaling exponents must satisfy the inequality $\zeta_{n+1} - \zeta_n < 2m + 1, \ \forall n > 2$, with m being the order of the hypodissipation, in order for *all* generalized structure functions F_n to have an inertial range.
- A possible small violation of the fusion rules can be compensated for by increasing the orders κ and m of hyperdiffusion and hypodiffusion correspondingly.