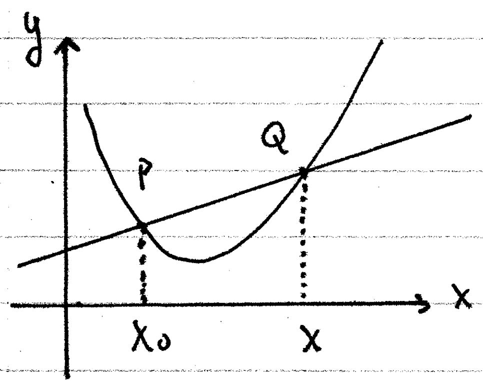
DIFFERENTIAL CALCULUS

Definition of differentiability

The derivative of a function is defined in the usual way as follows. Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $(c): y = f(x)$ be the graph of the function f. Choose $x_0, x \in A$ and consider the points $P(x_0, f(x_0))$ and $Q(x, f(x))$. We denote the slope of the segment PQ as $\lambda(f \mid x, x_0)$ and it is given by



$$\boxed{\forall x, x_0 \in A : \lambda(f \mid x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}}$$

The slope of the tangent line at x_0 is given by the limit $\lim_{x \to x_0} \lambda(f \mid x, x_0)$ and that motivates the following definitions:

Def: Let $f: A \to \mathbb{R}$ and $x_0 \in A$ and $S \subseteq A$. We say that

f differentiable on $x_0 \iff \exists l \in \mathbb{R} : \lim_{x \to x_0} \lambda(f \mid x, x_0) = l$

f differentiable on $S \iff \forall x_0 \in S :$ f differentiable on x_0

$\iff \forall x_0 \in S : \exists l \in \mathbb{R} : \lim_{x \to x_0} \lambda(f \mid x, x_0) = l$

Differentiability implies continuity

Prop: Let $f: A \to \mathbb{R}$ and $x_0 \in A$. Then, we have:
f differentiable at $x_0 \Rightarrow f$ continuous at x_0

Proof

Assume that f differentiable at x_0. Then,
f differentiable at $x_0 \Rightarrow \exists \ell \in \mathbb{R}: \lim_{x \to x_0} \lambda(f \mid x, x_0) = \ell$
Choose $\ell \in \mathbb{R}$ such that $\lim_{x \to x_0} \lambda(f \mid x, x_0) = \ell$. Then, we have:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} [f(x_0) + (f(x) - f(x_0))] =$$

$$= \lim_{x \to x_0} \left[f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right]$$

$$= \lim_{x \to x_0} [f(x_0) + \lambda(f \mid x, x_0)(x - x_0)]$$

$$= f(x_0) + \left[\lim_{x \to x_0} \lambda(f \mid x, x_0)\right]\left[\lim_{x \to x_0} (x - x_0)\right]$$

$$= f(x_0) + \ell(x_0 - x_0) = f(x_0) \Rightarrow f \text{ continuous at } x_0 \quad \square$$

The contrapositive statement reads:
f not continuous at $x_0 \Rightarrow f$ not differentiable at x_0.

Def: (Corner points). Let $f: A \to \mathbb{R}$ and let $x_0 \in A$. We say that
x_0 corner point of $f \Leftrightarrow \begin{cases} f \text{ continuous at } x_0 \\ f \text{ NOT differentiable at } x_0. \end{cases}$

Corner points can emerge from

a) Sudden change in the direction of the function
(example: $f(x) = |x|$ at $x = 0$)

b) When the graph of the function becomes momentarily vertical at a particular point
(example: $f(x) = \sqrt{x}$ at $x = 0$)

These two examples of corner points are elaborated upon in the following examples.

EXAMPLE

a) For $f(x)=|x|, \forall x \in \mathbb{R}$ show that $x_0=0$ is a corner point.

Solution

Since

$$f(0)=|0|=0 \qquad (1)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \qquad (2)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0 \qquad (3)$$

it follows that

$$\lim_{x\to 0} f(x) = 0 \qquad [\text{from Eq. } (2) \text{ and Eq. } (3)]$$

$$= f(0) \qquad [\text{from Eq. } (1)]$$

$\Rightarrow f$ continuous at $x_0=0$. (4)

Furthermore:

$$\lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{|x|-|0|}{x-0} = \frac{|x|}{x} =$$

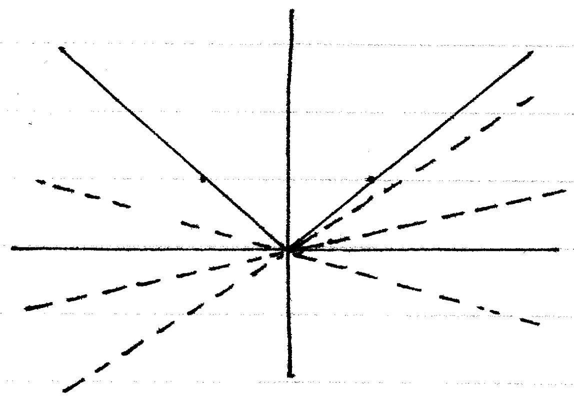
$$= \begin{cases} x/x, & \text{if } x>0 \\ -x/x, & \text{if } x<0 \end{cases} = \begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x<0 \end{cases} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0^+} \lambda(f|x,0) = 1 \wedge \lim_{x\to 0^-} \lambda(f|x,0) = -1$$

$$\Rightarrow \lim_{x\to 0} \lambda(f|x,0) \text{ does not exist} \Rightarrow \forall \ell \in \mathbb{R}: \lim \frac{f(x)-f(0)}{x-0} \neq \ell$$

$\Rightarrow f$ not differentiable at $x_0=0$. (5)

From Eq.(4) and Eq.(5): $x_0=0$ corner point of f.



↳ From the graph of $f(x)=|x|$, $\forall x \in \mathbb{R}$ we see that the corner point $x_0=0$, the function suddenly changes direction. As a result, we cannot draw a unique tangent line at $x_0=0$.

b) Show that $f(x)=\sqrt{x}$, $\forall x \in [0,+\infty)$ has a corner point at $x_0=0$.

Solution

$$f(0)=\sqrt{0}=0 \qquad (1)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\sqrt{x} = \lim_{x\to 0^+}\sqrt{x} = \sqrt{0}=0 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow f \text{ continuous at } x_0=0 \qquad (3)$$

Furthermore:

$$\lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{\sqrt{x}-\sqrt{0}}{x-0} = \frac{\sqrt{x}}{x} =$$

$$= \frac{\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{1}{\sqrt{x}}, \quad \forall x \in (0,+\infty)$$

Since:

$$\begin{cases} \sqrt{x} > 0\,,\ \forall x \in (0,+\infty) \\ \lim\limits_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0 \end{cases} \Rightarrow \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty \Rightarrow$$

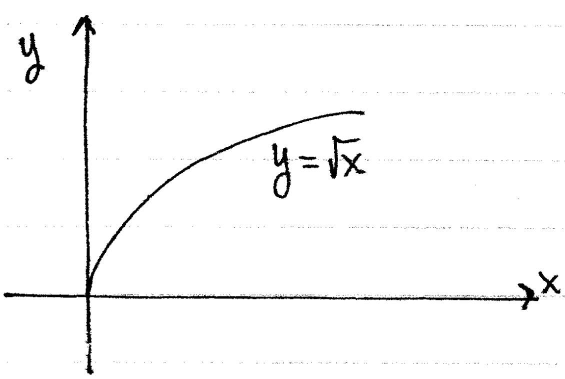
$$\Rightarrow \lim_{x\to 0^+} \lambda(f\,|\,x,0) = +\infty$$

$$\Rightarrow \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \neq \ell\,,\ \forall \ell \in \mathbb{R}$$

$\Rightarrow f$ not differentiable at $x_0 = 0$. (4)

From Eq. (3) and Eq. (4): $x_0 = 0$ corner point of f.

$\updownarrow\!\rightarrow$ From the graph of $f(x) = \sqrt{x}$, $\forall x \in [0,+\infty)$ we see that the graph becomes vertical at $x_0 = 0$. This is the second way one may lose differentiability without losing continuity.



c) Consider the function

$$f(x) = \begin{cases} x^2[\sin(\pi/x) + \cos(\pi/x)] &, \text{ if } x \in \mathbb{R} - \{0\} \\ 0 &, \text{ if } x = 0 \end{cases}$$

Show that f differentiable at $x_0 = 0$

<u>Solution</u>

Let $x \in \mathbb{R} - \{0\}$ be given. Then, we have:

$$\lambda(f|x,0) = \frac{f(x) - f(0)}{x - 0} = \frac{x^2[\sin(\pi/x) + \cos(\pi/x)] - 0}{x}$$

$$= x[\sin(\pi/x) + \cos(\pi/x)] \Rightarrow$$

$$\Rightarrow |\lambda(f|x,0)| = |x[\sin(\pi/x) + \cos(\pi/x)]| =$$

$$= |x| \cdot |\sin(\pi/x) + \cos(\pi/x)| \leq |x| [|\sin(\pi/x)| + |\cos(\pi/x)|]$$

$$\leq |x| \cdot (1 + 1) = 2|x| = |2x|$$

We have thus shown that

$$\begin{cases} \forall x \in \mathbb{R} - \{0\} : |\lambda(f|x,0)| \leq |2x| \\ \lim_{x \to 0} (2x) = 0 \end{cases} \Rightarrow \lim_{x \to 0} \lambda(f|x,0) = 0$$

$$\Rightarrow f \text{ differentiable at } x_0 = 0$$

c) Consider the function

$$f(x) = \begin{cases} x^2+2x & , x \in [0,+\infty) \\ ax+b & , x \in (-\infty,0) \end{cases}$$

Find all $a,b \in \mathbb{R}$ for which f differentiable at $x_0=0$.

Solution

We note that

$$\forall x \in (0,+\infty): \lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{(x^2+2x)-(0^2+2\cdot 0)}{x}$$

$$= \frac{x^2+2x}{x} = \frac{x(x+2)}{x} = x+2$$

$$\forall x \in (-\infty,0): \lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{ax+b-0}{x} = \frac{ax+b}{x}$$

$$\lim_{x\to 0^+} \lambda(f|x,0) = \lim_{x\to 0^+}(x+2) = 0+2 = 2$$

The limit $\lim_{x\to 0^-} \lambda(f|x,0)$ may or may not exist depending on whether $b=0$ or $b\neq 0$, so we leverage continuity but must do, as a result, a split argument:

($\Rightarrow$): Assume that f differentiable at $x_0=0$. Since:

$$f(0) = 0^2+2\cdot 0 = 0$$

$$\lim_{x\to 0^+} f(x) = 0^2+2\cdot 0 = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(ax+b) = a\cdot 0+b = b$$

it follows that:

$$f \text{ differentiable at } x_0=0 \Rightarrow f \text{ continuous at } x_0=0$$

$$\Rightarrow \lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \Rightarrow \underline{b=0}$$

For $b=0$:

$$\lim_{x\to 0^-} \lambda(f|x,0) = \lim_{x\to 0^-} \frac{ax+0}{x} = \lim_{x\to 0^-} a = a$$

and therefore:

f differentiable at $x_0=0 \Rightarrow \exists \ell \in \mathbb{R} : \lim_{x\to 0} \lambda(f|x,0) = \ell$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f|x,0) = \lim_{x\to 0^+} \lambda(f|x,0)$$

$$\Rightarrow a = 2.$$

We have thus shown that

f differentiable at $x_0=0 \Rightarrow (a=2 \land b=0)$

$(\Leftarrow)$: Assume that $a=2 \land b=0$. Then:

$a=2 \land b=0 \Rightarrow \forall x \in (-\infty,0): \lambda(f|x,0) = \dfrac{2x+0}{x} = \dfrac{2x}{x} = 2$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f|x,0) = 2 = \lim_{x\to 0^+} \lambda(f|x,0)$$

$$\Rightarrow \lim_{x\to 0} \lambda(f|x,0) = 2 \Rightarrow \exists \ell \in \mathbb{R}: \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow f \text{ differentiable at } x_0=0.$$

We have thus shown that:

f differentiable at $x_0=0 \Leftrightarrow a=2 \land b=0.$ $\square$

$\hookrightarrow$ Note that a direct argument of the form

f differentiable at $x_0=0 \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftrightarrow$

$\Leftrightarrow a=2 \land b=0$

is not possible if we wish to use continuity.

Consequently the forward ($\Rightarrow$) and backward ($\Leftarrow$) arguments need to be done separately.

THEORY QUESTIONS

① Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$ and $S \subseteq A$. State the definitions for

a) f differentiable at x_0

b) f differentiable on S

c) x_0 corner point of f

② Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$. Prove that:
f differentiable at $x_0 \Rightarrow f$ continuous at x_0.

EXERCISES

③ Show that the function

$$f(x) = \begin{cases} x^2+4x & , \text{ if } x \in [0,+\infty) \\ x^2-4x & , \text{ if } x \in (-\infty,0) \end{cases}$$

is continuous on $\mathbb{R}$ but not differentiable at $x_0 = 0$

④ Show that the function

$$f(x) = (x+|x|)^2 \;, \forall x \in \mathbb{R}$$

is continuous and differentiable at $x_0 = 0$

⑤ Define the function

$$f(x) = \begin{cases} x \sin(2x) \cos(\pi/x) [1+\sin(\pi/x)] & , \text{ if } x \in \mathbb{R}-\{0\} \\ 0 & , \text{ if } x=0 \end{cases}$$

Show that f is differentiable at $x_0 = 0$

⑥ Let $f: A \to \mathbb{R}$ be a function, and define $g: A \to \mathbb{R}$ such that

$\forall x \in A: \ g(x) = x f(x)$

Show that:

f continuous at $x_0 = 0 \Rightarrow g$ differentiable at $x_0 = 0$

⑦ Find all $a, b \in \mathbb{R}$ such that the following functions are differentiable at x_0:

a) $f(x) = \begin{cases} ax + b & , \text{ if } x \in (-\infty, 3) \\ x^2 & , \text{ if } x \in [3, +\infty) \end{cases}$ at $x_0 = 3$

b) $f(x) = \begin{cases} ax^2 + 2bx & , \text{ if } x \in [1, +\infty) \\ bx - a & , \text{ if } x \in (-\infty, 1) \end{cases}$ at $x_0 = 1$

⑧ Let $f: A \to \mathbb{R}$ be a function and define $g: A \to \mathbb{R}$ such that

$\forall x \in A: \ g(x) = |f(x)|$

Show that:

$\left.\begin{cases} f \text{ differentiable at } x_0 \in A \\ f(x_0) \neq 0 \end{cases}\right. \Rightarrow g$ differentiable at x_0

<u>Hint:</u> We write:

$$\delta(g|x, x_0) = \frac{(|f(x)| - |f(x_0)|)(|f(x)| + |f(x_0)|)}{(x - x_0)(|f(x)| + |f(x_0)|)}$$

and continue from there.

Derivative function

- Let $f: A \to \mathbb{R}$ be a function and let $S \subseteq A$. We say that
  f differentiable at $S \Leftrightarrow \forall x_0 \in S: f$ differentiable at x_0
- If $f: A \to \mathbb{R}$ is differentiable at S, then we define the derivative function $f': S \to \mathbb{R}$ as:
  $$\forall x_0 \in S: f'(x_0) = \lim_{x \to x_0} \lambda(f \mid x, x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
- The notation $f'(x)$ is attributed to Newton. The Leibnitz notation of the derivative is:
  $$\frac{df}{dx} = f' \quad \text{and} \quad \left.\frac{df}{dx}\right|_{x = x_0} = f'(x_0)$$
- If f' is also differentiable at S, then the derivative of f' is denoted as f'' and is called the <u>2nd derivative</u> of f. Likewise we define
  $$f'' = \frac{df'}{dx} = \frac{d^2 f}{dx^2}$$
  $$f''' = \frac{df''}{dx} = \frac{d^3 f}{dx^3}$$

  Beyond the 3rd derivative, we use the notation $f^{(4)}, f^{(5)}, \ldots, f^{(n)}$ and write:
  $$f^{(n)} = \frac{d f^{(n-1)}}{dx} = \frac{d^n f}{dx^n}$$

- If we can define $f^{(n)}$ at x_0 we say that f is n-times differentiable at x_0. Likewise, for $S \subseteq A$, we say that f n-times differentiable at $S \iff \forall x_0 \in S : f$ n-times differentiable at x_0.

Derivatives of basic functions

(1) $\boxed{f(x) = ax+b, \forall x \in \mathbb{R} \implies f'(x) = a, \forall x \in \mathbb{R}}$

Proof

Since

$$\forall x, x_0 \in \mathbb{R} : \lambda(f|x, x_0) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{(ax+b)-(ax_0+b)}{x-x_0}$$

$$= \frac{ax-ax_0}{x-x_0} = \frac{a(x-x_0)}{x-x_0} = a \implies$$

$$\implies \forall x_0 \in \mathbb{R} : f'(x_0) = \lim_{x \to x_0} \lambda(f|x, x_0) = a. \quad \square$$

For the next result we use the identity

$$\boxed{\forall a, b \in \mathbb{R} : \forall n \in \mathbb{N} - \{0\} : a^n - b^n = (a-b)\sum_{k=0}^{n-1}(a^{n-k-1} b^k)}$$

Note that:

$n=2$: $a^2 - b^2 = (a-b)(a+b)$

$n=3$: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$n=4$: $a^4 - b^4 = (a-b)(a^3+a^2b+ab^2+b^3)$

Proof

$$(a-b)\sum_{k=0}^{n-1} a^{n-k-1} b^k = \sum_{k=0}^{n-1} (a-b) a^{n-k-1} b^k =$$

$$= \sum_{k=0}^{n-1} (a^{n-k} b^k - a^{n-k-1} b^{k+1}) =$$

$$= \sum_{k=0}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-1} a^{n-k-1} b^{k+1} =$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-2} a^{n-k-1} b^{k+1} - a^{n-(n-1)-1} b^{(n-1)+1}$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=1}^{n-1} a^{n-k} b^k - b^n =$$

$$= a^n - b^n \qquad \square$$

(2) $\boxed{f(x) = ax^n ,\ \forall x \in \mathbb{R} \Rightarrow f'(x) = nax^{n-1} ,\ \forall x \in \mathbb{R}}$

Proof

Since:

$$\lambda(f \mid x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{ax^n - ax_0^n}{x - x_0} = \frac{a(x^n - x_0^n)}{x - x_0} =$$

$$= \frac{a(x - x_0) \sum_{k=0}^{n-1} x^{n-k-1} x_0^k}{x - x_0} =$$

$$= a \sum_{k=0}^{n-1} x^{n-k-1} x_0^k \Rightarrow$$

$$\Rightarrow f'(x_0) = \lim_{x\to x_0} \lambda(f|x,x_0) = \lim_{x\to x_0}\left[a\sum_{k=0}^{n-1} x^{n-k-1} x_0^k\right] =$$

$$= a \lim_{x\to x_0} \sum_{k=0}^{n-1} x^{n-k-1} x_0^k = a\sum_{k=0}^{n-1} \lim_{x\to x_0} (x^{n-k-1} x_0^k)$$

$$= a\sum_{k=0}^{n-1} x_0^{n-k-1} x_0^k = a\sum_{k=0}^{n-1} x_0^{n-1} = anx_0^{n-1} =$$

$$= nax_0^{n-1}, \quad \forall x_0 \in \mathbb{R}. \qquad \square$$

(3) $\boxed{f(x) = \sqrt{x}, \ \forall x \in [0,+\infty) \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}, \ \forall x \in (0,+\infty)}$

Proof

$$\forall x, x_0 \in [0,+\infty): \lambda(f|x,x_0) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} =$$

$$= \frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x})^2-(\sqrt{x_0})^2} = \frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})} = \frac{1}{\sqrt{x}+\sqrt{x_0}} \Rightarrow$$

$$\Rightarrow \forall x_0 \in (0,+\infty): f'(x_0) = \lim_{x\to x_0} \lambda(f|x,x_0) = \lim_{x\to x_0} \frac{1}{\sqrt{x}+\sqrt{x_0}} =$$

$$= \frac{1}{\sqrt{x_0}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}. \qquad \square$$

↳ Note that, as was shown previously, although the function $f(x)=\sqrt{x}$ is defined at $x=0$, it is <u>not</u> differentiable at $x=0$.

EXAMPLES

a) Let $f: \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at $x_0 \in \mathbb{R}$ and let $g: \mathbb{R} \to \mathbb{R}$ such that

$\forall x \in \mathbb{R} : g(x) = [f(x)]^2$

Show that g differentiable at x_0 with $g'(x_0) = 2f(x_0)f'(x_0)$, without using the chain rule

Solution

We have:

$$\forall x \in \mathbb{R} : \lambda(g|x,x_0) = \frac{g(x) - g(x_0)}{x - x_0} = \frac{[f(x)]^2 - [f(x_0)]^2}{x - x_0} =$$

$$= \frac{[f(x) - f(x_0)][f(x) + f(x_0)]}{x - x_0} =$$

$$= \lambda(f|x,x_0)[f(x) + f(x_0)]$$

and

$$f \text{ differentiable at } x_0 \Rightarrow \lim_{x \to x_0} \lambda(f|x,x_0) = f'(x_0)$$

and

$$f \text{ differentiable at } x_0 \Rightarrow f \text{ continuous at } x_0 \Rightarrow$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

$$\Rightarrow \lim_{x \to x_0} [f(x) + f(x_0)] = \lim_{x \to x_0} [f(x)] + f(x_0) =$$

$$= f(x_0) + f(x_0) = 2f(x_0)$$

therefore,

$$g'(x_0) = \lim_{x \to x_0} \lambda(g \mid x, x_0) = \lim_{x \to x_0} \{\lambda(f \mid x, x_0)[f(x) + f(x_0)]\} =$$

$$= \left[\lim_{x \to x_0} \lambda(f \mid x, x_0)\right]\left[\lim_{x \to x_0} (f(x) + f(x_0))\right]$$

$$= f'(x_0)[2f(x_0)] = 2f(x_0)f'(x_0) \qquad \square$$

b) Let $f:(0,+\infty)\to\mathbb{R}$ such that
$$\begin{cases} f \text{ differentiable at } x_0=1 \\ \forall a,b\in(0,+\infty): f(ab)=f(a)+f(b) \end{cases}$$
Show that:
$$\begin{cases} f \text{ differentiable on } (0,+\infty) \\ \forall x\in(0,+\infty): f'(x)=\dfrac{f'(1)}{x} \end{cases}$$

Solution

Choose some $b\in(0,+\infty)$. Then, we have:
$$f(b)=f(1b)=f(1)+f(b) \Longrightarrow f(1)=0$$
and therefore
$$\begin{aligned} f'(1) &= \lim_{x\to1}\frac{f(x)-f(1)}{x-1} \quad [\text{via } f \text{ differentiable at } x_0=1] \\ &= \lim_{x\to1}\frac{f(x)}{x-1} \quad [\text{via } f(1)=0] \qquad (1) \end{aligned}$$

Let $x_0\in(0,+\infty)$ be given. Then, we have:
$$\begin{aligned} \lim_{x\to x_0}\lambda(f\mid x,x_0) &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \\ &= \lim_{t\to1}\frac{f(tx_0)-f(x_0)}{tx_0-x_0} \quad [\text{via composition thm}] \\ &= \lim_{t\to1}\frac{f(t)+f(x_0)-f(x_0)}{x_0(t-1)} \quad [\text{via hypothesis}] \\ &= \lim_{t\to1}\frac{f(t)}{x_0(t-1)} = \frac{1}{x_0}\lim_{t\to1}\frac{f(t)}{t-1} \\ &= \frac{f'(1)}{x_0} \quad [\text{via Eq. (1)}] \end{aligned}$$

We have thus shown that

$$\begin{cases} f \text{ differentiable on } (0,+\infty) \\ \forall x \in (0,+\infty) : f'(x) = f'(1)/x. \end{cases}$$

EXERCISES

(9) Let $f: A \to \mathbb{R}$ with $x_0 \in A$ such that f differentiable at x_0. Show that:

$$\lim_{x \to x_0} \frac{x f(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0)$$

(10) Let $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{cases} f \text{ differentiable at } x_0 = 0 \\ \forall a, b \in \mathbb{R} : f(a+b) = f(a) + f(b) \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : f'(x) = f'(0) \end{cases}$$

(11) Let $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{cases} f \text{ differentiable at } x_0 = 0 \\ \forall x \in \mathbb{R} : f(x) \neq 0 \\ \forall a, b \in \mathbb{R} : f(a+b) = f(a) f(b) \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : f'(x) = f'(0) f(x) \end{cases}$$

(12) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \forall a, b \in \mathbb{R} : f(a+b) = f(a) f(b) \\ \forall x \in \mathbb{R} : f(x) = 1 + x g(x) \\ \lim_{x \to 0} g(x) = 1 \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : f'(x) = f(x). \end{cases}$$

(13) Let $f:\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} \forall a,b\in\mathbb{R}: f(a+b)+a+b=(f(a)+a)(f(b)+b) \\ \forall x\in\mathbb{R}: f(x)\neq 0 \end{cases}$$

Show that:

a) $f(0)=1$ b) $\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x\in\mathbb{R}: f'(x)=(f(x)+x)(f'(0)+1)-1 \end{cases}$

(14) Let $f:\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} \forall a,b\in\mathbb{R}: f(a+b)\leqslant f(a)+f(b) \\ f \text{ differentiable on } \mathbb{R} \\ f'(0)=f(0)=1 \end{cases}$$

Show that: $\forall x\in\mathbb{R}: f'(x)=f(x)$

(15) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ and let $a\in\mathbb{R}$ such that

$$\begin{cases} f,g \text{ differentiable on } \mathbb{R} \\ f(a)=g(a) \\ \forall x\in\mathbb{R}: f(x)+x\leqslant g(x)+a \end{cases}$$

Show that: $f'(a)+1=g'(a)$

(16) Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients and one variable. Show that:

$\forall f\in\mathbb{R}[x]: [(f')^2=f \iff \exists b\in\mathbb{R}: \forall x\in\mathbb{R}: f(x)=(1/4)x^2+bx+b^2]$

Basic differentiation rules

Let f, g be functions differentiable at a set $A \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$. Then:

$$\boxed{\begin{array}{l} h(x) = f(x) + g(x),\ \forall x \in A \Rightarrow h'(x) = f'(x) + g'(x),\ \forall x \in A \\ h(x) = af(x),\ \forall x \in A \Rightarrow h'(x) = af'(x),\ \forall x \in A \\ h(x) = f(x)g(x),\ \forall x \in A \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x),\ \forall x \in A \end{array}}$$

Proof

a) Assume that $h(x) = f(x) + g(x),\ \forall x \in A$. Then

$$\forall x, x_0 \in A : \lambda(h|x, x_0) = \frac{h(x) - h(x_0)}{x - x_0} =$$

$$= \frac{[f(x) + g(x)] - [f(x_0) + g(x_0)]}{x - x_0} =$$

$$= \frac{[f(x) - f(x_0)] + [g(x) - g(x_0)]}{x - x_0} =$$

$$= \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} = \lambda(f|x, x_0) + \lambda(g|x, x_0)$$

$$\Rightarrow \forall x_0 \in A : h'(x_0) = \lim_{x \to x_0} \lambda(h|x, x_0) =$$

$$= \lim_{x \to x_0} [\lambda(f|x, x_0) + \lambda(g|x, x_0)]$$

$$= \lim_{x \to x_0} \lambda(f|x, x_0) + \lim_{x \to x_0} \lambda(g|x, x_0)$$

$$= f'(x_0) + g'(x_0).$$

b) Assume that $h(x) = af(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \lambda(h|x,x_0) = \frac{h(x)-h(x_0)}{x-x_0} = \frac{af(x)-af(x_0)}{x-x_0} =$$

$$= \frac{a[f(x)-f(x_0)]}{x-x_0} = a\lambda(f|x,x_0) \Rightarrow$$

$$\Rightarrow \forall x_0 \in A: h'(x_0) = \lim_{x\to x_0} \lambda(h|x,x_0) = \lim_{x\to x_0} [a\lambda(f|x,x_0)]$$

$$= a \lim_{x\to x_0} \lambda(f|x,x_0) = af'(x_0)$$

c) Assume that $h(x) = f(x)g(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \lambda(h|x,x_0) = \frac{h(x)-h(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \frac{g(x)[f(x)-f(x_0)]+f(x_0)[g(x)-g(x_0)]}{x-x_0} =$$

$$= f(x_0)\frac{g(x)-g(x_0)}{x-x_0} + g(x)\frac{f(x)-f(x_0)}{x-x_0} =$$

$$= f(x_0)\lambda(g|x,x_0) + \lambda(f|x,x_0)g(x)$$

We note that:

g differentiable at $x_0 \Rightarrow g$ continuous at x_0

$$\Rightarrow \lim_{x\to x_0} g(x) = g(x_0)$$

and therefore:

$$\forall x_0 \in A: h'(x_0) = \lim_{x \to x_0} \lambda(h|x,x_0) =$$

$$= \lim_{x \to x_0} \left[f(x_0)\, \lambda(g|x,x_0) + \lambda(f|x,x_0)\, g(x) \right]$$

$$= f(x_0) \lim_{x \to x_0} \lambda(g|x,x_0) + \lim_{x \to x_0} \lambda(f|x,x_0) \lim_{x \to x_0} g(x)$$

$$= f(x_0)\, g'(x_0) + f'(x_0)\, g(x_0)$$

$$= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0). \qquad \square$$

THEORY QUESTIONS

(17) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be functions differentiable on A. Prove that:

$\forall x \in A: [f(x)+g(x)]' = f'(x) + g'(x)$

$\forall a \in \mathbb{R}: \forall x \in A: [af(x)]' = af'(x)$

$\forall x \in A: [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

EXERCISES

(18) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ such that:

$$\begin{cases} f, g \text{ 3-times differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R}: f'(x)g'(x) = a \\ \forall x \in \mathbb{R}: f(x)g(x) \neq 0 \end{cases}$$

and define $h: \mathbb{R} \to \mathbb{R}$ such that

$\forall x \in \mathbb{R}: h(x) = f(x)g(x)$

Show that:

a) $\forall x \in \mathbb{R}: \dfrac{h''(x)}{h(x)} = \dfrac{f''(x)}{f(x)} + \dfrac{2a}{f(x)g(x)} + \dfrac{g''(x)}{g(x)}$

b) $\forall x \in \mathbb{R}: \dfrac{h'''(x)}{h(x)} = \dfrac{f'''(x)}{f(x)} + \dfrac{g'''(x)}{g(x)}$

(19) Let $f \in \mathbb{R}[x]$ be a polynomial $f: \mathbb{R} \to \mathbb{R}$ with degree $\deg(f) = n \geq 2$. We say that:

p double zero of $f \Leftrightarrow \exists q \in \mathbb{R}[x]: \forall x \in \mathbb{R}: f(x) = (x-p)^2 q(x)$

Show that:

ρ double zero of $f \Leftrightarrow f(\rho) = 0 \wedge f'(\rho) = 0$

(20) Let $f \in \mathbb{R}[x]$ be a polynomial with degree 3 and three distinct roots $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$. Show that

$$\frac{\rho_1}{f'(\rho_1)} + \frac{\rho_2}{f'(\rho_2)} + \frac{\rho_3}{f'(\rho_3)} = 0$$

(21) Let $f \in \mathbb{R}[x]$ be a polynomial with degree $n \in \mathbb{N}^*$. Show that:

$$\forall x \in \mathbb{R}: f(x) = \sum_{a=0}^{n} \frac{f^{(a)}(0)\, x^a}{a!}$$

using proof by induction.

(22) Let $f_1 : A \to \mathbb{R}$, $f_2 : A \to \mathbb{R}$, $g_1 : A \to \mathbb{R}$, $g_2 : A \to \mathbb{R}$ be functions that are differentiable on $\mathbb{R}$ and let

$$\forall x \in A : h(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$$

Show that:

$$\forall x \in A : h'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$$

Chain rule

- The chain rule is a superrule that is used to generate differentiation rules that are then used in problems. We seldomly use the chain rule directly.
- Recall the definition of function composition:
  For $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$, we define $f \circ g: C \to \mathbb{R}$ with

$$\begin{cases} \operatorname{dom}(f\circ g) = \{x \in \operatorname{dom}(g) \mid g(x) \in \operatorname{dom}(f)\} \\ \qquad\qquad\quad = \{x \in B \mid g(x) \in A\} = C \\ \forall x \in C: (f\circ g)(x) = f(g(x)) \end{cases}$$

  Note that by definition, the belonging condition for $\operatorname{dom}(f\circ g)$ is:

$$x \in \operatorname{dom}(f\circ g) \Longleftrightarrow \begin{cases} x \in \operatorname{dom}(g) \\ g(x) \in \operatorname{dom}(f) \end{cases}$$

- The chain rule claims that:

$$\boxed{\begin{cases} g \text{ differentiable at } x_0 \\ f \text{ differentiable at } g(x_0) \end{cases} \Longrightarrow \begin{cases} f\circ g \text{ differentiable at } x_0 \\ (f\circ g)'(x_0) = f'(g(x_0))\, g'(x_0) \end{cases}}$$

We postpone the proof. Every choice of f generates a new generalized differentiation rule. For example:

1) For $f(x) = x^n$ with $n \in \mathbb{N}^*$, using

$$(x^n)' = nx^{n-1}$$

we obtain:

$$\boxed{\left([g(x)]^n\right)' = n\,[g(x)]^{n-1}\, g'(x)}$$

2) For $f(x) = \sqrt{x}$, using $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$, we obtain:

$$\boxed{\left(\sqrt{g(x)}\right)' = \frac{g'(x)}{2\sqrt{g(x)}}}$$

↳ Note that for each generalization, starting from the initial differentiation rule:

(a) All x are replaced with $g(x)$

(b) The entire result is then multiplied with $g'(x)$.

Step (a) corresponds to the $f'(g(x_0))$ factor

Step (b) corresponds to the $g'(x_0)$ factor.

We see therefore that every basic differentiation rule can give a more powerful generalized differentiation rule via the chain rule.

Proof of chain rule

Assume that g differentiable at x_0 and f differentiable at $g(x_0)$. It follows that $f\circ g$ can be defined on a neighborhood $N(x_0,\delta)$ for some $\delta>0$.

We define $y_0=g(x_0)$ and

$$F(y)=\begin{cases}\lambda(f\,|\,y,y_0), & \text{if } y\neq y_0\\ f'(y_0), & \text{if } y=y_0\end{cases}$$

We claim that $\lambda(f\circ g\,|\,x,x_0)=F(g(x))\lambda(g\,|\,x,x_0),\ \forall x\in N(x_0,\delta)$ (1)

To show the claim, let $x\in N(x_0,\delta)$ be given. We distinguish between the following cases:

Case 1: If $g(x)\neq g(x_0)$ then:

$$\lambda(f\circ g\,|\,x,x_0)=\frac{(f\circ g)(x)-(f\circ g)(x_0)}{x-x_0}=\frac{f(g(x))-f(g(x_0))}{x-x_0}=$$

$$=\frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)}\,\frac{g(x)-g(x_0)}{x-x_0}=$$

$$=\lambda(f\,|\,g(x),y_0)\,\lambda(g\,|\,x,x_0)=F(g(x))\,\lambda(g\,|\,x,x_0)$$

Case 2: If $g(x)=g(x_0)$, then:

$$\lambda(f\circ g\,|\,x,x_0)=\frac{(f\circ g)(x)-(f\circ g)(x_0)}{x-x_0}=\frac{f(g(x))-f(g(x_0))}{x-x_0}$$

$$=\frac{f(g(x_0))-f(g(x_0))}{x-x_0}=0$$

and

$$\lambda(g|x,x_0) = \frac{g(x)-g(x_0)}{x-x_0} = \frac{g(x_0)-g(x_0)}{x-x_0} = 0$$

and therefore $\lambda(f\circ g|x,x_0) = F(g(x))\lambda(g|x,x_0)$ holds trivially since both sides are zero.

This proves the claim.

Now, we note that

g differentiable at $x_0 \Rightarrow g$ continuous at $x_0 \Rightarrow$

$$\Rightarrow \lim_{x\to x_0} g(x) = g(x_0) = y_0 \qquad (2)$$

and

$$\lim_{y\to y_0} F(y) = \lim_{y\to y_0} \lambda(f|y,y_0) = \quad [\text{def of } F(y)]$$

$$= f'(y_0) = \quad [f \text{ differentiable at } y_0]$$

$$= F(y_0) \Rightarrow \quad [\text{def of } F(y)]$$

$\Rightarrow F$ continuous at y_0. (3)

Via the composition theorem, from Eq. (2) and Eq. (3):

$$\lim_{x\to x_0} F(g(x)) = F(\lim_{x\to x_0} g(x)) \quad [\text{via composition thm}]$$

$$= F(y_0) \quad [\text{via eq. (2)}]$$

$$= f'(y_0) \quad [\text{def of } F(y)]$$

$$= f'(g(x_0)) \quad (4) \quad [\text{def of } y_0]$$

and it follows that

$$[f(g(x_0))]' = \lim_{x\to x_0} \lambda(f\circ g|x,x_0) = \lim_{x\to x_0} [F(g(x))\lambda(g|x,x_0)] =$$

$$= \lim_{x\to x_0} F(g(x)) \cdot \lim_{x\to x_0} \lambda(g|x,x_0) = f'(g(x_0))\, g'(x_0). \quad \square$$

THEORY QUESTIONS

(23) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f\circ g: G \to \mathbb{R}$
and let $x_0 \in G$

a) Write the definition of G in terms of A and B

b) Prove that

$$\begin{cases} g \text{ differentiable at } x_0 \\ f \text{ differentiable at } g(x_0) \end{cases} \Rightarrow \begin{cases} f\circ g \text{ differentiable at } x_0 \\ (f\circ g)'(x_0) = f'(g(x_0))\, g'(x_0) \end{cases}$$

EXERCISES

(24) Let $f: \mathbb{R} \to \mathbb{R}$ such that f differentiable on $\mathbb{R}$.
Use the chain rule to show that.

a) f odd $\Rightarrow f'$ even

b) f periodic $\Rightarrow f'$ periodic

$\hookrightarrow$ Recall the following definitions

f even $\Leftrightarrow \forall x \in \mathbb{R}: f(-x) = f(x)$

f odd $\Leftrightarrow \forall x \in \mathbb{R}: f(-x) = -f(x)$

f periodic $\Leftrightarrow \exists a \in \mathbb{R}: \forall x \in \mathbb{R}: f(x+a) = f(x)$

(25) Let $f: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f \text{ 2-times differentiable on } \mathbb{R} \\ f \text{ odd} \\ \forall x \in \mathbb{R}: f(x) f'(x) \neq 0 \end{cases}$$

and let $g: \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R}: g(x) = f(x) f'(x)$
Show that:
a) $f(0) = f''(0) = 0$
b) g' even
c) $\forall x \in \mathbb{R}: \dfrac{g'(x)}{g(x)} = \dfrac{f'(x)}{f(x)} + \dfrac{f''(x)}{f'(x)}$

(26) Let $f \in \mathbb{R}[x]$ be a polynomial $f: \mathbb{R} \to \mathbb{R}$ and let $p \in \mathbb{R}$ and $n \in \mathbb{N}^*$. We say that
p zero of f with multiplicity $n \Leftrightarrow$
$\Leftrightarrow \exists q \in \mathbb{R}[x]: \forall x \in \mathbb{R}: f(x) = (x-p)^n q(x)$
Use proof by induction to show that
p zero of f with multiplicity $n \Leftrightarrow$
$\Leftrightarrow \forall k \in \{0\} \cup [n-1]: f^{(k)}(p) = 0$

(27) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. Show that
$$\left\{\begin{array}{l} \forall x \in \mathbb{R}: f(x) = x + a \\ g \text{ differentiable on } \mathbb{R} \\ f \circ g = g \circ f \end{array}\right. \Rightarrow g' \text{ periodic}$$

V The quotient rule

The quotient rule is derived from the chain rule as follows.

•1 First we show that

$$\boxed{\left(\forall x\in\mathbb{R}^*: f(x)=\frac{1}{x}\right) \Rightarrow \forall x\in\mathbb{R}^*: f'(x)=\frac{-1}{x^2}}$$

Proof

Since

$$\forall x,x_0\in\mathbb{R}-\{0\}: \lambda(f|x,x_0)=\frac{f(x)-f(x_0)}{x-x_0}=\frac{\frac{1}{x}-\frac{1}{x_0}}{x-x_0}=$$

$$=\frac{\left(\frac{x_0-x}{xx_0}\right)}{x-x_0}=\frac{-(x-x_0)}{xx_0(x-x_0)}=\frac{-1}{xx_0}\Rightarrow$$

$$\Rightarrow \forall x_0\in\mathbb{R}-\{0\}: f'(x_0)=\lim_{x\to x_0}\lambda(f|x,x_0)=\lim_{x\to x_0}\left(\frac{-1}{xx_0}\right)=$$

$$=\frac{-1}{x_0x_0}=\frac{-1}{x_0^2} \quad \square$$

•2 Via the chain rule, this result immediately generalizes to the reduced quotient rule:

$$\boxed{h(x)=\frac{1}{g(x)},\ \forall x\in A \Rightarrow h'(x)=\frac{-g'(x)}{[g(x)]^2},\ \forall x\in A}$$

• 3 Combined with the product rule, the reduced quotient rule gives the <u>quotient rule</u>:

$$\boxed{h(x) = \frac{f(x)}{g(x)}, \forall x \in A \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

<u>Proof</u>

$$h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \left[f(x) \cdot \frac{1}{g(x)}\right]' =$$

$$= f'(x)\frac{1}{g(x)} + f(x) \cdot \left[\frac{1}{g(x)}\right]' = \qquad \text{[product rule]}$$

$$= \frac{f'(x)}{g(x)} + f(x)\frac{-g'(x)}{[g(x)]^2} = \qquad \text{[reduced quotient rule]}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \square$$

EXERCISES

(28) Let $f : \mathbb{R}^* \to \mathbb{R}$ with $\forall x \in \mathbb{R}^* : f(x) = 1/x$.
Use proof by induction to show that:
$$\forall n \in \mathbb{N}^* : \forall x \in \mathbb{R}^* : f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

(29) Let $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R} : f(x) = \sqrt{x + \sqrt{1+x^2}}$
Show that.
a) $\forall x \in \mathbb{R} : f(x) = 2\sqrt{1+x^2}\, f'(x)$
b) $\forall x \in \mathbb{R} : 4(1+x^2) f''(x) + 4x f'(x) = f(x)$

(30) Let $f \in \mathbb{R}[x]$ be a polynomial with degree $n \in \mathbb{N}^*$ with distinct zeroes $\rho_1, \rho_2, \ldots, \rho_n \in \mathbb{R}$. Show that:
a) $\forall x \in \mathbb{R} - \{\rho_k \mid k \in [n]\} : \dfrac{f'(x)}{f(x)} = \displaystyle\sum_{k=1}^{n} \frac{1}{x - \rho_k}$
b) The function $g : \mathbb{R} \to \mathbb{R}$ given by
$\forall x \in \mathbb{R} : g(x) = f(x) f''(x) - [f'(x)]^2$
satisfies
$\forall x \in \mathbb{R} : g(x) \neq 0$

(31) Define $p : \mathbb{R} \to \mathbb{R}$ such that
$\forall x \in \mathbb{R} : p(x) = ax^2 + bx + c = a(x - \rho_1)(x - \rho_2)$
with ρ_1, ρ_2 the distinct zeroes of p. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall k \in \{1,2\} : \forall x \in \mathbb{R} : \; x f'(x) - f(x) \leqslant \rho_k f(x) \end{cases}$$

Show that:

a) $\forall x \in \mathbb{R} - \{\rho_1, \rho_2\} : \; \dfrac{p'(x)}{p(x)} = \dfrac{1}{x-\rho_1} + \dfrac{1}{x-\rho_2}$

b) $\forall x \in \mathbb{R} : \; f(x)\, p(x)\, p''(x) \leqslant p'(x) \begin{vmatrix} f(x) & f'(x) \\ p(x) & p'(x) \end{vmatrix}$

For part (b) introduce the function

$w(x) = \dfrac{p'(x)}{p(x)} f(x)$

Calculate $w'(x)$ and show, using (a), that $w'(x) \geqslant 0$.

▼ Trigonometric derivatives

- The derivative of $\sin x$ can be derived via the result

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

and the trigonometric identity for factoring the sum/difference of sine functions:

$$\sin a \pm \sin b = 2\sin\left(\frac{a\pm b}{2}\right)\cos\left(\frac{a\mp b}{2}\right)$$

The main result is:

① $\boxed{\forall x\in\mathbb{R} : (\sin x)' = \cos x}$

Proof

Let $x, x_0 \in \mathbb{R}$ be given with $x \neq x_0$.

$$\lambda(\sin \mid x, x_0) = \frac{\sin x - \sin x_0}{x - x_0} = \frac{2\sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)}{x-x_0} =$$

$$= \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cos\left(\frac{x+x_0}{2}\right), \ \forall x, x_0 \in \mathbb{R} \quad (1)$$

Since

$$\left.\begin{array}{l} \lim_{x\to x_0} \frac{x+x_0}{2} = \frac{x_0+x_0}{2} = x_0 \\ \cos \text{ continuous on } \mathbb{R} \end{array}\right\} \Rightarrow \lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right) = \cos x_0 \quad (2)$$

and

$$\left.\begin{array}{l} \lim\limits_{x \to x_0} \dfrac{x-x_0}{2} = 0 \\ \dfrac{x-x_0}{2} \neq 0, \forall x \in N(x_0, \delta) \\ \lim\limits_{y \to 0} \dfrac{\sin y}{y} = 1 \end{array}\right\} \Rightarrow \lim_{x \to 0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = 1 \quad (3)$$

From Eq. (1), Eq. (2), Eq. (3):

$$(\sin x_0)' = \lim_{x \to x_0} \Delta(\sin | x, x_0) =$$

$$= \lim_{x \to x_0} \left[\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cdot \cos\left(\frac{x+x_0}{2}\right) \right]$$

$$= \lim_{x \to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \lim_{x \to x_0} \cos\left(\frac{x+x_0}{2}\right) =$$

$$= 1 \cdot \cos\left(\frac{x_0+x_0}{2}\right) = \cos x_0 \qquad \square$$

↳ Note that the proof of this result depends on the continuity of cos and the limit $\lim\limits_{x \to 0} (\sin x)/x$. Consequently continuity has to be established first before establishing differentiability.

- For the derivative of cos we use the chain rule generalization of the above result

$$[\sin(g(x))]' = g'(x) \cos(g(x))$$

and the cofactor identities:

$$\forall x \in \mathbb{R}: \ \sin(\pi/2 - x) = \cos x$$
$$\forall x \in \mathbb{R}: \ \cos(\pi/2 - x) = \sin x$$

as follows:

(2) $\boxed{(\cos x)' = -\sin x \ , \forall x \in \mathbb{R}}$

<u>Proof</u>

$$(\cos x)' = [\sin(\pi/2 - x)]' = (\pi/2 - x)' \cos(\pi/2 - x)$$
$$= -\cos(\pi/2 - x) = -\sin x \ , \ \forall x \in \mathbb{R}.$$

(3) $\boxed{(\tan x)' = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x \ , \forall x \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}}$

<u>Proof</u>

$$(\tan x)' = \left[\frac{\sin x}{\cos x}\right]' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} =$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \quad (1)$$

From Eq. (1):

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} =$$

$$= 1 + \left(\frac{\sin x}{\cos x}\right)^2 = 1 + \tan^2 x \qquad \square$$

- Via the chain rule, we obtain the following generalized differentiation rules:

$(\sin x)' = \cos x$	$[\sin(g(x))]' = g'(x)\cos(g(x))$
$(\cos x)' = -\sin x$	$[\cos(g(x))]' = -g'(x)\sin(g(x))$
$(\tan x)' = \dfrac{1}{\cos^2 x}$	$[\tan(g(x))]' = \dfrac{g'(x)}{\cos^2(g(x))}$
$(\tan x)' = 1+\tan^2 x$	$[\tan(g(x))]' = [1+\tan^2(g(x))]\,g'(x)$

EXAMPLE

Consider the function

$$f(x) = \begin{cases} \sin(\pi x^2)/x & , \text{ if } x \in \mathbb{R}^* \\ 0 & , \text{ if } x=0 \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ f' \text{ continuous on } \mathbb{R}. \end{cases}$$

Solution

Since,

$$\forall x \in \mathbb{R}^* : f'(x) = \left[\frac{\sin(\pi x^2)}{x}\right]' = \frac{[\sin(\pi x^2)]' x - \sin(\pi x^2)(x)'}{x^2}$$

$$= \frac{(\pi x^2)' \cos(\pi x^2) x - \sin(\pi x^2)}{x^2} = \frac{2\pi x \cos(\pi x^2) x - \sin(\pi x^2)}{x^2}$$

$$= \frac{2\pi x^2 \cos(\pi x^2) - \sin(\pi x^2)}{x^2} = 2\pi \cos(\pi x^2) - \sin(\pi x^2)/x^2.$$

and

$$\lambda(f|x,0) = \frac{f(x) - f(0)}{x-0} = \frac{f(x)}{x} = \frac{\sin(\pi x^2)/x}{x} = \frac{\sin(\pi x^2)}{x^2}$$

$$\Rightarrow \lim_{x\to 0} \lambda(f|x,0) = \lim_{x\to 0} \frac{\sin(\pi x^2)}{x^2} = \pi \lim_{x \to 0} \frac{\sin(\pi x^2)}{\pi x^2}$$

$$= \pi \lim_{x\to 0} \frac{\sin x}{x} = \pi$$

it follows that f differentiable on $\mathbb{R}$ with

$$f'(x) = \begin{cases} 2\pi\cos(\pi x^2) - \sin(\pi x^2)/x^2 , & \text{if } x \in \mathbb{R}^* \\ \pi , & \text{if } x = 0 \end{cases}$$

Since,

$\forall x \in \mathbb{R}^* : f'(x) = 2\pi\cos(\pi x^2) - \sin(\pi x^2)/x^2$

$\Rightarrow f'$ continuous on $\mathbb{R}^*$

and

$$\lim_{x\to 0} f'(x) = \lim_{x \to 0} \left[2\pi\cos(\pi x^2) - \sin(\pi x^2)/x^2\right] =$$

$$= 2\pi\cos 0 - \pi \lim_{x\to 0} \frac{\sin(\pi x^2)}{\pi x^2} =$$

$$= 2\pi - \pi \lim_{x\to 0} \frac{\sin x}{x} = 2\pi - \pi \cdot 1 = \pi = f'(0)$$

$\Rightarrow f'$ continuous at $x = 0$.

we conclude that f' continuous on $\mathbb{R}$.

EXERCISES

(32) Consider the function

$$f(x) = \begin{cases} \sin^2(\pi x)/(x-1) & , \text{ if } x \in \mathbb{R}-\{1\} \\ 0 & , \text{ if } x=0 \end{cases}$$

Show that f differentiable on $\mathbb{R}$ and f' continuous on $\mathbb{R}$

(33) Let $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ and consider the function
$\forall x \in \mathbb{R}: f(x) = \sin(ax+b)$
Use proof by induction to show that
$\forall n \in \mathbb{N}^*: \forall x \in \mathbb{R}: f^{(n)}(x) = a^n \sin(ax+b+n\pi/2)$

(34) Let $a \in \mathbb{R}$ and consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) + ax & , \text{ if } x \in \mathbb{R}^* \\ 0 & , \text{ if } x=0 \end{cases}$$

Show that f differentiable on $\mathbb{R}$.

(35) Consider the function

$$\forall x \in \mathbb{R}: f(x) = \frac{\cos^2 t}{1+\sin^2 t}$$

Show that : $f(\pi/4) - 3f'(\pi/4) = 3$.

(36) Let $f:\mathbb{R}\to\mathbb{R}$ with $\forall x\in\mathbb{R}: f(x) = x\sin(ax)$. Show that

a) $\forall n\in\mathbb{N}^*: f^{(2n)}(x) = (-1)^n\left[a^{2n}x\sin(ax) - 2na^{2n-1}\cos(ax)\right]$

b) $|a|<1 \Rightarrow \lim\limits_{n\in\mathbb{N}^*} f^{(2n)}(x) = 0$

(37) Let $f:(0,1)\to\mathbb{R}$ be a function such that

$\forall x\in(0,\pi/2): f(\sin x) = \sin^2 x - \cos x$

Show that: $3f''(1/2) - 2f'(1/2) = 4+2\sqrt{3}$.