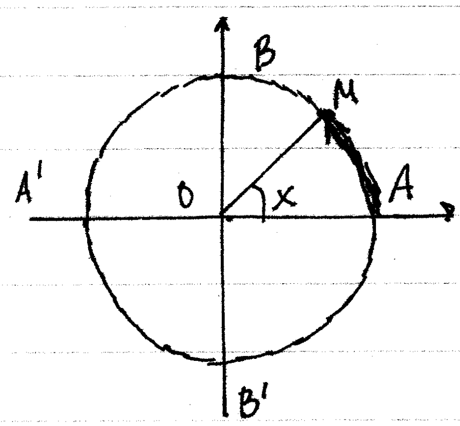
TRIGONOMETRIC FUNCTIONS

The trigonometric circle

- The trigonometric circle is an oriented circle with radius 1. An oriented circle is a circle with a well-defined initial point A and a positive (counterclockwise) and negative (clockwise) direction.



$OA = OM = 1$

- Let $x \in \mathbb{R}$ be given. Starting from the point A, we traverse the trigonometric circle counterclockwise (if $x > 0$) or clockwise (if $x < 0$) over an arc with total length x. We stop at the point M. Note that we could go around the whole circle multiple times.

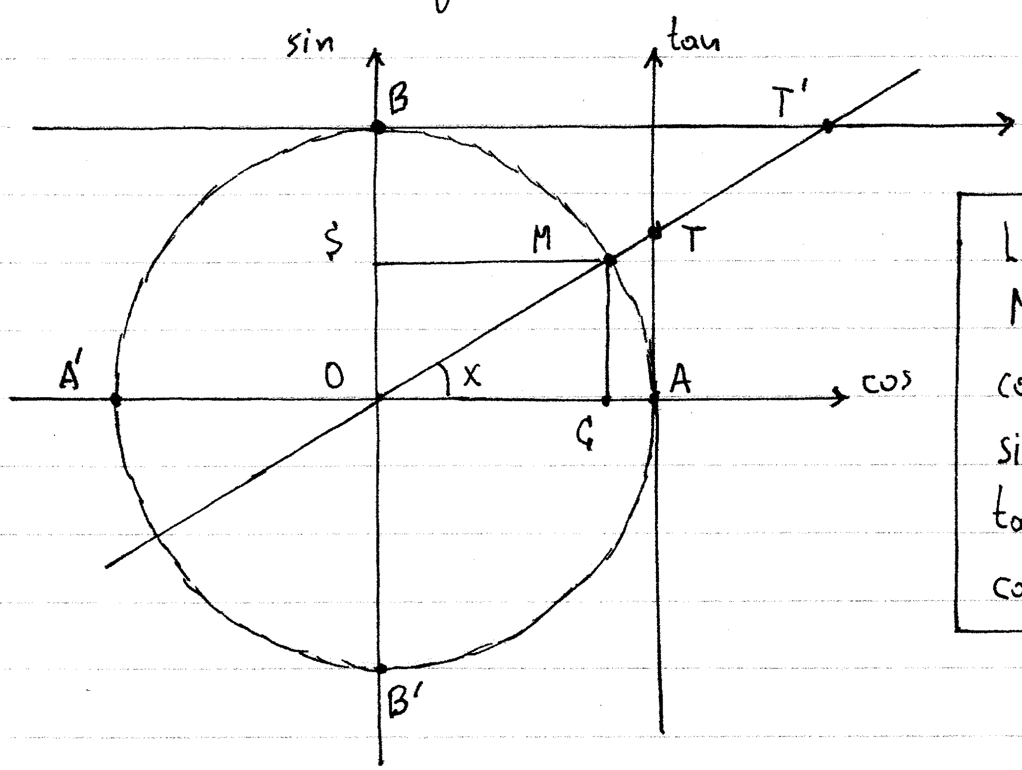
- We say that M is the terminal point of the arc x and define the winding function $\tau : \mathbb{R} \to \mathbb{R}^2$ such that $\tau(x) = M$.
- Consider the set $\mathbb{Z} = \{0, +1, -1, +2, -2, \dots\}$. Two arcs $x_1, x_2 \in \mathbb{R}$ have the same terminal point if and only if there exists a $k \in \mathbb{Z}$ such that $x_1 = x_2 + 2k\pi$. Symbolically, we write:

$$\boxed{\tau(x_1) = \tau(x_2) \Leftrightarrow \exists k \in \mathbb{Z} : x_1 = x_2 + 2k\pi}$$

- It is good to know the general form of arcs with terminal points at A, A', B, B', etc:

Terminal points	Arcs
A	$x = 2k\pi$
A'	$x = (2k+1)\pi$
B	$x = 2k\pi + \pi/2$
B'	$x = (2k+1)\pi + \pi/2$
A or A'	$x = k\pi$
B or B'	$x = k\pi + \pi/2$

Definition of trigonometric functions



Let $x \in \mathbb{R}$

$M = \mathcal{C}(x)$

$\cos x = \overline{OC}$

$\sin x = \overline{OS}$

$\tan x = \overline{AT}$

$\cot x = \overline{BT'}$

•1 On the trigonometric circle we define:

sin-axis: From A' to A

cos-axis: From B' to B

tan-axis: $\begin{cases} \text{Tangent to circle at } A \\ \text{Same direction as sin-axis} \end{cases}$

cot-axis: $\begin{cases} \text{Tangent to circle at } B \\ \text{Same direction as cos-axis.} \end{cases}$

•2 Let $x \in \mathbb{R}$ be an arc with terminal point $M = \mathcal{C}(x)$.

•3 We construct the following points:

C: projection of M to cos-axis

S: projection of M to sin-axis

T: intersection of line (OM) with tan-axis

T': intersection of line (OM) with cot-axis

•4 Now we define the trigonometric functions geometrically as follows:

$\forall x \in \mathbb{R}$: $\sin(x) = \overline{OS}$ = coordinate of S on sin-axis

$\forall x \in \mathbb{R}$: $\cos(x) = \overline{OC}$ = coordinate of C on cos-axis.

↳ (in both cases, O is the origin.)

$\forall x \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$: $\tan(x) = \overline{AT}$ = coordinate of T on tan-axis

↳ (A is the origin on tan-axis, and $\tan x$ is not defined at $M = B$ or $M = B'$ because then OM is parallel to tan-axis)

$\forall x \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$: $\cot(x) = \overline{BT'}$ = coordinate of T' on cot-axis

↳ (B is the origin, and $\cot x$ is not defined at $M = A$ or $M = A'$ because then $OM \parallel$ cot-axis).

Basic properties of trigonometric functions

Trigonometric Identities

$$\boxed{\sin^2 x + \cos^2 x = 1} \longrightarrow \begin{cases} \sin^2 x = 1 - \cos^2 x \\ \cos^2 x = 1 - \sin^2 x \end{cases}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad 1 + \tan^2 x = \frac{1}{\cos^2 x}$$

$$\cot x = \frac{\cos x}{\sin x} \qquad 1 + \cot^2 x = \frac{1}{\sin^2 x}$$

$$\boxed{(\tan x)(\cot x) = 1}$$

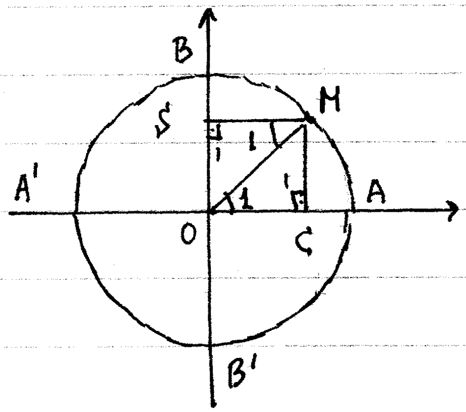
$$\cos^2 x = \frac{1}{1 + \tan^2 x} \qquad \sin^2 x = \frac{1}{1 + \cot^2 x}$$

Evaluation at standard angles

ϑ (radians)	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
ϑ (degrees)	0	$30°$	$45°$	$60°$	$90°$
$\sin\vartheta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\vartheta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\vartheta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	?
$\cot\vartheta$	?	$\sqrt{3}$	1	$\sqrt{3}/3$	0

"?" corresponds to "undefined".

Proof of $\sin^2 x + \cos^2 x = 1$



With no loss of generality, assume that the terminal point M is in the first quadrant. Then $\sin x = OC$ and $\cos x = OS$.

Define:

$\hat{O}_1 = C\hat{O}M \wedge \hat{C}_1 = O\hat{C}M \wedge \hat{M}_1 = S\hat{M}O \wedge \hat{S}_1 = O\hat{S}M$.

Since $\begin{cases} OC \perp BB' \\ MS \perp BB' \end{cases} \Rightarrow OC \parallel MS \Rightarrow \hat{O}_1 = \hat{M}_1$. (1)

Also have $\hat{C}_1 = \hat{S}_1 = 90^\circ$. (2)

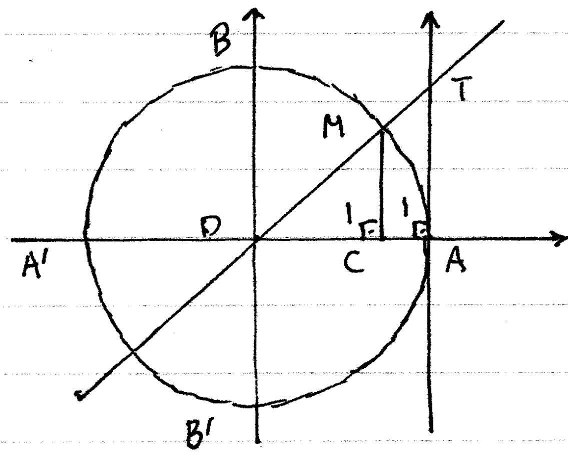
From Eq. (1) and Eq. (2):

$\overset{\triangle}{OCM} \sim \overset{\triangle}{MSO} \Rightarrow \dfrac{CM}{OM} = \dfrac{OS}{OM} \Rightarrow CM = OS$

and therefore, via the pythagorean theorem:

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= (OS)^2 + (OC)^2 \\
&= (CM)^2 + (OC)^2 && [\text{via } CM = OS] \\
&= OM^2 && [\text{pythagorean on } \overset{\triangle}{OCM}] \\
&= 1 && [\text{trig-circle radius}].
\end{aligned}
$$

Proof of $\tan x = \dfrac{\sin x}{\cos x}$



With no loss of generality assume that the terminal point M is in the first quadrant.
We previously showed that

$$CM = OS \qquad (1)$$

Compare $\overset{\Delta}{O\hat{C}M}$ with $\overset{\Delta}{O\hat{A}T}$.
Both share $\hat{O}$ and $\hat{A}_1 = \hat{C}_1 = 90^0$.

It follows that $\overset{\Delta}{O\hat{C}M} \sim \overset{\Delta}{O\hat{A}T}$ and therefore:

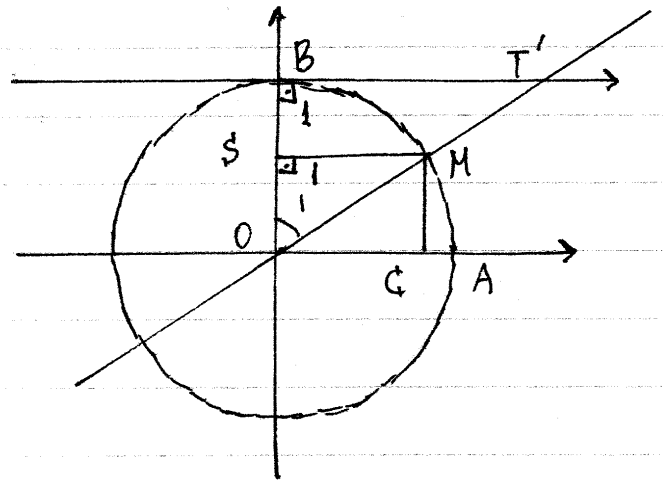
$$\tan x = AT = \frac{AT}{1} = \frac{AT}{OA} \quad [\text{because } OA = 1]$$

$$= \frac{CM}{OC} \quad [\text{because } \overset{\Delta}{O\hat{C}M} \sim \overset{\Delta}{O\hat{A}T}]$$

$$= \frac{OS}{OC} \quad [\text{because } CM = OS]$$

$$= \frac{\sin x}{\cos x}$$

Proof of $\cot x = \dfrac{\cos x}{\sin x}$



With no loss of generality assume that the terminal point M is in the first quadrant. Define $\hat{O}_1 = S\hat{O}M$. We have already shown that $\overset{\Delta}{OSM} \sim \overset{\Delta}{MCO} \Rightarrow \dfrac{MS}{OM} = \dfrac{OC}{OM} \Rightarrow$

$$\Rightarrow MS = OC. \quad (1)$$

Compare $\overset{\Delta}{OMS}$ with $\overset{\Delta}{OT'B}$. Both share $\hat{O}_1$ and $\hat{S}_1 = \hat{B}_1 = 90^\circ$ (with $\hat{S}_1 = O\hat{S}M$ and $\hat{B}_1 = O\hat{B}T'$), thus $\overset{\Delta}{OSM} \sim \overset{\Delta}{OBT'}$.

It follows that

$$\cot x = BT' = \frac{BT'}{1} = \frac{BT'}{OB} \qquad [\text{via } OB = 1]$$

$$= \frac{MS}{OS} \qquad [\text{via } \overset{\Delta}{OSM} \sim \overset{\Delta}{OBT'}]$$

$$= \frac{OC}{OS} \qquad [\text{via } MS = OC]$$

$$= \frac{\cos x}{\sin x}$$

Proof of other identities.

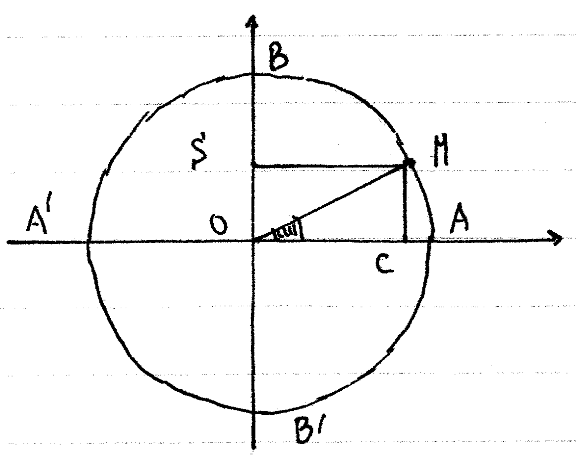
We have:

$$1+\tan^2 x = 1+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

and

$$1+\cot^2 x = 1+\frac{\cos^2 x}{\sin^2 x} = \frac{\cos^2 x+\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

Angle 30° - proof



Assume that $M\hat{O}C = 30^\circ$. Then:

$$\begin{cases} M\hat{O}C = 30^\circ \\ M\hat{C}O = 90^\circ \end{cases} \Rightarrow CM = \frac{OM}{2} = \frac{1}{2}$$

$$\Rightarrow \sin(30^\circ) = OS = CM = \frac{1}{2}$$

Since $0 < 30^\circ < 90^\circ \Rightarrow \cos 30^\circ > 0$ and it follows that

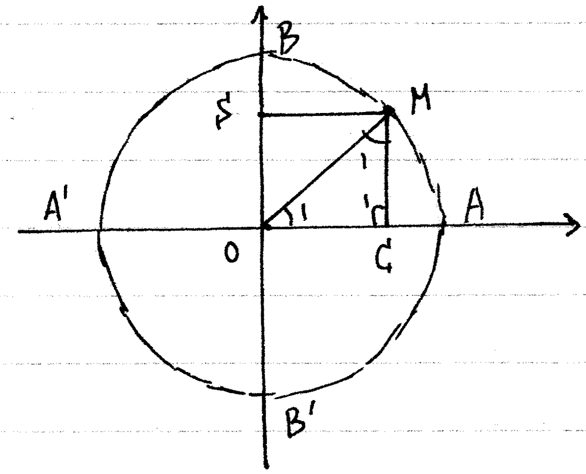
$$\cos^2(30^\circ) = 1-\sin^2(30^\circ) = 1-\left(\frac{1}{2}\right)^2 = 1-\frac{1}{4} = \frac{4-1}{4} = \frac{3}{4} \Rightarrow$$

$$\Rightarrow \cos(30^\circ) = \frac{\sqrt{3}}{2}$$, and also:

$$\tan(30^\circ) = \frac{\sin(30^\circ)}{\cos(30^\circ)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot(30^\circ) = \frac{\cos(30^\circ)}{\sin(30^\circ)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Angle 45° - Proof



Assume that $\hat{MOC} = 45°$.
Define $\hat{O}_1 = \hat{MOC}$ and
$\hat{M}_1 = \hat{OMC}$ and $\hat{C}_1 = \hat{OCM}$.

$$\hat{M}_1 = 180° - \hat{C}_1 - \hat{M}_1 =$$
$$= 180° - 90° - 45° =$$
$$= 45° = \hat{O}_1 \Rightarrow OC = CM$$
$$\Rightarrow OC = OS \Rightarrow$$
$$\Rightarrow \sin(45°) = \cos(45°)$$

Since

$$\sin^2(45°) + \cos^2(45°) = 1 \Rightarrow \sin^2(45°) + \sin^2(45°) = 1 \Rightarrow$$
$$\Rightarrow 2\sin^2(45°) = 1 \Rightarrow \sin^2(45°) = \frac{1}{2} \Rightarrow$$
$$\Rightarrow \sin(45°) = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos(45°)$$

we also have:

$$\tan(45°) = \frac{\sin(45°)}{\cos(45°)} = \frac{\cos(45°)}{\cos(45°)} = 1$$

$$\cot(45°) = \frac{\cos(45°)}{\sin(45°)} = \frac{\cos(45°)}{\cos(45°)} = 1$$

EXAMPLES

a) Simplify the following expression:

$A = [\sin(\pi/4) \cdot \cos(\pi/6) + \cot(\pi/3)] \tan(\pi/6)$

Solution

$$A = [\sin(\pi/4)\cos(\pi/6) + \cot(\pi/3)]\tan(\pi/6) =$$

$$= \left[\frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}\right]\frac{\sqrt{3}}{3} = \frac{\sqrt{2}(\sqrt{3})^2}{2\cdot 2\cdot 3} + \frac{(\sqrt{3})^2}{3^2} =$$

$$= \frac{3\sqrt{2}}{3\cdot 4} + \frac{3}{3^2} = \frac{\sqrt{2}}{4} + \frac{1}{3} = \frac{3\sqrt{2}+4}{4\cdot 3} = \frac{3\sqrt{2}+4}{12}$$

b) Simplify the following expression

$A = \dfrac{\sin(\pi/6) + \sin(\pi/3)}{\sin(\pi/6) - \sin(\pi/3)}$

Solution

$$A = \frac{\sin(\pi/6) + \sin(\pi/3)}{\sin(\pi/6) - \sin(\pi/3)} = \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}}{\frac{1}{2} - \frac{\sqrt{3}}{2}} = \frac{1+\sqrt{3}}{1-\sqrt{3}} =$$

$$= \frac{(1+\sqrt{3})^2}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{1^2 + 2\cdot 1\cdot\sqrt{3} + (\sqrt{3})^2}{1^2 - (\sqrt{3})^2} = \frac{1+2\sqrt{3}+3}{1-3} =$$

$$= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

$\downarrow\!\!\!\rightarrow$ When simplifying an arithmetic expression, we should remove radicals from the denominator.

a) To remove $\sqrt{a}$, multiply numerator and denominator with another $\sqrt{a}$

b) To remove $\sqrt{a} \pm \sqrt{b}$, multiply numerator and denominator with the conjugate $\sqrt{a} \mp \sqrt{b}$.

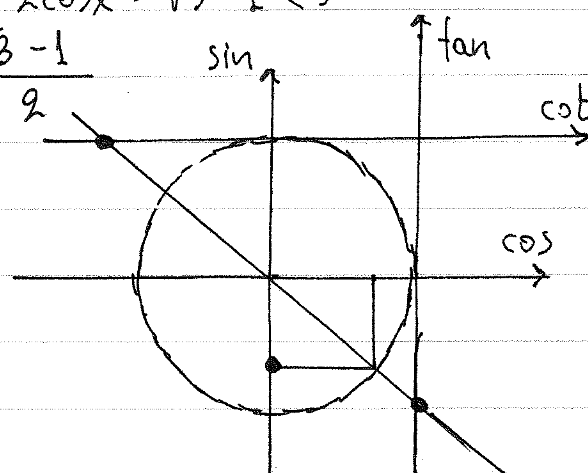
c) If $4\cos x + 1 = 2\cos x + \sqrt{3}$ and $3\pi/2 \leq x \leq 2\pi$, evaluate the expression $A = 2\sin x + 6\tan x$.

Solution

We note that

$4\cos x + 1 = 2\cos x + \sqrt{3} \Leftrightarrow 4\cos x - 2\cos x = \sqrt{3} - 1 \Leftrightarrow$

$\Leftrightarrow 2\cos x = \sqrt{3} - 1 \Leftrightarrow \cos x = \dfrac{\sqrt{3}-1}{2}$



and

$$3\pi/2 \leq x \leq 2\pi \Rightarrow \begin{cases} \sin x \leq 0 \\ \cot x \leq 0 \\ \tan x \leq 0 \end{cases}$$

so it follows that

$$\sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{\sqrt{3}-1}{2}\right)^2 =$$

$$= 1 - \frac{(\sqrt{3}-1)^2}{4} = 1 - \frac{(\sqrt{3})^2 - 2\sqrt{3} + 1^2}{4} =$$

$$= 1 - \frac{3 - 2\sqrt{3} + 1}{4} = 1 - \frac{4 - 2\sqrt{3}}{4} = 1 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \Rightarrow$$

$$\Rightarrow \sin x = \left(\frac{\sqrt{3}}{2}\right)^{1/2} \vee \sin x = -\left(\frac{\sqrt{3}}{2}\right)^{1/2} \overset{*}{\Rightarrow}$$

$$\Rightarrow \sin x = -\left(\frac{\sqrt{3}}{2}\right)^{1/2} = -\frac{\sqrt[4]{3}}{\sqrt{2}} = -\frac{\sqrt{2}\sqrt[4]{3}}{\sqrt{2}\sqrt{2}} = \frac{-\sqrt{2}\sqrt[4]{3}}{2}$$

and

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{-\sqrt{2}\sqrt[4]{3}}{2}}{\frac{\sqrt{3}-1}{2}} = \frac{-\sqrt{2}\sqrt[4]{3}}{\sqrt{3}-1} = \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} =$$

$$= \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{(\sqrt{3})^2-1^2} = \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{3-1} = \frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{2}$$

and therefore

$$A = 2\sin x + 6\tan x = 2\left[\frac{-\sqrt{2}\sqrt[4]{3}}{2}\right] + 6\left[\frac{-\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1)}{2}\right]$$

$$= -\sqrt{2}\sqrt[4]{3} - 3\sqrt{2}\sqrt[4]{3}(\sqrt{3}+1) = -\sqrt{2}\sqrt[4]{3}\left[1+3(\sqrt{3}+1)\right] =$$

$$= -\sqrt{2}\sqrt[4]{3}\left[1+3\sqrt{3}+3\right] = -\sqrt{2}\sqrt[4]{3}\left[4+3\sqrt{3}\right].$$

↳ Recall the following identities from algebra:

$$\boxed{\begin{aligned} a^2-b^2 &= (a-b)(a+b) \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a-b)^2 &= a^2-2ab+b^2 \end{aligned}}$$

EXERCISES

① Simplify the following expressions.

a) $5\cot^2(\pi/4) - \sin(\pi/6) - \dfrac{1}{\cos^2(\pi/3)}$

b) $\sin(\pi/6)\cos(\pi/3)\tan(\pi/4)$

c) $\tan(\pi/6)\sin^2(\pi/3) + \cos(\pi/4)$

d) $\tan(\pi/3)\left[\cos(\pi/6)\sin(\pi/3) + \cot(\pi/3)\right]$

② a) If $\sin x = 1/3$ and $\pi/2 < x < \pi$, then evaluate $A = \dfrac{5\tan x + 4\cos^2 x}{3\sin x}$

b) If $\cos x = 12/13$ and $3\pi/2 < x < 2\pi$, then evaluate $A = 5\cos x - 8\sin x + \tan^2 x$

c) If $\tan x = 3$ and $\pi < x < 3\pi/2$, then evaluate $A = 6\sin^2 x + \cos x - 3\tan x$

d) If $\cot x = 2$ and $0 < x < \pi/2$, then evaluate $A = 2\tan x - 4\sin x + \cos x$.

e) If $5\sin x + 4 = 2\sin x + 3$ and $3\pi/2 < x < 2\pi$ then evaluate $A = 2\sin x + 3\cos x - 5\tan x + \cot x$

③ Show that $\tan^2\left(\frac{\pi}{6}\right) + \tan^2\left(\frac{\pi}{4}\right) + \tan^2\left(\frac{\pi}{3}\right) = \frac{13}{3}$

Reduction to 1st quadrant

- The challenge is to rewrite a trigonometric function of the arc $k\pi/2 \pm x$ in terms of a trigonometric function of x. To do that we use the following properties:

1) Odd / Even property

$$\boxed{\begin{aligned} \sin(-x) &= -\sin x \\ \cos(-x) &= +\cos x \\ \tan(-x) &= -\tan x \\ \cot(-x) &= -\cot x \end{aligned}}$$

2) Periodicity

$$\boxed{\begin{aligned} \sin(x+2\pi) &= \sin x \\ \cos(x+2\pi) &= \cos x \\ \tan(x+\pi) &= \tan x \\ \cot(x+\pi) &= \cot x \end{aligned}}$$

3) Cofunction identities

$$\boxed{\begin{aligned} \sin(\pi/2-x) &= \cos x \\ \cos(\pi/2-x) &= \sin x \\ \tan(\pi/2-x) &= \cot x \\ \cot(\pi/2-x) &= \tan x \end{aligned}}$$

4) Angle $\pi+x$

$$\boxed{\begin{aligned} \sin(\pi+x) &= -\sin x \\ \cos(\pi+x) &= -\cos x \\ \tan(\pi+x) &= \tan x \\ \cot(\pi+x) &= \cot x \end{aligned}}$$

In general:

$$\boxed{\begin{aligned} \sin(k\pi+x) &= (-1)^k \sin x \\ \cos(k\pi+x) &= (-1)^k \cos x \end{aligned}}$$

EXAMPLES

a) Simplify the expression

$$A = \sin\left(\frac{19\pi}{6}\right)\cos\left(\frac{5\pi}{3}\right)\tan\left(\frac{14\pi}{3}\right)$$

Solution

Since

$$\sin\left(\frac{19\pi}{6}\right) = \sin\left(3\pi + \frac{\pi}{6}\right) = (-1)^3 \sin(\pi/6) = -\sin(\pi/6) = -1/2$$

$$\cos\left(\frac{5\pi}{3}\right) = \cos\left(\pi + \frac{2\pi}{3}\right) = -\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\pi - \frac{\pi}{3}\right) =$$

$$= +\cos(-\pi/3) = \cos(\pi/3) = 1/2$$

$$\tan\left(\frac{14\pi}{3}\right) = \tan\left(\frac{(15-1)\pi}{3}\right) = \tan(5\pi - \pi/3) = \tan(-\pi/3)$$

$$= -\tan(\pi/3) = -\sqrt{3}.$$

it follows that

$$A = \sin\left(\frac{19\pi}{6}\right)\cos\left(\frac{5\pi}{3}\right)\tan\left(\frac{14\pi}{3}\right) =$$

$$= \left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)(-\sqrt{3}) = \frac{\sqrt{3}}{4}$$

b) Simplify the expression

$$A = \frac{\cos(x - 5\pi/2)\sin(x+3\pi/2)}{\sin(x-3\pi)\cos(5\pi - x)}$$

Solution

Since

$\cos(x - 5\pi/2) = (-1)^3 \cos(3\pi + x - 5\pi/2) = -\cos(x + \pi/2) =$

$= -\cos(\pi/2 - (-x)) = -\sin(-x) = \sin x$

$\sin(x+3\pi/2) = \sin(x + 2\pi - \pi/2) = \sin(x - \pi/2) = -\sin(\pi/2 - x)$

$= -\cos x$

$\sin(x-3\pi) = (-1)^3 \sin(3\pi + x - 3\pi) = -\sin x$

$\cos(5\pi - x) = (-1)^5 \cos(-x) = -\cos(-x) = -\cos x$

it follows that

$$A = \frac{\cos(x-\pi/2)\sin(x+3\pi/2)}{\sin(x-3\pi)\cos(5\pi-x)} = \frac{\sin x\,[-\cos x]}{[-\sin x][-\cos x]} = -1$$

EXERCISES

(4) Simplify the following expressions

a) $A = \sin\left(\frac{7\pi}{3}\right)\cos\left(\frac{13\pi}{6}\right)\cos\left(-\frac{5\pi}{3}\right)\sin\left(\frac{11\pi}{6}\right)$

b) $A = \sin\left(-\frac{2\pi}{3}\right)\tan\left(\frac{5\pi}{3}\right)\cot\left(-\frac{4\pi}{3}\right)\cos\left(\frac{5\pi}{6}\right)$

c) $A = \sin\left(\frac{3\pi}{2}+x\right)\sin(\pi+x)+\sin\left(\frac{3\pi}{2}-x\right)\sin(\pi-x)$

d) $A = \sin\left(\frac{3\pi}{2}+x\right)+\cos\left(\frac{3\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}+x\right)$

e) $A = \dfrac{\sin(a-3\pi/2)\tan(b-\pi)}{\cot(3\pi/2-b)\sin(a+\pi/2)}$

f) $A = \dfrac{\sin(a+\pi/2)\tan(9\pi+a)\cos(a-\pi/2)}{\cos(11\pi-a)\sin(3\pi/2+a)\tan(2\pi+a)}$

g) $A = \dfrac{\sin(5\pi+a)\tan(3\pi+a)\cos(4\pi+a)}{\cos(7\pi-a)\tan(8\pi+a)\sin a}$

Simple trigonometric identities

- Recall that

$$\boxed{\sin^2 x + \cos^2 x = 1} \rightarrow \begin{cases} \sin^2 x = 1 - \cos^2 x \\ \cos^2 x = 1 - \sin^2 x \end{cases}$$

$$\boxed{\begin{array}{l} \tan x = \dfrac{\sin x}{\cos x} \\ \hline \cot x = \dfrac{\cos x}{\sin x} \end{array}} \rightarrow \tan x \cot x = 1 \rightarrow \begin{cases} \tan x = \dfrac{1}{\cot x} \\ \cot x = \dfrac{1}{\tan x} \end{cases}$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x} \qquad 1 + \cot^2 x = \frac{1}{\sin^2 x}$$

- Method: To show $A = B$

a) Direct Method

$A = \dots = \dots = B$

b) Indirect Method

$A = \dots = \dots = C$

$B = \dots = \dots = C$

It follows that $A = B$.

c) Method of Desperation

$A - B = \dots = \dots = 0 \Rightarrow$

$\Rightarrow A = B$

- Recall identities from intermediate algebra

$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$a^2 - b^2 = (a-b)(a+b)$
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$
$(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$

EXAMPLES

a) Show that:

$$\sin^6 x - \cos^6 x = (1-2\cos^2 x)(1-\sin^2 x\cos^2 x)$$

Solution

We have:

$$\sin^6 x - \cos^6 x = (\sin^3 x - \cos^3 x)(\sin^3 x + \cos^3 x) =$$

$$= (\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x)(\sin x + \cos x)$$

$$\times (\sin^2 x - \sin x\cos x + \cos^2 x) =$$

$$= [(\sin x - \cos x)(\sin x + \cos x)](1+\sin x\cos x)(1-\sin x\cos x) =$$

$$= (\sin^2 x - \cos^2 x)(1-\sin^2 x\cos^2 x) =$$

$$= [(1-\cos^2 x) - \cos^2 x](1-\sin^2 x\cos^2 x) =$$

$$= (1-2\cos^2 x)(1-\sin^2 x\cos^2 x).$$

Recall the following factorizations from algebra:

$$\boxed{\begin{aligned} a^2-b^2 &= (a-b)(a+b)\\ a^3-b^3 &= (a-b)(a^2+ab+b^2)\\ a^3+b^3 &= (a+b)(a^2-ab+b^2)\end{aligned}}$$

b) Show that $\dfrac{1-\sin x}{1+\sin x} = \left(\dfrac{1}{\cos x} - \tan x\right)^2$

<u>solution</u>

We have:

$$\frac{1-\sin x}{1+\sin x} = \frac{(1-\sin x)^2}{(1+\sin x)(1-\sin x)} = \frac{1-2\sin x+\sin^2 x}{1-\sin^2 x} =$$

$$= \frac{1-2\sin x+\sin^2 x}{\cos^2 x} =$$

$$= \frac{1}{\cos^2 x} - 2\,\frac{\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} =$$

$$= \left(\frac{1}{\cos x}\right)^2 - 2\frac{\sin x}{\cos x}\,\frac{1}{\cos x} + \left(\frac{\sin x}{\cos x}\right)^2 =$$

$$= \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)^2 = \left(\frac{1}{\cos x} - \tan x\right)^2$$

c) Show that: $\dfrac{\tan x+\tan y}{\cot x+\cot y} = \tan x\cdot\tan y$

<u>Solution</u>

We have:

$$\frac{\tan x+\tan y}{\cot x+\cot y} = \frac{\tan x+\tan y}{\dfrac{1}{\tan x}+\dfrac{1}{\tan y}} = \frac{\tan x+\tan y}{\left(\dfrac{\tan y+\tan x}{\tan x\tan y}\right)} =$$

$$= \frac{\tan x\tan y\,(\tan x+\tan y)}{\tan x+\tan y} = \tan x\tan y$$

EXERCISES

5. Show that

a) $\tan^2 a - \sin^2 a = \tan^2 a \sin^2 a$

b) $\cot^2 x - \cos^2 x = \cot^2 x \cdot \cos^2 x$

c) $(\sin\vartheta + \cos\vartheta)^4 - (\sin\vartheta - \cos\vartheta)^4 = 8\sin\vartheta\cos\vartheta$

d) $\tan\vartheta(1-\cot^2\vartheta) + \cot\vartheta(1-\tan^2\vartheta) = 0$

e) $\sin^2 x\tan x - \cos^2 x\cot x = \tan x - \cot x$

f) $(\sin x + \cos x + 1)(\sin x + \cos x - 1) = 2\sin x\cos x$

g) $\sin^2 a(1+\cot^2 a) + \cos^2 a(1+\tan^2 a) = 2$

h) $\sin^2 x\tan x + \cos^2 x\cot x + 2\sin x\cos x = \tan x + \cot x$

i) $4(\sin^6 x + \cos^6 x) - 3(\cos^4 x - \sin^4 x)^2 = 1$

6. Show that

a) $\dfrac{1+\tan^2 x}{1+\cot^2 x} = \tan^2 x$ b) $\dfrac{1-\tan^2 x}{1+\tan^2 x} = 1-2\sin^2 x$

c) $\dfrac{1-\sin\vartheta}{1+\sin\vartheta} - \dfrac{1+\sin\vartheta}{1-\sin\vartheta} = -4\dfrac{\tan\vartheta}{\cos\vartheta}$

d) $\dfrac{\sin x}{1-\cot x} + \dfrac{\cos x}{1-\tan x} = \sin x + \cos x$

e) $\dfrac{\cos^3 a - \cos a + \sin a}{\cos a} = \tan a - \sin^2 a$

f) $\dfrac{\sin a + \sin b}{\cos a + \cos b} + \dfrac{\cos a - \cos b}{\sin a - \sin b} = 0$

(7) Show that

a) $\dfrac{\sin^2 b - \sin^2 a}{\sin^2 a \sin^2 b} = \cot^2 a - \cot^2 b$

b) $\dfrac{(\sin a + \cos b)^2 + (\cos a + \sin b)(\cos a - \sin b)}{\sin a + \cos b} = 2\cos b$

c) $\dfrac{\cos^2 a - \sin^2 b}{\sin^2 a \sin^2 b} = \dfrac{1}{\tan^2 a}\left(\dfrac{1}{\sin^2 b} - \dfrac{1}{\cos^2 a}\right)$

d) $\dfrac{\tan x - \sin x}{\sin^3 x} = \dfrac{1}{\cos^2 x + \cos x}$

e) $\dfrac{\tan a}{1+\tan^2 a} + \dfrac{\cos^3 a}{\sin a} = \cot a$

f) $2\cos^2 x - \sin^2 x = \dfrac{2-\tan^2 x}{1+\tan^2 x}$

g) $\dfrac{1}{(\sin x \cos x)^2} - 4 = (\tan x - \cot x)^2$

h) $(\tan a - \sin a)^2 + (1-\cos a)^2 = \left(\dfrac{1}{\cos a} - 1\right)^2.$

(8) a) If $\dfrac{1+\cos^2 b}{1+2\sin^2 b} = \sin^2 a$, then show that $\sin^2 b = \dfrac{1+\cos^2 a}{1+2\sin^2 a}$

b) If $\left.\begin{array}{l} \alpha = x\cos\vartheta + y\sin\vartheta \\ b = x\sin\vartheta - y\cos\vartheta \end{array}\right\} \Rightarrow a^2 + b^2 = x^2 + y^2.$

c) Show that $A+B = \pi/2 \Rightarrow \cos^2 A + \cos^2 B = 1.$

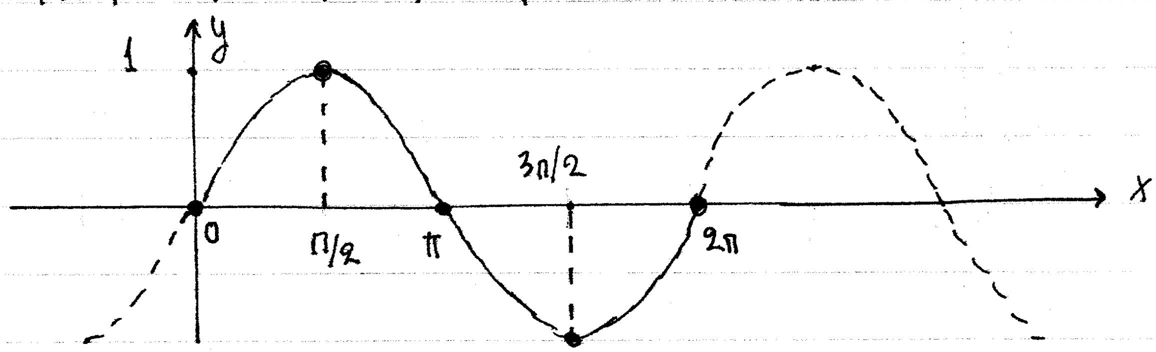
Graphs of sin and cos

1) $\boxed{f(x) = \sin(x)}$

Domain : $A = \mathbb{R}$

Range: $f(A) = [-1, 1]$

x	0	$\pi/2$	π	$3\pi/2$	2π
$f(x)$	0	1	0	-1	0

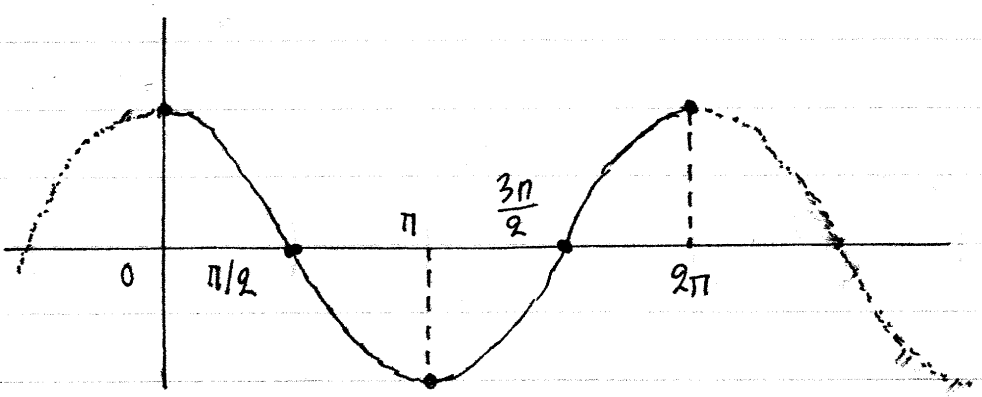


2) $\boxed{f(x) = \cos x}$

Domain : $A = \mathbb{R}$

Range: $f(A) = [-1, 1]$

x	0	$\pi/2$	π	$3\pi/2$	2π
$f(x)$	1	0	-1	0	1



Methodology: The problem is to graph the functions

$$f(t) = a\sin(\omega t + b) + c$$
$$f(t) = a\cos(\omega t + b) + c$$

▸ **Terminology**

ω = angular velocity (if t is time)
(we use kx instead of ωt for spacial dependence; k is the wavenumber)
b = phase shift
$\varphi = \omega t + b$ = phase
a = amplitude
c = vertical shift.

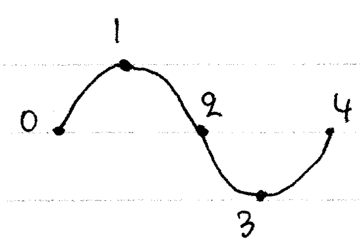
▸ To graph these functions.

• 1 Solve the equation $\omega t + b = k\pi/2$ with respect to t.

• 2 For $k = 0,1,2,3,4$ find the corresponding t_0, t_1, t_2, t_3, t_4 and note that:

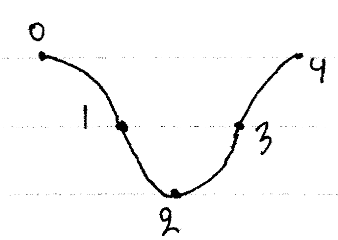
sin:

k	0	1	2	3	4
$\varphi = \omega t + b$	0	$\pi/2$	π	$3\pi/2$	2π
$f(t)$	c	$c+a$	c	$c-a$	c



cos:

k	0	1	2	3	4
$\varphi = \omega t + b$	0	$\pi/2$	π	$3\pi/2$	2π
$f(t)$	$c+a$	c	$c-a$	c	$c+a$



• 3 Given the points $(t_0, f(t_0))$, $(t_1, f(t_1))$, $(t_2, f(t_2))$, $(t_3, f(t_3))$, $(t_4, f(t_4))$ we construct the graph.

EXAMPLES

a) Graph the function $f(x) = 2\sin\left(\frac{2x+\pi}{5}\right) - 1$

Solution

Solve:

$$\frac{2x+\pi}{5} = \frac{k\pi}{2} \Leftrightarrow 2(2x+\pi) = 5k\pi \Leftrightarrow 4x + 2\pi = 5k\pi \Leftrightarrow$$

$$\Leftrightarrow 4x = (5k-2)\pi \Leftrightarrow x = \frac{(5k-2)\pi}{4}$$

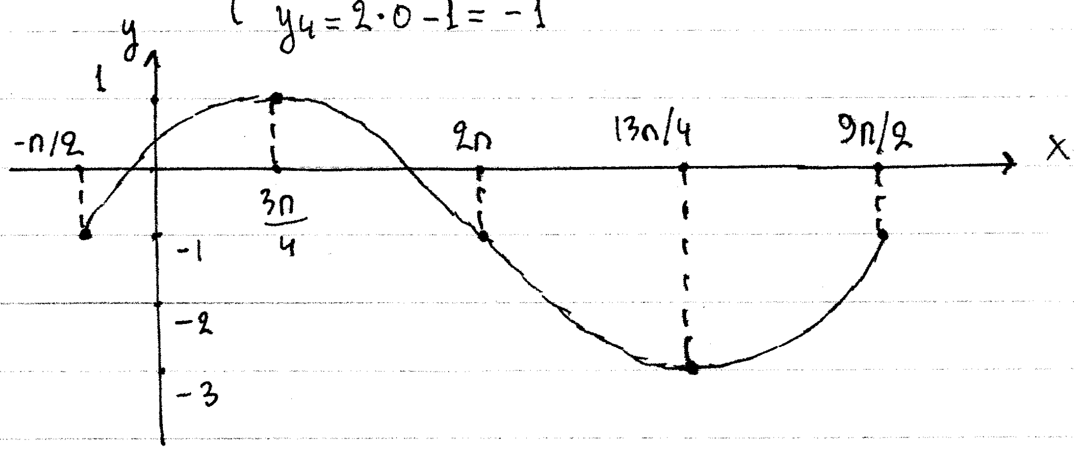
For $k=0$: $\begin{cases} x_0 = (0-2)\pi/4 = -\pi/2 \\ y_0 = 2\cdot 0 - 1 = -1 \end{cases}$

For $k=1$: $\begin{cases} x_1 = (5\cdot 1 - 2)\pi/4 = 3\pi/4 \\ y_1 = 2\cdot 1 - 1 = 1 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (5\cdot 2 - 2)\pi/4 = 8\pi/4 = 2\pi \\ y_2 = 2\cdot 0 - 1 = -1 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (5\cdot 3 - 2)\pi/4 = 13\pi/4 \\ y_3 = 2\cdot(-1) - 1 = -3 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (5\cdot 4 - 2)\pi/4 = 18\pi/4 = 9\pi/2 \\ y_4 = 2\cdot 0 - 1 = -1 \end{cases}$



b) Graph the function $f(x) = \frac{1}{2} - \cos\left(2x + \frac{\pi + x}{3}\right)$

Solution

Solve: $2x + \frac{\pi + x}{3} = \frac{k\pi}{2} \Leftrightarrow 6\left[2x + \frac{\pi + x}{3}\right] = 6 \cdot \frac{k\pi}{2} \Leftrightarrow$

$\Leftrightarrow 12x + 2(\pi + x) = 3k\pi \Leftrightarrow 12x + 2\pi + 2x = 3k\pi \Leftrightarrow 14x = (3k-2)\pi$

$\Leftrightarrow x = \frac{(3k-2)\pi}{14}$

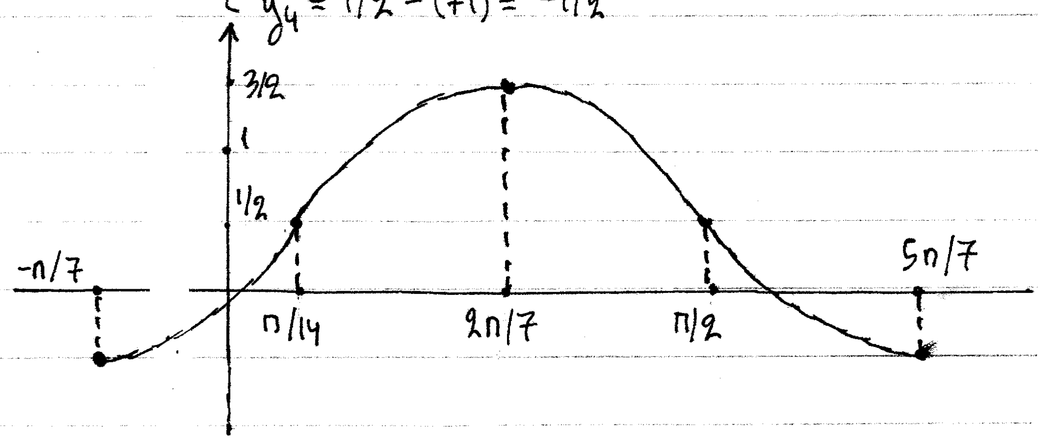
For $k=0$: $\begin{cases} x_0 = (3\cdot 0 - 2)\pi/14 = -2\pi/14 = -\pi/7 \\ y_0 = 1/2 - (+1) = -1/2 \end{cases}$

For $k=1$: $\begin{cases} x_1 = (3\cdot 1 - 2)\pi/14 = \pi/14 \\ y_1 = 1/2 - 0 = 1/2 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (3\cdot 2 - 2)\pi/14 = 4\pi/14 = 2\pi/7 \\ y_2 = 1/2 - (-1) = 3/2 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (3\cdot 3 - 2)\pi/14 = 7\pi/14 = \pi/2 \\ y_3 = 1/2 - 0 = 1/2 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (3\cdot 4 - 2)\pi/14 = 10\pi/14 = 5\pi/7 \\ y_4 = 1/2 - (+1) = -1/2 \end{cases}$



EXERCISES

⑨ Graph the following functions

a) $f(x) = \sin\left(\dfrac{2x+\pi}{3}\right)$

b) $f(x) = \dfrac{1}{2} + \sin\left(x - \dfrac{\pi+3x}{4}\right)$

c) $f(x) = 1 - \sin\left(\dfrac{\pi+x}{3} - \dfrac{x+4\pi}{6}\right)$

d) $f(x) = 2 - 3\cos\left(2x + \dfrac{x+\pi}{2} + \dfrac{2x-\pi}{4}\right)$

e) $f(x) = 1 + \cos\left(x - \dfrac{x+8\pi}{2} + \dfrac{\pi+3x}{6}\right)$

f) $f(x) = -1 + 2\cos\left(x(\pi x+1) - \pi(x+1)(x-1)\right)$

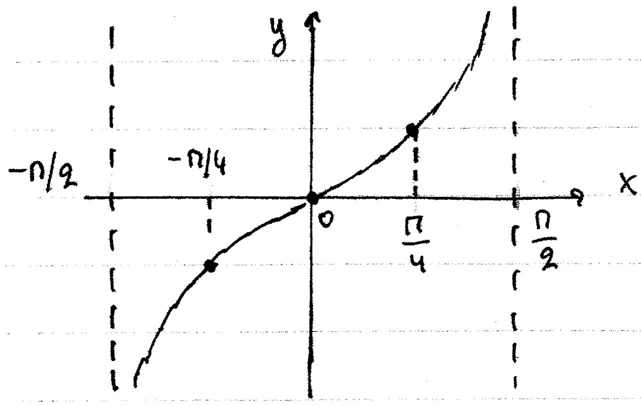
Graphs of tan and cot

1) $\boxed{f(x) = \tan(x)}$

Domain: $A = \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$

Range: $f(A) = \mathbb{R}$ Period: $T = \pi$

x	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
$f(x)$	$+\infty // -\infty$	-1	0	$+1$	$+\infty // -\infty$



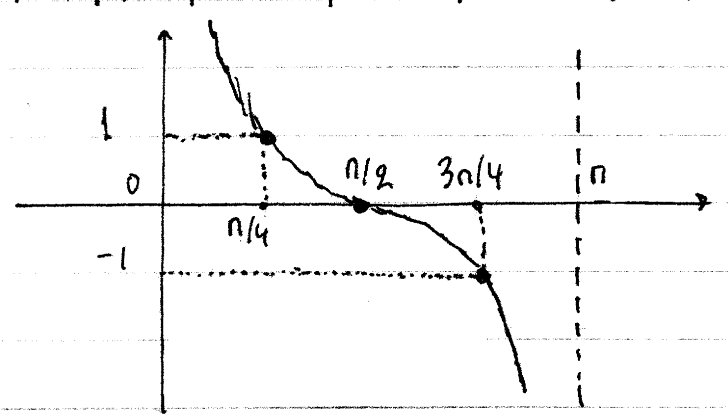
There are vertical asymptotes at $x = k\pi + \pi/2$

2) $\boxed{f(x) = \cot(x)}$

Domain: $A = \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

Range: $f(A) = \mathbb{R}$. Period $T = \pi$

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$f(x)$	$-\infty // +\infty$	1	0	-1	$-\infty // +\infty$



There are vertical asymptotes at $x = k\pi$

Methodology: The problem is to graph:

$\boxed{f(t) = a\tan(\omega t + b) + c}$

Solve $\boxed{\omega t + b = k\pi/4 - \pi/2}$

k	$\omega t + b$	$f(t)$
0	$-\pi/2^+$	$(-\infty)a$
1	$-\pi/4$	$-a+c$
2	0	c
3	$\pi/4$	$a+c$
4	$\pi/2^-$	$(+\infty)a$

$\boxed{f(t) = a\cot(\omega t + b) + c}$

Solve $\boxed{\omega t + b = k\pi/4}$

k	$\omega t + b$	$f(t)$
0	0^+	$(+\infty)a$
1	$\pi/4$	$a+c$
2	$\pi/2$	c
3	$3\pi/4$	$-a+c$
4	π^-	$(-\infty)a$

Then, graph the function!

EXAMPLES

a) Graph $f(x) = 2\tan(2x + \pi/2) - 1$

Solution

Solve: $2x + \pi/2 = k\pi/4 - \pi/2 \Leftrightarrow 4(2x + \pi/2) = 4(k\pi/4 - \pi/2) \Leftrightarrow$

$\Leftrightarrow 8x + 2\pi = k\pi - 2\pi \Leftrightarrow 8x = k\pi - 2\pi - 2\pi \Leftrightarrow 8x = (k-4)\pi \Leftrightarrow$

$\Leftrightarrow x = \dfrac{(k-4)\pi}{8}$

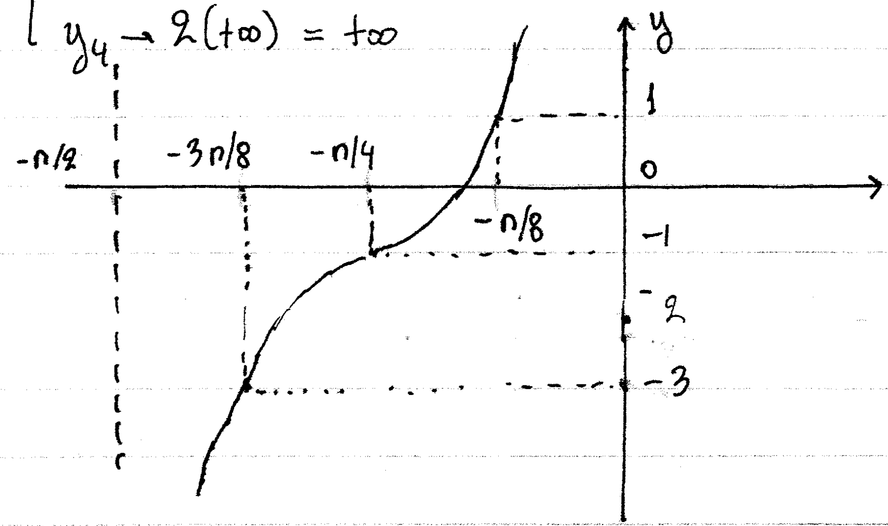
For $k=0$: $\begin{cases} x_0 = (0-4)\pi/8 = -\pi/2 \\ y_0 \to 2(-\infty) = -\infty \end{cases}$

For $k=1$: $\begin{cases} x_1 = (1-4)\pi/8 = -3\pi/8 \\ y_1 = 2(-1) - 1 = -3 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (2-4)\pi/8 = -2\pi/8 = -\pi/4 \\ y_2 = 2 \cdot 0 - 1 = -1 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (3-4)\pi/8 = -\pi/8 \\ y_3 = 2 \cdot (+1) - 1 = 2 - 1 = 1 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (4-4)\pi/8 = 0 \\ y_4 \to 2(+\infty) = +\infty \end{cases}$



b) Graph the function $f(x) = 2\cot(3x - \pi/3) + 1$

<u>Solution</u>

Solve: $3x - \pi/3 = k\pi/4 \Leftrightarrow 12(3x - \pi/3) = 12(k\pi/4) \Leftrightarrow$

$\Leftrightarrow \quad 36x - 4\pi = 3k\pi \Leftrightarrow 36x = (3k+4)\pi \Leftrightarrow$

$\Leftrightarrow \quad x = (3k+4)\pi/36$

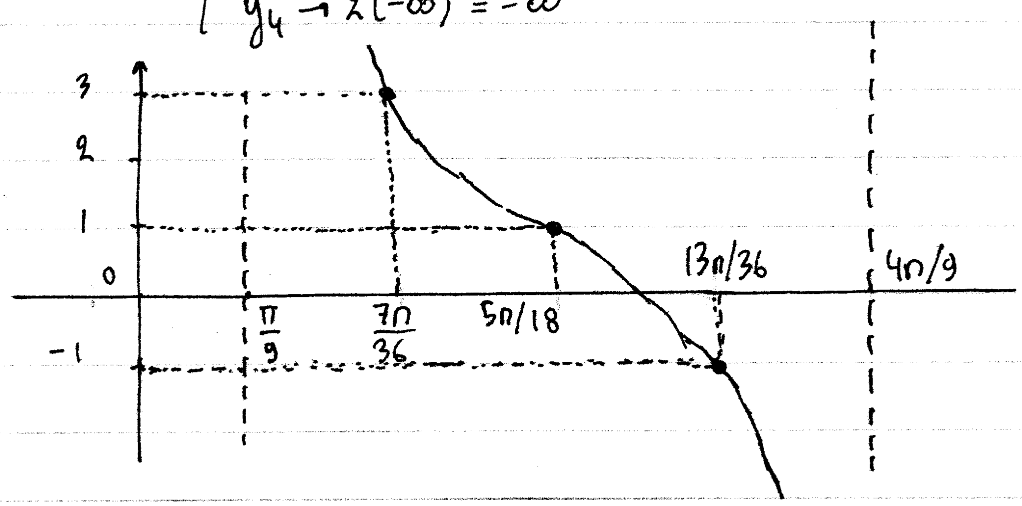
For $k=0$: $\begin{cases} x_0 = (3\cdot 0 + 4)\pi/36 = 4\pi/36 = \pi/9 \\ y_0 \to 2(+\infty) = +\infty \end{cases}$

For $k=1$: $\begin{cases} x_1 = (3\cdot 1 + 4)\pi/36 = 7\pi/36 \\ y_1 = 2\cdot 1 + 1 = 3 \end{cases}$

For $k=2$: $\begin{cases} x_2 = (3\cdot 2 + 4)\pi/36 = 10\pi/36 = 5\pi/18 \\ y_2 = 2\cdot 0 + 1 = 1 \end{cases}$

For $k=3$: $\begin{cases} x_3 = (3\cdot 3 + 4)\pi/36 = 13\pi/36 \\ y_3 = 2(-1) + 1 = -1 \end{cases}$

For $k=4$: $\begin{cases} x_4 = (3\cdot 4 + 4)\pi/36 = 16\pi/36 = 4\pi/9 \\ y_4 \to 2(-\infty) = -\infty \end{cases}$



EXERCISES

(10) Graph the following functions:

a) $f(x) = \tan\left(\frac{\pi - x}{4}\right)$

b) $f(x) = 2 - \tan\left(2x + \frac{\pi}{3}\right)$

c) $f(x) = 1 + \tan\left(\frac{\pi + x}{2} - \frac{\pi - 3x}{3}\right)$

d) $f(x) = \cot\left(x + 3(\pi - 2x)\right)$

e) $f(x) = 1 - \cot\left(2(x+\pi) - 3(2\pi - 3x)\right)$

f) $f(x) = 1 + \cot\left(x - \frac{\pi + 3x}{6}\right)$