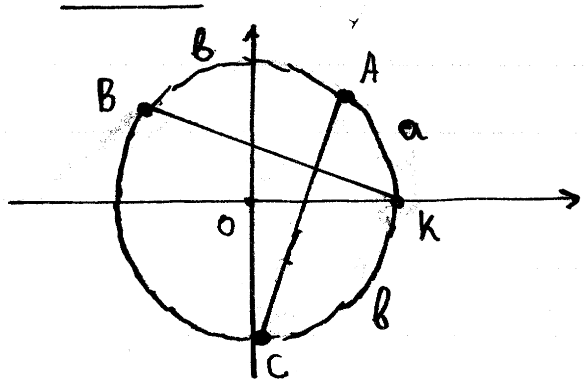
TRIGONOMETRIC IDENTITIES

Addition a±b identities

① $$\boxed{\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b}$$

Proof



Let $O(0,0)$ and $K(1,0)$. Choose A such that $A\hat{O}K = a$, and choose B such that $B\hat{O}A = b$. Also choose C such that $K\hat{O}C = b$ on the other side of the circle. It follows that:

$$x_A = \cos a, \quad y_A = \sin a$$
$$x_B = \cos(a+b), \quad y_B = \sin(a+b)$$
$$x_C = \cos(-b), \quad y_C = \sin(-b)$$
$$x_K = 1, \quad y_K = 0$$

Since

$$\left.\begin{array}{l} B\hat{O}K = B\hat{O}A + A\hat{O}K = b + a = a+b \\ A\hat{O}C = A\hat{O}K + K\hat{O}C = a+b \end{array}\right\} \Rightarrow B\hat{O}K = A\hat{O}C \Rightarrow$$

$$\Rightarrow BK = AC \Rightarrow \underline{BK^2 = AC^2} \quad (1)$$

We note that

$$BK^2 = (x_B - x_K)^2 + (y_B - y_K)^2 =$$
$$= (\cos(a+b) - 1)^2 + (\sin(a+b) - 0)^2 =$$

$$= \cos^2(a+b) - 2\cos(a+b) + 1 + \sin^2(a+b) =$$
$$= 1 - 2\cos(a+b) + [\cos^2(a+b) + \sin^2(a+b)] =$$
$$= 1 - 2\cos(a+b) + 1 = 2 - 2\cos(a+b)$$

and

$$AC^2 = (x_A - x_C)^2 + (y_A - y_C)^2 =$$
$$= (\cos a - \cos b)^2 + (\sin a + \sin b)^2 =$$
$$= \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a + 2\sin a \sin b + \sin^2 b$$
$$= (\sin^2 a + \cos^2 a) + (\sin^2 b + \cos^2 b) - 2(\cos a \cos b - \sin a \sin b)$$
$$= 1 + 1 - 2(\cos a \cos b - \sin a \sin b) =$$
$$= 2 - 2(\cos a \cos b - \sin a \cdot \sin b)$$

and from (1) it follows that:

$$BK^2 = AC^2 \Rightarrow 2 - 2\cos(a+b) = 2 - 2(\cos a \cos b - \sin a \sin b) \Rightarrow$$
$$\Rightarrow \cos(a+b) = \cos a \cos b - \sin a \sin b.$$

It follows that

$$\cos(a-b) = \cos a \cos b + \sin a \sin b. \quad \square$$

② $\boxed{\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a}$

<u>Proof</u>

$$\sin(a+b) = \cos\left(\frac{\pi}{2} - (a+b)\right) = \cos\left(\left(\frac{\pi}{2} - a\right) + (-b)\right) =$$
$$= \cos\left(\frac{\pi}{2} - a\right)\cos(-b) - \sin\left(\frac{\pi}{2} - a\right)\sin(-b)$$
$$= \sin a \cos b - \cos a[-\sin b] =$$

$= \sin a \cos b + \sin b \cos a$

It follows that

$$\sin(a-b) = \sin a \cos b - \sin b \cos a. \quad \square$$

③ $$\boxed{\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \cdot \tan b}}$$

Proof

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \cdot \sin b} =$$

$$= \frac{\cos a \cos b \left[\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b} \right]}{\cos a \cos b \left[1 - \frac{\sin a}{\cos a} \frac{\sin b}{\cos b} \right]} =$$

$$= \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

It follows that

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} \quad \square$$

④ $$\boxed{\cot(a \pm b) = \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a}} \quad (!!)$$

Proof

$$\cot(a+b) = \frac{1}{\tan(a+b)} = \frac{1-\tan a\tan b}{\tan a+\tan b} =$$

$$= \frac{1-\dfrac{1}{\cot a}\,\dfrac{1}{\cot b}}{\dfrac{1}{\cot a}+\dfrac{1}{\cot b}} = \frac{\dfrac{\cot a\cot b-1}{\cot a\cot b}}{\dfrac{\cot b+\cot a}{\cot a\cot b}} =$$

$$= \frac{\cot a\cot b-1}{\cot b+ cat\, a}$$

It follows that

$$\cot(a-b) = \frac{\cot a\cot(-b)-1}{\cot(-b)+\cot a} = \frac{-\cot a\cot b-1}{-\cot b+\cot a} =$$

$$= \frac{\cot a\cot b+1}{\cot b-\cot a} \qquad \square$$

EXAMPLES

a) Evaluate $\sin(7\pi/12)$, $\cos(7\pi/12)$, $\tan(7\pi/12)$

Solution

We have:

$$\sin(7\pi/12) = \sin(3\pi/12 + 4\pi/12) = \sin(\pi/4 + \pi/3) =$$
$$= \sin(\pi/4)\cos(\pi/3) + \sin(\pi/3)\cos(\pi/4) =$$
$$= \frac{\sqrt{2}}{2}\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$$

$$\cos(7\pi/12) = \cos(3\pi/12 + 4\pi/12) = \cos(\pi/4 + \pi/3) =$$
$$= \cos(\pi/4)\cos(\pi/3) - \sin(\pi/4)\sin(\pi/3) =$$
$$= \frac{\sqrt{2}}{2}\frac{1}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{2}(1-\sqrt{3})}{4}$$

$$\tan(7\pi/12) = \frac{\sin(7\pi/12)}{\cos(7\pi/12)} = \frac{\frac{\sqrt{2}(1+\sqrt{3})}{4}}{\frac{\sqrt{2}(1-\sqrt{3})}{4}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} =$$
$$= \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})} =$$
$$= \frac{1^2 + 2\sqrt{3} + (\sqrt{3})^2}{1^2 - (\sqrt{3})^2} = \frac{1 + 2\sqrt{3} + 3}{1-3} = \frac{4+2\sqrt{3}}{-2}$$
$$= -2 - \sqrt{3}.$$

b) Show that

$$\frac{\sin(a-b)}{\sin a \sin b} + \frac{\sin(b-c)}{\sin b \sin c} + \frac{\sin(c-a)}{\sin c \sin a} = 0$$

Solution

We note that

$$A = \frac{\sin(a-b)}{\sin a \sin b} + \frac{\sin(b-c)}{\sin b \sin c} + \frac{\sin(c-a)}{\sin c \sin a} =$$

$$= \frac{\sin(a-b)\sin c + \sin(b-c)\sin a + \sin(c-a)\sin b}{\sin a \sin b \sin c} \qquad (1)$$

and also that:

$\sin(a-b)\sin c + \sin(b-c)\sin a =$

$= [\sin a \cos b - \sin b \cos a]\sin c + [\sin b \cos c - \sin c \cos b]\sin a =$

$= \underline{\sin a \cos b \sin c} - \sin b \cos a \sin c + \sin b \cos c \sin a - \underline{\sin c \cos b \sin a} =$

$= \sin b \cos c \sin a - \sin b \cos a \sin c =$

$= \sin b [\sin a \cos c - \sin c \cos a] = \sin b \sin(a-c)$

$= \sin b [-\sin(c-a)] = -\sin(c-a)\sin b \Rightarrow$

$\Rightarrow \sin(a-b)\sin c + \sin(b-c)\sin a + \sin(c-a)\sin b = 0 \overset{(1)}{\Rightarrow}$

$$\Rightarrow A = \frac{0}{\sin a \sin b \sin c} = 0.$$

c) Show that

$$\tan(a-b)+\tan(b-c)+\tan(c-a)=\tan(a-b)\tan(b-c)\tan(c-a)$$

Solution

Define $x=a-b \wedge y=b-c \wedge z=c-a$ and note that

$x+y+z=(a-b)+(b-c)+(c-a)=a+b+b-c+c-a=0 \Rightarrow$

$\Rightarrow z=-x-y$

and therefore

$$A=\tan(a-b)+\tan(b-c)+\tan(c-a)=\tan x+\tan y+\tan z$$

$$=\tan x+\tan y+\tan(-x-y)=\tan x+\tan y-\tan(x+y)=$$

$$=\tan x+\tan y-\frac{\tan x+\tan y}{1-\tan x\tan y}=$$

$$=\frac{(\tan x+\tan y)(1-\tan x\tan y)-(\tan x+\tan y)}{1-\tan x\tan y}=$$

$$=\frac{(\tan x+\tan y)(1-\tan x\tan y-1)}{1-\tan x\tan y}=$$

$$=-(\tan x\tan y)\frac{\tan x+\tan y}{1-\tan x\tan y}=-\tan x\tan y\tan(x+y)$$

$$=\tan x\tan y\tan(-x-y)=\tan x\tan y\tan z=$$

$$=\tan(a-b)\tan(b-c)\tan(c-a)=B$$

d) Show that

$$\sin^2 x + \sin^2(x+2\pi/3) + \sin^2(x+4\pi/3) = 3/2$$

Solution

We note that

$$\sin(x+2\pi/3) = \sin(x+\pi-\pi/3) = -\sin(x-\pi/3) =$$
$$= -[\sin x\cos(\pi/3) - \cos x\sin(\pi/3)] =$$
$$= \cos x\sin(\pi/3) - \sin x\cos(\pi/3) =$$
$$= (\sqrt{3}/2)\cos x - (1/2)\sin x$$

and

$$\sin(x+4\pi/3) = \sin(x+\pi+\pi/3) = -\sin(x+\pi/3) =$$
$$= -[\sin x\cos(\pi/3) + \cos x\sin(\pi/3)] =$$
$$= -\sin x\cos(\pi/3) - \cos x\sin(\pi/3) =$$
$$= -(1/2)\sin x - (\sqrt{3}/2)\cos x$$

so it follows that

$$A = \sin^2 x + \sin^2(x+2\pi/3) + \sin^2(x+4\pi/3) =$$
$$= \sin^2 x + [(\sqrt{3}/2)\cos x - (1/2)\sin x]^2 + [-(1/2)\sin x - (\sqrt{3}/2)\cos x]^2$$
$$= \sin^2 x + [(\sqrt{3}/2)\cos x]^2 - 2[(\sqrt{3}/2)\cos x][(1/2)\sin x] + [(1/2)\sin x]^2$$
$$+ [(\sqrt{3}/2)\cos x]^2 + 2[(\sqrt{3}/2)\cos x][(1/2)\sin x] + [(1/2)\sin x]^2$$
$$= \sin^2 x + 2[(\sqrt{3}/2)\cos x]^2 + 2[(1/2)\sin x]^2$$
$$= \sin^2 x + 2(3/4)\cos^2 x + 2\cdot(1/4)\sin^2 x$$
$$= [1+1/2]\sin^2 x + (3/2)\cos^2 x = (3/2)\sin^2 x + (3/2)\cos^2 x$$
$$= (3/2)(\sin^2 x + \cos^2 x) = 3/2.$$

EXERCISES

① Show that

a) $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$

b) $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

c) $\sin(a-b)\cos b + \sin b\cos(a-b) = \sin a$

d) $\cos(a+b)\cos(a-b) - \sin(a+b)\sin(a-b) = \cos(2a)$

e) $$\frac{2\sin(a+b)}{\cos(a+b)+\cos(a-b)} = \tan a + \tan b$$

f) $$\frac{\sin(a-b)}{\cos a\cos b} + \frac{\sin(b-c)}{\cos b\cos c} + \frac{\sin(c-a)}{\cos c\cdot\cos a} = 0$$

g) $$\frac{\sin(a-b)}{\sin a\sin b} + \frac{\sin(b-c)}{\sin b\sin c} + \frac{\sin(c-a)}{\sin c\cdot\sin a} = 0$$

h) $$\frac{\tan^2(2a)-\tan^2(a)}{1-\tan^2(2a)\tan^2(a)} = \tan(a)\tan(3a)$$

i) $$\cos x + \cos\left(x+\frac{2\pi}{3}\right) + \cos\left(x+\frac{4\pi}{3}\right) = 0$$

j) $$\cos^2 x + \cos^2\left(\frac{\pi}{3}+x\right) + \cos^2\left(\frac{\pi}{3}-x\right) = \frac{3}{2}$$

② Calculate the trigonometric numbers $\sin x$, $\cos x$, $\tan x$, $\cot x$ for

a) $x = \pi/12$ b) $x = 5\pi/12$

③ If $\cos(a+b) = \cos a\cos b$, show that $\sin^2(a+b) = (\sin a + \sin b)^2$.

▼ 2a/3a identities

- The trigonometric numbers of $2a$ in terms of the trigonometric numbers of a

$\sin(2a) = 2\sin a\cos a$ $\cos(2a) = \cos^2 a - \sin^2 a$ $= 2\cos^2 a - 1$ $= 1 - 2\sin^2 a$	$\tan(2a) = \dfrac{2\tan a}{1-\tan^2 a}$ $\cot(2a) = \dfrac{\cot^2 a - 1}{2\cot a}$

- In terms of $\cos(2a)$:

$\sin^2 a = \dfrac{1-\cos(2a)}{2}$ $\cos^2 a = \dfrac{1+\cos(2a)}{2}$	$\tan^2 a = \dfrac{1-\cos(2a)}{1+\cos(2a)}$ $\cot^2 a = \dfrac{1+\cos(2a)}{1-\cos(2a)}$

Immediate consequence of $\cos(2a) = 2\cos^2 a - 1 = 1 - 2\sin^2 a$.

- In terms of $\tan(a/2)$

$\sin a = \dfrac{2\tan(a/2)}{1+\tan^2(a/2)}$ $\cos a = \dfrac{1-\tan^2(a/2)}{1+\tan^2(a/2)}$	$\tan a = \dfrac{2\tan(a/2)}{1-\tan^2(a/2)}$ $\cot a = \dfrac{1-\tan^2(a/2)}{2\tan(a/2)}$

Proof of $\tan(a/2)$ identitires

Since $\dfrac{1}{\cos^2 a} = 1 + \tan^2 a \Rightarrow \cos^2 a = \dfrac{1}{1+\tan^2 a}$

it follows that:

$$\sin a = 2\sin(a/2)\cos(a/2) = 2\,\frac{\sin(a/2)}{\cos(a/2)}\,\cos^2(a/2) =$$

$$= 2\tan(a/2)\,\frac{1}{1+\tan^2(a/2)} = \frac{2\tan(a/2)}{1+\tan^2(a/2)}$$

and

$$\cos a = 2\cos^2(a/2) - 1 = 2\,\frac{1}{1+\tan^2(a/2)} - 1 =$$

$$= \frac{2-(1+\tan^2(a/2))}{1+\tan^2(a/2)} = \frac{2-1-\tan^2(a/2)}{1+\tan^2(a/2)}$$

$$= \frac{1-\tan^2(a/2)}{1+\tan^2(a/2)}$$

and

$$\tan a = \frac{\sin a}{\cos a} = \frac{\left(\dfrac{2\tan(a/2)}{1+\tan^2(a/2)}\right)}{\left(\dfrac{1-\tan^2(a/2)}{1+\tan^2(a/2)}\right)} = \frac{2\tan(a/2)}{1-\tan^2(a/2)}$$

$$\cot a = \frac{\cos a}{\sin a} = \frac{\left(\dfrac{1-\tan^2(a/2)}{1+\tan^2(a/2)}\right)}{\left(\dfrac{2\tan(a/2)}{1+\tan^2(a/2)}\right)} = \frac{1-\tan^2(a/2)}{2\tan(a/2)}$$

• 3a identities: $\boxed{\begin{aligned} \sin(3a) &= -4\sin^3 a + 3\sin a \\ \cos(3a) &= 4\cos^3 a - 3\cos a \end{aligned}}$

Proof

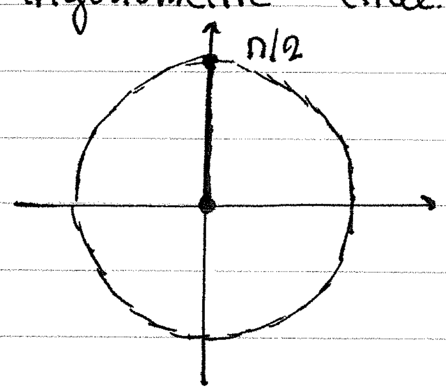
$$\sin(3a) = \sin(a+2a) = \sin a\cos(2a) + \sin(2a)\cos a =$$
$$= \sin a\,(1-2\sin^2 a) + (2\sin a\cos a)\cos a =$$
$$= \sin a\,(1-2\sin^2 a) + 2\sin a\,(1-\sin^2 a)$$
$$= \sin a - 2\sin^3 a + 2\sin a - 2\sin^3 a =$$
$$= (-2-2)\sin^3 a + (1+2)\sin a = -4\sin^3 a + 3\sin a$$

and

$$\cos(3a) = \sin(\pi/2 - 3a) = (-1)\sin(\pi + \pi/2 - 3a) = -\sin(3\pi/2 - 3a)$$
$$= -\sin(3(\pi/2 - a)) = -[-4\sin^3(\pi/2 - a) + 3\sin(\pi/2 - a)]$$
$$= -[-4\cos^3 a + 3\cos a] = 4\cos^3 a - 3\cos a \quad \square$$

APPLICATION

These identities can be used to find the trigonometric identities for various angles using $\cos(\pi/2) = 0$ as a starting point, which is shown geometrically via the trigonometric circle.



a) $\boxed{\text{Angle } \pi/4}$

$$\cos^2(\pi/4) = \frac{1+\cos(\pi/2)}{2} =$$
$$= \frac{1+0}{2} = \frac{1}{2} \quad (1)$$

and $\cos(\pi/4) > 0$ (2)

From Eq.(1) and Eq.(2):

$$\cos(\pi/4) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Likewise we can show $\sin(\pi/4) = \sqrt{2}/2$.

b) $\boxed{\text{Angle } \pi/6}$

Let $x = \cos(\pi/6)$ and note that

$4\cos^3(\pi/6) - 3\cos(\pi/6) = \cos(\pi/2) \Leftrightarrow 4x^3 - 3x = 0 \Leftrightarrow$

$\Leftrightarrow x(4x^2 - 3) = 0 \Leftrightarrow x(2x - \sqrt{3})(2x + \sqrt{3}) = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee 2x - \sqrt{3} = 0 \vee 2x + \sqrt{3} = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x = \sqrt{3}/2 \vee x = -\sqrt{3}/2$ (1)

Since $x = \cos(\pi/6) > 0$, it follows from Eq.(1) that

$\cos(\pi/6) = \sqrt{3}/2$.

Also:

$\sin^2(\pi/6) = 1 - \cos^2(\pi/6) = 1 - (\sqrt{3}/2)^2 = 1 - (3/4) = 1/4 \Rightarrow$

$\Rightarrow \sin(\pi/6) = 1/2 \vee \sin(\pi/6) = -1/2$

$\Rightarrow \sin(\pi/6) = 1/2$ (because $\sin(\pi/6) > 0$)

c) $\boxed{\text{Angle } \pi/3}$

$\sin(\pi/3) = 2\sin(\pi/6)\cos(\pi/6) = 2(1/2)(\sqrt{3}/2) = \sqrt{3}/2$

$\cos(\pi/3) = \cos^2(\pi/6) - \sin^2(\pi/6) = (\sqrt{3}/2)^2 - (1/2)^2 =$

$= (3/4) - (1/4) = 2/4 = 1/2$.

EXAMPLES

a) Evaluate $\sin(17\pi/12)$ and $\tan(3\pi/8)$.

Solution

We have

$$\sin(17\pi/12) = \sin((12+5)\pi/12) = \sin(\pi + 5\pi/12) = -\sin(5\pi/12)$$

$$\underset{5\pi/12 \in [0, \pi/2]}{=} -\sqrt{\frac{1-\cos(5\pi/6)}{2}} = -\sqrt{\frac{1-\cos(\pi-\pi/6)}{2}}$$

$$= -\sqrt{\frac{1+\cos(-\pi/6)}{2}} = -\sqrt{\frac{1+\cos(\pi/6)}{2}} =$$

$$= -\sqrt{\frac{1+(\sqrt{3}/2)}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}} = \frac{-\sqrt{2+\sqrt{3}}}{2}$$

and

$$\tan(3\pi/8) = \sqrt{\frac{1-\cos(3\pi/4)}{1+\cos(3\pi/4)}} = \sqrt{\frac{1-\cos(\pi-\pi/4)}{1+\cos(\pi-\pi/4)}} =$$

$$= \sqrt{\frac{1+\cos(-\pi/4)}{1-\cos(-\pi/4)}} = \sqrt{\frac{1+\cos(\pi/4)}{1-\cos(\pi/4)}} =$$

$$= \sqrt{\frac{1+\sqrt{2}/2}{1-\sqrt{2}/2}} = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = \sqrt{\frac{(2+\sqrt{2})^2}{(2-\sqrt{2})(2+\sqrt{2})}}$$

$$= \frac{2+\sqrt{2}}{\sqrt{2^2-(\sqrt{2})^2}} = \frac{2+\sqrt{2}}{\sqrt{4-2}} = \frac{2+\sqrt{2}}{\sqrt{2}} =$$

$$= \frac{(2+\sqrt{2})\sqrt{2}}{2}$$

b) Show that $\cos^4(\pi/8) + \cos^4(3\pi/8) = 3/4$

Solution

Since

$$\cos^2(\pi/8) = \frac{1+\cos(\pi/4)}{2} = \frac{1+(\sqrt{2}/2)}{2} = \frac{2+\sqrt{2}}{4} \Rightarrow$$

$$\Rightarrow \cos^4(\pi/8) = \frac{(2+\sqrt{2})^2}{4^2} = \frac{2^2+2\cdot 2\sqrt{2}+(\sqrt{2})^2}{16} = \frac{4+4\sqrt{2}+2}{16}$$

$$= \frac{6+4\sqrt{2}}{16} = \frac{3+2\sqrt{2}}{8}$$

and

$$\cos^2(3\pi/8) = \frac{1+\cos(3\pi/4)}{2} = \frac{1+\cos(\pi-\pi/4)}{2} = \frac{1-\cos(-\pi/4)}{2} =$$

$$= \frac{1-\cos(\pi/4)}{2} = \frac{1-(\sqrt{2}/2)}{2} = \frac{2-\sqrt{2}}{4} \Rightarrow$$

$$\Rightarrow \cos^4(3\pi/8) = \frac{(2-\sqrt{2})^2}{4^2} = \frac{2^2-2\cdot 2\sqrt{2}+(\sqrt{2})^2}{16} = \frac{4-4\sqrt{2}+2}{16}$$

$$= \frac{6-4\sqrt{2}}{16} = \frac{3-2\sqrt{2}}{8}$$

it follows that

$$A = \cos^4(\pi/8) + \cos^4(3\pi/8) = \frac{3+2\sqrt{2}}{8} + \frac{3-2\sqrt{2}}{8} =$$

$$= \frac{3+2\sqrt{2}+3-2\sqrt{2}}{8} = \frac{6}{8} = \frac{3}{4} = B.$$

c) Show that $\dfrac{\sin(3a)}{\sin a} - \dfrac{\cos(3a)}{\cos a} = 2$

Solution

We have:

$$A = \frac{\sin(3a)}{\sin a} - \frac{\cos(3a)}{\cos a} = \frac{\sin(3a)\cos a - \sin a\cos(3a)}{\sin a\cos a} =$$

$$= \frac{\sin(3a-a)}{\sin a\cos a} = \frac{\sin(2a)}{\sin a\cos a} = \frac{2\sin a\cos a}{\sin a\cos a} = 2 = B$$

d) Show that $\tan(\pi/6 + a)\tan(\pi/6 - a) = \dfrac{2\cos(2a) - 1}{2\cos(2a) + 1}$

Solution

We have:

$$A = \tan(\pi/6 + a)\tan(\pi/6 - a) =$$

$$= \frac{\tan(\pi/6) + \tan a}{1 - \tan(\pi/6)\tan a} \cdot \frac{\tan(\pi/6) - \tan a}{1 + \tan(\pi/6)\tan a} =$$

$$= \frac{\tan^2(\pi/6) - \tan^2 a}{1 - \tan^2(\pi/6)\tan^2 a} = \frac{(1/\sqrt{3})^2 - \tan^2 a}{1 - (1/\sqrt{3})^2\tan^2 a} =$$

$$= \frac{(1/3) - \tan^2 a}{1 - (1/3)\tan^2 a} = \frac{1 - 3\tan^2 a}{3 - \tan^2 a} =$$

$$= \frac{1 - 3\dfrac{1-\cos(2a)}{1+\cos(2a)}}{3 - \dfrac{1-\cos(2a)}{1+\cos(2a)}} = \frac{(1+\cos(2a)) - 3(1-\cos(2a))}{3(1+\cos(2a)) - (1-\cos(2a))}$$

$$= \frac{1+\cos(2a)-3+3\cos(2a)}{3+3\cos(2a)-1+\cos(2a)} = \frac{4\cos(2a)-2}{4\cos(2a)+2} =$$

$$= \frac{2[2\cos(2a)-1]}{2[2\cos(2a)+1]} = \frac{2\cos(2a)-1}{2\cos(2a)+1}$$

EXERCISES

④ Find the trigonometric numbers for the following angles:

a) $x = \pi/8 = 22.5^\circ$

b) $x = \pi/12 = 15^\circ$

c) $x = 5\pi/12 = 75^\circ$

⑤ Use the previous results to show that

a) $\cos^4(\pi/8) + \cos^4(3\pi/8) = 3/4$

b) $(1+\cos(\pi/8))(1+\cos(3\pi/8))(1+\cos(5\pi/8))$
$\times(1+\cos(7\pi/8)) = 1/8$

⑥ Show that:

a) $\cos(5a) = 16\cos^5 a - 20\cos^3 a + 5\cos a$

b) $\cos(\pi/10) = \frac{1}{4}\sqrt{10+2\sqrt{5}}$

c) $\cos(\pi/5) = \frac{1}{4}(\sqrt{5}+1)$

⑦ Show that

a) $\dfrac{\sin(2a)}{1+\cos(2a)} = \tan a$

b) $\dfrac{\sin(2a)}{1-\cos(2a)} = \cot a$

c) $\cos^4 a - \sin^4 a = \cos(2a)$

d) $\cot a - \tan a = 2\cot(2a)$

e) $\dfrac{1+\cot^2 a}{2\cot a} = \dfrac{1}{\sin(2a)}$

f) $\dfrac{\cot^2 a+1}{\cot^2 a-1} = \dfrac{1}{\cos(2a)}$

⑧ Show that

a) $\tan\left(\frac{\pi}{4}-a\right)=\frac{\cos(2a)}{1+\sin(2a)}$

b) $\cos^2\left(\frac{\pi}{4}-a\right)-\sin^2\left(\frac{\pi}{4}-a\right)=\sin(2a)$

c) $\tan\left(\frac{\pi}{4}+a\right)-\tan\left(\frac{\pi}{4}-a\right)=2\tan(2a)$

d) $\frac{\cos a+\sin a}{\cos a-\sin a}-\frac{\cos a-\sin a}{\cos a+\sin a}=2\tan(2a)$

e) $\frac{1-\cos(2a)+\sin(2a)}{1+\cos(2a)+\sin(2a)}=\tan a$

f) $\frac{\cot a+1}{\cot a-1}=\frac{\cos(2a)}{1-\sin(2a)}$

⑨ Show that

a) $3-4\cos 2a+\cos 4a=8\sin^4 a$

b) $\frac{2}{(1+\tan a)(1+\cot a)}=\frac{\sin(2a)}{1+\sin(2a)}$

c) $\tan x+\frac{1}{\cos x}=\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$

d) $\tan\left(\frac{a+b}{2}\right)=\frac{\sin a+\sin b}{\cos a+\cos b}$

e) $\frac{\sin(2a)}{1-\cos(2a)}\ \frac{1-\cos(a)}{\cos a}=\tan\left(\frac{a}{2}\right)$

f) $\tan\left(\frac{\pi}{6}+a\right)\tan\left(\frac{\pi}{6}-a\right)=\frac{2\cos(2a)-1}{2\cos(2a)+1}$

(10) Show that

a) $\dfrac{\sin(3a)}{\sin a} - \dfrac{\cos(3a)}{\cos a} = 2$

b) $\dfrac{3\cos a + \cos(3a)}{3\sin a - \sin(3a)} = \cot^3 a$

c) $4\sin a \cdot \sin\left(\frac{\pi}{3} + a\right)\sin\left(\frac{\pi}{3} - a\right) = \sin(3a)$

d) $\dfrac{\sin(3a) + \sin^3 a}{\cos^3 a - \cos(3a)} = \cot a$

e) $4\sin^3 a\cos 3a + 4\cos^3 a\sin 3a = 3\sin(4a)$

f) $\dfrac{\cos^3 a - \cos(3a)}{\cos a} + \dfrac{\sin^3 a + \sin(3a)}{\sin a} = 3.$

(11) Show that: $\cos(20^\circ)\cos(40^\circ)\cos(60^\circ)\cos(80^\circ) = \frac{1}{16}$
(Hint: Use $\sin(2x) = 2\sin x\cos x$)

(12) Show that

a) $\sin\left(\frac{\pi}{10}\right) = \dfrac{-1+\sqrt{5}}{4}$ (Hint: For $a = \pi/10$, solve $\sin(2a) = \sin(\pi/2 - 3a)$)

b) $\sin\left(\frac{3\pi}{10}\right) = \dfrac{1+\sqrt{5}}{4}$

c) $\tan\left(\frac{\pi}{20}\right) - \tan\left(\frac{3\pi}{20}\right) - \tan\left(\frac{7\pi}{20}\right) + \tan\left(\frac{9\pi}{20}\right) = 4$

(Hint: Switch to sin, cos and reduce to
$2/\sin(\pi/10) - 2/\sin(3\pi/10)$
which can be evaluated via (a), (b)).

Product-Sum identities

Product to sum

$$\boxed{\begin{aligned} 2\sin a\cos b &= \sin(a-b)+\sin(a+b)\\ 2\cos a\cos b &= \cos(a-b)+\cos(a+b)\\ 2\sin a\sin b &= \cos(a-b)-\cos(a+b) \end{aligned}} \quad (!!)$$

↳ These are immediate consequences of the $a \pm b$ identities.

Sum to product

$$\sin a \pm \sin b = 2\sin\left(\frac{a\pm b}{2}\right)\cos\left(\frac{a\mp b}{2}\right)$$

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right) \quad (!!)$$

$$\tan a \pm \tan b = \frac{\sin(a\pm b)}{\cos a\cos b}$$

$$\cot a \pm \cot b = \frac{\sin(b\pm a)}{\sin a\sin b} \quad (!!)$$

- Note that:

$$1 \pm \sin a = \sin(\pi/2) \pm \sin a = \dots$$

$$\sin a \pm \cos b = \sin a \pm \sin(\pi/2 - b) = \dots$$

$$1+\cos a = 2\cos^2(a/2), \quad 1-\cos a = 2\sin^2(a/2)$$

EXAMPLES

a) Show that $\dfrac{\sin x+\sin(3x)+\sin(5x)}{\cos x+\cos(3x)+\cos(5x)}=\tan(3x)$

Solution

We have:

$$A=\frac{\sin x+\sin(3x)+\sin(5x)}{\cos x+\cos(3x)+\cos(5x)}=\frac{[\sin x+\sin(5x)]+\sin(3x)}{[\cos x+\cos(5x)]+\cos(3x)}$$

$$=\frac{2\sin\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right)+\sin(3x)}{2\cos\left(\frac{x+5x}{2}\right)\cos\left(\frac{x-5x}{2}\right)+\cos(3x)}=$$

$$=\frac{2\sin(3x)\cos(2x)+\sin(3x)}{2\cos(3x)\cos(2x)+\cos(3x)}=\frac{\sin(3x)[2\cos(2x)+1]}{\cos(3x)[2\cos(2x)+1]}$$

$$=\frac{\sin(3x)}{\cos(3x)}=\tan(3x).=B$$

b) Show that $\sin(2x)+\cos(5x)=2\sin\left(\frac{\pi}{4}-\frac{3x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{7x}{2}\right)$

Solution

We have:

$$A=\sin(2x)+\cos(5x)=\sin(2x)+\sin(\pi/2-5x)=$$

$$=2\sin\left(\frac{2x+(\pi/2-5x)}{2}\right)\cos\left(\frac{2x-(\pi/2-5x)}{2}\right)=$$

$$=2\sin\left(\frac{2x+\pi/2-5x}{2}\right)\cos\left(\frac{2x-\pi/2+5x}{2}\right)=$$

$$= 2\sin\left(\frac{\pi}{4} - \frac{3x}{2}\right)\cos\left(\frac{7x}{2} - \frac{\pi}{4}\right) =$$

$$= 2\sin\left(\frac{\pi}{4} - \frac{3x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{7x}{2}\right) = B$$

c) Show that $\sin(3x)\cos(8x) - \sin(5x)\cos(6x) = -\sin(2x)\cos(3x)$

<u>Solution</u>

We have:

$A = \sin(3x)\cos(8x) - \sin(5x)\cos(6x) =$

$= (1/2)[\sin(3x+8x) + \sin(3x-8x)] - (1/2)[\sin(5x+6x) + \sin(5x-6x)]$

$= (1/2)[\sin(11x) - \sin(5x)] - (1/2)[\sin(11x) - \sin x] =$

$= (1/2)[\sin(11x) - \sin(5x) - \sin(11x) + \sin x]$

$= (1/2)[\sin x - \sin(5x)] = (1/2)[\sin x + \sin(-5x)] =$

$$= (1/2)\, 2\sin\left(\frac{x + (-5x)}{2}\right)\cos\left(\frac{x - (-5x)}{2}\right) =$$

$= \sin(-2x)\cos(3x) = -\sin(2x)\cos(3x) = B$

EXERCISES

(13) Write the following expressions as a sum or difference:

a) $2\sin(2a)\cos a$ c) $\cos(5a)\cos(7a)$

b) $2\sin a\cos(4a)$ d) $\sin a\cdot\sin(3a)$

(14) Evaluate the following expressions:

a) $2\cos 60°\cdot\sin 30°$ c) $\cos(150°)\cos(30°)$

b) $\sin 45°\cos 75°$ d) $2\sin(36°)\cos(54°)$

(15) Factor the following expressions

a) $\sin(4a)+\sin a$

b) $\sin(7a)-\sin(5a)$

c) $\cos(5a)-\cos(a)$

d) $\cos(3x)+\cos(5x)$

e) $\sin x-\sin 2x+\sin(3x)$

f) $\sin(3x)+\sin(7x)+\sin(10x)$

g) $\cos a+2\cos(2a)+\cos(3a)$

h) $\cos(7a)-\cos(5a)+\cos(3a)-\cos a$

(16) Show that

a) $$\frac{\cos(3a)-\cos(5a)}{\sin(5a)-\sin(3a)}=\tan(4a)$$

b) $$\frac{\sin(2a)+\sin(3a)}{\cos(2a)-\cos(3a)}=\cot\left(\frac{a}{2}\right)$$

c) $$\frac{\cos(2a)-\cos(4a)}{\sin(4a)-\sin(2a)}=\tan(3a)$$

d) $\dfrac{\cos(4a) - \cos a}{\sin a - \sin(4a)} = \tan\left(\dfrac{5a}{2}\right)$

e) $\dfrac{\sin(2a) + \sin(5a) - \sin a}{\cos(2a) + \cos(5a) + \cos a} = \tan(2a)$

f) $\dfrac{\sin a + \sin 3a + \sin 5a + \sin 7a}{\cos a + \cos 3a + \cos 5a + \cos 7a} = \tan(4a)$

g) $\dfrac{\sin a + \sin b}{\cos a + \cos b} = \tan\left(\dfrac{a+b}{2}\right)$

h) $\cos(5a)\cos(2a) - \cos(4a)\cos(3a) = -\sin 2a \sin a$

i) $\sin(4a)\cos a - \sin(3a)\cos(2a) = \sin a \cos 2a$

17 Show that

a) $(\cos a + \cos b)^2 + (\sin a - \sin b)^2 = 4\cos^2\left(\dfrac{a+b}{2}\right)$

b) $(\cos a + \cos b)^2 + (\sin a + \sin b)^2 = 4\cos^2\left(\dfrac{a-b}{2}\right)$

c) $(\cos a - \cos b)^2 + (\sin a - \sin b)^2 = 4\sin^2\left(\dfrac{a-b}{2}\right)$

d) $\dfrac{\sin(a+b)\sin(a-b)}{\cos^2 a \cos^2 b} = \tan^2 a - \tan^2 b$

e) $\cos a + \cos 2a + \cos 3a = \dfrac{\cos(2a)\sin(3a/2)}{\sin(a/2)}$