FUNCTION CONTINUITY

Definition of a continuous function

Function continuity is defined at a point $x_0 \in \mathbb{R}$ and over a subset $S \subseteq \mathbb{R}$ as follows:

Def: Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. We say that:

a) f continuous at $x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$

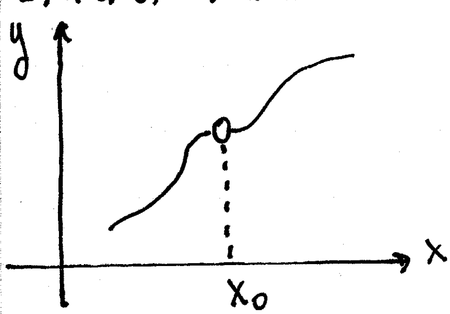
b) f continuous at $S \Leftrightarrow \forall x_0 \in S: \lim_{x \to x_0} f(x) = f(x_0)$

Note that, via the limit definition, the definition of continuity over a set S can be rewritten as follows:

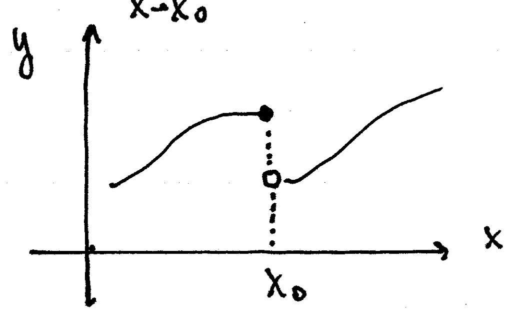
$$f \text{ continuous on } S \Leftrightarrow \forall x_0 \in S: \lim_{x \to x_0} f(x) = f(x_0) \Leftrightarrow$$
$$\Leftrightarrow \forall x_0 \in S: \forall \varepsilon \in (0, +\infty): \exists \delta \in (0, +\infty): \forall x \in A: (0 < |x - x_0| < \delta \Rightarrow$$
$$\Rightarrow |f(x) - f(x_0)| < \varepsilon)$$

- There are three ways continuity at a point x_0 may fail:

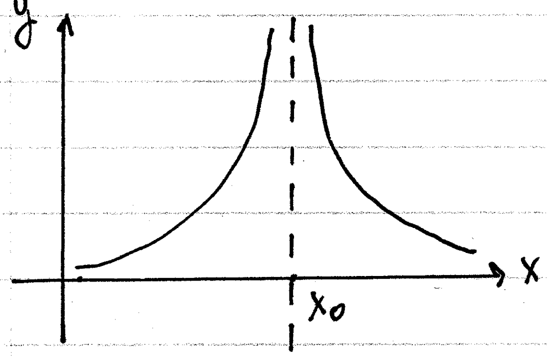
1) $f(x_0)$ is not defined



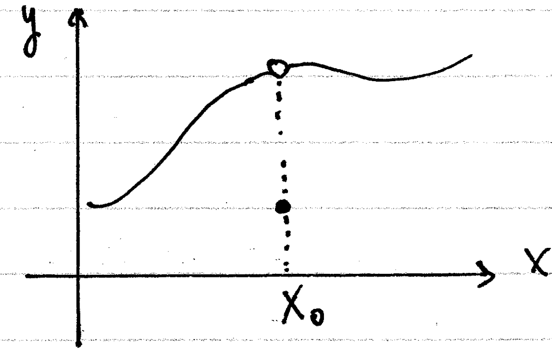
2) $\lim_{x \to x_0} f(x)$ does not exist



3) $\lim_{x \to x_0} f(x) \in \{+\infty, -\infty\}$



4) $\lim_{x \to x_0} f(x) \neq f(x_0)$



Continuity of basic functions

Let $\mathbb{R}[x]$ be the set of all polynomials functions $p: \mathbb{R} \to \mathbb{R}$ such that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad \forall x \in \mathbb{R}$$

Then, from the definition of contisiuity and the properties of limits, it follows that:

1) $\forall p \in \mathbb{R}[x]$: p continuous on $\mathbb{R}$
2) $\forall p, q \in \mathbb{R}[x]$: p/q continuous on $\mathbb{R} - \{x \in \mathbb{R} \mid q(x) = 0\}$
3) sin continuous on $\mathbb{R}$
4) cos continuous on $\mathbb{R}$
5) tan continuous on $\mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$
6) cot continuous on $\mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

Consequences of the composition theorems

From the function composition theorem, it follows that:

(1) If $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: G \to \mathbb{R}$, then:

$$\left\{ \begin{array}{l} g \text{ continuous on } x_0 \\ f \text{ continuous on } g(x_0) \end{array} \right. \Rightarrow f \circ g \text{ continuous on } x_0$$

(2) If $f: A \to \mathbb{R}$ with $(a,b) \subseteq A$ and (a_n) a sequence, then we have

$$\left\{ \begin{array}{l} f \text{ continuous on } (a,b) \\ \forall n \in \mathbb{N}^*: a_n \in (a,b) \\ \lim\limits_{n \in \mathbb{N}^*} a_n = x \end{array} \right. \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = x$$

Both results are immediate consequences of the composition theorem.

EXAMPLE

Consider the function

$$f(x) = \begin{cases} x\,g(x), & \text{with } x \in \mathbb{R} - \{0\} \\ a, & \text{with } x = 0 \end{cases}$$

with g continuous on $\mathbb{R}$. Show that

f continuous on $\mathbb{R} \Leftrightarrow a = 0$.

Solution

Assume that g continuous on $\mathbb{R}$. Let $x_0 \in \mathbb{R} - \{0\}$ be given. Then, we have:

g continuous on $\mathbb{R} \Rightarrow g$ continuous on x_0

$$\Rightarrow \lim_{x \to x_0} g(x) = g(x_0)$$

$$\Rightarrow \lim_{x \to x_0} f(x) = \lim_{x \to x_0} [x\,g(x)] = [\lim_{x \to x_0} x][\lim_{x \to x_0} g(x)]$$

$$= x_0\,g(x_0) = f(x_0) \Rightarrow$$

$\Rightarrow f$ continuous on x_0

We have thus shown that:

$\forall x_0 \in \mathbb{R} - \{0\} : f$ continuous on x_0

$\Rightarrow f$ continuous on $\mathbb{R} - \{0\}$. (1)

We also note that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} [x\,g(x)] = [\lim_{x \to 0} x][\lim_{x \to 0} g(x)] = 0\,g(0) = 0 \quad (2)$$

It follows that:

f continuous on $\mathbb{R} \Leftrightarrow f$ continuous on $x_0 = 0$ [via Eq. (1)]

$\Leftrightarrow \lim_{x \to 0} f(x) = f(0)$ [definition]

$\Leftrightarrow a = 0$. [via Eq. (2)]

THEORY QUESTIONS

① Let $f: A \to \mathbb{R}$ with $x_0 \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. Write the definition for the following statements

a) f continuous on x_0

b) f continuous on S

EXERCISES

② Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) + b & , \text{ if } x \in \mathbb{R} - \{0\} \\ a & , \text{ if } x = 0 \end{cases}$$

Show that:

f continuous on $\mathbb{R} \Leftrightarrow a = b$

③ Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ and define $h: \mathbb{R} \to \mathbb{R}$ such that

$\forall x \in \mathbb{R}: h(x) = \max\{f(x), g(x)\}$

Show that:

$$\begin{cases} f \text{ continuous on } \mathbb{R} \\ g \text{ continuous on } \mathbb{R} \end{cases} \Rightarrow h \text{ continuous on } \mathbb{R}.$$

↳ Hint: First, show that:

$\forall x \in \mathbb{R}: h(x) = (1/2)(f(x) + g(x)) + (1/2)|f(x) - g(x)|$

④ Let $f : [a,c] \to \mathbb{R}$ and $g : [c,b] \to \mathbb{R}$ such that

$$\begin{cases} f \text{ continuous on } [a,c] \\ f \text{ continuous on } [c,b] \end{cases}$$

Define $h : [a,b] \to \mathbb{R}$ such that

$$h(x) = \begin{cases} f(x), & \text{if } x \in [a,c] \\ g(x), & \text{if } x \in [c,b] \end{cases}$$

Show that:

h continuous on $[a,b] \iff f(c) = g(c)$.

▼ Continuity and dense sets

Let $\mathrm{Seq}(\mathcal{S})$ be the set of all sequences $a_n : \mathbb{N}^* \to \mathcal{S}$ with $\mathcal{S} \subseteq \mathbb{R}$ such that: $\forall n \in \mathbb{N}^* : a_n \in \mathcal{S}$

Def: Let $\mathcal{S} \subseteq \mathbb{R}$. We say that

$$\mathcal{S} \text{ dense in } \mathbb{R} \Leftrightarrow \forall x \in \mathbb{R} : \exists a \in \mathrm{Seq}(\mathcal{S}) : \lim_{n \in \mathbb{N}^*} a_n = x$$

Our main result is the following theorem:

Thm: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and let $\mathcal{S} \subseteq \mathbb{R}$. Then, we have:

$$\begin{cases} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in \mathcal{S} : f(x) = g(x) \\ \mathcal{S} \text{ dense on } \mathbb{R} \end{cases} \Rightarrow \forall x \in \mathbb{R} : f(x) = g(x)$$

Proof

Let $x \in \mathbb{R}$ be given. Since $\mathcal{S}$ dense on $\mathbb{R}$, choose $a \in \mathrm{Seq}(\mathcal{S})$ such that $\lim_{n \in \mathbb{N}^*} a_n = x$. Then, we have:

$$\begin{aligned} f(x) &= f\left(\lim_{n \in \mathbb{N}^*} a_n\right) && [\text{definition of } (a_n)] \\ &= \lim_{n \in \mathbb{N}^*} f(a_n) && [f \text{ continuous on } \mathbb{R}] \\ &= \lim_{n \in \mathbb{N}^*} g(a_n) && [\text{via } a_n \in \mathcal{S} \Rightarrow f(a_n) = g(a_n)] \end{aligned}$$

$= g(\lim_{n\in\mathbb{N}^+} a_n)$ [g continuous on $\mathbb{R}$]

$= g(x)$ [definition of (a_n)]

We have thus shown that

$\forall x\in\mathbb{R}: f(x) = g(x)$ $\square$

In order to put this theorem to use, we will now show that:

(1) $\boxed{\mathbb{Q} \text{ dense in } \mathbb{R}}$

Proof

Let $x\in\mathbb{R}$ be given. Choose $a, b\in \mathrm{Seq}(\mathbb{Q})$ such that the interval sequence $([a_n, b_n])$ is nested with

$\bigcap_{n\in\mathbb{N}^+} [a_n, b_n] = \{x\}$

It follows that $\lim_{n\in\mathbb{N}^+} a_n = x$. We have thus shown that

$(\forall x\in\mathbb{R}: \exists a\in \mathrm{Seq}(\mathbb{Q}): \lim_{n\in\mathbb{N}^+} a_n = x) \Rightarrow \mathbb{Q}$ dense in $\mathbb{R}$. $\square$

(2) $\boxed{\mathbb{R}-\mathbb{Q} \text{ dense in } \mathbb{R}}$

Proof

Let $x\in\mathbb{R}$ be given. Choose $a, b\in \mathrm{Seq}(\mathbb{Q})$ such that the interval sequence $([a_n, b_n])$ is nested with

$$\bigcap_{n\in\mathbb{N}^*} [a_n, b_n] = \{x\}$$

It follows that: $\lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} b_n = x$.

We define (c_n) such that
$\forall n\in\mathbb{N}^*: c_n = a_n + \sqrt{2}\,(b_n - a_n)$
and note that
$([a_n, b_n])$ nested $\Rightarrow \lim_{n\in\mathbb{N}^*} (b_n - a_n) = 0 \Rightarrow$

$$\Rightarrow \underline{\lim_{n\in\mathbb{N}^*} c_n} = \lim_{n\in\mathbb{N}^*} [a_n + \sqrt{2}(b_n - a_n)] =$$

$$= \lim_{n\in\mathbb{N}^*} a_n + \sqrt{2} \lim_{n\in\mathbb{N}^*} (b_n - a_n)$$

$$= x + \sqrt{2}\cdot 0 = \underline{x}$$

▸ We will show that $c \in \mathrm{Seq}(\mathbb{R}-\mathbb{Q})$.
Let $n\in\mathbb{N}^*$ be given. To show a contradiction, assume that $c_n \in \mathbb{Q}$. Then, choose $p, q \in \mathbb{Z}$ such that $c_n = p/q$.
It follows that

$$c_n = p/q \Rightarrow a_n + \sqrt{2}(b_n - a_n) = p/q \Rightarrow \sqrt{2}\,(b_n - a_n) = p/q - a_n$$

$$\Rightarrow \sqrt{2} = \frac{(p/q) - a_n}{b_n - a_n} \Rightarrow \sqrt{2} \in \mathbb{Q}$$

which is a contradiction. We have thus shown that
$(\forall n\in\mathbb{N}^*: c_n \notin \mathbb{Q}) \Rightarrow \underline{c \in \mathrm{Seq}(\mathbb{R}-\mathbb{Q})}$.
We conclude that:
$(\forall x\in\mathbb{R}: \exists c \in \mathrm{Seq}(\mathbb{R}-\mathbb{Q}) : \lim_{n\in\mathbb{N}^*} c_n = x) \Rightarrow \mathbb{R}-\mathbb{Q}$ dense on $\mathbb{R}$.

An immediate consequence of these results are the following statements:

① $\boxed{\begin{cases} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in \mathbb{Q} : f(x) = g(x) \end{cases} \Rightarrow \forall x \in \mathbb{R} : f(x) = g(x)}$

② $\boxed{\begin{cases} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in \mathbb{R} - \mathbb{Q} : f(x) = g(x) \end{cases} \Rightarrow \forall x \in \mathbb{R} : f(x) = g(x)}$

EXAMPLES

▸ The nowhere-continuous function

a) Consider the function

$$f(x) = \begin{cases} 0 & , \text{ if } x \in \mathbb{Q} \\ 1 & , \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that: $\forall x_0 \in \mathbb{R} : f$ not continuous on x_0

Solution

Let $x_0 \in \mathbb{R}$ be given. To show a contradiction, assume that f continuous on x_0. Since $\mathbb{Q}$ dense on $\mathbb{R}$, choose $(a_n): \mathbb{N}^* \to \mathbb{Q}$ such that $\lim_{n \in \mathbb{N}^*} a_n = x_0$. Likewise, since $\mathbb{R} - \mathbb{Q}$ dense on $\mathbb{R}$, choose $(b_n): \mathbb{N} \to \mathbb{R} - \mathbb{Q}$ such that $\lim_{n \in \mathbb{N}^*} b_n = x_0$. It follows that:

$$\begin{aligned} f(x_0) &= \lim_{x \to x_0} f(x) && [f \text{ continuous on } x_0] \\ &= \lim_{n \in \mathbb{N}^*} f(a_n) && [\text{via } \lim_{n \in \mathbb{N}^*} a_n = x_0] \\ &= \lim_{n \in \mathbb{N}^*} 0 && [a_n \in \mathbb{Q} \Rightarrow f(a_n) = 0] \\ &= 0 && (1) \end{aligned}$$

and

$$\begin{aligned} f(x_0) &= \lim_{x \to x_0} f(x) && [f \text{ continuous on } x_0] \\ &= \lim_{n \in \mathbb{N}^*} f(b_n) && [\text{via } \lim_{n \in \mathbb{N}^*} b_n = x_0] \end{aligned}$$

$$= \lim_{n \in \mathbb{N}^*} 1 \quad [\text{via } b_n \in \mathbb{R} - \mathbb{Q} \Rightarrow f(b_n) = 1]$$

$$= 1 \qquad (2)$$

Eq.(1) and Eq.(2), therefore <u>f not continuous on x_0</u>

We have thus shown that

$\forall x_0 \in \mathbb{R}$: f not continuous on x_0.

8) Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that

$$\begin{cases} f \text{ continuous on } \mathbb{R} \\ \forall x,y \in \mathbb{R} : f(x+y) = f(x) + f(y) \end{cases}$$

Show that: $\exists a \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = ax$

Solution

- We will show that: $\forall x \in \mathbb{R} : \forall n \in \mathbb{Z} : f(nx) = nf(x)$. using proof by induction.

Let $x \in \mathbb{R}$ be given. For $n = 0$, we have:

$f(0x) = f(0x + 0x) = f(0x) + f(0x) \Rightarrow f(0x) = 0 = 0f(x)$

For $n = k$, we assume that: $f(kx) = kf(x)$.

For $n = k+1$, we will show that: $f((k+1)x) = (k+1)f(x)$.

We have:

$$f((k+1)x) = f(kx + x) = f(kx) + f(x) = kf(x) + f(x) =$$
$$= (k+1)f(x)$$

For $n = k-1$, we will show that: $f((k-1)x) = (k-1)f(x)$

We have:

$$kf(x) = f(kx) = f((k-1)x + x) = f((k-1)x) + f(x) \Rightarrow$$
$$\Rightarrow f((k-1)x) = kf(x) - f(x) = (k-1)f(x)$$

We have thus shown, by induction, that $\forall n \in \mathbb{Z} : f(nx) = nf(x)$ and conclude that:

$\forall x \in \mathbb{R} : \forall n \in \mathbb{Z} : f(nx) = nf(x)$

- Let $x \in \mathbb{Q}$ be given. Choose $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$ such that $x = p/q$. It follows that

$$\begin{cases} f(p) = f(q(p/q)) = f(qx) = qf(x) \\ f(p) = f(p \cdot 1) = pf(1) \end{cases} \Rightarrow qf(x) = pf(1) \Rightarrow$$
$$\Rightarrow f(x) = (p/q)f(1) = xf(1).$$

We have thus shown that
$\forall x \in \mathbb{Q} : f(x) = x f(1)$
Define: $\forall x \in \mathbb{R} : g(x) = x f(1)$. Then, we have:

$$\begin{cases} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in \mathbb{Q} : f(x) = g(x) \end{cases} \Rightarrow$$

$$\Rightarrow \forall x \in \mathbb{R} : f(x) = g(x) = x f(1)$$

$$\Rightarrow \exists a \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = ax \quad (\text{for } a = f(1)) \qquad \square$$

$\hookrightarrow$ Our methodology here is to first establish the claim on $\mathbb{Z}$ using proof by induction. Then, we generalize by proving the claim on $\mathbb{Q}$. Continuity is then used to rapidly extend the claim on $\mathbb{R}$.

THEORY QUESTIONS

(5) State the definition of the statement:
S dense on $\mathbb{R}$.

(6) Prove that:
a) $\mathbb{Q}$ dense on $\mathbb{R}$
b) $\mathbb{R}-\mathbb{Q}$ dense on $\mathbb{R}$

(7) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two functions. Prove that:

$$\begin{cases} S \text{ dense on } \mathbb{R} \\ f,g \text{ continuous on } \mathbb{R} \\ \forall x \in S : f(x) = g(x) \end{cases} \Rightarrow \forall x \in \mathbb{R} : f(x) = g(x)$$

EXERCISES

(8) Show that the set $S = \{a\sqrt{2} \mid a \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.

(9) Let S be a set dense in $\mathbb{R}$ and let $a \in \mathbb{R}$ be some number. Show that the set
$T = \{x + a \mid x \in S\}$
is also dense in $\mathbb{R}$.

(10) Consider the function

$$f(x) = \begin{cases} x & , \text{ if } x \in \mathbb{Q} \\ x+1 & , \text{ if } x \in \mathbb{R}-\mathbb{Q} \end{cases}$$

Show that: $\forall x_0 \in \mathbb{R}$: f not continuous on x_0

(11) Consider the function

$$f(x) = \begin{cases} x & , \text{ if } x \in \mathbb{Q} \\ 0 & , \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that:

$\forall x_0 \in \mathbb{R} : (f \text{ continuous on } x_0 \Leftrightarrow x_0 = 0)$

↳ Hint: The "$\Rightarrow$" proof uses rational and irrational sequences. However, the "$\Leftarrow$" requires proof by limit definition or properties of limits.

(12) Consider the functions

$$\begin{cases} f(x) = \begin{cases} x & , \text{ if } x \in \mathbb{Q} \\ 2-x & , \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases} \\ \forall x \in \mathbb{R} : g(x) = f(x) f(2-x) \end{cases}$$

Show that:

a) g continuous on $\mathbb{R}$

b) $\forall x_0 \in \mathbb{R} : (f \text{ continuous on } x_0 \Leftrightarrow x_0 = 1)$

(13) Let $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ be two functions such that g_1, g_2 continuous on $\mathbb{R}$. Let $S \subseteq \mathbb{R}$ be the set $S = \{x \in \mathbb{R} \mid g_1(x) = g_2(x)\}$

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by:

$$f(x) = \begin{cases} g_1(x) & , \text{ if } x \in \mathbb{Q} \\ g_2(x) & , \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that:
$\forall x_0 \in \mathbb{R} : (f \text{ continuous on } x_0 \Leftrightarrow x_0 \in S)$

(14) Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that
$\forall x,y \in \mathbb{R} : f(x+y) = f(x) + f(y)$
Show that:
f continuous at $x_0 = 0 \Rightarrow f$ continuous on $\mathbb{R}$.

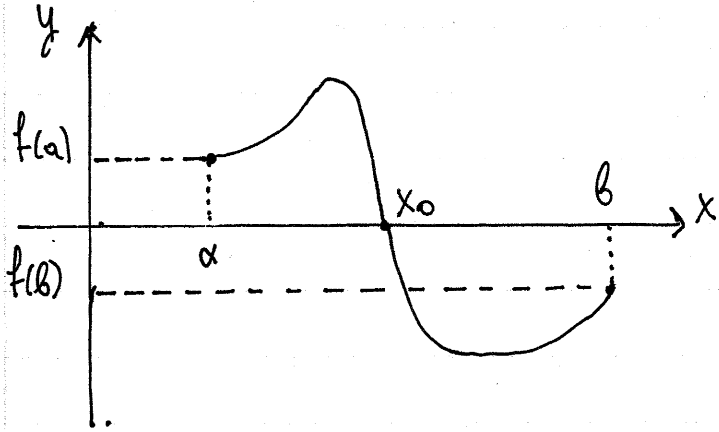
(15) Let $f: (0,+\infty) \to \mathbb{R}$ such that
$\forall x,y \in (0,+\infty) : f(xy) = f(x) + f(y)$
Assume that f continuous on $x_0 = 1$
Show that:
a) $f(1) = 0$
b) $\forall x,y \in (0,+\infty) : f(x/y) = f(x) - f(y)$
c) f continuous on $(0,+\infty)$
d) $\forall a \in \mathbb{R} : \forall x \in (0,+\infty) : f(x^a) = af(x)$.

Bolzano theorem

Thm: Let $f: A \to \mathbb{R}$ and let $[a,b] \subseteq A$. Then, we have:

$$\begin{cases} f \text{ continuous on } [a,b] \\ f(a)f(b) < 0 \end{cases} \Rightarrow \exists x_0 \in (a,b) : f(x_0) = 0$$

interpretation: When a function f is continuous on $[a,b]$, then if the values of f at $x=a$ and $x=b$ have opposite signs, then it has to pass through zero for some $x_0 \in (a,b)$. This result connects the formal definition of continuity with our intuitive understanding of the geometrical meaning of continuity.



Proof

Assume that f continuous on $[a,b] \wedge f(a)f(b) < 0$.
Since $f(a), f(b)$ are heterosigned, assume with no loss of generality that $f(a) < 0$ and $f(b) > 0$.

- We construct an interval sequence $([a_n, b_n])$ such that

$$\begin{cases} ([a_n, b_n]) \text{ nested} \\ \forall n \in \mathbb{N}^* : (f(a_n) \leq 0 \wedge f(b_n) \geq 0) \end{cases}$$

Define $[a_1, b_1] = [a,b]$ and note that trivially, we have:

$f(a_1) \leq 0 \wedge f(b_1) \geq 0$

Assume that $[a_n, b_n]$ has been defined such that

$f(a_n) \leq 0 \wedge f(b_n) \geq 0$

Let $c_n = (a_n + b_n)/2$ and define

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, c_n] & \text{, if } f(c_n) \geq 0 \\ [c_n, b_n] & \text{, if } f(c_n) < 0 \end{cases}$$

By construction, it follows that $([a_n, b_n])$ nested. We may therefore choose $\underline{x_0 \in [a,b]}$ such that

$$\bigcap_{n \in \mathbb{N}^*} [a_n, b_n] = \{x_0\}$$

Then, we have:

$$\begin{cases} f \text{ continuous on } [a,b] \\ \lim\limits_{n \in \mathbb{N}^*} a_n = \lim\limits_{n \in \mathbb{N}^*} b_n = x_0 \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = \lim_{n \in \mathbb{N}^*} f(b_n) = f(x_0)$$

and therefore

$$\forall n \in \mathbb{N}^* : \begin{cases} f(a_n) \leq 0 \\ f(b_n) \geq 0 \end{cases} \Rightarrow$$

$$\Rightarrow f(x_0) = \lim_{n \in \mathbb{N}^*} f(a_n) \leq 0 \wedge f(x_0) = \lim_{n \in \mathbb{N}^*} f(b_n) \geq 0$$

$$\Rightarrow \underline{f(x_0) = 0}$$

We also note that

$$\begin{cases} f(a) < 0 \\ f(b) > 0 \end{cases} \Rightarrow \begin{cases} x_0 \neq a \\ x_0 \neq b \end{cases} \Rightarrow \underline{x_0 \in (a,b)}$$

We have thus shown that

$\exists x_0 \in (a,b) : f(x_0) = 0.$ □

EXAMPLES

a) Show that the equation

$$\sin(\cos 3x) = 0$$

has at least one solution on $(0, \pi)$.

Solution

Define $f(x) = \sin(\cos(3x))$, $\forall x \in \mathbb{R}$.

We note that f continuous on $[0, \pi]$ (1).

and also:

$f(0) = \sin(\cos(3 \cdot 0)) = \sin(\cos 0) = \sin 1$ (2)

$f(\pi) = \sin(\cos(3\pi)) = \sin(\cos \pi) = \sin(-1) = -\sin 1$ (3)

From Eq. (2) and Eq. (3):

$f(0)f(\pi) = (\sin 1)(-\sin 1) = -\sin^2 1 < 0$ (3)

From Eq. (1) and Eq. (3):

$(\exists x_0 \in (0, \pi) : f(x_0) = 0) \Rightarrow x_0$ solves $\sin(\cos(3x)) = 0$.

b) If $a, b \in \mathbb{R}$ with $0 < a < b < \pi/2$, show that the equation

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0$$

has at least one solution $x_0 \in (a, b)$.

Solution

We note that for $x \in (a,b)$, we have $(x-a)(x-b) \neq 0$, and therefore:

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0 \Leftrightarrow (x-b)\sin x + (x-a)\cos x = 0$$

Define $f(x) = (x-b)\sin x + (x-a)\cos x$, $\forall x \in \mathbb{R}$

Then: f continuous on $[a,b]$ (1)

$f(a) = (a-b)\sin a + (a-a)\cos a = (a-b)\sin a$ (2)

$f(b) = (b-b)\sin b + (b-a)\cos b = (b-a)\cos b$ (3)

From Eq.(2) and Eq.(3):

$$f(a)f(b) = [(a-b)\sin a][(b-a)\cos b] = (a-b)(b-a)\sin a \cos b$$
$$= -(a-b)^2 \sin a \cos b.$$

We note that $a \neq b \Rightarrow (a-b)^2 > 0$

and $0 < a < \pi/2 \Rightarrow \sin a > 0$

and $0 < b < \pi/2 \Rightarrow \cos b > 0$.

It follows that

$f(a)f(b) = -(a-b)^2 \sin a \cos b < 0$ (4)

From Eq.(1) and Eq.(4), via Bolzano theorem,

$(\exists x_0 \in (a,b) : f(x_0) = 0) \Rightarrow x_0$ solves $\dfrac{\sin x}{x-a} + \dfrac{\cos x}{x-b} = 0$

THEORY QUESTIONS

(16) Let $f: A \to \mathbb{R}$ and $[a,b] \subseteq A$. Prove the Bolzano theorem:

$$\begin{cases} f \text{ continuous on } [a,b] \\ f(a)f(b) < 0 \end{cases} \Rightarrow \exists x_0 \in (a,b) : f(x_0) = 0$$

EXERCISES

(17) Let $a \in (0,+\infty)$. Show that the equation $x^n - a = 0$ has at least one solution on $(0,+\infty)$ using the Bolzano theorem.

↳ Note that if one also establishes uniqueness, then we have an alternate proof of the existence and uniqueness of $\sqrt[n]{a}$

(18) Let $a,b,c \in \mathbb{R}$ with $a<b<c$. Show that the equation
$(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$
has at least one solution in (a,b) and at least one additional solution in (b,c).

(19) Show that the equation $9x^3 - 6x^2 - 11x + 4 = 0$ has at least two solutions in $(0,2)$

(20) Show that the equation $x = \sin(x)$ has at least one solution on $(-\pi/2, \pi/2)$.

(21) Let $a, b \in \mathbb{R}$ with $a < b$. Show that the equation

$$\frac{x^2+1}{x-a} + \frac{x^4+1}{x-b} = 0$$

has at least one solution on (a, b).

(22) Let $a, b, c \in \mathbb{R}$ with $a < b < c$. Show that the equation

$$\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c} = 0$$

has at least one solution in (a, b) and at least one additional solution in (b, c).

(23) Let $f : [0,1] \to \mathbb{R}$ be a function such that

$$\begin{cases} f \text{ continuous on } [0,1] \\ \forall x \in [0,1] : 0 < f(x) < 1 \end{cases}$$

Show that the equation $f(x) = x$ has at least one solution on $(0,1)$

(24) Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f, g \text{ continuous on } [a,b] \\ f(a) = g(b) \land f(b) = g(a) \end{cases}$$

Show that $\exists c \in [a,b] : f(c) = g(c)$

(25) Let $f : A \to \mathbb{R}$ with $[0, 2\pi] \subseteq A$ such that $f(0) = f(2\pi)$.

Show that:

f continuous on $[0, 2\pi] \Rightarrow \exists x_0 \in [0, \pi] : f(x_0 + \pi) = f(x_0)$.

(26) Show that the equation $ax^3 + x^2 + x = 1$ with $a \neq -1$ has at least one solution in the interval $(-1, 1)$.
What happens when $a = -1$?

(27) Let $a, b \in \mathbb{R}$ with $a < b$. Show that the equation

$$\frac{x^2+1}{x-a} + \frac{x^6+1}{x-b} = 0$$

has a solution in (a, b).

▼ Continuity and function bounds

Similarly to bounded nets (a_n) on a directed set $(D,<)$ we define bounded functions as follows:

Def: Let $f: A \to \mathbb{R}$ and let $S \subseteq A$. We say that

f upper bounded on $S \Leftrightarrow \exists b \in \mathbb{R}: \forall x \in S: f(x) \leq b$

f lower bounded on $S \Leftrightarrow \exists b \in \mathbb{R}: \forall x \in S: f(x) \geq b$

f bounded on $S \Leftrightarrow \begin{cases} f \text{ upper bounded on } S \\ f \text{ lower bounded on } S \end{cases}$

and also show the following proposition:

Prop: Let $f: A \to \mathbb{R}$ and let $S \subseteq A$. Then, we have:

f bounded on $S \Leftrightarrow \exists p \in (0, +\infty): \forall x \in S: |f(x)| \leq p$

The following statements are also immediate consequences of the definition:

f bounded on $S \Leftrightarrow f(S)$ bounded

$\begin{cases} f \text{ bounded on } S_1 \\ f \text{ bounded on } S_2 \end{cases} \Rightarrow f$ bounded on $S_1 \cup S_2$

The contrapositive of the last statement reads:

f not bounded on $S_1 \cup S_2 \Rightarrow$
$\Rightarrow (f$ not bounded on $S_1 \vee f$ not bounded on $S_2)$

Our main results are needed later for differential calculus and are the following theorems:

1 → Bounded property of continuous functions on a closed interval

$\boxed{\begin{array}{l}\underline{\text{Thm}}\text{: let } f: A \to \mathbb{R} \text{ with } [a,b] \subseteq A. \text{ Then, we have:} \\ \quad f \text{ continuous on } [a,b] \Rightarrow f \text{ bounded on } [a,b]\end{array}}$

<u>Proof</u>

Assume that f continuous on $[a,b]$. To show a contradiction, assume that f not bounded on $[a,b]$. We will construct an interval sequence $([a_n,b_n])$ such that

$$\begin{cases} ([a_n,b_n]) \text{ nested} \\ \forall n \in \mathbb{N}^*: f \text{ not bounded on } [a_n,b_n] \end{cases}$$

as follows:

Choose $[a_1,b_1] = [a,b]$. By hypothesis, f not bounded on $[a_1,b_1]$.

Assume that $[a_k,b_k]$ has been constructed such that f not bounded on $[a_k,b_k]$.

Define $c_k = (a_k+b_k)/2$. Then, we have:

f not bounded on $[a_k,b_k] \Rightarrow$
$\Rightarrow f$ not bounded on $[a_k,c_k] \vee f$ not bounded on $[c_k,b_k]$

and we choose

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, c_k], & \text{if } f \text{ not bounded on } [a_k, c_k] \\ [c_k, b_k], & \text{otherwise} \end{cases}$$

By construction, it follows that f not bounded on $[a_{k+1}, b_{k+1}]$. The ~~w~~ resulting interval sequence $([a_n, b_n])$ is nested and satisfies

$$\forall n \in \mathbb{N}^* : f \text{ not bounded on } [a_n, b_n]$$

Consequently, for each $n \in \mathbb{N}^*$ we can choose $q_n \in [a_n, b_n]$ such that $f(q_n) \geq n$. Then, we have:

$$\begin{cases} \forall n \in \mathbb{N}^* : f(q_n) \geq n \\ \lim_{n \in \mathbb{N}^*} n = +\infty \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} f(q_n) = +\infty$$

$$\Rightarrow f(q_n) \text{ not convergent} \quad (1)$$

Since $([a_n, b_n])$ nested, choose $x_0 \in \mathbb{R}$ such that $\forall n \in \mathbb{N}^* : x_0 \in [a_n, b_n]$

Then, we have:

$$\begin{cases} \forall n \in \mathbb{N}^* : a_n \leq q_n \leq b_n \\ \lim_{n \in \mathbb{N}^*} a_n = \lim_{n \in \mathbb{N}^*} b_n = x_0 \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} q_n = x_0 \quad \text{[via squeeze thm]}$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} f(q_n) = f(x_0) \quad \text{[via } f \text{ continuous on } [a, b]\text{]}$$

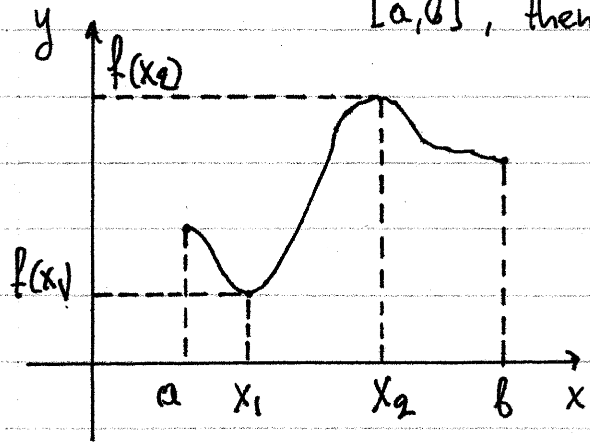
$$\Rightarrow f(q_n) \text{ convergent} \quad (2)$$

Eq. (1) contradicts Eq. (2). We have thus shown that: f bounded on $[a, b]$.

(9) Extremum Value Theorem

Thm: Let $f: A \to \mathbb{R}$ with $[a,b] \subseteq A$. Then, we have:
f continuous on $[a,b] \Rightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

► <u>interpretation</u>: If f is continuous on a closed interval $[a,b]$, then f has a global minimum at $x=x_1$ and a global maximum at $x=x_2$ when the domain of f is restricted to the closed interval $[a,b]$.



Proof

Assume that f continuous on $[a,b]$. Then, we have:

f continuous on $[a,b] \Rightarrow f$ bounded on $[a,b]$
$\Rightarrow f([a,b])$ bounded
$\Rightarrow f([a,b])$ upper bounded

thus we can define $M = \sup(f([a,b]))$.

To show that $M \in f([a,b])$, we assume that $M \notin f([a,b])$ in order to show a contradiction. Then, we have:

$M \notin f([a,b]) \Rightarrow \forall x \in [a,b] : (f(x) \neq M \land f(x) \leq M)$
$\Rightarrow \forall x \in [a,b] : f(x) < M$
$\Rightarrow \forall x \in [a,b] : M - f(x) > 0$

Define $\forall x \in [a,b] : g(x) = 1/(M - f(x))$. Then, we have:

f continuous on $[a,b] \Rightarrow g$ continuous on $[a,b]$

$\Rightarrow g$ bounded on $[a,b]$

$\Rightarrow \exists p \in (0,+\infty) : \forall x \in [a,b] : |g(x)| \leq p$

Choose $p \in (0,+\infty)$ such that $\forall x \in [a,b] : |g(x)| \leq p$.

Let $\underline{x \in [a,b]}$ be given. Then, we have:

$$|g(x)| \leq p \Rightarrow \left| \frac{1}{M - f(x)} \right| \leq p \Rightarrow \frac{1}{|M - f(x)|} \leq p$$

$$\Rightarrow \frac{1}{M - f(x)} \leq p \qquad [\text{via } M - f(x) > 0]$$

$$\Rightarrow 1 \leq p(M - f(x)) \qquad [\text{via } M - f(x) > 0]$$

$$\Rightarrow M - f(x) \geq 1/p \qquad [\text{via } p > 0]$$

$$\Rightarrow -f(x) \geq 1/p - M \Rightarrow \underline{f(x) \leq M - 1/p}$$

We have thus shown that:

$(\forall x \in [a,b] : f(x) \leq M - 1/p) \Rightarrow M - 1/p$ upper bound of $f([a,b])$

$\Rightarrow M - 1/p \geq \sup(f([a,b])) = M$

$\Rightarrow -1/p \geq 0 \Rightarrow p \leq 0$

which is a contradiction, since $p > 0$.

We have thus shown that

$M \in f([a,b]) \Rightarrow \exists x_2 \in [a,b] : f(x_2) = M = \sup(f([a,b]))$

$\Rightarrow \exists x_2 \in [a,b] : \forall x \in [a,b] : f(x) \leq f(x_2)$

With a similar argument, we can show that

$\exists x_1 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x)$

Combining the two statements, we conclude that

$\exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$. $\square$

THEORY QUESTIONS

(28) Let $f : A \to \mathbb{R}$ and let $S \subseteq A$. Write the definitions for the following statements.

a) f upper bounded on S

b) f lower bounded on S

c) f bounded on S

(29) Let $f : A \to \mathbb{R}$ with $[a,b] \subseteq A$. Prove that:

a) f continuous on $[a,b] \Rightarrow f$ bounded on $[a,b]$.

b) f continuous on $[a,b] \Longrightarrow$

$$\Longrightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$$

EXERCISES

(30) Let $f : A \to \mathbb{R}$ and let S, S_1, S_2 be subsets of A. Show the following statements

a) f bounded on $S \Leftrightarrow \exists p \in (0, +\infty) : \forall x \in S : |f(x)| \leq p$

b) f bounded on $S \Leftrightarrow f(S)$ bounded

c) $\left.\begin{array}{l} f \text{ bounded on } S_1 \\ f \text{ bounded on } S_2 \end{array}\right\} \Rightarrow f$ bounded on $S_1 \cup S_2$.

(31) Produce a counterexample to show that it is not possible to prove for all functions f the statement f continuous on $(a,b) \Rightarrow f$ bounded on (a,b).