8. Turbulence in Fluid Flows

Turbulence is defined as the unstable state of 3-dimensional, non-homogeneous and anisotropic fluid flows.

Based on assumptions of self similarity homogeneous and isotropic fluid flows, Kolmogorov (1941) developed the theory of turbulence with famous Kolmogorov energy spectrum

$$E(k) = C_k \, \varepsilon^{2/3} k^{-5/3}, \tag{8.1}$$

for *isotropic* and *homogeneous* turbulence. It has strong experimental support in oceanic turbulence. Thousands of high quality papers were published in many high quality journals.

Unfortunately, no solution for *anisotropic* and *non-homogeneous* turbulence flows during the last 75 years.

The Kolmogorov theory deals with only *isotropic* and *homogeneous* turbulence which is just one of more kind of turbulence, by no means generic. Kolmogorov kinetic energy spectrum $E(k)$ satisfies the *isotropic* and *homogeneous turbulence law*. The major concern is why every kind of turbulence becomes *isotropic* at relatively high wavenumbers and obeys the same law (8.1). See Debnath (2008) *Sir James Lighthill - Modern Fluid Mechanics*, Imperial College Press.

see page 120

Turbulence is studied based on the nonlinear Navier-Stokes equations and continuity equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} \;=\; -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \qquad (8.2)$$

$$div\mathbf{u} \;=\; 0 \qquad (8.3)$$

$\mathbf{u}(\mathrm{x}, t)$**-velocity field,** $p(\mathrm{x}, t)$ **is the pressure and** ρ **is the density and** $\nu = \mu/\rho$ **is kinematic viscocity.**

In non-dimension form (8.2)-(8.3) $\Rightarrow$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} \;=\; -\nabla p + \frac{1}{R}\nabla^2 \mathbf{u} \qquad (8.4)$$

$$div\mathbf{u} \;=\; 0 \qquad (8.5)$$

where $R = \left(\frac{U\ell}{\nu}\right)$ **is the Reynolds number. The problem is to find the unstable velocity** $\mathbf{u}(\mathrm{x}, t)$ **as** $\nu \to 0$ **or** $R \to \infty$ **with a complex random small-scale structure at some macroscopic scale. Turbulence occurs often in atmosphere and ocean.**

Major difficulties are

(i) no existence of solutions of (8.2)-(8.3) or (8.4)-(8.5), uniqueness, regularity and continuous dependent of solutions on the initial data.

(ii) strong indications are that solutions of (8.2)-(8.3) are singular at certain places and in certain times.

(iii) nonlinear convection term

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u}$$

This leads to an infinite number of equations for all possible moments of the velocity field.

(iv) closure problem in the statistical theory of turbulence, dynamical equation for second order moments involve third-order moments, third order moments involve 4th order moments, etc. It contains more unknowns than the number of equations

(v) the nature of the N-S equations changes because the nonlinear convective term dominates over the linear viscous term. In the limit as $R \to \infty$ (fully developed turbulent flows), the *second-order* N-S equation reduce to the *first order nonlinear Euler equations*.

So, a slightly viscous fluid flow leads to a *singular perturbation* of the inviscid fluid motions. - singular perturbation problem.

Debnath (1998) in a volume by Debnath & Riahi - vol. I (*Nonlinear Instability, Chaos and Turbulence*)

Because of some major difficulties associated with the N-S equations, some people believe that the N-S equations are not adequate for the real description of real turbulent flows. New equations are needed — only one person proposed 4th order PDE with some artificial term without success.

Ladyzhanskaya (1963) proposed a new biharmonic damping term, $-\lambda\nabla^4$u to the right hand of the NS equation (8.2) and proved the existence and uniqueness of the new equation for all $\lambda > 0$. However, applied mathematicians become *turbulent* because of artificial nature of the damping term included.

No Success!!

Foias and Temam (1987) theory

Mandelbrot - fractals
- wavelet transforms
Direct Numerical Simulations (DNS)

Farge et al. (1992-2002) wavelet transform analysis of coherent structures of 2D turbulence (Indeed, turbulence is 3 Dimensional).

Turbulence problem is still unsolved except for the statistical 1941 Kolmogorov. Approximate statistical theory of isotropic and homogeneous turbulent flows.

Debnath (2002) ***Wavelet Transforms and Their Applications,*** **Birkhauser Verlag.**

3. Wavelet Transforms and Turbulence

Turbulence is a highly unstable state of fluid flows, and characterized by the Reynolds Number, $Re\left(Re = \frac{\textbf{nonlinear convec.}}{\textbf{viscous Dissi.}}\right)$ **which is responsible for the flow instability.**

All methods in TFs rely on the Fourier Spectral Analysis. Viscous dissipative term is represented in Fourier space because the Fourier modes diagonalize the Laplace operator $(\nu\nabla^2\mathbf{u})$**. The *nonlinear convection term* is very *complicated* in Fourier space where it becomes a *convolution*, that is, all *Fourier modes* are involved.**

Fully developed TFs corresponds to flows where nonlinear convection is *dominant*. FTA does not seem to be adequate for TFs as $Re \to \infty$**.**

We need a *new method* to deal with Nonlinear Convection term in the same way as the FT which is suitable for *linear viscous term*.

Taylor Hypothesis and Kolmogorov Theory

1. **Taylor's new concept of eddy cascade (1919) which enables the energy to be transferred to smaller and smaller eddy sizes until viscosity has significant effects ⇒universal equilibrium. Theory of small scales - formulated by Kolmogorov and Obukov (1941) and then by Heisenberg, Onsager, Chandrasekar, Batch-**

elor, Sen and others.

2. He characterized TFs by their *correlation functions*, that is, by FT of their two-point correction function which gives the *energy spectrum*.

* Similarity arguments for TFs.

* Relation between various mean products of vel. gradients in *isotropic* and *homogeneous turbulence* (?)

To simplify the computation of the correlation functions, Taylor made the hypothesis of statistical homogenity and isotropy of TFs, supposing that essemble averages are invariant under both translation and rotation.

Equivalence of the correlation description of TFs and spectrum analysis ⇒mathematical form by Karman and Howarth for isotropic turbulence.

Kolmogorov developed the *statistical* Theory of fully developed isotropic homogeneous turbulence and studied the way in which the N-S equations in $3D$ distribute energy among the *different scales* of the flow.

The external forces act only on the *largest scales* where frictional forces act only on the *smallest scales* which in the limit as $Re \to \infty$, leaves an intermediate range scales, called the *inertial range*. In

this range, ***energy is constant*** **and only transferred from** ***large*** **to** ***small scales*** **at a constant rate ε which is supposed to be constant.**

Kolmogorov Theory assumed ***skewness is constant*** **and flow is** ***non-intermittent*****.**

$\Rightarrow k^{-5/3}$ **law as**

$$E(k) = \underset{\substack{\uparrow \\ \text{Kolmogorov} \\ \text{const.}}}{C} \quad \underset{\substack{\uparrow \\ \text{rate of energy} \\ \text{tranfer}}}{\in^{2/3}} \quad \underset{\substack{\uparrow \\ \text{wave} \\ \text{no}}}{k^{-5/3}}$$

Landau (1944) criticized Kolmogorov's and Taylor hypothesis.

ε **is constant, dissipation is** ***not random*****.**

Other forms of ***Energy Spectrum*****.**

Various other criticism of K-theory

Debnath (1978) ***Oceanic Turbulence***

Liepmann (1961) wrote in the ***Proc.*** **of the Turbulence Conf:**

"The success of the spectral representation of turbulence fields is due, after all, not to the belief in the existence of definite waves but to the possibility of representing quite general functions as Fourier integrals. In the application to stochastic problems the usefulness of the Fourier representation, stems essentially from their translational in-

variance. Consequently, really successful models for representing turbulence motion will require far broader invariance considerations. It is clear that the essence of turbulent motion is vortex interaction. In the particular case of homogeneous isotropic turbulence this fact is largely masked, since the vorticity fluctuation appear as simple derivatives of the velocity fluctuation.

In general, *this is not the case*, and a *Fourier representation* is *probably not the ultimate answer*..."

These remarks, written 45 years ago, are still very pertinent and hence *definite direction is needed for future research in turbulence*.

Turbulence contains *coherent structures* even at very large Re.

Examples of coherent structures include

(i) *Karman vortices* observed by Roshko (1961)

(ii) *Horseshoe vortices* observed in Turbulence B.L. and mixing layers (Cantwell, 1981)

(iii) *Vortices tubes* or *vorticity filaments* observed by Cadol et. al (1995).

In fact, structures are defined as *local condensations* of the vorticity field which exists for times much larger than eddy turnover time.

Coherent structures are responsible for *non-Gaussian* *behavior of TFs*, which *contradicts* the Gaussian hypothesis of Kolmogorov.

All exceptions results measuring the turbulent energy spectrum rely on the Taylor hypothesis and may *not be valid* as long as the flow *is intermittent*.

Today we still do *not* have a complete theory to explain the *formation* and *persistence of coherent structures* at a very high Re, and they probably play an essential role in their intermittency.

Several wind tunnel exceptions (Anselent *et al*, 1984) have shown that the energy associated with the smallest scales of TFs *is not* distributed *densely* in space and time $\Rightarrow$ conjectures that the dissipation occurs should be fractals or multifractals (Mandelbrot, 1975; Frisch, 1978; Parisi and Frisch, 1985).

It is now thought that the time and space *intermittency* of TFs is related to the presence of *coherent structures*.

It is clear from the above that *wavelet transform analysis (WTA) can answer* some of these *unresolved problems* and *open questions*.

Wavelet transform plays an important role in separating the coherent (non-Gaussian) components from the incoherent (Gaussian) components of TFs in order to devise a new *conditional averages* (*nonlinear procedures*) to replace the classical *ensemble averages*.

During the last several years, wavelet transform analysis is proposed as a new method for studying TFs. This is found to be very useful in calculating quantities, such as:

(i) Energy spectra, (ii) structure functions

(iii) singularity spectra, (iv) fractal dimensions

Major advantages of WTA include:

1. **WTA is able to analyze** ***locally singular behavior*** **of the TFs.**
2. **Local information can be found and it is then used to construct statistics describing the** ***distribution*** **and** ***types*** **of singularities (fractals, multifractals, cusps, isolated, etc)**
3. ***Local conditional average versions*** **of traditional measures such as the** ***energy spectra*** **and** ***structure functions*****.**

Energy Spectra

Traditional Fourier energy spectrum $E(k)$ **of a real function is defined by**

$$E(k) = \frac{1}{2\pi} \left|\widehat{f}(k)\right|^2, \qquad k \geq 0$$

where

$$\widehat{f}(k) = \mathcal{F}\{f(x)\}.$$

Major Weakness of FTA**.**

Modulus **of FT is used, phase information is lost. It neglects *local behavior* and any *organization* of the turbulent velocity field.**

WT extends the concept of energy spectrum so that one can define a *local energy spectrum* $E(k,x)$ **using** L^2 **norm wavelet**

$$\widetilde{E}(k,x) = \frac{1}{2C_\psi k_0} \left|\widetilde{f}\left(x, \frac{k_0}{k}\right)\right|^2, \qquad k \geq 0$$

where k_0 **is the peak wavenumber of the wavelet** ψ **and**

$$C_\psi = \int_0^\infty \frac{\left|\widetilde{\psi}(k)\right|^2}{k}\, dk < \infty.$$

We measure $\widetilde{E}(k,x)$ **at different places of TFs in order to determine**

- **(i) What parts of the flow contribute most to the *overall Fourier energy spectrum*.**
- **(ii) How the energy spectrum depends on the *local flow conditions*.**
- **(iii) Types of energy spectrum contributed by *coherent structures*, such as *isolated vortices*, and type of energy spectrum contributed by the *unorganized part* of the flow.**

Since the WT analyzes the flow into wavelets rather than sine waves, it is possible to find the *mean wavelet*

energy spectrum $\widetilde{E}(k)$

$$\widetilde{E}(k) = \int_0^\infty \widetilde{E}(k,x)\,dx$$

$\Rightarrow$**correct Fourier exponent for a power-law Fourier energy spectrum $E(k) \sim k^{-\beta}$ provided the analyzing wavelet has at least $n > \frac{1}{2}(\beta - 1)$ vanishing moments.**

For isolated cusps, $\beta = 1+2\alpha$, where α is a Hölder exponent such that

$$|f(x) - f(x_0)| \leq C\ |x - x_0|^\alpha \quad \textbf{as} \quad x \to x_0$$

thus the steeper the energy spectrum the more vanishing moments of the wavelet we need. The nth moment is defined by

$$\int_{-\infty}^{\infty} x^n \underset{\substack{\uparrow \\ \textbf{wavelet} \\ \textbf{involved}}}{\psi(x)}\,dx \ .$$

For all isolated cusps singularities,

$$E(k) = \underset{\substack{\nwarrow \\ \textbf{const.}}}{C} \quad k^{-2(\alpha+1)} \ .$$

For spirals and/or non-isolated singularities,

$$E(k) \leq C\ k^{-2\alpha}.$$

The mean wavelet energy spectrum is a smoothed version of the Fourier energy spectrum as can be

seen from the relation between two spectra

$$\widetilde{E}(k) = \frac{1}{C_\psi k} \int_0^\infty E(\omega) \left| \widehat{\psi}\left(\frac{k_0 \omega}{k}\right) \right|^2 d\omega$$

$\uparrow$ **weighted average of** $E(\omega)$ $\uparrow$ **weighted . at** k

This result is very useful for TFs.
Examples:
1. The Mexican wavelet

$$\widetilde{\psi}(k) = k^2 \exp\left(-\frac{k^2}{2}\right)$$

has only 2 vanishing moments and can *correctly* measure energy spectrum upto $\beta < 5$**.**
2. The Morlet wavelet

$$\widetilde{\psi}(k) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(k - k_\psi)^2\right], \qquad k > 0,$$

$$= 0, \qquad k \leq 0$$

$\Rightarrow$**accurate estimate of the power-law exponent of the energy spectrum for** $\beta < 7$ **if** $k_\psi = 6$**.**
3. Perrier and Basdevant (1995) present a family of new wavelet

$$\widetilde{\Pi}_n(k) = \alpha_n \exp\left[-\frac{1}{2}\left(k^2 + k^{-2n}\right)\right], \qquad n \geq 1.$$

This gives any power-law energy spectrum, and these wavelets can detect the differences between

a power law energy spectrum and a *Gaussian energy spectrum* $E(k) \sim \exp\left(-\frac{k^2}{k_0^2}\right)$. These are used to determine *at what wavenumber* the GES begins.

Meneveau (1991) *first* used the *wavelet* representation to measure *local energy spectra* in TFs. He used the *discrete wavelet transform* to measure local energy spectra in *experimental* and direct numerical simulation (DNS) flows.

Results are very *accurate* and 100% *correct* throughtout the inertial range.

This *local spectral information*, which links the physical and Fourier space view of TFs, can *only* be obtained using the *wavelet transform*.

Using wavelet transforms, we can discuss other results for TFs such as:

1. **Structure Functions**
2. **Singularity spectrum for multifractals**
3. **Isolated and dense singularities**
4. **New Diagnostics in $2D$ or $3D$ TFs**
5. **Dynamical role of coherent structures**
6. **$2D$ or $3D$ Turbulence Modeling**
7. **Turbulence Computation**

Comparison between DNS and wavelet based numerical scheme

Remarkable Difference between *Fourier Analysis* & *Wavelet Transfer Analysis*

FTA	*WTA*
Local FT spectrum cannot be defined because Fourier modes are nonlocal.	Local wavelet transform spectrum can be defined because wavelets are localized functions.
Only the modulars of the Fourier energy spectrum E(k) is used. Phase information is completely lost.	Wavelet transform extends the concept of energy spectrum by defining local energyspectrum E (k, x) using L^2 norm wavelet.
FTA is unable to separate coherent and incoherent structures.	WTA provided such separation.

FTA	*WTA*
Cannot provide appropriate functional representation space for analyzing the physical structures of TFs because it averages over space and time and thus *loses all spatial form* .	Does provide appropriate functional spaces (Sobolev, Hölder and Besov spaces
Cannot describe intermittency of TFs - indeed FTA is based on non-intermittency of TFs.	Wavelet Analysis can relate the intermittency to the presence of coherent structures.
FTA cannot compute the transfers of energy and of enstrophy between coherent and incoherent components of TFs.	WTA can be used effectively to compute the energy and enstrophy transfers between coherent and incoherent components of TFs.

FTA	*WTA*
Fourier energy spectrum is sensitive to only the *strongest* isolated singularities in fht TFs. *No* information about the *form* and *location* of other singularities.	It analyzes *locally* the singular behavior and provides information about *types* of singularities (cusp, isolated, fractals, multifractals, spinals, etc.)

Conclusions:

1. WTA has recently been used to analyze, model, and compute turbulent flows.

2. All experiments have shown that wavelet techniques are mathematically valid and *superior* to existing numerical methods.

3. Wavelet analysis may be *right* method for developing a statistical theory of turbulence based on identification of elementary dynamical structures from *known observational data.*

4. Wavelet transform theory *is likely* to replace Fourier transform. Statistical theory of turbulence is based on the symmetries of the *N-S equations*.

5. Recent success suggests that the N-S equations are *not* appropriate model equations to compute *large Reynolds number flows*. Indeed, in the limit, $Re \to \infty$, there is some symmetry breaking associated with the production of coherent structures out of the *random background* flow.

6. WTA is a new mathematical tool that will bring *new* insights to help understanding turbulent flows.

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