STRUCTURE OF THE SET OF REAL NUMBERS

Preliminaries

Let A be a set and $p(x)$ a statement about x. We will use the following notation for quantified statements:

- Universal quantifier
  $\forall x \in A : p(x)$
  "For all $x \in A$, $p(x)$ is satisfied"
- Existential quantifier
  $\exists x \in A : p(x)$
  "There exists at least one $x \in A$ such that $p(x)$ is satisfied"
- Unique existential quantifier
  $\exists ! \, x \in A : p(x)$
  "There exists a unique $x \in A$ such that $p(x)$ is satisfied"

We define $\mathbb{R}$ to be the set of real numbers. Although there are several constructions of $\mathbb{R}$ from the set of natural numbers, we sidestep the construction, and assume that $\mathbb{R}$ exists and satisfies 3 axioms

1) The field axioms
2) The axiom of order
3) The axiom of completeness

All properties of real numbers are then derived as a consequence of these axioms.

▼ The field axiom

Axiom: (Field axiom)
The set $\mathbb{R}$ is endowed with two operations: addition ("+") and multiplication ("·") such that:

1) $\forall x, y \in \mathbb{R} : (x+y = y+x \wedge xy = yx)$ — Commutative
2) $\forall x, y, z \in \mathbb{R} : \begin{cases} (x+y)+z = x+(y+z) \\ (xy)z = x(yz) \end{cases}$ — Associative
3) $\forall x \in \mathbb{R} : \begin{cases} x+0 = 0+x = x \\ 1x = x1 = x \end{cases}$ — Neutral elements
4) $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x+y = y+x = 0$ — Additive inverse
5) $\forall x \in \mathbb{R} - \{0\} : \exists y \in \mathbb{R} : xy = yx = 1$ — Multiplicative inverse
6) $\forall x, y, z \in \mathbb{R} : x(y+z) = xy + xz$ — Distributive

Remark
An immediate consequence of the field axiom is that

$$\begin{cases} (\mathbb{R}, +) \text{ abelian group} \\ (\mathbb{R} - \{0\}, \cdot) \text{ abelian group} \end{cases}$$

It follows that the $y \in \mathbb{R}$ claimed to exist in items 3,4 has to be unique (see my Linear Algebra Lecture notes for more details). As a result, items 3,4 can be strengthened to read:

$\forall x \in \mathbb{R} : \exists! y \in \mathbb{R} : x+y = y+x = 0$
$\forall x \in \mathbb{R} - \{0\} : \exists! y \in \mathbb{R} : xy = yx = 1$

and that leads to the following notation:

notation :

Let $x \in \mathbb{R}$ be given. Then we introduce the following notation.

a) $-x$ is the unique number such that $x+(-x)=(-x)+x=0$

b) If $x \neq 0$, then $x^{-1}=1/x$ is the unique number such that $xx^{-1}=x^{-1}x=1$

c) Subtraction: $\forall x,y \in \mathbb{R}: x-y = x+(-y)$

d) Division: $\forall x \in \mathbb{R}: \forall y \in \mathbb{R}-\{0\}: x/y = xy^{-1}$

e) 0 is the zero element

f) 1 is the unit element.

→ Immediate consequences of the field axioms

1 $\boxed{\begin{array}{l}\text{Uniqueness of zero element}\\ (\forall x \in \mathbb{R}: x+z=z+x=x) \Rightarrow z=0\end{array}}$

Proof

Assume that $\forall x \in \mathbb{R}: x+z=z+x=x$. Then:

$$\begin{aligned} z &= 0+z && [0 \text{ zero element}] \\ &= 0 && [\text{hypothesis}]. \end{aligned}$$

2 $\boxed{\begin{array}{l}\text{Uniqueness of unit number}\\ (\forall x \in \mathbb{R}: xz=zx=x) \Rightarrow z=1\end{array}}$

Proof

Assume that $\forall x \in \mathbb{R}: xz=zx=x$. Then

$$\begin{aligned} z &= 1z && [1 \text{ unit element}] \\ &= 1 && [\text{hypothesis}]. \end{aligned}$$

③ **Addition cancellation law**
$\forall x,y,z \in \mathbb{R} : (x+z = y+z \Leftrightarrow x=y)$

Proof

Let $x,y,z \in \mathbb{R}$ be given.

($\Rightarrow$) : Assume that $x+z = y+z$. Then:

$$
\begin{aligned}
x &= x+0 && [\text{zero element}]\\
&= x+[z+(-z)] && [-z \text{ inverse of } z]\\
&= (x+z)+(-z) && [\text{associative}]\\
&= (y+z)+(-z) && [\text{hypothesis}]\\
&= y+[z+(-z)] && [\text{associative}]\\
&= y+0 && [-z \text{ inverse of } z]\\
&= y && [\text{zero element}]
\end{aligned}
$$

($\Leftarrow$) : Assume that $x=y$. Then, it immediately follows that $x+z=y+z$. □

④ **Multiplication cancellation law**
a) $\forall x,y,z \in \mathbb{R} : (x=y \Rightarrow xz = yz)$
b) $\forall x,y,z \in \mathbb{R} : \begin{cases} xz = yz \\ z \neq 0 \end{cases} \Rightarrow x=y$

Proof

Let $x,y,z \in \mathbb{R}$ be given.

a) Assume that $x=y$. It immediately follows that $xz=yz$.

b) Assume that $xz=yz$ and $z\neq 0$. Then:

$z \neq 0 \Rightarrow \exists z' \in \mathbb{R} : zz' = z'z = 1$.

Choose $z' \in \mathbb{R}$ such that $zz' = z'z = 1$. Then:

$$
\begin{aligned}
x &= x1 && [\text{unit element}] \\
&= x(zz') && [z' \text{ inverse of } z] \\
&= (xz)z' && [\text{associative}] \\
&= (yz)z' && [\text{hypothesis}] \\
&= y(zz') && [\text{associative}] \\
&= y1 && [z' \text{ inverse of } z] \\
&= y && [\text{unit element}].
\end{aligned}
$$

$\square$

(5) **Multiplication law**

$\forall x \in \mathbb{R}: \ 0x = x0 = 0$

Proof

Let $x \in \mathbb{R}$ be given. Choose some $y \in \mathbb{R}$. Then:

$$
\begin{aligned}
xy + 0 &= xy && [\text{zero element}] \\
&= x(y+0) && [\text{zero element}] \\
&= xy + x0 && [\text{distributive}] \\
\Rightarrow 0 &= x0 && [\text{addition cancellation law}]
\end{aligned}
$$

It follows, via commutative, that $0x = x0 = 0$.

(6) **Law of signs**

$\forall x, y \in \mathbb{R}: \ (-x)y = x(-y) = -xy$

Proof

Let $x, y \in \mathbb{R}$ be given. Then:

$$
\begin{aligned}
xy + (-x)y &= [x + (-x)]y && [\text{distributive}] \\
&= 0y && [-x \text{ inverse of } x] \\
&= 0 \Rightarrow && [\text{nullification}]
\end{aligned}
$$

$\Rightarrow (-x)y$ additive inverse of xy

$\Rightarrow -xy = (-x)y$.

and

$$
\begin{aligned}
xy + x(-y) &= x[y + (-y)] && [\text{distributive}] \\
&= x0 && [-y \text{ inverse of } y] \\
&= 0 && [\text{nullification}]
\end{aligned}
$$

$\Rightarrow x(-y)$ additive inverse of xy

$\Rightarrow -xy = x(-y)$.

We conclude that $(-x)y = x(-y) = -xy$.

$\to$ The following are immediate corollaries of the law of signs:

$\forall x \in \mathbb{R} : -(-x) = x$

$\forall x, y \in \mathbb{R} : (-x)(-y) = xy$.

(7) $\boxed{\begin{array}{l} \text{Zero product property} \\ \forall x, y \in \mathbb{R} : (xy = 0 \Leftrightarrow (x = 0 \lor y = 0)) \end{array}}$

Proof

Let $x, y \in \mathbb{R}$ be given.

$(\Rightarrow)$: Assume that $xy = 0$. We distinguish between the following cases.

Case 1: Assume that $x=0$. Then

$$x=0 \Rightarrow \underline{x=0 \lor y=0}.$$

Case 2: Assume that $x \neq 0$. Then

$$\begin{aligned} xy &= 0 && \text{[hypothesis]} \\ &= x0 && \text{[zero element]} \\ \Rightarrow y &= 0 && \text{[via } x\neq 0\text{, cancellation law]} \\ &\Rightarrow \underline{x=0 \lor y=0} \end{aligned}$$

In both cases we get $x=0 \lor y=0$.

($\Leftarrow$): Assume that $x=0 \lor y=0$. We distinguish between the following cases:

Case 1: Assume that $x=0$. Then $xy = 0y = 0$

Case 2: Assume that $y=0$. Then $xy = x0 = 0$

In both cases we find that $xy=0$.

(8) $\boxed{\begin{array}{l} \text{Adding / Multiplying equations} \\ \forall x,y,a,b \in \mathbb{R}: \begin{cases} x=y \\ a=b \end{cases} \Rightarrow \begin{cases} x+a = y+b \\ ax = by \end{cases} \end{array}}$

The proof is trivial.

(9) $\boxed{\begin{array}{l} \text{Sum of squares} \\ \forall a,b \in \mathbb{R}: (a^2+b^2=0 \Leftrightarrow (a=0 \land b=0)) \\ \forall a_1,\dots,a_n \in \mathbb{R}: (a_1^2+a_2^2+\dots+a_n^2=0 \Leftrightarrow \\ \qquad \Leftrightarrow \forall k \in \{1,2,\dots,n\}: a_k=0) \end{array}}$

The proof requires the axiom of order.

THEORY QUESTIONS

① State the field axiom of $\mathbb{R}$.

② Show the following properties of real numbers using the field axiom:

a) $\forall x,y,z \in \mathbb{R}: (x+z = y+z \Leftrightarrow x=y)$

b) $\forall x,y,z \in \mathbb{R}: \begin{cases} xz = yz \\ z \neq 0 \end{cases} \Rightarrow x=y$

③ Show the following using the field axioms and the cancellation law.

a) $\forall x \in \mathbb{R}: (0x = x0 = 0)$ (nullification law)

④ Show the following using the field axioms, the cancellation law and the nullification law

a) $\forall x,y \in \mathbb{R}: (-x)y = x(-y) = -xy$

b) $\forall x,y \in \mathbb{R}: xy = 0 \Leftrightarrow (x=0 \vee y=0)$

EXERCISE

⑤ Use proof by induction to show that

$\forall x_1, x_2, \ldots, x_n \in \mathbb{R}: (x_1 x_2 \cdots x_n = 0 \Leftrightarrow$
$\Leftrightarrow (x_1 = 0 \vee x_2 = 0 \vee \cdots \vee x_n = 0))$.

Integer powers

We define the following number sets:

a) Set of natural numbers

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

$\mathbb{N}^* = \mathbb{N} - \{0\}$

b) Set of integers

$\mathbb{Z} = \{x, -x \mid x \in \mathbb{N}\} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

$\mathbb{Z}^* = \mathbb{Z} - \{0\}$.

Then we define integer powers as follows

Def: Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we define

$$a^n = \begin{cases} 1 & , \text{if } n = 0 \\ a^{n-1} a & , \text{if } n \in \mathbb{N}^* \end{cases}$$

$$a^{-n} = \frac{1}{a^n} \quad , \text{if } a \neq 0$$

Using proof by induction on $\mathbb{Z}$, we can show that:

Prop:

a) $\forall a \in \mathbb{R} - \{0\} : \forall x, y \in \mathbb{Z} : a^x a^y = a^{x+y}$

b) $\forall a \in \mathbb{R} - \{0\} : \forall x, y \in \mathbb{Z} : (a^x)^y = a^{xy}$

c) $\forall a, b \in \mathbb{R} - \{0\} : \forall x \in \mathbb{Z} : (ab)^x = a^x b^x$

Proof of (a)

Let $a \in \mathbb{R} - \{0\}$ and $x, \in \mathbb{Z}$ be given.
For $y = 0$:
$a^x a^y = a^x a^0 = a^x 1 = a^x = a^{x+0} = a^{x+y}$
For $y = n$, we assume that $a^x a^n = a^{x+n}$.
For $y = n+1$, it follows that:
$$a^x a^y = a^x a^{n+1} = a^x (a^n a) = (a^x a^n) a = $$
$$= a^{x+n} a = a^{x+n+1} = a^{x+y}$$
For $y = n-1$, it follows that
$$a^x a^y = a^x a^{n-1} = \frac{(a^x a^{n-1}) a}{a} = \frac{a^x (a^{n-1} a)}{a} =$$
$$= \frac{a^x a^n}{a} = \frac{a^{x+n}}{a} = a^{x+n-1} = a^{x+y}$$
Statement (a) follows by induction, for all $y \in \mathbb{Z}$.

Proof of (b), (c) $\blacktriangleright$ $\boxed{\text{Homework.}}$ You can use (a) to prove (b) and (c), again via induction.

THEORY QUESTIONS

⑥ Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. State the definition of a^n and a^{-n}.
⑦ Show that $\forall a \in \mathbb{R}-\{0\}: \forall x,y \in \mathbb{Z}: a^x a^y = a^{x+y}$.

EXERCISES

⑧ Show, using proof by induction on $\mathbb{Z}$, the following:
a) $\forall a \in \mathbb{R}-\{0\}: \forall x,y \in \mathbb{Z}: (a^x)^y = a^{xy}$
b) $\forall a,b \in \mathbb{R}-\{0\}: \forall x \in \mathbb{Z}: (ab)^x = a^x b^x$
⑨ Note that by definition $0^0 = 1$. Explain why we cannot define 0^{-1} if we wish the main properties of powers (see questions 7,8) to be satisfied. Generalize the argument for 0^{-x} for all $x \in \mathbb{N}^*$.

The order axiom

Let $\mathcal{P}(\mathbb{R})$ be the set of all subsets of $\mathbb{R}$ such that
$A \in \mathcal{P}(\mathbb{R}) \Leftrightarrow A \subseteq \mathbb{R}$.

Axiom : (order axiom)

$$\boxed{\exists!\ \mathbb{R}_+^* \in \mathcal{P}(\mathbb{R}) : \begin{cases} \forall x,y \in \mathbb{R}_+^* : (x+y) \in \mathbb{R}_+^* \wedge xy \in \mathbb{R}_+^* \\ \forall x \in \mathbb{R} \quad : x = 0 \ \underline{\vee}\ x \in \mathbb{R}_+^* \ \underline{\vee} -x \in \mathbb{R}_+^* \end{cases}}$$

Here $\underline{\vee}$ represents an exclusive "or"
$\mathbb{R}_+^*$ is the set of strictly positive numbers.
We now define the relations "$<$", "$>$", "$\leq$", "$\geq$".

Def: (Inequalities)
Let $x,y \in \mathbb{R}$ be given. Then

$$x < y \Leftrightarrow (y-x) \in \mathbb{R}_+^*$$
$$x > y \Leftrightarrow (x-y) \in \mathbb{R}_+^*$$
$$x \leq y \Leftrightarrow (x<y \vee x=y)$$
$$x \geq y \Leftrightarrow (x>y \vee x=y).$$

notation : We write:

$\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$ $\qquad \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$

$\mathbb{R}_-^* = \{x \in \mathbb{R} \mid x < 0\}$ $\qquad \mathbb{R}_- = \{x \in \mathbb{R} \mid x < 0\}$

$\mathbb{R}^* = \mathbb{R} - \{0\}$

- The following statements are immediate consequences of the order axiom

a) $\forall x, y \in \mathbb{R} : (x > 0 \land y > 0) \Rightarrow (x + y > 0 \land xy > 0)$
b) $\forall x, y \in \mathbb{R} : (x < 0 \land y < 0) \Rightarrow (x + y < 0 \land xy > 0)$
c) $\forall x, y \in \mathbb{R} : (x = y \veebar x < y \veebar x > y)$

→ <u>Equisigned and heterosigned numbers</u>

Def: Let $x, y \in \mathbb{R}$ be given. Then:
x, y equisigned $\Leftrightarrow x, y \in \mathbb{R}_+^* \lor x, y \in \mathbb{R}_-^*$
x, y heterosigned $\Leftrightarrow (x \in \mathbb{R}_+^* \land y \in \mathbb{R}_-^*) \lor (x \in \mathbb{R}_-^* \land y \in \mathbb{R}_+^*)$

We now show that:

Thm: (Law of signs)
$\forall x, y \in \mathbb{R} : x, y$ equisigned $\Leftrightarrow xy > 0$
$\forall x, y \in \mathbb{R} : x, y$ heterosigned $\Leftrightarrow xy < 0$

Immediate consequences of the law of signs:

a) $\forall a \in \mathbb{R} : (a \neq 0 \Rightarrow a^2 > 0)$
b) $1 > 0 \longleftarrow$ Show that $1 \neq 0$. Then $1 = 1 \cdot 1 = 1^2 > 0$.
c) $\forall a \in \mathbb{R} : (a \neq 0 \Rightarrow a, 1/a$ equisigned).
→ $\forall a \in \mathbb{R} : (a > 0 \Rightarrow 1/a > 0)$
→ $\forall a \in \mathbb{R} : (a < 0 \Rightarrow 1/a < 0)$
d) $\forall x, y \in \mathbb{R} : (xy > 0 \Rightarrow x/y > 0)$
$\forall x, y \in \mathbb{R} : (xy < 0 \Rightarrow x/y < 0)$

Proof of law of signs

$(\Rightarrow)$:

Assume that x, y equisigned. Then:

$\underline{x, y \text{ equisigned}} \Rightarrow (x>0 \wedge y>0) \vee (x<0 \wedge y<0)$
$\Rightarrow xy>0 \vee (-x>0 \wedge -y>0)$
$\Rightarrow xy>0 \vee (-x)(-y)>0$
$\Rightarrow xy>0 \vee xy>0 \Rightarrow \underline{xy>0}.$

Assume that x, y heterosigned. Then

$\underline{x, y \text{ heterosigned}} \Rightarrow (x>0 \wedge y<0) \vee (x<0 \wedge y>0)$
$\Rightarrow (x>0 \wedge -y>0) \vee (-x>0 \wedge y>0)$
$\Rightarrow x(-y)>0 \vee (-x)y>0$
$\Rightarrow -xy>0 \vee -xy>0 \Rightarrow -xy>0 \Rightarrow \underline{xy<0}.$

$(\Leftarrow)$

We note that

$xy \neq 0 \Rightarrow x \neq 0 \wedge y \neq 0$
$\Rightarrow (x>0 \veebar x<0) \wedge (y>0 \veebar y<0)$
$\Rightarrow x, y \text{ equisigned} \veebar x, y \text{ heterosigned}$

and it follows that

$\underline{xy>0} \Rightarrow xy \neq 0 \wedge \overline{xy<0}$
$\Rightarrow \begin{cases} x, y \text{ equisigned} \veebar x, y \text{ heterosigned} \\ x, y \text{ not heterosigned} \end{cases}$
$\Rightarrow \underline{x, y \text{ equisigned}}$

and

$\underline{xy<0} \Rightarrow xy \neq 0 \wedge \overline{xy>0}$
$\Rightarrow \begin{cases} x, y \text{ equisigned} \veebar x, y \text{ heterosigned} \\ x, y \text{ not equisigned} \end{cases}$
$\Rightarrow \underline{x, y \text{ heterosigned}}$ $\square$

In this argument we used the contrapositive of
$\forall a,b \in \mathbb{R} : (ab = 0 \Leftarrow (a = 0 \vee b = 0))$
which is given by
$\forall a,b \in \mathbb{R} : (ab \neq 0 \Leftrightarrow (a \neq 0 \wedge b \neq 0))$

Transitive property.

<u>Prop</u> : $\boxed{\forall x,y,z \in \mathbb{R} : ((x > y \wedge y > z) \Rightarrow x > z)}$

<u>Proof</u>

Let $x,y,z \in \mathbb{R}$ be given and assume that $x > y \wedge y > z$.
Then:

$$\begin{cases} x > y \\ y > z \end{cases} \Rightarrow \begin{cases} x - y > 0 \\ y - z > 0 \end{cases} \Rightarrow (x-y)+(y-z) > 0$$

$$\Rightarrow x - z > 0 \Rightarrow x > z. \qquad \square$$

Immediate corollaries of the transitive property is:
$\forall x,y,z \in \mathbb{R} : ((x < y \wedge y < z) \Rightarrow x < z)$
$\forall x \in \mathbb{R}_+^* : \forall y \in \mathbb{R}_-^* : x > y$

Order and operations on $\mathbb{R}$.

(1) $\boxed{\forall x,y,z \in \mathbb{R} : (x > y \Leftarrow x + z > y + z)}$

<u>Proof</u>

Let $x, y, z \in \mathbb{R}$ be given. Then

$$\underline{x+z > y+z} \Leftrightarrow (x+z)-(y+z) > 0$$
$$\Leftrightarrow x+z-y-z > 0$$
$$\Leftrightarrow x-y > 0 \Leftrightarrow \underline{x > y}$$

$\square$

(2) $\boxed{\forall a, x, y \in \mathbb{R} : ((x > y \wedge a > 0) \Rightarrow ax > ay)}$

Proof

Let $a, x, y \in \mathbb{R}$ be given and assume that $x > y \wedge a > 0$.

Then:

$$\begin{cases} x > y \\ a > 0 \end{cases} \Rightarrow \begin{cases} x-y > 0 \\ a > 0 \end{cases} \Rightarrow a(x-y) > 0 \Rightarrow ax - ay > 0$$
$$\Rightarrow ax > ay.$$

$\square$

(3) $\boxed{\forall a, x, y \in \mathbb{R} : ((x > y \wedge a < 0) \Rightarrow ax < ay)}$

Proof

Let $a, x, y \in \mathbb{R}$ be given and assume that $x > y \wedge a < 0$

Then:

$$\begin{cases} x > y \\ a < 0 \end{cases} \Rightarrow \begin{cases} x-y > 0 \\ a < 0 \end{cases} \Rightarrow a(x-y) < 0 \Rightarrow ax - ay < 0$$
$$\Rightarrow ax < ay.$$

$\square$

(4) $\boxed{\forall a, b, x, y \in \mathbb{R} : ((x > y \wedge a > b) \Rightarrow x+a > y+b)}$

Proof

Let $a, b, x, y \in \mathbb{R}$ be given and assume that $x > y \wedge a > b$. Then:

$$\begin{cases} x > y \\ a > b \end{cases} \Rightarrow \begin{cases} x - y > 0 \\ a - b > 0 \end{cases} \Rightarrow (x-y) + (a-b) > 0 \Rightarrow$$
$$\Rightarrow (x+a) - (y+b) > 0 \Rightarrow x + a > y + b. \qquad \square$$

⑤ $\boxed{\forall a, b, x, y \in \mathbb{R} : \left(\begin{cases} x > a > 0 \\ y > b > 0 \end{cases} \Rightarrow xy > ab \right)}$

Proof

Let $a, b, x, y \in \mathbb{R}$ be given and assume that $x > a > 0$ and $y > b > 0$. Then, we note that

$$xy - ab = xy - ay + ay - ab = (x-a)y + (y-b)a \qquad (1)$$

and it follows that

$$\begin{cases} x > a > 0 \\ y > b > 0 \end{cases} \Rightarrow \begin{cases} x - a > 0 \wedge y > 0 \\ y - b > 0 \wedge a > 0 \end{cases} \Rightarrow \begin{cases} (x-a)y > 0 \\ (y-b)a > 0 \end{cases}$$
$$\Rightarrow (x-a)y + (y-b)a > 0$$
$$\Rightarrow xy - ab > 0 \qquad \text{[via Eq. (1)]}$$
$$\Rightarrow xy > ab. \qquad \square$$

→ Statements ④ and ⑤ show that

a) We can always add two inequalities that have the same direction

b) We can always multiply two inequalities that have the same direction if both sides on both inequalities are positive.

c) Using the method of induction, we can show that

$$\boxed{\begin{array}{l}\forall x,y\in\mathbb{R}_+ : \forall n\in\mathbb{N}^* : (x>y \Leftrightarrow x^n>y^n)\\ \forall x,y\in\mathbb{R}_+^* : \forall n\in\mathbb{N}^* : (x>y \Leftrightarrow x^{-n}<y^{-n})\end{array}}$$

⑥ $\boxed{\forall a,b\in\mathbb{R} : a^2+b^2=0 \Leftrightarrow a=0 \land b=0}$

Proof

($\Rightarrow$): Assume that $a^2+b^2=0$. To show that $a=0$, assume that $a\neq 0$, in order to show a contradiction. It follows that

$$\begin{cases} a\neq 0 \\ b^2\geq 0\end{cases} \Rightarrow \begin{cases} a^2>0 \\ b^2\geq 0\end{cases} \Rightarrow a^2+b^2\geq a^2>0 \Rightarrow$$

$$\Rightarrow a^2+b^2>0 \leftarrow \text{contradiction with hypothesis}$$

It follows that $a=0$. Similarly, we show that $b=0$. We conclude that $\underline{a=0 \land b=0}$.

($\Leftarrow$): Assume that $a=0\land b=0$. Then $a^2+b^2=0^2+0^2=0$.

THEORY QUESTIONS

(10) State the order axiom of $\mathbb{R}$

(11) Let $x, y \in \mathbb{R}$ be given. State the definition for

a) x, y equisigned

b) x, y heterosigned

(12) Use the order axiom and the law of signs to show that

a) $\forall x, y, z \in \mathbb{R}: ((x > y \wedge y > z) \Rightarrow x > z)$

b) $\forall x, y, z \in \mathbb{R}: (x > y \Leftrightarrow x + z > y + z)$

c) $\forall a, x, y \in \mathbb{R}: ((x > y \wedge a > 0) \Rightarrow ax > ay)$

d) $\forall a, x, y \in \mathbb{R}: ((x > y \wedge a < 0) \Rightarrow ax < ay)$

e) $\forall a, b, x, y \in \mathbb{R}: ((x > y \wedge a > b) \Rightarrow x + a > y + b)$

f) $\forall a, b, x, y \in \mathbb{R}: \left(\begin{cases} x > a > 0 \\ y > b > 0 \end{cases} \Rightarrow xy > ab \right)$

g) $\forall a, b \in \mathbb{R}: (a^2 + b^2 = 0 \Leftrightarrow a = 0 \wedge b = 0)$

EXERCISE

(12a) Use the properties ~~defined~~ derived from the order axiom to prove that

a) $\forall x, y \in \mathbb{R}_+ : \forall n \in \mathbb{N}^* : (x > y \Leftrightarrow x^n > y^n)$

b) $\forall x, y \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : (x > y \Leftrightarrow x^{-n} < y^{-n})$

The Bernoulli inequality

$$\boxed{\forall a \in \mathbb{R} : \forall n \in \mathbb{N} : (a > -1 \Rightarrow (1+a)^n \geq 1+na)}$$

Proof

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$ be given and assume that $a > -1$. Then, we have $1+a > 0$. We use proof by induction on $n \in \mathbb{N}$.

For $n = 0$:

$$\begin{cases} (1+a)^n = (1+a)^0 = 1 \\ 1+na = 1+0a = 1 \end{cases} \Rightarrow (1+a)^n \geq 1+na$$

For $n = k$, assume that $(1+a)^k \geq 1+ka$.

For $n = k+1$, we will show that $(1+a)^{k+1} \geq 1+(k+1)a$

We have:

$$\begin{cases} (1+a)^k \geq 1+ka \\ 1+a > 0 \end{cases} \Rightarrow (1+a)^k(1+a) \geq (1+ka)(1+a) \Rightarrow$$

$$\begin{aligned} \Rightarrow (1+a)^{k+1} &= (1+a)^k(1+a) \\ &\geq (1+ka)(1+a) \\ &= 1+a+ka+ka^2 \\ &= 1+(k+1)a+ka^2 \\ &\geq 1+(k+1)a \Rightarrow (1+a)^{k+1} \geq 1+(k+1)a \end{aligned}$$

By induction, we conclude that

$\forall a \in \mathbb{R} : \forall n \in \mathbb{N} : (a > -1 \Rightarrow (1+a)^n \geq 1+na)$ $\square$

EXERCISES

(13) Use the Bernoulli inequality or proof by induction to show that

a) $\forall n \in \mathbb{N}^* : 5^n > 1+4n$

b) $\forall n \in \mathbb{N}^* : 3^n > 2^n (n+1)$

c) $\forall n \in \mathbb{N}^* : (1+1/n)^n \geqslant 2$

d) $\forall n \in \mathbb{N}^* : \left(\dfrac{2n}{n+1}\right)^n \geqslant \dfrac{n+1}{2}$

e) $\forall n \in \mathbb{N}^* : 2^{n+2} > 2n+5$

f) $\forall n \in \mathbb{N}^* : 3^{2n} > 2^{2n+1}$

g) $\forall n \in \mathbb{N}^* : (n \geqslant 4 \Rightarrow 3^{n-1} > n^2)$

h) $\forall n \in \mathbb{N}^* : (n \geqslant 10 \Rightarrow 2^n > n^3)$

i) $\forall a, b \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : (n \geqslant 2 \Rightarrow (a+b)^n > a^n + na^{n-1}b)$.

Intervals and absolute values

Def: Let $a, b \in \mathbb{R}$ be given. We define:

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$(a,+\infty) = \{x \in \mathbb{R} \mid x > a\}$

$[a,+\infty) = \{x \in \mathbb{R} \mid x \geq a\}$

$(-\infty,b) = \{x \in \mathbb{R} \mid x < b\}$

$(-\infty,b] = \{x \in \mathbb{R} \mid x \leq b\}$

Def: Let $x \in \mathbb{R}$ be given. We define the absolute value $|x|$ such that

$$|x| = \begin{cases} x & , \text{ if } x \in [0,+\infty) \\ -x & , \text{ if } x \in (-\infty,0) \end{cases}$$

The following are immediate consequences of the absolute value definition:

a) $\forall x \in \mathbb{R} : (x = 0 \Leftrightarrow |x| = 0)$

b) $\forall x \in \mathbb{R} : (|x| \geq x \wedge |x| \geq -x)$

c) $\forall x \in \mathbb{R} : |x| = \max\{x, -x\}$

d) $\forall x \in \mathbb{R} : -|x| \leq x \leq |x|$

e) $\forall x \in \mathbb{R} : |-x| = |x|$

f) $\forall x \in \mathbb{R} : |x| \geq 0$

g) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : |x|^{2n} = x^{2n}$

Properties of the absolute value

① Let $x \in \mathbb{R}$ and $p \in (0, +\infty)$ be given. Then

a) $|x| < p \Leftrightarrow x \in (-p, p)$

b) $|x| > p \Leftrightarrow x \in (-\infty, -p) \cup (p, +\infty)$

c) $|x| = p \Leftrightarrow (x = p \lor x = -p)$.

Proof

Let $x \in \mathbb{R}$ and $p \in (0, +\infty)$ be given.

a) $\underline{|x| < p} \Leftrightarrow |x|^2 < p^2$ [via $|x| \geq 0$ and $p > 0$]

$\Leftrightarrow x^2 < p^2 \Leftrightarrow x^2 - p^2 < 0 \Leftrightarrow (x-p)(x+p) < 0$

$\Leftrightarrow x-p,\ x+p$ heterosigned

$\Leftrightarrow \begin{cases} x - p < 0 \\ x + p > 0 \end{cases} \Leftrightarrow \begin{cases} x < p \\ x > -p \end{cases} \Leftrightarrow -p < x < p$

$\Leftrightarrow \underline{x \in (-p, p)}$

b) $\underline{|x| > p} \Leftrightarrow |x|^2 > p^2$ [via $|x| \geq 0$ and $p > 0$]

$\Leftrightarrow x^2 > p^2 \Leftrightarrow x^2 - p^2 > 0 \Leftrightarrow (x-p)(x+p) > 0$

$\Leftrightarrow x-p,\ x+p$ equisigned

$\Leftrightarrow \begin{cases} x - p > 0 \\ x + p > 0 \end{cases} \lor \begin{cases} x - p < 0 \\ x + p < 0 \end{cases}$

$\Leftrightarrow \begin{cases} x > p \\ x > -p \end{cases} \lor \begin{cases} x < p \\ x < -p \end{cases} \Leftrightarrow$

$\Leftrightarrow x > p \lor x < -p \Leftrightarrow x \in (p, +\infty) \lor x \in (-\infty, -p)$

$\Leftrightarrow \underline{x \in (-\infty, -p) \cup (p, +\infty)}$

c) $\underline{|x| = p} \Leftrightarrow |x|^2 = p^2 \Leftrightarrow x^2 = p^2 \Leftrightarrow x^2 - p^2 = 0$
$\Leftrightarrow (x-p)(x+p) = 0 \Leftrightarrow x - p = 0 \lor x + p = 0$
$\Leftrightarrow \underline{x = p \lor x = -p.}$

② $\boxed{\forall x, y \in \mathbb{R} : ||x| - |y|| \leq |x+y| \leq |x| + |y|}$

<u>Proof</u>

Let $x, y \in \mathbb{R}$ be given. Then:

$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases} \Rightarrow -(|x|+|y|) \leq x+y \leq |x|+|y|$

$\Rightarrow |x+y| \leq |x|+|y| \quad$ [via ①]

We conclude that $\forall x, y \in \mathbb{R} : |x+y| \leq |x|+|y|$.

Furthermore, we have:

$|x| = |x+y-y| \leq |x+y| + |-y| = |x+y| + |y| \Rightarrow$
$\Rightarrow |x+y| \geq |x| - |y| \quad (1)$

and

$|y| = |y+x-x| \leq |y+x| + |-x| = |x+y| + |x| \Rightarrow$
$\Rightarrow |x+y| \geq |y| - |x| \quad (2)$

From Eq. (1) and Eq. (2):

$|x+y| \geq \max\{|x|-|y|, |y|-|x|\} =$
$= \max\{|x|-|y|, -(|x|-|y|)\} = ||x|-|y||$
$\Rightarrow ||x|-|y|| \leq |x+y|.$

We conclude that

$\forall x, y \in \mathbb{R} : ||x|-|y|| \leq |x+y| \leq |x|+|y|. \qquad \square$

(3) $\boxed{\forall x,y\in\mathbb{R}: |xy| = |x||y|}$

Proof

Let $x,y\in\mathbb{R}$ be given. Then,

$|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 = (|x||y|)^2 \Rightarrow$

$\Rightarrow |xy| = |x||y|$ [via $|xy|\geq 0 \wedge |x||y|\geq 0$]. $\square$

↳ Immediate consequences of these properties are the following statements

a) $\forall x\in\mathbb{R}: \forall y\in\mathbb{R}^*: \left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$

b) $\forall x_1,x_2,\ldots,x_n\in\mathbb{R}: |x_1+x_2+\cdots+x_n| \leq |x_1|+|x_2|+\cdots+|x_n|$

c) $\forall x_1,x_2,\ldots,x_n\in\mathbb{R}: |x_1x_2\cdots x_n| = |x_1||x_2|\cdots|x_n|$

(4) $\boxed{\forall x,y\in\mathbb{R}: (|x|+|y|=0 \Leftrightarrow (x=0 \wedge y=0))}$

Proof

Let $x,y\in\mathbb{R}$ be given.

($\Rightarrow$): Assume that $|x|+|y|=0$. Then, we have:

$$\begin{cases} x\leq |x| \leq |x|+|y| = 0 \\ x \geq -|x| \geq -|x|-|y| = -(|x|+|y|) = -0 = 0 \end{cases} \Rightarrow$$

$\Rightarrow x\leq 0 \wedge x\geq 0 \Rightarrow x=0$.

Similarly, we show that $y=0$. We conclude that $x=0\wedge y=0$.

($\Leftarrow$): Assume that $x=0\wedge y=0$. Then

$|x|+|y| = |0|+|0| = 0+0 = 0$. $\square$

An immediate consequence of 4 is the following statement:

$$\forall x_1, x_2, \dots, x_n \in \mathbb{R} : \left(|x_1| + |x_2| + \dots + |x_n| = 0 \iff \right.$$
$$\left. \iff (x_1 = 0 \wedge x_2 = 0 \wedge \dots \wedge x_n = 0) \right)$$

THEORY QUESTIONS

(14) Let $x \in \mathbb{R}$ be given. State the definition of $|x|$.

(15) Show that:

a) $\forall x \in \mathbb{R} : \forall p \in (0,+\infty) : (|x| < p \Leftrightarrow x \in (-p,p))$

b) $\forall x \in \mathbb{R} : \forall p \in (0,+\infty) : (|x| > p \Leftrightarrow x \in (-\infty,-p) \cup (p,+\infty))$

c) $\forall x \in \mathbb{R} : \forall p \in (0,+\infty) : (|x| = p \Leftrightarrow (x = p \lor x = -p))$

d) $\forall x,y \in \mathbb{R} : ||x|-|y|| \leq |x+y| \leq |x|+|y|$

e) $\forall x,y \in \mathbb{R} : |xy| = |x||y|$

f) $\forall x,y \in \mathbb{R} : (|x|+|y| = 0 \Leftrightarrow (x=0 \land y=0))$

EXERCISES

(16) Let $a,b,x,y \in \mathbb{R}$ be given. Show that

a) $x,y \in (a,b) \Rightarrow |x-y| < |a-b|$

b) $a<x<b \Rightarrow ||a-x|-|b-x|| = |a+b-2x|$

c) $x<a<b \lor a<b<x \Rightarrow ||a-x|-|b-x|| = b-a$

d) $a<x<1 \Rightarrow ||x-1|+|x-a|| \geq |1-x|-|a-x|$

(17) Show that

a) $\forall a,b \in \mathbb{R}^* : \left(\left| \dfrac{a|b|+b|a|}{ab} \right| = 2 \Rightarrow a,b \text{ equisigned} \right)$

b) $\forall a,b \in \mathbb{R} : (||a|-|b|| = |a+b| \Rightarrow ab \leq 0)$

(18) Show that

a) $\forall x,y\in\mathbb{R}: \max\{x,y\} = (1/2)(x+y+|x-y|)$

b) $\forall x,y\in\mathbb{R}: \min\{x,y\} = (1/2)(x+y-|x-y|)$

c) $\forall x,y,z\in\mathbb{R}: \max\{x,y,z\} = (1/4)(2x+y+z+|y-z|+$
$+|2x-y-z-|y-z||)$

d) $\forall x,y,z\in\mathbb{R}: \min\{x,y,z\} = (1/4)(2x+y+z-|y-z|-$
$-|2x-y-z+|y-z|)$

(19) Show that

a) $(a<b \wedge |x-a|<|x-b|) \Rightarrow x<(1/2)(a+b)$

b) $\begin{cases} |x-x_0|<\varepsilon/2 \\ |y-y_0|<\varepsilon/2 \end{cases} \Rightarrow \begin{cases} |x+y-(x_0+y_0)|<\varepsilon \\ |x-y-(x_0+y_0)|<\varepsilon \end{cases}$

c) $\begin{cases} |x-x_0|<\min\left\{1, \dfrac{\varepsilon}{2(y_0+1)}\right\} \\ |y-y_0|<\dfrac{\varepsilon}{2(y_0+1)} \end{cases} \Rightarrow |xy-x_0y_0|<\varepsilon$

d) $\begin{cases} |y-y_0|<(1/2)\min\{|y_0|, \varepsilon|y_0|^2\} \\ y_0\neq 0 \end{cases} \Rightarrow \begin{cases} y\neq 0 \\ |1/y-1/y_0|<\varepsilon. \end{cases}$

▼ Axiom of completeness and well-ordering principle

We begin with the following definition:

Def: Let $a \in \mathbb{R}$ be given and let S be a set such that $S \subseteq \mathbb{R} \wedge S \neq \emptyset$. Then:

a) a upper bound of $S \Leftrightarrow \forall x \in S: x \leq a$
b) a lower bound of $S \Leftrightarrow \forall x \in S: x \geq a$
c) S upper bounded $\Leftrightarrow \exists b \in \mathbb{R}$: b upper bound of S
d) S lower bounded $\Leftrightarrow \exists b \in \mathbb{R}$: b lower bound of S
e) $a = \max S \Leftrightarrow a \in S \wedge$ (a upper bound of S)
f) $a = \min S \Leftrightarrow a \in S \wedge$ (a lower bound of S)
g) $a = \sup S \Leftrightarrow \begin{cases} a \text{ upper bound of } S \\ \forall \varepsilon \in (0, +\infty): a - \varepsilon \text{ not upper bound of } S \end{cases}$
h) $a = \inf S \Leftrightarrow \begin{cases} a \text{ lower bound of } S \\ \forall \varepsilon \in (0, +\infty): a + \varepsilon \text{ not lower bound of } S \end{cases}$

- $\sup S$ is the "least upper bound" of S, if it exists.
- $\inf S$ is the "greatest lower bound" of S, if it exists.

EXAMPLES

a) $\inf (a, +\infty) = a$, but $\min (a, +\infty)$ undefined
b) $\sup (-\infty, b) = b$, but $\max (-\infty, b)$ undefined
c) $\inf [a, b] = \min [a, b] = a$
d) $\sup [a, b] = \max [a, b] = b$

Well-ordering principle

We introduce the following notation:

a) Let $n \in \mathbb{N}$. We define
$[n] = \{x \in \mathbb{N} \mid 1 \leq x \leq n\} = \{1, 2, \ldots, n\}$
and note that $[0] = \emptyset$.

b) Let A, B be two sets. We define $\mathrm{Map}(A, B)$ as the set of all mappings $\varphi : A \to B$.

Def: Let S be a set. We say that
$$S \text{ finite} \Leftrightarrow \exists n \in \mathbb{N} : \exists \varphi \in \mathrm{Map}(S, [n]) : \varphi \text{ bijection}$$

Axiom: Let S be a set. Then
$$\begin{cases} S \subseteq \mathbb{R} \\ S \text{ finite} \end{cases} \Rightarrow \begin{cases} \exists a \in S : \max S = a \\ \exists a \in S : \min S = a \end{cases}$$

The well-ordering principle is a fundamental axiom of set theory, although it can also be derived from the "axiom of choice".

Axiom of completeness

Axiom:

a) $$\begin{cases} \emptyset \neq S \subseteq \mathbb{R} \\ S \text{ upper bounded} \end{cases} \Rightarrow \exists a \in \mathbb{R} : \sup S = a$$

b) $$\begin{cases} \emptyset \neq S \subseteq \mathbb{R} \\ S \text{ lower bounded} \end{cases} \Rightarrow \exists a \in \mathbb{R} : \inf S = a$$

Consequences of the completeness axiom

(1) $\boxed{\begin{array}{l}\underline{\text{Thm}}: \text{(Archimedes theorem)} \\ \forall x \in \mathbb{R} : \exists n \in \mathbb{N}^* : n > x\end{array}}$

Proof

To show a contradiction, assume that the negation of the claim, which reads:

$$\exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : n \leq x$$

is satisfied. Choose some $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}^* : n \leq x$.

Then, we have:

$$\begin{cases} x \text{ upper bound of } \mathbb{N} \\ \emptyset \neq \mathbb{N} \subseteq \mathbb{R} \end{cases} \Rightarrow \begin{cases} \mathbb{N} \text{ upper bounded} \\ \emptyset \neq \mathbb{N} \subseteq \mathbb{R} \end{cases} \Rightarrow$$

$$\Rightarrow \exists b \in \mathbb{R} : \sup \mathbb{N} = b$$

Choose $b \in \mathbb{N}$ such that $\sup \mathbb{N} = b$. Then:

$$b - 1 < b = \sup \mathbb{N} \Rightarrow b - 1 < \sup \mathbb{N} \Rightarrow$$

$$\Rightarrow b-1 \text{ not upper bound of } \mathbb{N}$$

$$\Rightarrow \overline{\forall n_0 \in \mathbb{N} : n_0 \leq b-1}$$

$$\Rightarrow \exists n_0 \in \mathbb{N} : n_0 > b-1$$

Choose an $n_0 \in \mathbb{N}$ such that $n_0 > b-1$. It follows that

$$n_0 + 1 > (b-1) + 1 = b = \sup \mathbb{N} \Rightarrow n_0 + 1 > \sup \mathbb{N} \quad (1)$$

and $n_0 + 1 \in \mathbb{N} \Rightarrow n_0 + 1 \leq \sup \mathbb{N}$ (2)

Eq. (1) and Eq. (2) contradict. We conclude that $\forall x \in \mathbb{R} : \exists n \in \mathbb{N}^* : n > x$.

② **Thm:** (Approximation property)

$$\boxed{\begin{array}{l} \left\{\begin{array}{l} A \text{ upper bounded} \\ \emptyset \neq A \subseteq \mathbb{R} \end{array}\right. \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists a \in A : \sup A - \varepsilon \leq a \\ \left\{\begin{array}{l} A \text{ lower bounded} \\ \emptyset \neq A \subseteq \mathbb{R} \end{array}\right. \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists a \in A : \inf A + \varepsilon \geq a \end{array}}$$

Proof

a) Assume that A upper bounded and $\emptyset \neq A \subseteq \mathbb{R}$.
Let $\varepsilon \in (0,+\infty)$ be given. To show a contradiction, assume that $\forall a \in A : \sup A - \varepsilon > a$. Then, we have:

$(\forall a \in A : a < \sup A - \varepsilon) \Rightarrow \sup A - \varepsilon$ upper bound on A
$\Rightarrow \sup A \leq \sup A - \varepsilon \Rightarrow -\varepsilon \geq 0 \Rightarrow \varepsilon \leq 0 \leftarrow$ Contradiction.

It follows that $\exists a \in A : \sup A - \varepsilon \leq a$.

b) Assume that A lower bounded and $\emptyset \neq A \subseteq \mathbb{R}$.
Let $\varepsilon \in (0,+\infty)$ be given. To show a contradiction, assume that $\forall a \in A : \inf A + \varepsilon < a$. Then, we have:

$(\forall a \in A : a > \inf A + \varepsilon) \Rightarrow \inf A + \varepsilon$ lower bound of A
$\Rightarrow \inf A \geq \inf A + \varepsilon \Rightarrow \varepsilon \leq 0 \longleftarrow$ Contradiction

It follows that $\exists a \in A : \inf A + \varepsilon \geq a$. □

Characterization of intervals

The following characterization of intervals is used later in differential calculus. We begin with the definition:

Def: Let I be a set with $\emptyset \neq I \subseteq \mathbb{R}$. Then
I interval $\Leftrightarrow \exists a,b \in \mathbb{R} : (I = [a,b] \vee I = [a,b) \vee I = (a,b] \vee$
$\vee I = (a,b) \vee I = [a,+\infty) \vee I = (a,+\infty) \vee I = (-\infty,b) \vee I = (-\infty,b])$

Now we derive the following equivalent characterization

Thm: Let I be a set with $\emptyset \neq I \subseteq \mathbb{R}$ Then:
I interval $\Leftrightarrow \forall a,b \in I : (a < b \Rightarrow [a,b] \subseteq I)$

Proof

($\Rightarrow$): Easy to show but tedious. (Homework).

($\Leftarrow$): Assume that $\forall a,b \in I : (a<b \Rightarrow [a,b] \subseteq I)$.

Since $I \neq \emptyset$, choose some $t \in I$ and define $A = [t,+\infty) \cap I$ and $B = (-\infty,t] \cap I$, and note that

$$A \cup B = [I \cap [t,+\infty)] \cup [I \cap (-\infty,t]] = I \cap [[t,+\infty) \cup (-\infty,t]]$$
$$= I \cap \mathbb{R} = I$$

and

$$A = I \cap [t,+\infty) \subseteq [t,+\infty) \Rightarrow (\forall x \in A : x \in [t,+\infty)) \Rightarrow$$
$$\Rightarrow \forall x \in A : t \leq x$$

We distinguish between the following cases:

Case 1: Assume that A is not upper bounded. We obviously have: $A = [t, +\infty) \cap I \subseteq [t, +\infty)$. (1)
We will now show that $[t, +\infty) \subseteq A$. Let $x \in [t, +\infty)$ be given. It follows that $t \leq x$. Furthermore, since
A not upper bounded $\Rightarrow \exists \xi \in A : x \leq \xi$
Choose $\xi \in A$ such that $x \leq \xi$. Then, we have:

$$\begin{cases} t, \xi \in A \\ t \leq x \leq \xi \end{cases} \Rightarrow \begin{cases} t, \xi \in I \\ x \in [t, \xi] \end{cases} \Rightarrow x \in I \quad \text{[via hyp]}.$$

and: $x \in I \wedge x \in [t, +\infty) \Rightarrow x \in I \cap [t, +\infty) \Rightarrow x \in A$.
It follows that $(\forall x \in [t, +\infty) : x \in A) \Rightarrow [t, +\infty) \subseteq A$. (2)
From Eq.(1) and Eq.(2): $A = [t, +\infty)$.

Case 2: Assume that A is upper bounded. Then, we can define $p = \sup A$, and it follows that:

$$\begin{aligned} p = \sup A &\Rightarrow p \text{ upper bound of } A \Rightarrow \\ &\Rightarrow \forall x \in A : t \leq x \leq p \quad \text{[via previous result } t \leq x] \\ &\Rightarrow (\forall x \in A : x \in [t, p]) \Rightarrow A \subseteq [t, p] \quad (3) \end{aligned}$$

We will now show that $[t, p) \subseteq A$.
Let $x \in [t, p)$ be given. By the approximation property:
$\forall \varepsilon \in (0, +\infty) : \exists \xi \in A : \sup A - \varepsilon < \xi$
$\Rightarrow \exists \xi \in A : \sup A - (p - x) < \xi \quad [\text{via } \varepsilon = p - x > 0]$
Choose some $\xi \in A$ such that $\sup A - (p - x) < \xi$. Then, we have:

$$x = p - (p - x) = \sup A - (p - x) < \xi \Rightarrow x < \xi$$

therefore:

$$\begin{cases} t \leq x \leq s \\ t, s \in A \end{cases} \Rightarrow \begin{cases} x \in [t, s] \\ t, s \in I \end{cases} \Rightarrow x \in I.$$

and it follows that

$$\begin{cases} x \in I \\ x \in [t, p) \end{cases} \Rightarrow \begin{cases} x \in I \\ x \in [t, +\infty) \end{cases} \Rightarrow x \in I \cap [t, +\infty) \rightarrow x \in A.$$

We have thus shown that

$$(\forall x \in [t, p) : x \in A) \Rightarrow [t, p) \subseteq A.$$

We conclude that

$$[t, p) \subseteq A \subseteq [t, p] \Rightarrow A = [t, p) \vee A = [t, p].$$

In both cases, we find that

$$\exists p \in \mathbb{R} : (A = [t, p) \vee A = [t, p] \vee A = [t, +\infty))$$

Similarly, we can show that

$$\exists q \in \mathbb{R} : (B = (q, t] \vee B = [q, t] \vee B = (-\infty, t]).$$

Since $I = A \cup B$, it immediately follows that I follows one of the 8 interval forms. $\square$

THEORY QUESTIONS

(20) State the axiom of completeness.

(21) State the definition for the following statements

a) p upper bound of A

b) p lower bound of A

c) A upper bounded

d) A lower bounded

e) $p = \sup(A)$

f) $p = \inf(A)$.

(22) Use quantifier algebra to write out the detailed definition for the following negated statements.

a) A not upper bounded

b) A not lower bounded

c) $p \neq \sup(A)$

d) $p \neq \inf(A)$

(23) State and prove the Archimedes theorem:
$(\forall x \in \mathbb{R}_+^* : \exists n \in \mathbb{N}^* : n > x)$.

EXERCISE – PROJECT

(24) Let I be a set with $\emptyset \neq I \subseteq \mathbb{R}$. Write the complete proof of the theorem:

I interval $\Leftrightarrow \forall a, b \in I : (a < b \Rightarrow [a, b] \subseteq I)$.

EXERCISES

(25) Let A, B be sets such that $\emptyset \neq A \subseteq B \subseteq \mathbb{R}$.
Show that

a) $\sup(A) \leqslant \sup(B)$

b) $\inf(A) \geqslant \inf(B)$

(26) Let A, B be sets such that $\emptyset \neq A \subseteq \mathbb{R}$ and $\emptyset \neq B \subseteq \mathbb{R}$ and consider the definitions

$$A+B = \{x+y \mid x \in A \wedge y \in B\}$$
$$-B = \{-y \mid y \in B\}$$

Show that

a) $\sup(A+B) = \sup(A) + \sup(B)$

b) $\inf(-A) = -\sup(A)$

c) $\sup(-A) = -\inf(A)$

d) $\sup(A+(-B)) = \sup(A) - \inf(B)$.

Rational and real numbers

Def: Let $x \in \mathbb{R}$ be given. We say that
$$x \text{ rational} \Leftrightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N}^* : x = a/b$$

notation: The set of all rational numbers is denoted as
$$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \wedge b \in \mathbb{N}^*\} \text{ and } \mathbb{Q}^* = \mathbb{Q} - \{0\}$$

We also define:
$$\mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geq 0\} \qquad \mathbb{Q}_- = \{x \in \mathbb{Q} \mid x \leq 0\}$$
$$\mathbb{Q}_+^* = \{x \in \mathbb{Q} \mid x > 0\} \qquad \mathbb{Q}_-^* = \{x \in \mathbb{Q} \mid x < 0\}$$

$\to$ From the axiom of completeness we can show that
$$\forall x \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : \exists y \in \mathbb{R}_+^* : y^n = x$$

The unique $y \in \mathbb{R}_+^*$ such that $y^n = x$ is denoted as $\sqrt[n]{x}$. We can also write $\sqrt{x} = \sqrt[2]{x}$. We can then argue that $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$. The details are as follows:

Lemma: $\boxed{\forall a, b \in \mathbb{R} : \forall n \in \mathbb{N}^* : (0 < a < b \Rightarrow b^n - a^n < n(b-a) b^{n-1})}$

Proof

Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}^*$ be given and assume that $0 < a < b$.
Then, we have
$$b^n - a^n = (b-a) \sum_{k=0}^{n-1} (b^k a^{n-1-k})$$
$$< (b-a) \sum_{k=0}^{n-1} (b^k b^{n-1-k}) \quad [\text{via } b - a > 0 \wedge n-1-k \geq 0]$$
$$= (b-a) \sum_{k=0}^{n-1} b^{n-1} = (b-a)(n b^{n-1}) \Rightarrow b^n - a^n < n(b-a) b^{n-1} \quad \square$$

<u>Thm</u> : $\boxed{\forall x \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : \exists y \in \mathbb{R}_+^* : y^n = x}$

<u>Proof</u>

Let $x \in \mathbb{R}_+^*$ and $n \in \mathbb{N}^*$ be given. To construct $y \in \mathbb{R}_+^*$, we define

$$S = \{t \in \mathbb{R}_+^* : t^n \leq x\}$$

▸ We claim that $S \neq \emptyset$.

Let $p = x/(x+1)$. Then, it follows that:

$$\begin{cases} 0 < p < x \\ 0 < p < 1 \end{cases} \Rightarrow \begin{cases} 0 < p < x \\ 0 < p^{n-1} < 1 \end{cases} \Rightarrow 0 < p^n < x \Rightarrow$$

$$\Rightarrow p \in \mathbb{R}_+^* \wedge p^n < x \Rightarrow p \in S.$$

and therefore $S \neq \emptyset$. This proves the claim.

▸ Claim: S is upper bounded.

It is sufficient to show that: $\exists M \in \mathbb{R} : \forall t \in S : t \leq M$.

Choose $\underline{M = x+1}$. Let $\underline{t \in S}$ be given. Then, we have:

$$M = x+1 \Rightarrow \begin{cases} 1 < M \\ 0 < x < M \end{cases} \Rightarrow \begin{cases} 1 < M^{n-1} \\ 0 < x < M \end{cases} \Rightarrow x < M^n$$

and it follows that

$$t \in S \Rightarrow t^n < x < M^n \Rightarrow \begin{cases} t^n < M^n \\ t > 0 \wedge M > 0 \end{cases} \Rightarrow \underline{t < M}$$

We have thus shown that

$(\exists M \in \mathbb{R} : \forall t \in S : t \leq M) \Rightarrow S$ upper bounded.

and this proves the claim.

▸ Since $0 \neq S \subseteq \mathbb{R} \wedge S$ upper bounded, via the axiom of completeness, we choose $y = \sup S$. We will now show that $y^n = x$.

To show a contradiction, assume that $y^n \neq x$. Then, we distinguish between the following cases:

<u>Case 1</u>: Assume that $y^n < x$. Choose $h \in \left(0, \min\left\{1, \frac{x-y^n}{n(y+1)^{n-1}}\right\}\right)$ noting that the interval is not empty because $x - y^n > 0$. It follows that:

$$
\begin{aligned}
(y+h)^n &= [(y+h)^n - y^n] + y^n \\
&< n[(y+h) - y](y+h)^{n-1} + y^n && \text{[via Lemma]} \\
&= nh(y+h)^{n-1} + y^n \\
&< n \frac{x-y^n}{n(y+1)^{n-1}} (y+h)^{n-1} + y^n && \left[\text{via } h < \frac{x-y^n}{n(y+1)^{n-1}}\right] \\
&= (x-y^n)\frac{(y+h)^{n-1}}{(y+1)^{n-1}} + y^n \\
&< (x-y^n)\frac{(y+1)^{n-1}}{(y+1)^{n-1}} + y^n && \text{[via } h < 1] \\
&= (x-y^n) + y^n = x \Rightarrow (y+h)^n < x \Rightarrow
\end{aligned}
$$

$\Rightarrow y+h \in S \Rightarrow y+h \leq \sup S \Rightarrow y+h \leq y \Rightarrow \underline{h \leq 0}$

which is a contradiction, since h was chosen to satisfy $h > 0$. Therefore, this case does not materialize.

<u>Case 2</u>: Assume that $y^n > x$. We now define

$$h = \frac{y^n - x}{ny^{n-1}}$$

and note that $0 < h < y^n/(ny^{n-1}) = y$

$0 < h < y^n/(ny^{n-1}) = y \Rightarrow 0 < y - h < y \Rightarrow$

$$
\begin{aligned}
\Rightarrow y^n - (y-x)^n &< n[y - (y-h)]\,y^{n-1} && \text{[via Lemma]} \\
&= nhy^{n-1} = n\frac{y^n - x}{ny^{n-1}}\, y^{n-1} = y^n - x \Rightarrow
\end{aligned}
$$

$\Rightarrow y^n - (y-h)^n < y^n - x \Rightarrow -(y-h)^n < -x \Rightarrow (y-h)^n > x$
$\Rightarrow (\forall t \in S : t^n < x < (y-h)^n) \Rightarrow (\forall t \in S : t < y-h)$
$\Rightarrow y-h$ upper bound of $S \Rightarrow y-h \geq \sup S \Rightarrow$
$\Rightarrow y-h \geq y \Rightarrow -h \geq 0 \Rightarrow h \leq 0$
which is a contradiction, since h was chosen to satisfy $h > 0$. Therefore Case 2 does not materialize.
From the above argument, we conclude that $y^n = x$.
We have thus proved the theorem $\square$

$\boxed{\underline{\text{Thm:}} \quad \sqrt{2} \notin \mathbb{Q} \quad \text{(Hipassos of Metapontum)}}$

<u>Proof</u>
To show a contradiction, assume that $\sqrt{2} \in \mathbb{Q}$. Then, we have:

$$\sqrt{2} \in \mathbb{Q} \Rightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N}^* : \sqrt{2} = a/b$$

Choose some $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$ such that $\sqrt{2} = a/b$ so that the ratio a/b has no further simplifications.
It follows that

$$a = b\sqrt{2} \Rightarrow a^2 = (b\sqrt{2})^2 = 2b^2 \Rightarrow a^2 \text{ even} \Rightarrow a \text{ even}$$
$$\Rightarrow \exists \lambda \in \mathbb{Z} : a = 2\lambda.$$

Choose $\lambda \in \mathbb{Z}$ such that $a = 2\lambda$. Then, we have:

$$2b^2 = a^2 = (2\lambda)^2 = 4\lambda^2 \Rightarrow b^2 = 2\lambda^2 \Rightarrow b^2 \text{ even} \Rightarrow$$
$$\Rightarrow b \text{ even.}$$

This is a contradiction, because (a even $\wedge$ b even) implies that the fraction a/b can be simplified in contradiction with our choice above. We conclude that $\sqrt{2} \notin \mathbb{Q}$. $\square$

THEORY QUESTIONS

(27) Show that $\sqrt{2} \notin \mathbb{Q}$.

(28) Write the definition of "x is rational" using quantifier notation.

EXERCISES

(29) Show that

a) $\sqrt{3} \notin \mathbb{Q}$

b) $\sqrt{6} \notin \mathbb{Q}$

c) $\sqrt{2}+\sqrt{3} \notin \mathbb{Q}$ (Hint: use (a) and (b))

(30) Let $a, b \in \mathbb{Q}_+^*$ be given. Show that:
$(\sqrt{a} \notin \mathbb{Q} \wedge a \neq b) \Rightarrow (\sqrt{a}-\sqrt{b}) \notin \mathbb{Q}$

(31) Let $a, b, c, d \in \mathbb{Q}$ with $b>0$ and $d>0$ and $\sqrt{b} \notin \mathbb{Q}$ and $\sqrt{d} \notin \mathbb{Q}$. Show that

a) $a+\sqrt{b} = c+\sqrt{d} \Leftrightarrow (a=c \wedge b=d)$

b) $a-\sqrt{b} = c-\sqrt{d} \Leftrightarrow (a=c \wedge b=d)$

(Hint: for exercise 31 use the result from exercise 30).