SEQUENCES AND NETS

Sequences and nets - definitions

Def: A sequence (a_n) is a mapping $a: \mathbb{N} \to \mathbb{R}$ or $a: \mathbb{N}^* \to \mathbb{R}$.

- The net is a generalization of the sequence definition that allows us to define limits and study their properties in a general way which specializes to
  a) Limits of sequences
  b) Limits of functions
  c) Limits of partitions (used to define integrals).

Def: A directed set $(D, <)$ consists of a set D and a relation "$<$" such that

$$\begin{cases} \forall x \in D: \ x < x \\ \forall x, y, z \in D: \ ((x<y \land y<z) \Rightarrow x<z) \\ \forall x, y \in D: \exists z \in D: \ (x<z \land y<z) \end{cases}$$

Def: A net (a_n) is a mapping $a: D \to \mathbb{R}$ where $(D, <)$ is a directed set

- Note that $(\mathbb{N}, \leq)$ and $(\mathbb{N}^*, \leq)$ are directed sets, so a sequence is a special case of a net. Thus all definitions given on nets also apply to sequences.

Basic properties of nets

Def : Let (a_n) be a net on $(D,<)$. We say that

a) (a_n) increasing $\Leftrightarrow \forall p,q \in D : (p<q \Rightarrow a_p \leq a_q)$

b) (a_n) decreasing $\Leftrightarrow \forall p,q \in D : (p<q \Rightarrow a_p \geq a_q)$

c) (a_n) upper bounded $\Leftrightarrow \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n \leq b)$

d) (a_n) lower bounded $\Leftrightarrow \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n \geq b)$

e) (a_n) bounded $\Leftrightarrow \begin{cases} (a_n) \text{ lower bounded} \\ (a_n) \text{ upper bounded} \end{cases}$

f) (a_n) negatively upper bounded $\Leftrightarrow$

$\Leftrightarrow \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n \leq b < 0)$

g) (a_n) positively " lower bounded $\Leftrightarrow$

$\Leftrightarrow \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n \geq b > 0)$

- For sequences, some of these definitions simplify as follows:

(a_n) increasing $\Leftrightarrow \forall n \in \mathbb{N}^* : a_{n+1} \geq a_n$

(a_n) decreasing $\Leftrightarrow \forall n \in \mathbb{N}^* : a_{n+1} \leq a_n$

(a_n) upper bounded $\Leftrightarrow \exists b \in \mathbb{R} : \forall n \in \mathbb{N}^* : a_n \leq b$

(a_n) lower bounded $\Leftrightarrow \exists b \in \mathbb{R} : \forall n \in \mathbb{N}^* : a_n \geq b$

(a_n) negatively upper bounded $\Leftrightarrow \exists b \in \mathbb{R} : \forall n \in \mathbb{N}^* : a_n \leq b < 0$

(a_n) positively lower bounded $\Leftrightarrow \exists b \in \mathbb{R} : \forall n \in \mathbb{N}^* : a_n \geq b > 0$

The following proposition is a convenient criterion for showing that (a_n) is bounded:

Prop: Let (a_n) be a net on $(D,<)$. Then, we have:
(a_n) bounded $\iff \exists b\in(0,+\infty):\exists n_0\in D:\forall n\in D:(n>n_0\Rightarrow |a_n|\leqslant b)$

Proof

($\Rightarrow$): Assume that (a_n) bounded. Then, we have:

$$(a_n) \text{ bounded} \Rightarrow \begin{cases}(a_n) \text{ upper bounded}\\ (a_n) \text{ lower bounded}\end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases}\exists b\in\mathbb{R}:\exists n_0\in D:\forall n\in D:(n>n_0\Rightarrow a_n\leqslant b)\\ \exists b\in\mathbb{R}:\exists n_0\in D:\forall n\in D:(n>n_0\Rightarrow a_n\geqslant b)\end{cases}$$

Choose $b_1,b_2\in\mathbb{R}$ and $n_1,n_2\in D$ such that:

$$\begin{cases}\forall n\in D:(n>n_1\Rightarrow a_n\leqslant b_1)\\ \forall n\in D:(n>n_2\Rightarrow a_n\geqslant b_2)\end{cases}$$

Choose $n_0\in D$ such that $n_0>n_1\wedge n_0>n_2$. Let $b=\max\{|b_1|,|b_2|\}>0$.
We will show that $\forall n\in D:(n>n_0\Rightarrow |a_n|\leqslant b)$.
Let $n\in D$ be given and assume that $n>n_0$. Then, we have:

$$n>n_0\Rightarrow\begin{cases}n>n_1\\ n>n_2\end{cases}\Rightarrow\begin{cases}a_n\leqslant b_1\leqslant |b_1|\leqslant\max\{|b_1|,|b_2|\}=b\\ a_n\geqslant b_2\geqslant -|b_2|\geqslant-\max\{|b_1|,|b_2|\}=-b\end{cases}$$

$$\Rightarrow -b\leqslant a_n\leqslant b\Rightarrow |a_n|\leqslant b.$$

We have thus shown that

$$\exists b\in(0,+\infty):\exists n_0\in D:\forall n\in D:(n>n_0\Rightarrow |a_n|\leqslant b). \quad (1)$$

($\Leftarrow$): Assume that Eq. (1) is satisfied. Choose $b\in(0,+\infty)$ and $n_0\in D$ such that

$\forall n\in D: (n>n_0 \Rightarrow |a_n|\leq B)$

Let $n\in D$ be given and assume that $n>n_0$. Then, we have:

$n>n_0 \Rightarrow |a_n|\leq B \Rightarrow -B\leq a_n\leq B \Rightarrow a_n\leq B \wedge a_n\geq -B$.

We have thus shown that

$$\begin{cases} \forall n\in D: (n>n_0 \Rightarrow a_n\leq B) \\ \forall n\in D: (n>n_0 \Rightarrow a_n\geq -B) \end{cases} \Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ lower bounded} \end{cases}$$

$\Rightarrow (a_n)$ bounded. $\square$

Note that for sequences, the proposition simplifies to

$$\boxed{(a_n) \text{ bounded} \Leftrightarrow \exists B\in(0,+\infty): \forall n\in\mathbb{N}^k: (|a_n|\leq B)}$$

EXAMPLES

a) Show that (a_n) given by

$$\forall n \in \mathbb{N}^*: \quad a_n = \frac{3n^2 - 4n\cos(n^2+1) + 3}{4n^2+3n-2}$$

is bounded.

Solution

Let $n \in \mathbb{N}^*$ be given. Then we have:

$$|a_n| = \left| \frac{3n^2 - 4n\cos(n^2+1)+3}{4n^2+3n-2} \right| = \frac{|3n^2 - 4n\cos(n^2+1)+3|}{4n^2+3n-2}$$

$$\leqslant \frac{|3n^2| + |4n||\cos(n^2+1)| + 3}{4n^2+3n-2} \leqslant \frac{3n^2+4n+3}{4n^2+3n-2} \leqslant$$

$$\leqslant \frac{3n^2+4n^2+3n^2}{4n^2+3n-2} \leqslant \frac{10n^2}{4n^2} = \frac{10}{4} \Rightarrow |a_n| \leqslant 10/4.$$

We have thus shown that:

$(\forall n \in \mathbb{N}^*: |a_n| \leqslant 10/4) \Rightarrow (a_n)$ bounded. $\square$

b) Show that (a_n) given by

$$\forall n \in \mathbb{N}^*: \quad a_n = 3n^2 - n - \cos(n^2-n)$$

is not bounded.

Solution

To show a contradiction, assume that (a_n) is not bounded.

Then, we have:

(a_n) not bounded $\Rightarrow \exists p \in (0, +\infty): \forall n \in \mathbb{N}^*: |a_n| \leqslant p$.

Choose a $p \in (0,+\infty)$ such that $\forall n \in \mathbb{N}^*: |a_n| \leqslant p$.

Let $\underline{n \in \mathbb{N}^*}$ be given. Then, we have:

$$p \geq |a_n| = |3^{n^2-n} - \cos(n^2-n)| \geq ||3^{n^2-n}| - |\cos(n^2-n)||$$
$$\geq |3^{n^2-n}| - |\cos(n^2-n)| \geq 3^{n^2-n} - 1 = (1+2)^{n^2-n} - 1$$
$$\geq 1 + 2(n^2-n) - 1 = 2n^2 - n = n(2n-1) \geq 2n-1 \Rightarrow$$
$$\Rightarrow p \geq 2n-1 \Rightarrow 2n \leq p-1 \Rightarrow \underline{n \leq (p-1)/2}$$

We have thus shown that

$\forall n \in \mathbb{N}^*: n \leq (p-1)/2$

which is a contradiction with the Archimedes theorem

We conclude that (a_n) not bounded. $\square$

THEORY QUESTIONS

① Let (a_n) be a net on $(D,<)$. Write the definitions for the following statements:

a) (a_n) increasing
b) (a_n) decreasing
c) (a_n) upper bounded
d) (a_n) lower bounded
e) (a_n) bounded
f) (a_n) negatively upper bounded
g) (a_n) positively lower bounded.

② Let (a_n) be a net on $(D,<)$. Show that:

(a_n) bounded $\Leftrightarrow \exists B \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow |a_n| \leq B)$

EXERCISES

③ Show that the following sequences are bounded

a) $a_n = \dfrac{n}{n^2+8}$

b) $a_n = \dfrac{n\cos n + \sin n}{n^2}$

c) $a_n = \dfrac{1}{n} \sin\left(\dfrac{\pi n}{10}\right)$

d) $a_n = \dfrac{5\sin(3n)}{4n}$

e) $a_n = \dfrac{4n+5}{5^n}$

f) $a_n = \dfrac{3n^2-1}{\sqrt{4^n}}$

④ Show that the following sequences are not bounded

a) $a_n = \dfrac{4n^2+1}{5n}$

b) $a_n = -4n^2+3n+1$

c) $a_n = \dfrac{2n^2+5}{3n+n\sin n}$

d) $a_n = (-2)^{n+1} + (-2)^n + 2.$

(5) Let (a_n) and (b_n) be two sequences. Show that:

a) $\begin{cases} (a_n) \text{ bounded} \\ \forall n \in \mathbb{N}^*: b_n = a_n/n \end{cases} \Rightarrow (b_n) \text{ bounded}$

b) $\begin{cases} (a_n), (b_n) \text{ increasing} \\ \forall n \in \mathbb{N}^*: c_n = a_n + b_n \end{cases} \Rightarrow (c_n) \text{ increasing}$

c) $\begin{cases} (a_n), (b_n) \text{ bounded} \\ \forall n \in \mathbb{N}^*: c_n = a_n (a_n + b_n)^2 \end{cases} \Rightarrow (c_n) \text{ bounded}$

▼ Definition of limit of nets and sequences

Def: Let (a_n) be a net on $(D, \leq)$ and let $l \in \mathbb{R}$.
We say that

$$\lim a_n = l \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow |a_n - l| < \varepsilon)$$

$$\lim a_n = +\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n > 1/\varepsilon)$$

$$\lim a_n = -\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n < -1/\varepsilon)$$

(a_n) convergent $\Leftrightarrow \exists l \in \mathbb{R} : \lim a_n = l$

(a_n) divergent $\Leftrightarrow \begin{cases} (a_n) \text{ not convergent} \\ \lim a_n \neq +\infty \wedge \lim a_n \neq -\infty \end{cases}$

▸ Note that when (a_n) is a sequence, we introduce the notation
$[n_0] = \{x \in \mathbb{N} \mid 1 \leq x \leq n_0\} = \{1, 2, \ldots, n_0\}$
and note that the limit definitions simplify as follows:

$$\lim_{n \in \mathbb{N}^*} a_n = l \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - l| < \varepsilon$$

$$\lim_{n \in \mathbb{N}^*} a_n = +\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n > 1/\varepsilon$$

$$\lim_{n \in \mathbb{N}^*} a_n = -\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n < -1/\varepsilon$$

Zero sequences and nets

Let (a_n) be a net on $(D,<)$. We recall the definition
$\lim a_n = 0 \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$
The following is an immediate consequence of this definition:
$\lim a_n = 0 \Leftrightarrow \lim(-a_n) = 0 \Leftrightarrow \lim |a_n| = 0$

Properties of zero nets

Let $(a_n), (b_n)$ be nets on $(D,<)$. We show the following properties:

① $\boxed{\lim a_n = 0 \Rightarrow (a_n) \text{ bounded}}$

Proof

Assume that $\lim a_n = 0$. Then, we have:

$$\begin{aligned}\lim a_n = 0 &\Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon) \\ &\Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < 1) \quad [\text{via } \varepsilon = 1] \\ &\Rightarrow (a_n) \text{ bounded}\end{aligned}$$

$\square$

② $\boxed{\begin{cases} \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| \leq |b_n|) \\ \lim b_n = 0 \end{cases} \Rightarrow \lim a_n = 0}$

Proof

From hypothesis, choose $n_1 \in D$ such that
$\forall n \in D : (n > n_1 \Rightarrow |a_n| \leq |b_n|)$.

We note that

$\lim b_n = 0 \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |b_n| < \varepsilon)$

Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow |b_n| < \varepsilon)$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $\underline{n \in D}$ be given and assume that $\underline{n > n_0}$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |a_n| \leq |b_n| \\ |b_n| < \varepsilon \end{cases} \Rightarrow \underline{|a_n| < \varepsilon}$$

We have thus shown that:

$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$

$\Rightarrow \lim a_n = 0$ $\square$

③ $\boxed{\begin{cases} \lim a_n = 0 \\ (b_n) \text{ bounded} \end{cases} \Rightarrow \lim (a_n b_n) = 0}$

<u>Proof</u>

We have:

$(b_n) \text{ bounded} \Rightarrow \exists p \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |b_n| \leq p)$

Choose $p \in (0,+\infty)$ and $n_1 \in D$ such that

$\forall n \in D : (n > n_1 \Rightarrow |b_n| \leq p)$

We also have:

$\lim a_n = 0 \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$

Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Since $\varepsilon / p > 0$, choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow |a_n| < \varepsilon / p)$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $\underline{n \in D}$ be given and assume that $\underline{n > n_0}$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |b_n| \leq p \\ |a_n| < \varepsilon/p \end{cases} \Rightarrow$$

$$\Rightarrow |a_n b_n| = |a_n||b_n| \leq |a_n| p < (\varepsilon/p) p = \varepsilon$$

$$\Rightarrow \underline{|a_n b_n| < \varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n b_n| < \varepsilon)$$

$$\Rightarrow \lim (a_n b_n) = 0$$ $\square$

▸ Immediate consequence of (3) is the statement

$$\boxed{\lim a_n = 0 \Rightarrow \forall \lambda \in \mathbb{R} : \lim (\lambda a_n) = 0}$$

(4) $$\boxed{\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \begin{cases} \lim (a_n + b_n) = 0 \\ \lim (a_n b_n) = 0 \end{cases}}$$

<u>Proof</u>

a) We have:

$$\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \begin{cases} \lim a_n = 0 \\ (b_n) \text{ bounded} \end{cases} \Rightarrow \lim (a_n b_n) = 0$$

b) We have:

$$\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \begin{cases} \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon) \\ \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |b_n| < \varepsilon) \end{cases}$$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $n_1 \in D$ and $n_2 \in D$ such that

$$\begin{cases} \forall n \in D : (n > n_1 \Rightarrow |a_n| < \varepsilon/2) \\ \forall n \in D : (n > n_2 \Rightarrow |b_n| < \varepsilon/2) \end{cases}$$

Choose $n_0 \in D$ such that $n_0 \geq n_1$ and $n_0 \geq n_2$. Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |a_n| < \varepsilon/2 \\ |b_n| < \varepsilon/2 \end{cases} \Rightarrow$$

$$\Rightarrow |a_n + b_n| \leq |a_n| + |b_n| < \varepsilon/2 + |b_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\Rightarrow |a_n + b_n| < \varepsilon$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow |a_n + b_n| < \varepsilon)$$

$$\Rightarrow \lim (a_n + b_n) = 0 \qquad \square$$

(5) $$\boxed{\begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n \geq 0) \end{cases} \Rightarrow \forall k \in \mathbb{N}^* : \lim \sqrt[k]{a_n} = 0}$$

Proof: Homework

EXAMPLE

a) Use the limit definition to show that

$$\lim_{n\in\mathbb{N}^*} \frac{\sin(n)+\cos(n)}{n^2+1} = 0$$

Solution

Define $\forall n\in\mathbb{N}^*: a_n = \dfrac{\sin(n)+\cos(n)}{n^2+1}$

Let $\varepsilon\in(0,+\infty)$ be given. We have:

$$|a_n| = \left|\frac{\sin(n)+\cos(n)}{n^2+1}\right| = \frac{|\sin(n)+\cos(n)|}{n^2+1} \leqslant$$

$$\leqslant \frac{|\sin(n)|+|\cos(n)|}{n^2+1} \leqslant \frac{1+1}{n^2+1} = \frac{2}{n^2+1}$$

$$\leqslant \frac{2}{n^2} < \frac{1}{\varepsilon} \Leftrightarrow n^2 > \varepsilon \Leftrightarrow n > \sqrt{\varepsilon}. \quad (1)$$

Choose $n_0\in\mathbb{N}^*$ such that $n_0 > \sqrt{\varepsilon}$, via the Archimedes theorem. Let $n\in\mathbb{N}^* - [n_0]$ be given. Then, we have:

$n > n_0 \Rightarrow n > \sqrt{\varepsilon} \Rightarrow |a_n| < \varepsilon$ [via Eq. (1)]

We have thus shown that

$\forall\varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^* - [n_0]: |a_n| < \varepsilon$

$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = 0.$ $\square$

THEORY QUESTIONS

(6) Prove the following properties, with $(a_n), (b_n)$ nets on $(D, <)$.

a) $\lim a_n = 0 \Rightarrow (a_n)$ bounded

b) $\begin{cases} \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| \leq |b_n|) \\ \lim b_n = 0 \end{cases} \Rightarrow \lim a_n = 0$

c) $\begin{cases} \lim a_n = 0 \\ (b_n) \text{ bounded} \end{cases} \Rightarrow \lim (a_n b_n) = 0$

d) $\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \lim (a_n + b_n) = 0$

e) $\begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \geq 0) \end{cases} \Rightarrow \forall k \in \mathbb{N}^* : \lim \sqrt[k]{a_n} = 0$

EXERCISES

(7) Use the limit definition to show that $\lim a_n = 0$ for the following sequences:

a) $a_n = \dfrac{(-1)^n}{(n+1)^2}$

b) $a_n = \dfrac{1+\sqrt{n}}{n^3}$

c) $a_n = \dfrac{5}{3n^2 - 1}$

d) $a_n = \dfrac{\sin(n) - \cos(n)}{n+4}$

e) $a_n = \dfrac{n-1}{n^2+1}$

f) $a_n = \dfrac{(-1)^n}{3^n}$

g) $a_n = \dfrac{n^2+5n-1}{n^3+n+3}$

h) $a_n = \dfrac{\sin(2n) + 4\cos(3n)}{n+3}$

(8) Let $(a_n), (b_n)$ be sequences such that

$$\begin{cases} \forall n \in \mathbb{N}^*: (a_n > 0 \wedge b_n > 0) \\ \lim\limits_{n\in\mathbb{N}^*} a_n = 0 \wedge \lim\limits_{n\in\mathbb{N}^*} b_n = 0 \end{cases}$$

Show that $\lim\limits_{n\in\mathbb{N}^*} \dfrac{a_n^2+b_n^2}{a_n+b_n} = 0$

using the limit properties.
(Hint: Use the Cauchy identity $x^2+y^2 = (x+y)^2 - 2xy$.)

(9) Given the sequence

$$\forall n \in \mathbb{N}^*: a_n = \frac{5n}{6n+7}$$

show via the limit definition that $\lim\limits_{n\in\mathbb{N}^*} a_n \neq 0$.

(10) Let $(a_n), (b_n)$ be sequences such that

$$\begin{cases} \forall n \in \mathbb{N}^*: (a_n > 0 \wedge b_n > 0) \\ \lim\limits_{n\in\mathbb{N}^*} a_n = 0 \wedge \lim\limits_{n\in\mathbb{N}^*} b_n = 0 \end{cases}$$

Show that $\lim\limits_{n\in\mathbb{N}^*} \dfrac{a_n^3+b_n^3}{a_n+b_n} = 0$

using the limit properties.
(Hint: Use the Cauchy identity $x^3+y^3 = (x+y)^3 - 3xy(x+y)$)

Basic zero sequences

The limits of the following sequences can be used as theorems for other exercises.

(1) $\boxed{\forall p \in (0,+\infty) : \lim\limits_{n\in\mathbb{N}^*} \frac{1}{n^p} = 0}$

Proof

Define $\forall n\in\mathbb{N}^* : a_n = 1/n^p$. Let $\varepsilon \in (0,+\infty)$ be given. We note that

$$|a_n| < \varepsilon \Leftrightarrow |1/n^p| < \varepsilon \Leftrightarrow 1/n^p < \varepsilon \Leftrightarrow n^p > 1/\varepsilon \Leftrightarrow$$
$$\Leftrightarrow n > (1/\varepsilon)^{1/p}$$

Choose $n_0 \in \mathbb{N}^*$ such that $n_0 > (1/\varepsilon)^{1/p}$, via the Archimedes theorem. Let $n\in\mathbb{N}^* - [n_0]$ be given. Then, we have:

$$n > n_0 \Rightarrow n > (1/\varepsilon)^{1/p} \Rightarrow |a_n| < \varepsilon$$

We have thus shown that

$$\forall \varepsilon\in(0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n\in\mathbb{N}^* - [n_0] : |a_n| < \varepsilon$$
$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \frac{1}{n^p} = 0 \qquad \square$$

(2) $\boxed{\begin{cases} \forall n\in\mathbb{N}^* : a_n = p^n \\ |p| < 1 \end{cases} \Rightarrow \lim\limits_{n\in\mathbb{N}^*} a_n = 0}$

Proof

We distinguish between the following cases.

Case 1: Assume that $p=0$. Then, we have:

$$(\forall n\in\mathbb{N}^* : a_n = 0^n = 0) \Rightarrow \lim_{n\in\mathbb{N}^*} a_n = 0.$$

Case 2 : Assume that $p \neq 0$ and $|p| < 1$. Then $1/|p| > 1$, and we can choose $b \in (0, +\infty)$ such that $1/|p| = 1 + b$.
Let $n \in \mathbb{N}^*$ be given. Then, we have:

$1/|p|^n = (1+b)^n \geq 1 + nb > nb > 0 \Rightarrow 1/|p|^n > nb > 0 \Rightarrow$
$\Rightarrow |a_n| = |p^n| = |p|^n < 1/(nb) = |1/(nb)| \Rightarrow |a_n| < |1/(nb)|$

and conclude that

$$\forall n \in \mathbb{N}^* : |a_n| < |1/(nb)|. \qquad (1)$$

We also have:

$$\lim_{n \in \mathbb{N}^*} \frac{1}{n} = 0 \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{1}{nb} = 0 \qquad (2)$$

From Eq. (1) and Eq. (2): $\lim_{n \in \mathbb{N}^*} a_n = 0$ $\square$

THEORY QUESTIONS

(11) Show that

a) $\forall p \in (0, +\infty) : \lim_{n \in \mathbb{N}^*} \frac{1}{n^p} = 0$

b) $\begin{cases} \forall n \in \mathbb{N}^* : a_n = p^n \\ |p| < 1 \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} a_n = 0$

EXERCISES

(12) Use the limit properties to show that $\lim_{n \in \mathbb{N}^*} a_n = 0$ for the following sequences:

a) $a_n = \frac{4n^2 + 3}{n^3}$

b) $a_n = \frac{5 + \cos(n)}{3n^4}$

c) $a_n = \frac{n}{(-3)^n (n^2 + 2)}$

d) $a_n = \frac{n!}{n^n}$

e) $a_n = \frac{1^2 + 2^2 + \cdots + n^2}{n^4 + 5n + 2}$

f) $a_n = \frac{1^3 + 2^3 + \cdots + n^3}{3n^5 + 2n}$

g) $a_n = \sum_{a=1}^{n} \frac{\sin(a)}{a^2 + 1}$

Convergent nets and sequences

Let (a_n) be a net on $(D,\prec)$ and recall the definitions

$$\boxed{\begin{array}{l}\lim a_n = \ell \Leftrightarrow \forall \varepsilon \in (0,+\infty): \exists n_0 \in D: \forall n \in D: (n \succ n_0 \Rightarrow |a_n - \ell| < \varepsilon) \\ (a_n) \text{ convergent} \Leftrightarrow \exists \ell \in \mathbb{R}: \lim a_n = \ell\end{array}}$$

When (a_n) is a sequence, the limit definition simplifies to

$$\boxed{\lim_{n\in\mathbb{N}^*} a_n = \ell \Leftrightarrow \forall \varepsilon \in (0,+\infty): \exists n_0 \in \mathbb{N}^*: \forall n \in \mathbb{N}^* - [n_0]: |a_n - \ell| < \varepsilon}$$

Uniqueness of convergent limit

Thm: Let $(a_n), (b_n)$ be nets on $(D,\prec)$ and let $\ell_1, \ell_2 \in \mathbb{R}$. Then, we have

$$\begin{cases} \lim a_n = \ell_1 \\ \lim b_n = \ell_2 \end{cases} \Rightarrow \ell_1 = \ell_2$$

Proof

To show a contradiction, assume that $\ell_1 \neq \ell_2$. Since,

$$\begin{cases} \lim a_n = \ell_1 \\ \lim b_n = \ell_2 \end{cases} \Rightarrow \begin{cases} \forall \varepsilon \in (0,+\infty): \exists n_0 \in D: \forall n \in D: (n \succ n_0 \Rightarrow |a_n - \ell_1| < \varepsilon) \\ \forall \varepsilon \in (0,+\infty): \exists n_0 \in D: \forall n \in D: (n \succ n_0 \Rightarrow |a_n - \ell_2| < \varepsilon) \end{cases}$$

For $\varepsilon = |\ell_1 - \ell_2|/2$, choose $n_1, n_2 \in D$ such that

$$\begin{cases} \forall n \in D: (n \succ n_1 \Rightarrow |a_n - \ell_1| < |\ell_1 - \ell_2|/2) \\ \forall n \in D: (n \succ n_2 \Rightarrow |a_n - \ell_2| < |\ell_1 - \ell_2|/2) \end{cases}$$

Choose $n \in D$ such that $n > n_1$ and $n > n_2$. It follows that

$$\begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |a_n - \ell_1| < |\ell_1 - \ell_2|/2 \\ |a_n - \ell_2| < |\ell_1 - \ell_2|/2 \end{cases} \Rightarrow$$

$$\Rightarrow |\ell_1 - \ell_2| = |(a_n - \ell_2) - (a_n - \ell_1)| \leq |a_n - \ell_2| + |a_n - \ell_1|$$

$$< |\ell_1 - \ell_2|/2 + |\ell_1 - \ell_2|/2 = |\ell_1 - \ell_2| \Rightarrow$$

$$\Rightarrow |\ell_1 - \ell_2| < |\ell_1 - \ell_2|$$

which is a contradiction. We conclude that $\ell_1 = \ell_2$ $\square$

General properties

Let (a_n) be a net on $(D, <)$. We have the following general properties:

① $\boxed{(a_n) \text{ convergent} \Rightarrow (a_n) \text{ bounded}}$

Proof

Choose $\ell \in \mathbb{R}$ such that $\lim a_n = \ell$. Then, we have:

$$\lim a_n = \ell \Rightarrow \lim (a_n - \ell) = 0 \Rightarrow (a_n - \ell) \text{ bounded}$$

$$\Rightarrow \exists b \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n - \ell| \leq b)$$

Choose $b \in (0, +\infty)$ and $n_0 \in D$ such that:

$$\forall n \in D : (n > n_0 \Rightarrow |a_n - \ell| \leq b)$$

Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow |a_n - \ell| \leq b \Rightarrow -b \leq a_n - \ell \leq b \Rightarrow$$

$$\Rightarrow -b + \ell \leq a_n \leq b + \ell$$

We have thus shown that:

$\forall n \in D: (n > n_0 \Rightarrow -b+l \leq a_n \leq b+l)$

$\Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow (a_n) \text{ bounded}$ $\square$

(2) $\boxed{\lim a_n = l \neq 0 \Rightarrow \exists n_0 \in D: \forall n \in D: (n > n_0 \Rightarrow a_n, l \text{ equisigned})}$

Proof

We have:

$\lim a_n = l \Rightarrow \lim (a_n - l) = 0 \Rightarrow$

$\Rightarrow \exists n_0 \in D: \forall n \in D: (n > n_0 \Rightarrow |a_n - l| < |l|/2)$

using $\varepsilon = |l|/2$. Choose $n_0 \in D$ such that

$\forall n \in D: (n > n_0 \Rightarrow |a_n - l| < |l|/2)$

Let $n \in D$ be given and assume that $n > n_0$. It follows that

$n > n_0 \Rightarrow |a_n - l| < |l|/2 \Rightarrow -|l|/2 < a_n - l < |l|/2 \Rightarrow$

$\Rightarrow l - |l|/2 < a_n < l + |l|/2$.

We distinguish between the following cases.

Case 1: Assume that $l > 0$. Then, we have:

$a_n > l - |l|/2 = l - l/2 = l/2 > 0 \Rightarrow a_n > l/2 > 0 \Rightarrow$

$\Rightarrow a_n, l$ equisigned.

Case 2: Assume that $l < 0$. Then, we have:

$a_n < l + |l|/2 = l - l/2 = l/2 < 0 \Rightarrow a_n < l/2 < 0 \Rightarrow$

$\Rightarrow a_n, l$ equisigned

We have thus shown that

$\exists n_0 \in D: \forall n \in D: (n > n_0 \Rightarrow a_n, l \text{ equisigned})$ $\square$

A corollary of property 2 is the following statement:

$$\boxed{\lim a_n = \ell \neq 0 \Rightarrow \exists n_0 \in D : \forall n \in D : \left(n > n_0 \Rightarrow \frac{|\ell|}{2} < |a_n| < \frac{3|\ell|}{2}\right)}$$

③ $$\boxed{\begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} \Rightarrow \lim a_n \geq 0}$$

Proof

To show a contradiction, assume that $\lim a_n < 0$. Then, via property 2 we have:

$\lim a_n < 0 \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n < 0)$

Choose $n_1 \in D$ such that

$\forall n \in D : (n > n_1 \Rightarrow a_n < 0)$

From the hypothesis, choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow a_n > 0)$

Choose $n \in D$ such that $n > n_1$ and $n > n_2$. Then, we have:

$$\begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n < 0 \\ a_n > 0 \end{cases}$$

which is a contradiction. We conclude that $\lim a_n \geq 0$. □

EXAMPLE

Use the limit definition to show that $\lim\limits_{n\in\mathbb{N}^*} \dfrac{n^2+3n-1}{2n^2+n+1} = \dfrac{1}{2}$

Solution

Define $\forall n\in\mathbb{N}^*: a_n = \dfrac{n^2+3n-1}{2n^2+n+1}$. Let $\varepsilon\in(0,+\infty)$ be given.

Then, we have:

$$|a_n - 1/2| = \left|\frac{n^2+3n-1}{2n^2+n+1} - \frac{1}{2}\right| = \left|\frac{2(n^2+3n-1)-(2n^2+n+1)}{2(2n^2+n+1)}\right|$$

$$= \frac{|2n^2+6n-2-2n^2-n-1|}{2(2n^2+n+1)} = \frac{|(2-2)n^2+(6-1)n+(-2-1)|}{2(2n^2+n+1)}$$

$$= \frac{|5n-3|}{2(2n^2+n+1)} \leq \frac{|5n|+|3|}{2(2n^2+n+1)} = \frac{5n+3}{2(2n^2+n+1)}$$

$$< \frac{5n+3}{4n^2} \leq \frac{5n+3n}{4n^2} = \frac{8n}{4n^2} = \frac{2}{n} < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow n/2 > 1/\varepsilon \Leftrightarrow n > 2/\varepsilon$$

Via the Archimedes theorem, choose $n_0\in\mathbb{N}^*$ such that $n_0 > 2/\varepsilon$. Let $n\in\mathbb{N}^*-[n_0]$ be given. Then, we have:

$n > n_0 \Rightarrow n > 2/\varepsilon \Rightarrow |a_n - 1/2| < \varepsilon$

We have thus shown that

$\forall\varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: |a_n-1/2|<\varepsilon$

$\Rightarrow \lim\limits_{n\in\mathbb{N}^*} a_n = 1/2$ $\square$

THEORY QUESTIONS

(13) Show that if $(a_n), (b_n)$ are nets on $(D, <)$:

a) $\begin{cases} \lim a_n = l_1 \\ \lim a_n = l_2 \end{cases} \Rightarrow l_1 = l_2$

b) (a_n) convergent $\Rightarrow (a_n)$ bounded.

c) $\lim a_n = l \neq 0 \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n, l$ equisigned$)$

d) $\begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} \Rightarrow \lim a_n \geq 0$

EXERCISES

(14) Use the limit definition to show that

a) $\lim\limits_{n \in \mathbb{N}^*} \dfrac{2n}{3n+1} = \dfrac{2}{3}$

b) $\lim\limits_{n \in \mathbb{N}^*} \dfrac{n^2 - n}{(n+1)^2} = 1$

c) $\lim\limits_{n \in \mathbb{N}^*} \dfrac{2n^3}{n^3 - 1} = 2$

d) $\lim\limits_{n \in \mathbb{N}^*} \left(3 + \dfrac{1}{n}\right)^2 = 9$

e) $\lim\limits_{n \in \mathbb{N}^*} \left[\left(\dfrac{3}{4}\right)^n - 1\right] = -1$

f) $\lim\limits_{n \in \mathbb{N}^*} \left(1 + \dfrac{1}{n}\right)^5 = 1$

Limits and operations

(1) $$\boxed{(a_n), (b_n) \text{ convergent} \Rightarrow \begin{cases} \lim (a_n+b_n) = \lim a_n + \lim b_n \\ \lim (a_n b_n) = \lim a_n \lim b_n \end{cases}}$$

Proof

Since $(a_n), (b_n)$ convergent, we define $a = \lim a_n$ and $b = \lim b_n$.

a) It follows that

$$\begin{cases} \lim (a_n - a) = 0 \\ \lim (b_n - b) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \lim [(a_n+b_n) - (a+b)] = \lim [(a_n-a) + (b_n-b)] = 0$$

$$\Rightarrow \lim (a_n+b_n) = a+b = \lim a_n + \lim b_n.$$

b) Since

$$\forall n \in \mathbb{N}^*: a_n b_n - ab = a_n b_n - a b_n + a b_n - ab =$$
$$= (a_n - a) b_n + a(b_n - b) \qquad (1)$$

and

$$\lim (b_n - b) = 0 \Rightarrow \lim [a(b_n - b)] = 0 \qquad (2)$$

and

$$\begin{cases} \lim (a_n - a) = 0 \\ b_n \text{ convergent} \end{cases} \Rightarrow \begin{cases} \lim (a_n - a) = 0 \\ b_n \text{ bounded} \end{cases} \Rightarrow$$

$$\Rightarrow \lim [(a_n - a) b_n] = 0 \qquad (3)$$

it follows from Eq.(1), Eq.(2), Eq.(3) that

$$\lim (a_n b_n - ab) = 0 \Rightarrow \lim (a_n b_n) = ab = \lim a_n \lim b_n \quad \square$$

$$\text{②} \boxed{\begin{cases} (a_n) \text{ convergent} \\ \lim a_n \neq 0 \end{cases} \Rightarrow \lim \frac{1}{a_n} = \frac{1}{\lim a_n}}$$

Proof

Define $a = \lim a_n$. We note that:

$\lim a_n \neq 0 \Rightarrow \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : (n > n_0 \Rightarrow |a|/2 < |a_n| < 3|a|/2)$

Let $n \in \mathbb{N}$ be given and assume that $n > n_0$. Note that since $a \neq 0$, we have:

$|a_n| > |a|/2 > 0 \Rightarrow 1/|a_n| < 2/|a|$

and therefore

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{a - a_n}{a_n a} \right| = \frac{|a - a_n|}{|a_n||a|} < \frac{2|a - a_n|}{|a|^2}$$

We have thus shown that

$$\forall n \in \mathbb{N} : (n > n_0 \Rightarrow \left| \frac{1}{a_n} - \frac{1}{a} \right| < \frac{2|a - a_n|}{|a|^2} \qquad (1)$$

We also have:

$$\lim a_n = a \Rightarrow \lim (a_n - a) = 0 \Rightarrow \lim |a_n - a| = 0 \Rightarrow$$
$$\Rightarrow \lim \frac{2|a - a_n|}{|a|^2} = 0 \qquad (2)$$

From Eq. (1) and Eq. (2):

$$\lim \left(\frac{1}{a_n} - \frac{1}{a} \right) = 0 \Rightarrow \lim \frac{1}{a_n} = \frac{1}{a} = \frac{1}{\lim a_n} \qquad \square$$

→ The following result is an immediate consequence of the limit operation properties

Prop: Let $(a_n), (b_n)$ be nets on $(D, <)$. Then:

$$\begin{cases} (a_n), (b_n) \text{ converges} \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n) \end{cases} \Rightarrow \lim a_n \leq \lim b_n$$

<u>Proof</u>

Choose $n_0 \in D$ such that

$\forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n)$

Then, we have:

$$\begin{cases} (a_n), (b_n) \text{ convergent} \\ \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n) \end{cases} \Rightarrow \begin{cases} (a_n - b_n) \text{ convergent} \\ \forall n \in D : (n > n_0 \Rightarrow a_n - b_n \leq 0) \end{cases}$$

$$\Rightarrow \lim (a_n - b_n) \leq 0 \Rightarrow \lim a_n - \lim b_n \leq 0$$

$$\Rightarrow \lim a_n \leq \lim b_n \qquad \square.$$

EXAMPLES

Use the limit properties to evaluate the following limits:

a) $a_n = \dfrac{n^2+3n-1}{3n^2+5n+2} \longleftarrow \lim\limits_{n\in\mathbb{N}^*} a_n$

Solution

$$a_n = \frac{n^2+3n-1}{3n^2+5n+2} = \frac{n^2(1+3n^{-1}-n^{-2})}{n^2(3+5n^{-1}+2n^{-2})} =$$

$$= \frac{1+3n^{-1}-n^{-2}}{3+5n^{-1}+2n^{-2}}, \quad \forall n\in\mathbb{N}^* \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \frac{1+3n^{-1}-n^{-2}}{3+5n^{-1}+2n^{-2}} =$$

$$= \frac{1+3\cdot 0-0}{3+5\cdot 0+2\cdot 0} = \frac{1}{3}$$

b) $a_n = \dfrac{2^{n+1}+3^{2n}}{9^n+5^{n+1}}, \quad \forall n\in\mathbb{N}^*$

Solution

$$a_n = \frac{2^{n+1}+3^{2n}}{9^n+5^{n+1}} = \frac{2\cdot 2^n+9^n}{9^n+5\cdot 5^n} = \frac{9^n[2(2/9)^n+1]}{9^n[1+5(5/9)^n]}$$

$$= \frac{2(2/9)^n+1}{1+5(5/9)^n}, \quad \forall n\in\mathbb{N}^* \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \frac{2(2/9)^n+1}{1+5(5/9)^n} = \frac{2\cdot 0+1}{1+5\cdot 0} =$$

$$= \frac{0+1}{1+0} = 1$$

THEORY QUESTIONS

(15) Let $(a_n), (b_n)$ be nets on $(D, <)$. Show that

a) $(a_n), (b_n)$ convergent $\Rightarrow \lim(a_n + b_n) = \lim a_n + \lim b_n$

b) $(a_n), (b_n)$ convergent $\Rightarrow \lim(a_n b_n) = \lim a_n \lim b_n$

c) $\begin{cases} (a_n) \text{ convergent} \\ \lim a_n \neq 0 \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = \dfrac{1}{\lim a_n}$

EXERCISES

(16) Use the limit properties to evaluate the limit $\lim\limits_{n \in \mathbb{N}^*} a_n$ for the following sequences

a) $a_n = \left(1 + \dfrac{4}{n^2} - \dfrac{5}{n^3}\right)^9$

b) $a_n = \dfrac{2n^3 + 4n^2 - 2}{3n^3 + 6n - 5}$

c) $a_n = \dfrac{n^2 + 2n + 3}{3n^3 + n^2 - 1}$

d) $a_n = \dfrac{2^n + 5^n}{4^n + 7^n}$

e) $a_n = \dfrac{5 + 3^n + 5^{n+1}}{7 + 2^n + 5^{n+4}}$

f) $a_n = \dfrac{2 \cdot 5^n - 3^{2n}}{6 + 4^{2n+1}}$

Squeeze theorem and n-root limits

Thm : (Squeeze theorem)
Let $(a_n), (b_n), (c_n)$ be nets on $(D, <)$. Then, we have:
$$\left\{\begin{array}{l} \exists n_0 \in D: \forall n \in D: (n > n_0 \Rightarrow a_n \leq b_n \leq c_n) \\ \lim a_n = \lim c_n = \ell \end{array}\right. \Rightarrow \lim b_n = \ell$$

Proof

Choose $n_0 \in D$ such that: $\forall n \in D: (n > n_0 \Rightarrow a_n \leq b_n \leq c_n)$

Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$$a_n \leq b_n \leq c_n \Rightarrow 0 \leq b_n - a_n \leq c_n - a_n \Rightarrow$$
$$\Rightarrow |b_n - a_n| \leq c_n - a_n \leq |c_n - a_n|$$

We have thus shown that

$$\forall n \in D: (n > n_0 \Rightarrow |b_n - a_n| \leq |c_n - a_n| \quad (1)$$

We also have:

$$\lim (c_n - a_n) = \lim c_n - \lim a_n = \ell - \ell = 0 \quad (2)$$

From Eq.(1) and Eq.(2), it follows that

$$\lim (b_n - a_n) = 0 \Rightarrow$$
$$\Rightarrow \lim b_n = \lim [(b_n - a_n) + a_n] = \lim (b_n - a_n) + \lim a_n$$
$$= 0 + \ell = \ell$$

$\square$

Results on n-root limits

① $\boxed{\lim_{n \in \mathbb{N}^*} \sqrt[n]{n} = 1}$

Proof

Since $(\forall n\in\mathbb{N}^*: n\geq 1) \Rightarrow (\forall n\in\mathbb{N}^*: \sqrt[n]{n}\geq 1)$, we define a sequence (p_n) with $\forall n\in\mathbb{N}^*: p_n\geq 0$ such that
$\forall n\in\mathbb{N}^*: \sqrt[n]{n} = (1+p_n)^2$
Let $n\in\mathbb{N}^*$ be given. Then, we have:

$$\sqrt[n]{n} = (1+p_n)^2 \Rightarrow n=(1+p_n)^{2n} \Rightarrow$$
$$\Rightarrow \sqrt{n} = (1+p_n)^n \geq 1+np_n > np_n \Rightarrow \sqrt{n} > np_n$$
$$\Rightarrow |p_n| = p_n < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = \left|\frac{1}{\sqrt{n}}\right| \Rightarrow$$
$$\Rightarrow \underline{|p_n| < |1/\sqrt{n}|}.$$

We have thus shown that

$$\begin{cases} \forall n\in\mathbb{N}^*: |p_n| < |1/\sqrt{n}| \\ \lim_{n\in\mathbb{N}^*} (1/\sqrt{n}) = 0 \end{cases} \Rightarrow \lim_{n\in\mathbb{N}^*} p_n = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} (\sqrt[n]{n}-1) = \lim_{n\in\mathbb{N}^*} [(1+p_n)^2-1] = \lim_{n\in\mathbb{N}^*} (1+2p_n+p_n^2-1)$$
$$= \lim_{n\in\mathbb{N}^*} (2p_n+p_n^2) = 2\lim_{n\in\mathbb{N}^*} p_n + (\lim_{n\in\mathbb{N}^*} p_n)^2$$
$$= 2\cdot 0+0 = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{n} = 1. \qquad \square$$

2 $$\boxed{\forall a\in(0,+\infty): \lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = 1}$$

<u>Proof</u>

Let $a\in(0,+\infty)$ be given. We distinguish between the following cases.

<u>Case 1</u>: Assume that $a=1$. Then, we have:

$$\lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = \lim_{n\in\mathbb{N}^*} \sqrt[n]{1} = \lim_{n\in\mathbb{N}^*} 1 = 1$$

Case 2: Assume that $a>1$. Then, we have

$\forall n\in\mathbb{N}^* : \sqrt[n]{a} > 1$

and we can therefore define a sequence (p_n) such that

$\forall n\in\mathbb{N}^* : (\sqrt[n]{a} = 1+p_n \wedge p_n > 0)$

Let $n\in\mathbb{N}^*$ be given. Then, we have:

$$\sqrt[n]{a} = 1+p_n \Rightarrow a = (1+p_n)^n \geq 1+np_n > np_n \Rightarrow$$
$$\Rightarrow |p_n| = p_n < a/n = |a/n| \Rightarrow \underline{|p_n| < |a/n|}$$

We have thus shown that

$$\begin{cases} \forall n\in\mathbb{N}^* : |p_n| < |a/n| \\ \lim_{n\in\mathbb{N}^*} (a/n) = 0 \end{cases} \Rightarrow \lim_{n\in\mathbb{N}^*} p_n = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = \lim_{n\in\mathbb{N}^*} (1+p_n) = 1 + \lim_{n\in\mathbb{N}^*} p_n = 1+0 = 1$$

Case 3: Assume that $0<a<1$. Then, we have:

$0<a<1 \Rightarrow 1/a > 1 \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{1/a} = 1$ [via case 2]

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = \lim_{n\in\mathbb{N}^*} \frac{1}{\sqrt[n]{1/a}} = \frac{1}{\lim_{n\in\mathbb{N}^*} \sqrt[n]{1/a}}$$

$$= \frac{1}{1} = 1$$

□

③ $$\boxed{\lim_{n\in\mathbb{N}^*} a_n = a \in (0,+\infty) \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a_n} = 1}$$

<u>Proof</u>

Since

$$\lim_{n\in\mathbb{N}^*} a_n = a > 0 \Rightarrow \exists n_0\in\mathbb{N}^* : \forall n\in\mathbb{N}^*-[n_0] : a_n, a \text{ equisigned}$$
$$\Rightarrow \exists n_0\in\mathbb{N}^* : \forall n\in\mathbb{N}^*-[n_0] : a_n > 0 \qquad (1)$$

and

$$\lim_{n\in\mathbb{N}^*} a_n = a > 0 \Rightarrow \exists n_0 \in\mathbb{N}^* : \forall n\in\mathbb{N}^*-[n_0] : \frac{|a|}{2} < |a_n| < \frac{3|a|}{2} \qquad (2)$$

Via Eq. (1), choose $n_1\in\mathbb{N}^*$ such that

$\forall n\in\mathbb{N}^*-[n_1] : a_n>0$

Via Eq. (2) choose $n_2\in\mathbb{N}^*$ such that

$\forall n\in\mathbb{N}^*-[n_2] : |a|/2 < |a_n| < 3|a|/2$

Define $n_0 = \max\{n_1, n_2\}$. Let <u>$n\in\mathbb{N}^*-[n_0]$</u> be given.

Then, we have:

$$\begin{cases} |a|/2 < |a_n| < 3|a|/2 \\ a_n > 0 \end{cases} \Rightarrow |a|/2 < a_n < 3|a|/2 \Rightarrow \underline{\sqrt[n]{|a|/2} < \sqrt[n]{a_n} < \sqrt[n]{3|a|/2}}$$

We have thus shown that

$$\begin{cases} \forall n\in\mathbb{N}^*-[n_0] : \sqrt[n]{|a|/2} < \sqrt[n]{a_n} < \sqrt[n]{3|a|/2} \\ \lim_{n\in\mathbb{N}^*} \sqrt[n]{|a|/2} = \lim_{n\in\mathbb{N}^*} \sqrt[n]{3|a|/2} = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a_n} = 1 \qquad \square$$

THEORY QUESTIONS

(17) Let (a_n), (b_n), (c_n) be nets on $(D,<)$. Show that

$$\left\{\begin{array}{l} \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n \leq c_n) \\ \lim a_n = \lim c_n = \ell \end{array}\right. \Rightarrow \lim b_n = \ell$$

(18) Show that:

a) $\lim\limits_{n \in \mathbb{N}} \sqrt[n]{n} = 1$ b) $\forall a \in (0,+\infty) : \lim\limits_{n \in \mathbb{N}^*} \sqrt[n]{a} = 1$

c) $\lim\limits_{n \in \mathbb{N}^*} a_n = a > 0 \Rightarrow \lim\limits_{n \in \mathbb{N}^*} \sqrt[n]{a_n} = 1$

EXERCISES

(19) Use limit properties to evaluate the limit of the following sequences

a) $a_n = \sqrt[n]{n^2+1}$ b) $a_n = \sqrt[n]{2n^3 - n + 5}$

c) $a_n = \sqrt[n]{2^n + 3^n + 5^n}$ d) $a_n = \sqrt[n]{3 + 1/n}$

e) $a_n = \sqrt[n]{\dfrac{7n+1}{3n+2}}$ f) $a_n = \sqrt[n]{\dfrac{5^{n+1}+2}{5^n+4^n}}$

(20) Use the squeeze theorem and limit properties to evaluate the limit of the following sequences

a) $a_n = \dfrac{n^3}{n^4+1} + \dfrac{n^3}{n^4+2} + \dots + \dfrac{n^3}{n^4+n}$

b) $a_n = \dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \dots + \dfrac{n}{n^2+n}$

c) $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}$

d) $a_n = \frac{\sin(1)}{n^2+1} + \frac{\sin(2)}{n^2+2} + \dots + \frac{\sin(n)}{n^2+n}$

e) $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$

Convergent recursive sequences

> **Thm:** Let (a_n) be a net on $(D,<)$. Then, we have:
> $$\begin{cases} (a_n) \text{ increasing} \\ (a_n) \text{ upper bounded} \end{cases} \Rightarrow \begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D : \lim a_n = \sup\{a_n \mid n \in D \wedge n > n_0\} \end{cases}$$

Proof

Since,

(a_n) upper bounded $\Rightarrow \exists n_0 \in D : \exists b \in \mathbb{R} : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b)$

choose $n_0 \in D$ and $b \in \mathbb{R}$ such that

$\forall n \in D : (n > n_0 : a_n \leq b)$

It follows that the set $A = \{a_n \mid n \in D \wedge n > n_0\}$ is upper bounded, so by the axiom of completeness we can define $x = \sup A$. We will now show that $\lim a_n = x$.

Let $\varepsilon \in (0, +\infty)$ be given. From the approximation theorem choose $n_1 \in D$ such that $n_1 > n_0$ and $x - \varepsilon < a_{n_1} \leq x$.

Let $n \in D$ be given and assume that $n > n_1$. It follows that

$$n > n_1 \Rightarrow \begin{cases} x - \varepsilon < a_{n_1} \leq x \\ a_{n_1} \leq a_n \leq \sup A = x \end{cases} \Rightarrow$$

$$\Rightarrow x - \varepsilon < a_{n_1} \leq a_n \leq x < x + \varepsilon$$

$$\Rightarrow x - \varepsilon < a_n < x + \varepsilon \Rightarrow$$

$$\Rightarrow -\varepsilon < a_n - x < \varepsilon \Rightarrow |a_n - x| < \varepsilon$$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_1 \in D : \forall n \in D : (n > n_1 \Rightarrow |a_n - x| < \varepsilon)$

$\Rightarrow \lim a_n = x \Rightarrow (a_n)$ convergent $\square$

Similarly, we can show that

Thm: Let (a_n) be a net on $(D,<)$. Then, we have:

$$\left\{\begin{array}{l}(a_n)\text{ decreasing}\\(a_n)\text{ lower bounded}\end{array}\right. \Rightarrow \left\{\begin{array}{l}(a_n)\text{ convergent}\\ \exists n_0\in D:\ \lim a_n = \inf\{a_n \mid n\in D \wedge n>n_0\}\end{array}\right.$$

Note that when (a_n) is a sequence, we define:

$$\inf a_n = \inf\{a_n \mid n\in\mathbb{N}^*\}$$
$$\sup a_n = \sup\{a_n \mid n\in\mathbb{N}^*\}$$

and the previous theorems simplify to the following statements:

$$\left\{\begin{array}{l}(a_n)\text{ increasing}\\(a_n)\text{ upper bounded}\end{array}\right. \Rightarrow \left\{\begin{array}{l}(a_n)\text{ convergent}\\ \lim\limits_{n\in\mathbb{N}^*} a_n = \sup a_n\end{array}\right.$$

$$\left\{\begin{array}{l}(a_n)\text{ decreasing}\\(a_n)\text{ lower bounded}\end{array}\right. \Rightarrow \left\{\begin{array}{l}(a_n)\text{ convergent}\\ \lim\limits_{n\in\mathbb{N}^*} a_n = \inf a_n\end{array}\right.$$

EXAMPLE

Evaluate the limit of the sequence (a_n) defined recursively by:

$$\begin{cases} a_1 = 5 \\ \forall n \in \mathbb{N}^* : a_{n+1} = \dfrac{2(a_n - 12)}{a_n - 8} \end{cases}$$

Solution

We note that

$$\forall n \in \mathbb{N}^* : a_{n+1} - a_n = \frac{2(a_n - 12)}{a_n - 8} - a_n = \frac{2(a_n - 12) - a_n(a_n - 8)}{a_n - 8}$$

$$= \frac{2a_n - 24 - a_n^2 + 8a_n}{a_n - 8} = \frac{-a_n^2 + 10a_n - 24}{a_n - 8} =$$

$$= \frac{-(a_n^2 - 10a_n + 24)}{a_n - 8} = \frac{-(a_n - 6)(a_n - 4)}{a_n - 8}$$

- We need to compare (a_n) with $4, 6, 8$.
- We claim that $\forall n \in \mathbb{N}^* : a_n < 6$.

For $n = 1$, $a_1 = 5 < 6$. For $n = k$, assume that $a_k < 6$.
For $n = k+1$, we will show that $a_{k+1} < 6$. We have:

$$a_{k+1} - 6 = \frac{2(a_k - 12)}{a_k - 8} - 6 = \frac{2(a_k - 12) - 6(a_k - 8)}{a_k - 8}$$

$$= \frac{2a_k - 24 - 6a_k + 48}{a_k - 8} = \frac{-4a_k + 24}{a_k - 8} =$$

$$= \frac{-4(a_k - 6)}{a_k - 8}$$

and therefore,

$$a_k < 6 \Rightarrow \begin{cases} a_k - 6 < 0 \\ a_k - 8 < 0 \end{cases} \Rightarrow a_{k+1} - 6 < 0 \Rightarrow a_{k+1} < 6$$

We have thus shown the claim

▶ We claim that $\forall n \in \mathbb{N}^*: a_n > 4$.

For $n = 1$, we have $a_1 = 5 > 4$. For $n = k$, assume that $a_k > 4$. For $n = k+1$, we will show that $a_{k+1} > 4$. We have:

$$a_{k+1} - 4 = \frac{2(a_k - 12)}{a_k - 8} - 4 = \frac{2(a_k - 12) - 4(a_k - 8)}{a_k - 8} =$$

$$= \frac{2a_k - 24 - 4a_k + 32}{a_k - 8} = \frac{-2a_k + 8}{a_k - 8} = \frac{-2(a_k - 4)}{a_k - 8}$$

and therefore

$$4 < a_k < 6 \Rightarrow \begin{cases} a_k - 4 > 0 \\ a_k - 8 < 0 \end{cases} \Rightarrow a_{k+1} - 4 > 0 \Rightarrow a_{k+1} > 4.$$

We have thus shown the claim.

We conclude that

$$\forall n \in \mathbb{N}^*: 4 < a_n < 6 \Rightarrow \begin{cases} \forall n \in \mathbb{N}^*: a_{n+1} - a_n < 0 \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} (a_n) \text{ decreasing} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow (a_n) \text{ convergent}$$

$$\Rightarrow \exists x \in \mathbb{R}: \lim_{n \in \mathbb{N}^*} a_n = x$$

Choose $x \in \mathbb{R}$ such that $\lim_{n \in \mathbb{N}^*} a_n = x$. Then, we have:

$$x = \lim_{n \in \mathbb{N}^*} a_{n+1} = \lim_{n \in \mathbb{N}^*} \frac{2(a_n - 12)}{a_n - 8} = \frac{2(x - 12)}{x - 8} \Leftrightarrow$$

$$\Leftrightarrow x(x-8) = 2(x-12) \Leftrightarrow x^2 - 8x = 2x - 24 \Leftrightarrow$$
$$\Leftrightarrow x^2 - 8x - 2x + 24 = 0 \Leftrightarrow x^2 - 10x + 24 = 0$$
$$\Leftrightarrow (x-6)(x-4) = 0 \Leftrightarrow x - 6 = 0 \lor x - 4 = 0$$
$$\Leftrightarrow x = 6 \lor x = 4 \Leftrightarrow x \in \{4, 6\}.$$

Since

$$\begin{cases} a_1 = 5 \\ (a_n) \text{ decreasing} \end{cases} \Rightarrow (\forall n \in \mathbb{N}^* : a_n \leqslant 5) \Rightarrow$$
$$\Rightarrow x = \lim_{n \in \mathbb{N}^*} a_n \leqslant 5 \Rightarrow x \neq 6.$$

we conclude that $\lim_{n \in \mathbb{N}^*} a_n = x = 4$ $\square$

THEORY QUESTIONS

(21) Let (a_n) be a net on $(D,<)$. Show that

$$\begin{cases}(a_n) \text{ increasing}\\ (a_n) \text{ upper bounded}\end{cases} \Rightarrow \begin{cases}(a_n) \text{ convergent}\\ \lim a_n = \sup\{a_n \mid n\in D \wedge n>n_0\}, \exists n_0\in D.\end{cases}$$

EXERCISES

(22) Let (a_n) be a net on $(D,<)$. Write the proof for the statement

$$\begin{cases}(a_n) \text{ decreasing}\\ (a_n) \text{ lower bounded}\end{cases} \Rightarrow \begin{cases}(a_n) \text{ convergent}\\ \exists n_0\in D: \lim a_n = \inf\{a_n \mid n\in D \wedge n>n_0\}.\end{cases}$$

(23) Show that the following sequences are convergent and evaluate their limit:

a) $\begin{cases}a_1 = 1\\ \forall n\in\mathbb{N}^*: a_{n+1} = \sqrt{1+a_n}\end{cases}$

b) $\begin{cases}a_1 = 3\\ a_{n+1} = (3a_n - 4)/5\end{cases}$

c) $\begin{cases}a_1 = 2\\ a_{n+1} = (2a_n - 3)/4\end{cases}$

d) $\begin{cases}a_1 = 1/4\\ a_{n+1} = (1/2)a_n^2 + (1/8)\end{cases}$

e) $\begin{cases}a_1 = 1\\ a_{n+1} = (1/3)a_n + 2\end{cases}$

f) $\begin{cases}a_1 = 0\\ a_{n+1} = (3a_n + 1)/4\end{cases}$

g) $\begin{cases}a_1 = 3\\ a_{n+1} = (a_n^2 + 4)/5\end{cases}$

h) $\begin{cases}a_1 = 2\\ a_{n+1} = \sqrt{1+2a_n} - 1\end{cases}$

i) $\begin{cases}a_1 = 2\\ a_{n+1} = \sqrt{a_n + 6}\end{cases}$

j) $\begin{cases}a_1 = 3\\ a_{n+1} = (1/2)(a_n + 2/a_n)\end{cases}$

Nested intervals

Nested intervals of rational numbers can be used to approximate and define real numbers.

Def: Let $([a_n, b_n]) : [a_1, b_1], [a_2, b_2], \ldots$ be a sequence of closed intervals. We say that

$([a_n, b_n])$ nested $\Leftrightarrow \begin{cases} \forall n \in \mathbb{N}^* : [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \\ \lim\limits_{n\in\mathbb{N}^*} (a_n - b_n) = 0 \end{cases}$

We show that every nested interval sequence $([a_n, b_n])$ has at least one common element $x \in \mathbb{R}$.

Thm: $([a_n, b_n])$ nested $\Rightarrow \exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : x \in [a_n, b_n]$

Proof

$([a_n, b_n])$ nested $\Rightarrow \forall n \in \mathbb{N}^* : [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

$\Rightarrow \forall n \in \mathbb{N}^* : a_1 \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b_1$

$\Rightarrow \begin{cases} (a_n) \text{ increasing} \\ (a_n) \text{ upper bounded} \\ (b_n) \text{ decreasing} \\ (b_n) \text{ lower bounded} \end{cases} \Rightarrow \begin{cases} (a_n), (b_n) \text{ convergent} \\ \lim\limits_{n\in\mathbb{N}^*} a_n = \sup a_n \\ \lim\limits_{n\in\mathbb{N}^*} b_n = \inf b_n \end{cases}$

Choose $x = \lim\limits_{n\in\mathbb{N}^*} a_n = \sup a_n$ and note that

$$\inf b_n = \lim_{n\in\mathbb{N}^*} b_n = \lim_{n\in\mathbb{N}^*} [a_n - (a_n - b_n)] =$$

$$= \lim_{n\in\mathbb{N}^*} a_n - \lim_{n\in\mathbb{N}^*} (a_n - b_n) = x - 0 = x$$

It follows that

$$\begin{cases} x = \sup a_n \\ x = \inf b_n \end{cases} \Rightarrow \begin{cases} \forall n\in\mathbb{N}^* : a_n \leq x \\ \forall n\in\mathbb{N}^* : b_n \geq x \end{cases} \Rightarrow \forall n\in\mathbb{N}^* : a_n \leq x \leq b_n$$

$$\Rightarrow \forall n\in\mathbb{N}^* : x \in [a_n, b_n]$$

We have thus shown that $\exists x\in\mathbb{R} : \forall n\in\mathbb{N}^* : x\in[a_n,b_n]$. □

▸ We will now show that this element is unique:

Thm: $\begin{cases} ([a_n,b_n]) \text{ nested} \\ x_1, x_2 \in \bigcap_{n\in\mathbb{N}^*} [a_n,b_n] \end{cases} \Rightarrow x_1 = x_2$

Proof

To show a contradiction, assume that $x_1 \neq x_2$, and with no loss of generality assume that $x_1 < x_2$. Then, we have:

$([a_n,b_n])$ nested $\Rightarrow \lim_{n\in\mathbb{N}^*} (a_n - b_n) = 0 \Rightarrow$

$\Rightarrow \forall \varepsilon\in(0,+\infty) : \exists n_0\in\mathbb{N}^* : \forall n\in\mathbb{N}^* - [n_0] : |a_n - b_n| < \varepsilon$

$\Rightarrow \exists n_0\in\mathbb{N}^* : \forall n\in\mathbb{N}^* - [n_0] : |a_n - b_n| < |x_2 - x_1|$

Choose $n_0\in\mathbb{N}^*$ such that

$$\forall n\in\mathbb{N}^* - [n_0] : |a_n - b_n| < |x_2 - x_1| \qquad (1)$$

Choose an $n\in\mathbb{N}^* - [n_0]$. Then:

$$x_1, x_2 \in [a_n, b_n] \Rightarrow |a_n - b_n| \geq |x_2 - x_1| \qquad (2)$$

Eq.(1) and Eq.(2) contradict. It follows that $x_1 = x_2$ □

$\to$ Let $Seq(A)$ be the set of all mappings $a: \mathbb{N}^* \to A$.
Thus $Seq(\mathbb{Q})$ is the set of all rational sequences.

We will now show that every real number can be approximated using nested intervals with rational end points

Thm: $\forall x \in \mathbb{R}: \exists a, b \in Seq(\mathbb{Q}): \begin{cases} ([a_n, b_n]) \text{ nested} \\ \{x\} = \bigcap_{n \in \mathbb{N}^*} [a_n, b_n] \end{cases}$

Proof

Let $x \in \mathbb{R}$ be given.

- Construction of $[a_1, b_1]$: From Archimedes theorem, choose $a_1, b_1 \in \mathbb{Z}$ such that $b_1 > x$ and $a_1 > -x$.
It follows that: $a_1 < x < b_1 \Rightarrow x \in [a_1, b_1]$.
- Assume that $[a_k, b_k]$ has been constructed. To construct $[a_{k+1}, b_{k+1}]$ we define:

$$a_{k+1} = \begin{cases} (1/2)(a_k + b_k), & \text{if } x > (1/2)(a_k + b_k) \\ a_k, & \text{if } x \leq (1/2)(a_k + b_k) \end{cases}$$

$$b_{k+1} = \begin{cases} b_k, & \text{if } x > (1/2)(a_k + b_k) \\ (1/2)(a_k + b_k), & \text{if } x \leq (1/2)(a_k + b_k) \end{cases}$$

By construction, we have

$$\forall n \in \mathbb{N}^*: \begin{cases} [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \\ b_{n+1} - a_{n+1} = (1/2)(b_n - a_n) \end{cases} \Rightarrow$$

$$\Rightarrow \forall n \in \mathbb{N}^*: \begin{cases} [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \Rightarrow ([a_n, b_n]) \text{ nested.} \\ \lim_{n \in \mathbb{N}^*} (a_n - b_n) = 0 \end{cases}$$

We also have:

$$\begin{cases} x \in [a_1, b_1] \\ \forall n \in \mathbb{N}^*: (x \in [a_n, b_n] \Rightarrow x \in [a_{n+1}, b_{n+1}]) \end{cases} \Rightarrow$$

$$\Rightarrow \forall n \in \mathbb{N}^*: x \in [a_n, b_n]$$

$$\Rightarrow \{x\} = \bigcap_{n \in \mathbb{N}^*} [a_n, b_n]. \qquad \square$$

THEORY QUESTIONS

(24) Let $([a_n, b_n])$ be a sequence of intervals. State the necessary and sufficient conditions for the statement: "$([a_n,b_n])$ nested".

(25) Prove the following theorems

a) $([a_n, b_n])$ nested $\Rightarrow \exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : x \in [a_n, b_n]$

b) $\begin{cases} ([a_n,b_n]) \text{ nested} \\ x_1, x_2 \in \bigcap\limits_{n\in\mathbb{N}^*} [a_n,b_n] \end{cases} \Rightarrow x_1 = x_2$

c) $\forall x \in \mathbb{R} : \exists a, b \in Seq(\mathbb{Q}) : \begin{cases} ([a_n,b_n]) \text{ nested} \\ \{x\} = \bigcap\limits_{n\in\mathbb{N}^*} [a_n,b_n] \end{cases}$

EXERCISES

(26) Let $([a_n,b_n])$ and $([c_n,d_n])$ be nested interval sequences such that

$\{x\} = \bigcap\limits_{n\in\mathbb{N}^*} [a_n, b_n]$ and $\{y\} = \bigcap\limits_{n\in\mathbb{N}^*} [c_n, d_n]$

Show that:

a) $\{x+y\} = \bigcap\limits_{n\in\mathbb{N}^*} [a_n + c_n, b_n + d_n]$

b) $\{xy\} = \bigcap\limits_{n\in\mathbb{N}^*} [a_n c_n, b_n d_n]$

(27) Let $([a_n, b_n])$ be a sequence of intervals such that $\lim_{n\in\mathbb{N}^*} (a_n - b_n) = 0$. Explain why it is not possible to show that

$\exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : x \in [a_n, b_n]$

without the additional assumption that

$\forall n \in \mathbb{N}^* : [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

↳ <u>Hint:</u> Construct a counterexample $([a_n, b_n])$ such that $\bigcap_{n\in\mathbb{N}^*} [a_n, b_n] = \emptyset \wedge \lim_{n\in\mathbb{N}^*} (a_n - b_n) = 0$

<u>Hint 2:</u> Drift, drift, drift away
gently down the stream...

Cauchy sequences

> **Def:** Let (a_n) be a sequence. We say that
> (a_n) Cauchy $\Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}|$
> $: |a_{n_1} - a_{n_2}| < \varepsilon$

▸ Our main result is that:

(a_n) convergent $\Leftrightarrow$ (a_n) Cauchy.

We can therefore use the negation of the definition above to show that a sequence (a_n) is not convergent:

(a_n) not convergent $\Leftrightarrow$ (a_n) not Cauchy $\Leftrightarrow$

$\Leftrightarrow \exists \varepsilon \in (0,+\infty) : \forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geq \varepsilon$

The details are given in the following:

Properties of Cauchy sequences

① $\boxed{(a_n) \text{ Cauchy} \Rightarrow (a_n) \text{ Bounded}}$

Proof

Assume that (a_n) Cauchy. Then, we have:

(a_n) Cauchy $\Rightarrow$

$\Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < \varepsilon$

$\Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < 1$

Choose $n_0 \in \mathbb{N}^*$ such that

$\forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < 1$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then, we have:

$|a_n| - |a_{n_0+1}| \leq ||a_n| - |a_{n_0+1}|| \leq |a_n - a_{n_0+1}| < 1 \Rightarrow$

$\Rightarrow |a_n| < 1 + |a_{n_0+1}|$.

Choose $b = 1 + |a_{n_0+1}|$. We have thus shown that

$(\exists b \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n| \leq b) \Rightarrow$

$\Rightarrow$ $\underline{(a_n) \text{ bounded}}$ $\square$

② $\boxed{(a_n) \text{ convergent} \Rightarrow (a_n) \text{ Cauchy.}}$

Proof

Assume that (a_n) convergent. Define $\ell = \lim_{n \in \mathbb{N}^*} a_n$.

It follows that

$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon$

Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Choose $\underline{n_0 \in \mathbb{N}^*}$ such that

$\forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon/2$.

Let $\underline{n_1, n_2 \in \mathbb{N}^* - [n_0]}$ be given. Then, we have:

$$\begin{cases} n_1 > n_0 \\ n_2 > n_0 \end{cases} \Rightarrow \begin{cases} |a_{n_1} - \ell| < \varepsilon/2 \\ |a_{n_2} - \ell| < \varepsilon/2 \end{cases} \Rightarrow$$

$$\Rightarrow |a_{n_1} - a_{n_2}| = |(a_{n_1} - \ell) - (a_{n_2} - \ell)| \leq |a_{n_1} - \ell| + |a_{n_2} - \ell|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow$$

$\Rightarrow \underline{|a_{n_1} - a_{n_2}| < \varepsilon}$

We have thus shown that

$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < \varepsilon$

$\Rightarrow (a_n)$ Cauchy. $\square$

③ $\boxed{(a_n) \text{ Cauchy} \Rightarrow (a_n) \text{ convergent}}$

Proof

Assume that (a_n) Cauchy. We will construct a nested $([b_n, c_n])$ interval sequence such that

$$\begin{cases} \forall k \in \mathbb{N}^* : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n \in [b_k, c_k] \\ \bigcap_{k \in \mathbb{N}^*} [b_k, c_k] = \{\lim_{n \in \mathbb{N}^*} a_n\} \end{cases}$$

▶ Construction of $[b_1, c_1]$: Since

(a_n) Cauchy $\Rightarrow$ (a_n) bounded $\Rightarrow$

$$\Rightarrow \exists b_1, c_1 \in \mathbb{R} : \forall n \in \mathbb{N}^* : a_n \in [b_1, c_1]$$

Choose $b_1, c_1 \in \mathbb{R}$ such that $\forall n \in \mathbb{N}^* : a_n \in [b_1, c_1]$.

We have thus constructed $[b_1, c_1]$.

▶ Assume that $[a_k, b_k]$ has been constructed such that

$\exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n \in [b_k, c_k]$

▶ Construction of $[b_{k+1}, c_{k+1}]$:

Choose $p_1 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [p_1] : a_n \in [b_k, c_k]$.

Since, (a_n) Cauchy $\Rightarrow$

$$\Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < |b_k - c_k|/4$$

choose $p_2 \in \mathbb{N}^*$ such that

$$\forall n_1, n_2 \in \mathbb{N}^* - [p_2] : |a_{n_1} - a_{n_2}| < |b_k - c_k|/4$$

Define $n_0 = \max\{p_1, p_2\} + 1$ and choose

$$\begin{cases} b_{k+1} = \max\{b_k, a_{n_0} - |b_k - c_k|/4\} \\ c_{k+1} = \min\{c_k, a_{n_0} + |b_k - c_k|/4\} \end{cases}$$

thus constructing $[b_{k+1}, c_{k+1}]$.

▶ Claim: $\forall n \in \mathbb{N}^* - [n_0] : a_n \in [b_{k+1}, c_{k+1}]$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then, we have:

$n > n_0 \Rightarrow n > p_1 \Rightarrow a_n \in [b_k, c_k]$

and

$$n > n_0 \Rightarrow \begin{cases} n > p_2 \\ n_0 > p_2 \end{cases} \Rightarrow |a_n - a_{n_0}| < |b_k - c_k|/4 \Rightarrow$$

$$\Rightarrow -|b_k - c_k|/4 < a_n - a_{n_0} < |b_k - c_k|/4$$

$$\Rightarrow a_{n_0} - |b_k - c_k|/4 < a_n < a_{n_0} + |b_k - c_k|/4$$

and therefore

$$\max\{b_k, a_{n_0} - |b_k - c_k|/4\} < a_n < \min\{c_k, a_{n_0} + |b_k - c_k|/4\}$$

$$\Rightarrow b_{k+1} < a_n < c_{k+1} \Rightarrow a_n \in [b_{k+1}, c_{k+1}].$$

and this proves the claim.

▸ Claim: $[b_{k+1}, c_{k+1}] \subseteq [b_k, c_k]$.

We have:

$$a_{n_0} \in [b_k, c_k] \Rightarrow b_k \leq a_{n_0} \leq c_k \Rightarrow \begin{cases} b_k < a_{n_0} + |b_k - c_k|/4 \\ a_{n_0} - |b_k - c_k|/4 < c_k \end{cases}$$

$$\Rightarrow \max\{b_k, a_{n_0} - |b_k - c_k|/4\} < \min\{c_k, a_{n_0} + |b_k - c_k|/4\}$$

$$\Rightarrow b_{k+1} < c_{k+1}$$

noting that all other pairwise combinations in the definition of b_{k+1} and c_{k+1} also satisfy the same inequality

We also note that, by definition, we have $b_{k+1} \geq b_k$ and $c_{k+1} \leq c_k$. It follows that

$$b_k \leq b_{k+1} \leq c_{k+1} \leq c_k \Rightarrow [b_{k+1}, c_{k+1}] \subseteq [b_k, c_k]$$

thus proving the claim.

▸ <u>Claim</u>: $\lim_{n \in \mathbb{N}^*} (b_n - c_n) = 0$

We note that:

$b_{k+1} = \max\{b_k, a_{n_0} - |b_k - c_k|/4\} \geq a_{n_0} - |b_k - c_k|/4 \Rightarrow$
$\Rightarrow -b_{k+1} \leq -a_{n_0} + |b_k - c_k|/4$
and
$c_{k+1} = \min\{c_k, a_{n_0} + |b_k - c_k|/4\} \leq a_{n_0} + |b_k - c_k|/4$
and therefore
$c_{k+1} - b_{k+1} \leq [-a_{n_0} + |b_k - c_k|/4] + [a_{n_0} + |b_k - c_k|/4]$
$= |b_k - c_k|/2 \Rightarrow |b_{k+1} - c_{k+1}| \leq |b_k - c_k|/2.$
We have thus shown that
$\forall n \in \mathbb{N}^* : |b_{n+1} - c_{n+1}| \leq |b_n - c_n|/2$
$\Rightarrow \forall n \in \mathbb{N}^* : |b_{n+1} - c_{n+1}| \leq |b_1 - c_1|/2^n$
$\Rightarrow \lim_{n \in \mathbb{N}^*} (b_n - c_n) = 0$

- We conclude from the above that $([b_n, c_n])$ is nested.
- Define $\{\ell\} = \bigcap_{n \in \mathbb{N}^*} [b_n, c_n]$. We will now show that

$$\lim_{n \in \mathbb{N}^*} a_n = \ell.$$

Let $\varepsilon \in (0, +\infty)$ be given. Since $\lim_{n \in \mathbb{N}^*} (b_n - c_n) = 0$, choose $k \in \mathbb{N}^*$ such that $|b_k - c_k| < \varepsilon$. Choose $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : a_n \in [b_k, c_k]$. Let $n \in \mathbb{N}^* - [n_0]$ be given. It follows that

$$\begin{cases} a_n \in [b_k, c_k] \\ \ell \in [b_k, c_k] \end{cases} \Rightarrow |a_n - \ell| \leq |b_k - c_k| < \varepsilon \Rightarrow \underline{|a_n - \ell| < \varepsilon}$$

We have thus shown that
$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon$
$\Rightarrow \lim_{n \in \mathbb{N}^*} a_n = \ell \Rightarrow (a_n)$ convergent. $\square$

Methodology

We can use the contrapositive of property 2 to show that a sequence (a_n) is not convergent by proving the statement
$\exists \varepsilon \in (0,+\infty): \forall n_0 \in \mathbb{N}^*: \exists n_1, n_2 \in \mathbb{N}^* - [n_0]: |a_{n_1} - a_{n_2}| \geq \varepsilon$

EXAMPLE

Show that the sequence (a_n) with

$$\forall n \in \mathbb{N}^*: \quad a_n = \frac{(-1)^n n}{n+2}$$

is not convergent.

Solution

Let $n_0 \in \mathbb{N}^*$ be given. Via the Archimedes theorem, choose $n_1 = 2k > n_0$ and $n_2 = 2k+1 > n_0$ with $k \in \mathbb{N}^*$. Then, we have:

$$|a_{n_1} - a_{n_2}| = \left| \frac{(-1)^{2k}(2k)}{2k+2} - \frac{(-1)^{2k+1}(2k+1)}{(2k+1)+2} \right| =$$

$$= \left| \frac{k}{k+1} + \frac{2k+1}{2k+3} \right| = \frac{k}{k+1} + \frac{2k+1}{2k+3} \geq$$

$$\geq \frac{k}{k+1} \geq \frac{k}{k+k} = \frac{k}{2k} = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \underline{|a_{n_1} - a_{n_2}| \geq 1/2}$$

We have thus shown that:

$\forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geq 1/2$

$\Rightarrow \exists \varepsilon \in (0, +\infty) : \forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geq \varepsilon$

$\Rightarrow (a_n)$ not Cauchy $\Rightarrow (a_n)$ not convergent. $\square$

THEORY QUESTIONS

(28) State the definition for

a) (a_n) Cauchy.

b) (a_n) not Cauchy

(29) Let (a_n) be a sequence. Prove that

a) (a_n) Cauchy $\Rightarrow$ (a_n) bounded

b) (a_n) convergent $\Rightarrow$ (a_n) Cauchy.

(30)* Let (a_n) be a sequence. Prove that

a) (a_n) Cauchy $\Rightarrow$ (a_n) convergent.
(optional).

EXERCISES

(31) Let $(a_n), (b_n)$ be two sequences. Use the Cauchy sequence definition to show that

a) $(a_n), (b_n)$ Cauchy $\Rightarrow$ $(a_n + b_n)$ Cauchy

b) $(a_n), (b_n)$ Cauchy $\Rightarrow$ $(a_n b_n)$ Cauchy

c) (a_n) Cauchy $\Rightarrow$ $(|a_n|)$ Cauchy.

(32) Show that the following sequences are not convergent.

a) $a_n = \dfrac{1+(-1)^n}{2}$

b) $a_n = \sin(n\pi/2)$

c) $a_n = \dfrac{(-1)^n (n+2)}{3n}$

d) $a_n = (-1)^n (3n+2)$

e) $a_n = \dfrac{2(-2)^n + 2^n}{(-2)^n - 3.2^{n-1}}$

f) $a_n = \dfrac{n^2 + (-1)^n n^2}{n+1}$

g) $a_n = \dfrac{n\cos(3n\pi/4)}{n+1}$

▼ Sequences / nets with limit going to infinity.

Let (a_n) be a net on $(D,<)$. We recall the following definitions:

$$\boxed{\begin{array}{l} \lim a_n = +\infty \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in D: \forall n\in D: (n>n_0 \Rightarrow a_n > 1/\varepsilon) \\ \lim a_n = -\infty \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in D: \forall n\in D: (n>n_0 \Rightarrow a_n < -1/\varepsilon) \end{array}}$$

An immediate consequence of these definitions (via choosing $\varepsilon=1$) are the following statements:

$$\boxed{\begin{array}{l} \lim a_n = +\infty \Rightarrow (a_n) \text{ lower bounded} \\ \lim a_n = -\infty \Rightarrow (a_n) \text{ upper bounded} \\ \lim a_n = \pm\infty \Leftrightarrow \lim(-a_n) = \mp\infty \\ (\lim a_n = +\infty \vee \lim a_n = -\infty) \Rightarrow \lim |a_n| = +\infty \end{array}}$$

Note that when (a_n) is a sequence, these definitions simplify to:

$$\boxed{\begin{array}{l} \lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \Leftrightarrow \forall\varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: a_n > 1/\varepsilon \\ \\ \lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \Leftrightarrow \forall\varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: a_n < -1/\varepsilon \end{array}}$$

We also note that when (a_n) is a sequence, then:

$$\boxed{\lim_{n\in\mathbb{N}^*} a_n = \pm\infty \Leftrightarrow \lim_{n\in\mathbb{N}^*} a_{n+k} = \pm\infty}$$

▸ Uniqueness: To establish uniqueness, we first prove the following statements:

① $\boxed{\lim a_n = +\infty \Rightarrow (a_n) \text{ not upper bounded}}$

Proof

To show a contradiction, assume that (a_n) upper bounded. Then, we have:

(a_n) bounded $\Rightarrow \exists b \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b)$

Choose $b \in (0, +\infty)$ and $n_1 \in D$ such that

$\forall n \in D : (n > n_1 \Rightarrow a_n \leq b)$

We also have:

$\lim a_n = +\infty \Rightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

$\Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > b)$.

Choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow a_n > b)$

Choose $n_0 \in D$ such that $n_0 > n_1$ and $n_0 > n_2$. Then, we have

$$\begin{cases} n_0 > n_1 \\ n_0 > n_2 \end{cases} \Rightarrow \begin{cases} a_{n_0} \leq b \\ a_{n_0} > b \end{cases}$$

which is a contradiction. We conclude that (a_n) not upper bounded. □

▸ An immediate consequence of property 1 is:

② $\boxed{\lim a_n = -\infty \Rightarrow (a_n) \text{ not lower bounded}}$

Uniqueness of the limit is established by noting that

$\lim a_n = +\infty \Rightarrow (a_n)$ not upper bounded $\Rightarrow \lim a_n \neq -\infty$

$\lim a_n = -\infty \Rightarrow (a_n)$ not lower bounded $\Rightarrow \lim a_n \neq +\infty$

and

$\lim a_n = +\infty \vee \lim a_n = -\infty \Rightarrow (a_n)$ not bounded $\Rightarrow$
$\Rightarrow (a_n)$ not convergent $\Rightarrow \forall l \in \mathbb{R} : \lim a_n \neq l$.

and

$\lim a_n = l \Rightarrow (a_n)$ convergent $\Rightarrow (a_n)$ bounded $\Rightarrow$

$$\Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow \begin{cases} \lim a_n \neq +\infty \\ \lim a_n \neq -\infty \end{cases}$$

We conclude that if $\lim a_n$ exists, it has a unique evaluation in the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

Properties of nets with limit to infinity.

Let $(a_n), (b_n)$ be nets on $(D, <)$. Then, we have:

(1)

$$\boxed{\begin{array}{l} \text{If } \lim a_n = +\infty, \text{ then:} \\ \text{a) } b_n \text{ lower bounded} \Rightarrow \lim (a_n + b_n) = +\infty \\ \text{b) } b_n \text{ positively lower bounded} \Rightarrow \lim (a_n b_n) = +\infty \\ \text{c) } b_n \text{ negatively upper bounded} \Rightarrow \lim (a_n b_n) = -\infty \end{array}}$$

Proof

a) Assume that $\lim a_n = +\infty$ and b_n lower bounded. Then,
b_n lower bounded $\Rightarrow \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow b_n \geq b)$
Choose $b \in \mathbb{R}$ and $n_1 \in D$ such that $\forall n \in D : (n > n_1 \Rightarrow b_n \geq b)$
Let $\varepsilon \in (0, +\infty)$ be given. Then, we have
$\lim a_n = +\infty \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon - b)$
Choose $n_2 \in D$ such that $\forall n \in D : (n > n_2 \Rightarrow a_n > 1/\varepsilon - b)$

Choose $n_0 \in D$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $n \in D$ be given and assume that $n > n_0$. Then we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} b_n \geq b \\ a_n > 1/\varepsilon - b \end{cases} \Rightarrow \underline{a_n + b_n > 1/\varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n + b_n > 1/\varepsilon)$$

$$\Rightarrow \lim (a_n + b_n) = +\infty$$

b) Assume that $\lim a_n = +\infty$ and (b_n) positively lower bounded. Then, we have:

b_n positively lower bounded $\Rightarrow$

$$\Rightarrow \exists b \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow b_n \geq b)$$

Choose $b \in (0, +\infty)$ and $n_1 \in D$ such that

$$\forall n \in D : (n > n_1 \Rightarrow b_n \geq b)$$

Let $\underline{\varepsilon \in (0, +\infty)}$ be given. Since,

$$\lim a_n = +\infty \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/(\varepsilon b))$$

choose $n_2 \in D$ such that

$$\forall n \in D : (n > n_2 \Rightarrow a_n > 1/(\varepsilon b))$$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $\underline{n \in D}$ be given and $\underline{\text{assume that } n > n_0}$. Then, we have

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n > 1/(\varepsilon b) > 0 \\ b_n \geq b > 0 \end{cases} \Rightarrow$$

$$\Rightarrow a_n b_n \geq a_n b > [1/(\varepsilon b)] b = 1/\varepsilon \Rightarrow \underline{a_n b_n > 1/\varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n b_n > 1/\varepsilon)$$

$$\Rightarrow \lim (a_n b_n) = +\infty$$

c) Homework.

Similarly we can show that

② $\boxed{\begin{array}{l}\text{If } \lim a_n = -\infty \text{, then} \\ \text{a) } (b_n) \text{ upper bounded} \Rightarrow \lim(a_n + b_n) = -\infty \\ \text{b) } (b_n) \text{ negatively upper bounded} \Rightarrow \lim(a_n b_n) = +\infty \\ \text{c) } (b_n) \text{ positively lower bounded} \Rightarrow \lim(a_n b_n) = -\infty\end{array}}$

The proof is to simply apply property 1 on the nets $-a_n$ and $-b_n$.

③ $\boxed{\begin{cases} \lim a_n \in \{+\infty, -\infty\} \\ \forall n \in D : a_n \neq 0 \end{cases} \Rightarrow \lim \frac{1}{a_n} = 0}$

Proof

We distinguish between the following cases.

Case 1: Assume that $\lim a_n = +\infty$. Then, we have:

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $n_0 \in D$ such that

$\forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$n > n_0 \Rightarrow a_n > 1/\varepsilon > 0 \Rightarrow 0 < 1/a_n < \varepsilon \Rightarrow |1/a_n| < \varepsilon$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |1/a_n| < \varepsilon)$

$\Rightarrow \lim (1/a_n) = 0$.

Case 2: Assume that $\lim a_n = -\infty$. Then, we have:

$\lim a_n = -\infty \Rightarrow \lim(-a_n) = +\infty \Rightarrow \lim \dfrac{1}{-a_n} = 0 \Rightarrow$

$$\Rightarrow \lim \frac{1}{a_n} = 0 \qquad \square$$

$$\boxed{\begin{array}{ll} \text{④ a)} \begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} & \Rightarrow \lim \dfrac{1}{a_n} = +\infty \\ \text{b)} \begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n < 0) \end{cases} & \Rightarrow \lim \dfrac{1}{a_n} = -\infty \end{array}}$$

Proof

a) Let $\varepsilon \in (0, +\infty)$ be given. Choose $n_1 \in D$ such that $\forall n \in D : (n > n_1 \Rightarrow a_n > 0)$

Since $\lim a_n = 0 \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$, choose $n_2 \in D$ such that $\forall n \in D : (n > n_2 \Rightarrow |a_n| < \varepsilon)$. Choose $n_0 \in D$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n > 0 \\ |a_n| < \varepsilon \end{cases} \Rightarrow 0 < a_n < \varepsilon \Rightarrow$$

$$\Rightarrow \frac{1}{a_n} > \frac{1}{\varepsilon}$$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow 1/a_n > 1/\varepsilon)$

$\Rightarrow \lim(1/a_n) = +\infty$

b) Homework! $\square$

(5) $\boxed{\begin{array}{ll} \left\{\begin{array}{l}\lim a_n = +\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n)\end{array}\right. & \Rightarrow \lim b_n = +\infty \\ \left\{\begin{array}{l}\lim a_n = -\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \geq b_n)\end{array}\right. & \Rightarrow \lim b_n = -\infty \end{array}}$

Proof

a) Let $\underline{\varepsilon \in (0, +\infty)}$ be given. Choose $n_1 \in D$ such that
$\forall n \in D : (n > n_1 \Rightarrow a_n \leq b_n)$

Since $\lim a_n = +\infty \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

choose $n_2 \in D$ such that
$\forall n \in D : (n > n_2 \Rightarrow a_n > 1/\varepsilon)$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $\underline{n \in D}$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n \leq b_n \\ a_n > 1/\varepsilon \end{cases} \Rightarrow \underline{b_n > 1/\varepsilon}$$

We have thus shown that
$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow b_n > 1/\varepsilon)$
$\Rightarrow \lim b_n = +\infty$

b) Homework

THEORY QUESTIONS

(33) Let (a_n), (b_n) be nets. Prove that

a) $\lim a_n = +\infty \Rightarrow (a_n)$ not upper bounded

b) $\begin{cases} \lim a_n = +\infty \\ (b_n) \text{ lower bounded} \end{cases} \Rightarrow \lim (a_n + b_n) = +\infty$

c) $\begin{cases} \lim a_n = +\infty \\ b_n \text{ positively lower bounded} \end{cases} \Rightarrow \lim (a_n b_n) = +\infty$

d) $\begin{cases} \lim a_n \in \{+\infty, -\infty\} \\ \forall n \in D : a_n \neq 0 \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = 0$

e) $\begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = +\infty$

f) $\begin{cases} \lim a_n = +\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n) \end{cases} \Rightarrow \lim b_n = +\infty$

EXERCISES

(34) Use the limit definition to write complete proofs for the following statements

a) $\lim a_n = -\infty \Rightarrow (a_n)$ not lower bounded

b) $\begin{cases} \lim a_n = +\infty \\ b_n \text{ negatively upper bounded} \end{cases} \Rightarrow \lim (a_n b_n) = -\infty$

c) $\begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n < 0) \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = -\infty$

d) $\begin{cases} \lim a_n = -\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \geq b_n) \end{cases} \Rightarrow \lim b_n = -\infty$

Basic sequence limits

① $$\boxed{\forall p\in(0,+\infty):\lim_{n\in\mathbb{N}^*} n^p=+\infty}$$

Proof

Let $p\in(0,+\infty)$ be given and define $\forall n\in\mathbb{N}^*: a_n=n^p$.
Let $\varepsilon\in(0,+\infty)$ be given. Then, we have:
$a_n>1/\varepsilon \Leftrightarrow n^p>1/\varepsilon>0 \Leftrightarrow n>(1/\varepsilon)^{1/p}$
Via the Archimedes theorem, choose $n_0\in\mathbb{N}^*$ such that $n_0>(1/\varepsilon)^{1/p}$. Let $n\in\mathbb{N}^*$ be given and assume that $n>n_0$. Then, we have
$n>n_0 \Rightarrow n>(1/\varepsilon)^{1/p} \Rightarrow a_n>1/\varepsilon$.
We have thus shown that
$\forall\varepsilon\in(0,+\infty):\exists n_0\in\mathbb{N}^*:\forall n\in\mathbb{N}^*:(n>n_0\Rightarrow a_n>1/\varepsilon)$
$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n=\lim_{n\in\mathbb{N}^*} n^p=+\infty \qquad \square$$

- Note that combining this result with Property 3 immediately gives the following result:

② $$\forall p\in(-\infty,0):\lim_{n\in\mathbb{N}^*} n^p=0$$

③ $$\forall a\in(1,+\infty):\lim_{n\in\mathbb{N}^*} a^n=+\infty$$

Proof

Let $a \in (1, +\infty)$ be given and define $p \in (0, +\infty)$ such that $a = 1+p$. Let $\varepsilon \in (0, +\infty)$ be given. Then, we have:
$a^n = (1+p)^n \geq 1 + np > np > 1/\varepsilon \Leftarrow n > 1/(\varepsilon p)$.
Via the Archimedes theorem, choose $n_0 \in \mathbb{N}^*$ such that $n_0 > 1/(\varepsilon p)$. Let $n \in \mathbb{N}^*$ be given and assume that $n > n_0$. Then, we have:
$n > n_0 \Rightarrow n > 1/(\varepsilon p) \Rightarrow a^n > 1/\varepsilon$
We have thus shown than
$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > n_0 \Rightarrow a^n > 1/\varepsilon)$
$\Rightarrow \lim\limits_{n \in \mathbb{N}^*} a^n = +\infty$

We conclude that $\forall a \in (1, +\infty) : \lim\limits_{n \in \mathbb{N}^*} a^n = +\infty$ $\square$

$\hookrightarrow$ Result 1 can be combined with the following result, derived from the limit properties, to find the limits of polynomial and rational sequences

$$\boxed{\begin{aligned} &\forall n \in \mathbb{N}^* : c_n = \frac{b_p n^p + b_{p-1} n^{p-1} + \cdots + b_1 n + b_0}{a_q n^q + a_{q-1} n^{q-1} + \cdots + a_1 n + a_0} \Rightarrow \\ &\Rightarrow \lim_{n \in \mathbb{N}^*} c_n = \lim_{n \in \mathbb{N}^*} \frac{b_p n^p}{a_q n^q} \end{aligned}}$$

For $q = 0$ and $a_0 = 1$, this result reduces to a polynomial sequence as well.

EXAMPLES

a) Use the limit definition to show that $\lim_{n\in\mathbb{N}^*} a_n = +\infty$ for $\forall n\in\mathbb{N}^*: a_n = \sin(3n) + 3n^2 - n$

Solution

Let $\varepsilon\in(0,+\infty)$ be given. Then, we have:

$$a_n = \sin(3n) + 3n^2 - n \geq -1 + 3n^2 - n \geq -n^2 + 3n^2 - n^2$$
$$= n^2 > 1/\varepsilon > 0 \Leftrightarrow n > 1/\sqrt{\varepsilon}.$$

Via the Archimedes theorem, choose $n_0\in\mathbb{N}^*$ such that $n_0 > 1/\sqrt{\varepsilon}$. Let $n\in\mathbb{N}^*$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow n > 1/\sqrt{\varepsilon} \Rightarrow a_n > 1/\varepsilon$$

We have thus shown that

$$\forall\varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*: (n > n_0 \Rightarrow a_n > 1/\varepsilon)$$
$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = +\infty \qquad \square$$

b) Use the limit definition to show that $\lim_{n\in\mathbb{N}^*} a_n = +\infty$ for $\forall n\in\mathbb{N}^*: a_n = n^5 + 3n^3 - 9n^2$

Solution

Let $\varepsilon\in(0,+\infty)$ be given. Then, we have:

$$a_n = n^5 + 3n^3 - 9n^2 = n^5 + 3n^2(n-3).$$

If we restrict $n > 3$, then we have:

$$a_n = n^5 + 3n^2(n-3) \geq n^5 + 3n^2 \geq n^5 > 1/\varepsilon > 0 \Leftrightarrow$$
$$\Leftrightarrow n > (1/\varepsilon)^{1/5}$$

Via the Archimedes theorem, choose $n_0\in\mathbb{N}^*$ such that $n_0 > \max\{3, (1/\varepsilon)^{1/5}\}$. Let $n \in \mathbb{N}^*$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow n > \max\{3, (1/\varepsilon)^{1/5}\} \Rightarrow$$

$$\Rightarrow \begin{cases} n > 3 \\ n > (1/\varepsilon)^{1/5} \end{cases} \Rightarrow a_n > 1/\varepsilon$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} a_n = +\infty$$

□

THEORY QUESTIONS

(35) Prove that

a) $\forall p \in (0, +\infty): \lim_{n \in \mathbb{N}^*} n^p = +\infty$

b) $\forall a \in (1, +\infty): \lim_{n \in \mathbb{N}^*} a^n = +\infty$

(36) Use the limit definition to show that $\lim_{n \in \mathbb{N}^*} a_n = +\infty$ for the following sequences

a) $a_n = 3n^3 + n^2 + 5n + 1$ b) $a_n = n^5 + n^4 - 10n^3$

c) $a_n = \dfrac{n^3 + \sin(2n)}{n^2 + 1}$ d) $a_n = n 3^n + n^2$

e) $a_n = 3^n + 6^n + n\cos(5n)$ f) $a_n = (5n + \sin(5n)) 3^n$

(37) Use the limit definition to show that $\lim_{n \in \mathbb{N}^*} a_n = -\infty$ for the following sequences

a) $a_n = (1 - 2n)^3$ b) $a_n = \sin(5n) - 3n^2$

c) $a_n = -n^2 - n + 1$ d) $a_n = -5n^3 + n - 2$

e) $a_n = \dfrac{4 - n^3}{4n}$ f) $a_n = \cos(3n) - n^2 5^n$

g) $a_n = n\sin(3n) - 2^n - 5^n$ h) $a_n = 3^n - 5^n + \cos(2n)$

i) $a_n = (\cos(3n) - 3n) 7^n$

(38) Use the properties of the limit to evaluate the limit of the following sequences

a) $a_n = 3n^3 - 5n^2 + 2n - 1$

b) $a_n = \dfrac{-2n^3 + 3n - 5}{n^2 - 3n - 2}$

c) $a_n = 2^n + 3^n - 5^n$

d) $a_n = (3 + \cos(5n))(2^n - 5^n)$

e) $a_n = (\sin(9n) - 3)(2^n - 5^n + 3^n + \cos(3n))$

f) $a_n = \sum_{k=1}^{n} \dfrac{n^2}{n^2 + k^2}$

g) $a_n = \dfrac{7^n - 3^n}{5^n + 2^n}$

h) $a_n = \dfrac{4^n + 3^n - 6^n}{5^n + 2^n}$

i) $a_n = \sum_{k=1}^{n} k^n$

j) $a_n = \sum_{k=1}^{n} \dfrac{n^3 + \sin(n)}{n^3 + k^2}$

k) $a_n = \sum_{k=1}^{n} \dfrac{n5^n + \cos(5n)}{n2^n + k}$

l) $a_n = \sum_{k=1}^{n} \dfrac{n^2 + \cos(2n)}{(n+k)^2}$

m) $a_n = \dfrac{1}{(n+1)^2} \sum_{k=1}^{n} (k^2 + \cos(k) + \sin(k))$

Divergent sequences

In order to show that a sequence (a_n) (or more generally any net) is <u>divergent</u> (i.e. that <u>$\lim a_n$ does not exist</u>), we have to show that

$$\begin{cases} (a_n) \text{ not convergent} \\ \lim a_n \neq +\infty \wedge \lim a_n \neq -\infty \end{cases}$$

To do that, it is helpful to use the following results:

$$\boxed{\begin{array}{l} (a_n) \text{ not bounded} \Rightarrow (a_n) \text{ not convergent} \\ (|a_n|) \text{ not convergent} \Rightarrow (a_n) \text{ not convergent} \\ \lim a_n = +\infty \vee \lim a_n = -\infty \Rightarrow (a_n) \text{ not bounded} \\ (a_n) \text{ not Cauchy} \Rightarrow (a_n) \text{ not convergent.} \end{array}}$$

Specifically for a sequence (a_n), we can also use the following results:

$$\boxed{\begin{array}{l} \lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \Rightarrow \begin{cases} (a_n) \text{ lower bounded} \\ (a_n) \text{ not upper bounded} \end{cases} \\ \lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ not lower bounded} \end{cases} \\ (a_{pn+k}) \text{ not upper bounded} \Rightarrow (a_n) \text{ not upper bounded} \\ (a_{pn+k}) \text{ not lower bounded} \Rightarrow (a_n) \text{ not lower bounded} \end{array}}$$

with $p, k \in \mathbb{N}^*$

EXAMPLES

a) Show that $\forall n \in \mathbb{N}^*: a_n = (-1)^n (2n+3)$ is divergent

Solution

Since

$$|a_n| = |(-1)^n (2n+3)| = |2n+3| = 2n+3 > 2n, \ \forall n \in \mathbb{N}^*$$

and

$$\lim_{n \in \mathbb{N}^*} (2n) = +\infty$$

it follows that

$$\lim_{n \in \mathbb{N}^*} |a_n| = +\infty \Rightarrow (|a_n|) \text{ not bounded} \Rightarrow$$

$$\Rightarrow (|a_n|) \text{ not convergent} \Rightarrow$$

$$\Rightarrow \underline{(a_n) \text{ not convergent}}$$

To show that $\lim_{n \in \mathbb{N}^*} a_n \neq +\infty$, assume that $\lim_{n \in \mathbb{N}^*} a_n = +\infty$ in order to show a contradiction. Then, we have:

$$\lim_{n \in \mathbb{N}^*} a_n = +\infty \Rightarrow \underline{(a_n) \text{ lower bounded}}$$

and

$$a_{2n+1} = (-1)^{2n+1} (2(2n+1)+3) = -(4n+2+3) = -4n-5, \ \forall n \in \mathbb{N}^*$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} a_{2n+1} = \lim_{n \in \mathbb{N}^*} (-4n-5) = \lim_{n \in \mathbb{N}^*} (-4n) = -\infty \Rightarrow$$

$$\Rightarrow (a_{2n+1}) \text{ not lower bounded} \Rightarrow$$

$$\Rightarrow \underline{(a_n) \text{ not lower bounded}}$$

which is a contradiction. It follows that $\underline{\underline{\lim_{n \in \mathbb{N}^*} a_n \neq +\infty}}$

To show that $\lim\limits_{n\in\mathbb{N}^*} a_n \neq -\infty$, we assume that $\lim a_n = -\infty$ in order to show a contradiction. Then, we have

$\lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \Rightarrow (a_n)$ upper Bounded

and

$a_{2n} = (-1)^{2n}(2(2n)+3) = 4n+3,\ \forall n\in\mathbb{N}^* \Rightarrow$

$\Rightarrow \lim\limits_{n\in\mathbb{N}^*} a_{2n} = \lim\limits_{n\in\mathbb{N}^*}(4n+3) = \lim\limits_{n\in\mathbb{N}^*} 4n = +\infty \Rightarrow$

$\Rightarrow (a_{2n})$ not upper bounded

$\Rightarrow (a_n)$ not upper Bounded

It follows that $\underline{\underline{\lim\limits_{n\in\mathbb{N}^*} a_n \neq -\infty}}$, and we conclude that

$\begin{cases} (a_n) \text{ not convergent} \\ \lim\limits_{n\in\mathbb{N}^*} a_n \neq +\infty \wedge \lim\limits_{n\in\mathbb{N}^*} a_n \neq -\infty \end{cases} \Rightarrow (a_n)$ divergent.

b) Show that $\forall n \in \mathbb{N}^*: a_n = \dfrac{(-1)^n (3n)}{n+1}$ is divergent

Solution

Since

$$|a_n| = \left|\frac{(-1)^n (3n)}{n+1}\right| = \frac{3n}{n+1} \leq \frac{3n}{n} = 3, \ \forall n \in \mathbb{N}^*$$

$\Rightarrow (a_n)$ bounded $\Rightarrow \underline{\lim\limits_{n \in \mathbb{N}^*} a_n \neq +\infty \wedge \lim\limits_{n \in \mathbb{N}^*} a_n \neq -\infty}$

it is sufficient to show that

$\exists \varepsilon \in (0,+\infty): \forall n_0 \in \mathbb{N}^*: \exists n_1, n_2 \in \mathbb{N}^* - [n_0]: |a_{n_1} - a_{n_2}| \geq \varepsilon$

Choose $\varepsilon = 1$. Let $n_0 \in \mathbb{N}^*$ be given. Since:

$$\lim_{n \in \mathbb{N}^*} a_{2n} = \lim_{n \in \mathbb{N}^*} \frac{(-1)^{2n} 3(2n)}{2n+1} = \lim_{n \in \mathbb{N}^*} \frac{6n}{2n+1} =$$

$$= \lim_{n \in \mathbb{N}^*} \frac{6n}{2n} = \frac{6}{2} = 3 \Rightarrow$$

$\Rightarrow \exists p \in \mathbb{N}^*: \forall n \in \mathbb{N}^*: (n > p \Rightarrow |a_{2n} - 3| < 1)$

and

$$\lim_{n \in \mathbb{N}^*} a_{2n+1} = \lim_{n \in \mathbb{N}^*} \frac{(-1)^{2n+1} 3(2n+1)}{(2n+1)+1} = \lim_{n \in \mathbb{N}^*} \frac{-6n-3}{2n+2}$$

$$= \lim_{n \in \mathbb{N}^*} \frac{-6n}{2n} = \frac{-6}{2} = -3 \Rightarrow$$

$\Rightarrow \exists q \in \mathbb{N}^*: \forall n \in \mathbb{N}^*: (n > q \Rightarrow |a_{2n+1} - (-3)| < 1)$

Choose $p, q \in \mathbb{N}^*$ such that

$$\begin{cases} \forall n \in \mathbb{N}^*: (n > p \Rightarrow |a_{2n} - 3| < 1) \\ \forall n \in \mathbb{N}^*: (n > q \Rightarrow |a_{2n+1} + 3| < 1) \end{cases}$$

Choose $n_1 = \max\{n_0, p\} + 1$ and $n_2 = \max\{n_0, q\} + 1$

which implies that $n_1, n_2 \in \mathbb{N}^* - [n_0]$. Then, we have:

$$\begin{cases} n_1 > p \\ n_2 > q \end{cases} \Rightarrow \begin{cases} |a_{n_1} - 3| < 1 \\ |a_{n_2} + 3| < 1 \end{cases} \Rightarrow \begin{cases} -1 < a_{n_1} - 3 < 1 \\ -1 < a_{n_2} + 3 < 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 2 < a_{n_1} < 4 \\ -4 < a_{n_2} < -2 \end{cases} \Rightarrow a_{n_2} < -2 < 2 < a_{n_1}$$

$$\Rightarrow |a_{n_1} - a_{n_2}| > |2 - (-2)| = 4$$

$$\Rightarrow |a_{n_1} - a_{n_2}| > 4$$

We have thus shown that

$\forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| > 4$

$\Rightarrow (a_n)$ not Cauchy $\Rightarrow$ <u>(a_n) not convergent</u>

We conclude that

$$\begin{cases} (a_n) \text{ not convergent} \\ \lim\limits_{n \in \mathbb{N}^*} a_n \neq +\infty \wedge \lim\limits_{n \in \mathbb{N}^*} a_n \neq -\infty \end{cases} \Rightarrow (a_n) \text{ divergent.}$$

EXERCISES

(39) Show that the following sequences are divergent.

a) $a_n = \sin(n\pi/7)$

b) $a_n = \sin(n\pi/4) + \cos(n\pi/2)$

c) $a_n = \dfrac{(-1)^n n^2}{3n+1}$

d) $a_n = (-1)^n 7^n$

e) $a_n = \dfrac{(-1)^n 7^n}{7^n + 5^n}$

f) $a_n = \dfrac{6(-1)^n 3^n}{3^n + 2^n}$

g) $a_n = (-1)^n \sqrt[n]{2^n + 3^n}$

h) $a_n = (-1)^n \sqrt[n]{n^2 + 3n + 2}$

(40) Let (a_n) be a sequence. Show that

a) $\lim\limits_{n\in\mathbb{N}^*} a_n = l \in \mathbb{R} \Leftrightarrow \lim\limits_{n\in\mathbb{N}^*} a_{2n} = l \wedge \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} = l$

b) $\lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \Leftrightarrow \lim\limits_{n\in\mathbb{N}^*} a_{2n} = +\infty \wedge \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} = +\infty$

c) $\lim\limits_{n\in\mathbb{N}^*} a_{2n} = l_1 \in \mathbb{R} \wedge \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} = l_2 \in \mathbb{R} \wedge l_1 \neq l_2 \Rightarrow$

$\Rightarrow (a_n)$ divergent

d) $\begin{cases} (a_{2n+1}) \text{ convergent} \\ \lim\limits_{n\in\mathbb{N}^*} a_{2n} \in \{+\infty, -\infty\} \end{cases} \Rightarrow (a_n)$ divergent

e) $\begin{cases} (a_{2n}) \text{ convergent} \\ \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} \in \{+\infty, -\infty\} \end{cases} \Rightarrow (a_n)$ divergent.