2D Navier-Stokes equations with singular forcing

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Support: NSF

In this talk:

- Motivation: Nonlinear Fokker-Planck Equation

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- Navier-Stokes Results

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- Minima of Free Energy: Onsager Equation

$$f = Z^{-1}e^{-U[f]}.$$

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Embedding in Fluid: Passive

$$\partial_t f + u \cdot \nabla_x f + \mathrm{div}_g(Wf) = \mathrm{div}_g\left(f\nabla_g\left(\log f + U[f]\right)\right)$$

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Macro-Micro Effect: from first principles, if scales are separated.

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added stress tensor.

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$$\sigma(x) = \int_M \{c \cdot \nabla_g U[f](x,m) - \mathrm{div}_g c\} f(x,m) d\mu(m) \quad *$$

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$f \geq 0,\ \sup_x \int_M f d\mu(m) < \infty \Rightarrow \sigma \in L^\infty$

Theorem

*3DNS + NL Fokker-Planck eqns with *. Then*

$$E(t) = \frac{1}{2}\int |u|^2 dx + \\ + \int \left\{ f \log f + \frac{1}{2}(U[f])f \right\} dx d\mu.$$

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If the smooth solution is time independent, then $u = 0$ *and* f *solves the Onsager equation*

$$f = Z^{-1} e^{-U[f]}.$$

NLFP + 2D time dependent Navier-Stokes

Theorem

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$$\|\nabla_x \nabla_x u\|_{L^p(0,T;L^q(\mathbb{T}^2))} \leq K,$$

$$\sup_{t \leq T} \|\nabla_x u(\cdot, t)\|_{L^\infty} \leq K,$$

and

$$\sup_{t \leq T} \|f(\cdot, t)\|_{W^{1,q}(\mathbb{T}^2; H^{-\alpha}(M)))} \leq K.$$

hold.

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Previous result, with Masmoudi: no apriori bounds and proof by contradiction, controlled loss of regularity.

Related Results

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Lin, Zhang: a corotational model with noncompact target.

Navier Stokes Equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \nabla \cdot \sigma$$
$$\nabla \cdot u = 0$$

The tensor $\sigma_{ij}(x, t)$: driving stress.

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$$\int_0^T \|u\|^2_{L^\infty(dx)}dt < \infty$$

Exponent of the amplification factor of tracers, key quantity

$$\int_0^T \|\nabla u\|_{L^\infty(dx)}dt < \infty$$

2D, Bounded stress

Theorem

Let $\sigma \in L^\infty(dtdx)$. Let $u_0 \in L^2(dx)$. There exists a unique weak solution of the forced 2D NS eqns, with

$$u \in L^\infty(dt)(L^2(dx)) \cap L^2(dt)(W^{1,2}(dx))$$

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Moreover,

$$\int_0^T \|\nabla u\|_{L^q(dx)}^{\frac{q}{q-1}} dt < \infty, \quad \forall\, q \geq 2$$

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Navier-Stokes with nearly singular forces

$$\partial_t u + u \cdot \nabla_x u - \nu \Delta_x u + \nabla_x p = \mathrm{div}_x \sigma, \quad \nabla_x \cdot u = 0$$

Theorem

Let u be a solution of the 2D Navier-Stokes system with divergence-free initial data $u_0 \in W^{1,2}(\mathbb{R}^2) \cap W^{1,r}(\mathbb{R}^2)$. Let $T > 0$ and let the forces $\nabla \cdot \sigma$ obey

$$\sigma \in L^1(0, T; L^\infty(\mathbb{R}^2)) \cap L^2(0, T; L^2(\mathbb{R}^2))$$
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with $r > 2$.

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with $r>2$. Let

$$\|\sigma\|_{L^\infty}\sim K, \quad \|\nabla\cdot\sigma\|_{L^r}\sim B$$

Then

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$$u = \sum_{q=-1}^{\infty} \Delta_q(u)$$

Littlewood-Paley decomposition

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Parabolic balls:

Hölder regularity of weak solution

$\Omega_1 \Subset \Omega$ domains in $\mathbb{R}^2$, $0 < T_1 < T$.

$$Q = \Omega \times (-T, 0), \qquad Q_1 = \Omega_1 \times (-T_1, 0).$$

Parabolic balls: $Q(z_0, R) = B(x_0, R) \times (t_0 - R^2, t_0)$, where $z_0 = (x_0, t_0)$, $x_0 \in \mathbb{R}^2$, $t_0 \in \mathbb{R}$.

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$$(f)_{z_0,R} = \frac{1}{|Q(z_0,R)|} \int\limits_{Q(z_0,R)} f(z)dz,$$
$$[p]_{x_0,R} = \frac{1}{|B(x_0,R)|} \int\limits_{B(x_0,R)} p(x)dx.$$

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Note

$$L^r(Q) \subset M_{2,\gamma}(Q)$$

for $r \geq \frac{4}{1-\gamma}$.

Theorem

(C-Seregin) Let

$$u \in L^4(Q;\mathbb{R}^2),\ p \in L^2(Q),\ \sigma \in M_{2,\gamma}(Q;\mathbb{M}^{2\times 2})$$

with $0 \le \gamma < 1$, *satisfying the Navier-Stokes equations*

$$\partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p + \mathrm{div}\,\sigma, \quad \mathrm{div}\, u = 0$$

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$$u \in C^{\gamma}(\overline{Q}_1)$$

if $0 < \gamma < 1$ *and*

$$u \in BMO(Q_1)$$

if $\gamma = 0$.

Additional results

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Theorem

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$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = \operatorname{div} \sigma, \qquad \operatorname{div} u = 0,$$

$$u(\cdot, 0) = u_0(\cdot) \in H,$$

where $\sigma \in L^r(\mathbb{T}^2 \times (0,T); \mathbb{M}^{2\times 2})$ *with* $r \geq 4$.

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where $\sigma \in L^r(\mathbb{T}^2 \times (0,T); \mathbb{M}^{2\times 2})$ with $r \geq 4$. Then, given $s > 0$, there exists a constant C_s depending only on s, ν, the norm of u_0 in H, the norm of σ in $L^r(\mathbb{T}^2)$, such that

$$\|u\|_{L^\infty(\mathbb{T}^2 \times (s,T))} \leq C_s.$$

Moreover, the function u is Hölder continuous in $\mathbb{T}^2 \times [s,T]$ with exponent $\gamma = 1 - \frac{4}{r}$.

Idea of proof of Hölder continuity for NS

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The iteration relates integrals on smaller parabolic cubes to integrals on larger ones. For the iterative procedure to succeed, the modulus of absolute continuity of the map

$$\Omega \subset \{\mathbb{T}^2 \times (0, T)\} \mapsto \int_\Omega |u(x,t)|^4 dxdt,$$

needs to be controlled uniformly apriori, to guarantee that such an integral is arbitrarily small, if the parabolic Lebesgue measure of Ω is small enough.

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Let the function $v \in L^4(Q(z_0, R))$ *satisfy the heat equation*

$$\partial_t v - \Delta v = 0$$

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$$\Phi(v; z_0, \varrho) \leq c \left(\frac{\varrho}{R} \right)^4 \Phi(v; z_0, R)$$

for all $0 < \varrho \leq R$.

Lemma

Given $G \in L^2(Q(z_0, R); \mathbb{M}^{2\times 2})$, there exists a unique function

$$w \in C([t_0 - R^2, t_0]; L^2(B(x_0, R); \mathbb{R}^2)) \cap L^2([t_0 - R^2, t_0]; W^{1,2}(B(x_0, R); \mathbb{R}^2))$$

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on the parabolic boundary of $Q(z_0, R)$. Moreover, the function w satisfies the estimates:

$$|w|^2_{2,Q(z_0,R)} \equiv \sup_{t_0 - R^2 < t < t_0} \|w(\cdot, t)\|^2_{2,B(x_0,R)} + \|\nabla w\|^2_{2,Q(z_0,R)}$$

$$\leq 2\|G\|^2_{2,Q(z_0,R)},$$

$$\Phi(w; z_0, R) \leq c|w|^2_{2,Q(z_0,R)}.$$

Lemma

For solutions of NSE, we have

$$\Phi(u; z_0, \varrho) \leq c\left\{\left[\left(\frac{\varrho}{R}\right)^4 + \Psi(u; z_0, R)\right]\Phi(u; z_0, R) + \right.$$

$$\left. + D(p; z_0, R) + MR^{2+2\gamma}\right\}$$

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The following result is used to control the modulus of absolute continuity.

Theorem

Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ be a solution of the 2D Navier-Stokes equations with initial data $u_0 \in H \cap L^r(\mathbb{T}^2)$ and $\sigma \in L^r(\mathbb{T}^2 \times (0,T); \mathbb{M}^{2\times 2})$ with $r \geq 4$. There exists a constant K depending only on the norm $\|\sigma\|_{L^r(\mathbb{T}^2\times(0,T))}$, ν, T and the norm of u_0 in $H \cap L^r(\mathbb{T}^2)$ such that

$$\sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^r(\mathbb{T}^2)} \leq K.$$

Generalized Ladyzhenskaya inequalities

The previous result is based on the inequality

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valid for all $r \ge \frac{n}{2}$, with $B_2^{0,n}$ the Besov space with norm

$$\|f\|_{B_q^{s,p}(\mathbb{R}^n)} = \left[\sum_{j=-\infty}^{\infty} \lambda_j^{qs} \|\Delta_j f\|^q_{L^p(\mathbb{R}^n)}\right]^{\frac{1}{q}}$$

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