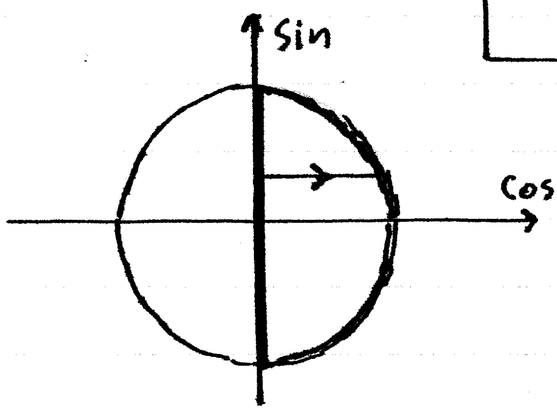
TRIGONOMETRIC EQUATIONS AND INEQUALITIES

Inverse trigonometric functions

1) Inverse sine :

$$y = \text{Arcsin}x \Leftrightarrow \begin{cases} x = \sin y \\ -\pi/2 \leq y \leq \pi/2 \end{cases}$$

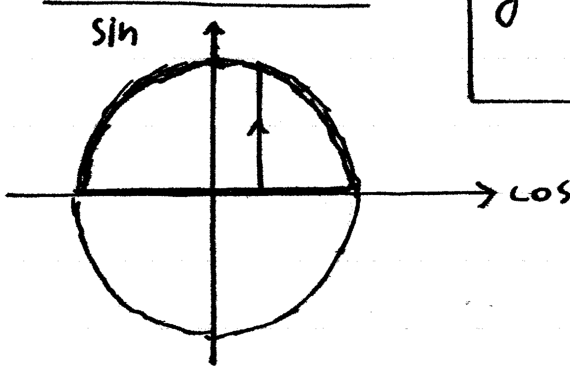


Domain: $A = [-1, 1]$

Range: $f(A) = [-\pi/2, \pi/2]$

2) Inverse cosine :

$$y = \text{Arccos}x \Leftrightarrow \begin{cases} x = \cos y \\ 0 \leq y \leq \pi \end{cases}$$

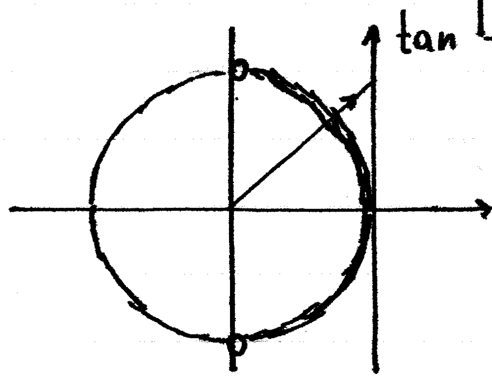


Domain: $A = [-1, 1]$

Range: $f(A) = [0, \pi]$

3) Inverse tangent

$$y = \text{Arctan}x \Leftrightarrow \begin{cases} x = \tan y \\ -\pi/2 \leq y \leq \pi/2 \end{cases}$$

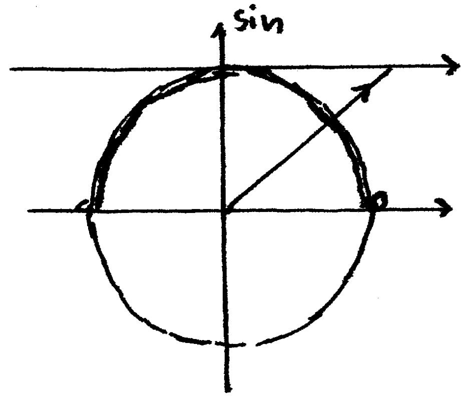


Domain : $A = (-\infty, +\infty)$

Range: $f(A) = [-\pi/2, \pi/2)$

4) <u>Inverse cotangent</u> :

$$\boxed{y = \operatorname{Arccot} x \Leftrightarrow \begin{cases} x = \cot y \\ 0 < y < \pi \end{cases}}$$



Domain: $A = (-\infty, +\infty)$
Range: $f(A) = (0, \pi)$

$\mapsto$ By definition, it follows that

$$\boxed{\begin{array}{ll} \sin(\operatorname{Arcsin} x) = x & , \forall x \in [-1,1] \\ \cos(\operatorname{Arccos} x) = x & , \forall x \in [-1,1] \\ \tan(\operatorname{Arctan} x) = x & , \forall x \in \mathbb{R} \\ \cot(\operatorname{Arccot} x) = x & , \forall x \in \mathbb{R} \end{array}}$$

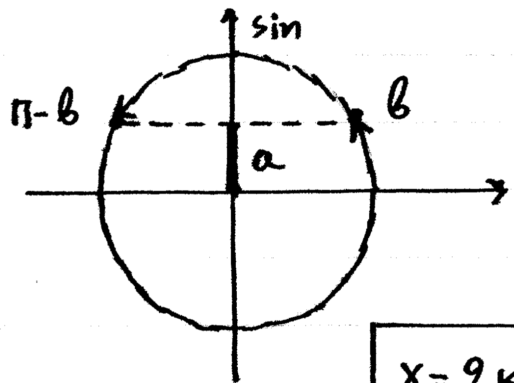
and

$$\boxed{\begin{array}{ll} \operatorname{Arcsin}(\sin x) = x, & \forall x \in [-\pi/2, \pi/2] \\ \operatorname{Arccos}(\cos x) = x, & \forall x \in [0, \pi] \\ \operatorname{Arctan}(\tan x) = x, & \forall x \in (-\pi/2, \pi/2) \\ \operatorname{Arccot}(\cot x) = x, & \forall x \in (0, \pi) \end{array}}$$

V Fundamental Trigonometric Equations

(1) $\boxed{\sin x = a} \Leftrightarrow \sin x = \sin b$ with $b = \text{Arcsin}(a)$

▶ We assume $|a| \leq 1$.



Solutions:

$b \quad , \quad \pi - b$

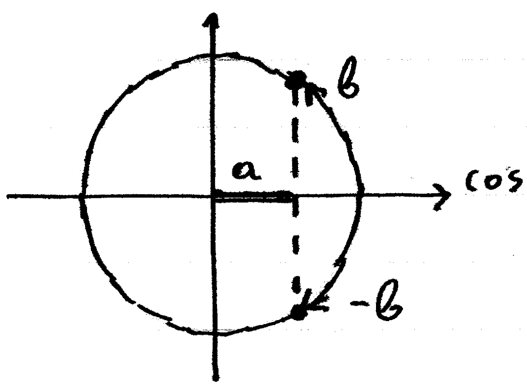
$2\pi + b \quad , \quad 3\pi - b$

$4\pi + b \quad , \quad 5\pi - b$

$$\boxed{x = 2k\pi + b \vee x = (2k+1)\pi - b}$$

(2) $\boxed{\cos x = a} \Leftrightarrow \cos x = \cos b$ with $b = \text{Arccos}\, b$

▶ We assume $|a| \leq 1$



Solutions:

$b \qquad -b$

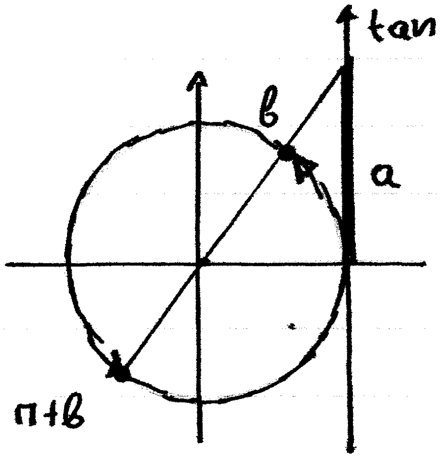
$2\pi + b \qquad 2\pi - b$

$4\pi + b \qquad 4\pi - b$

$$\boxed{x = 2k\pi \pm b}$$

③ $\boxed{\tan x = a} \Leftrightarrow \tan x = \tan b$ with $b = \text{Arctan}(a)$

- Assume $a \in \mathbb{R}$.



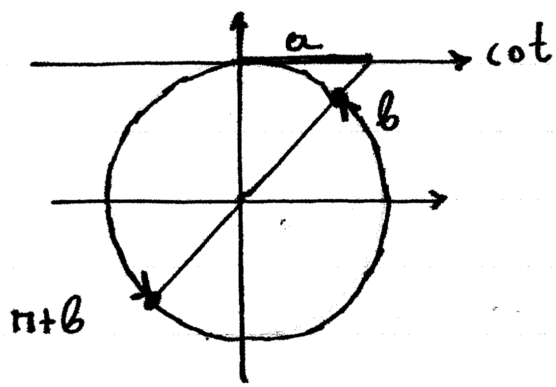
Solutions

b	$\pi + b$
$2\pi + b$	$3\pi + b$
$4\pi + b$	$5\pi + b$

$$\boxed{x = k\pi + b}$$

④ $\boxed{\cot x = a} \Leftrightarrow \cot x = \cot b$ with $b = \text{Arccot}(a)$

- Assume $a \in \mathbb{R}$.



Solutions

b	$\pi + b$
$2\pi + b$	$3\pi + b$
$4\pi + b$	$5\pi + b$

$$\boxed{x = k\pi + b}$$

Special cases

1) $a=0$

$$\boxed{\begin{array}{l} \sin x = 0 \Leftrightarrow x = k\pi \\ \cos x = 0 \Leftrightarrow x = k\pi + \pi/2 \end{array}}$$

2) $a=1$

$$\boxed{\begin{array}{l} \sin x = 1 \Leftrightarrow x = 2k\pi + \pi/2 \\ \cos x = 1 \Leftrightarrow x = 2k\pi \end{array}}$$

3) $a=-1$

$$\boxed{\begin{array}{l} \sin x = -1 \Leftrightarrow x = 2k\pi - \pi/2 \\ \cos x = -1 \Leftrightarrow x = (2k+1)\pi \end{array}}$$

Forms reducible to fundamental trigonometric equations

1) Forms:

$\sin f(x) = \sin g(x)$	$\tan f(x) = \tan g(x)$
$\cos f(x) = \cos g(x)$	$\cot f(x) = \cot g(x)$

EXAMPLES

a) $2\sin\left(3x+\frac{\pi}{3}\right) - 1 = 0 \Leftrightarrow \sin\left(3x+\frac{\pi}{3}\right) = \frac{1}{2} = \sin\frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow 3x + \frac{\pi}{3} = 2k\pi + \frac{\pi}{6} \vee 3x + \frac{\pi}{3} = (2k+1)\pi - \frac{\pi}{6} \Leftrightarrow$

$\Leftrightarrow 3x = 2k\pi + \frac{\pi}{6} - \frac{\pi}{3} \vee 3x = (2k+1)\pi - \frac{\pi}{6} - \frac{\pi}{3} \Leftrightarrow$

$$\Leftrightarrow 3x = 2k\pi - \frac{\pi}{6} \vee 3x = (2k+1)\pi - \frac{\pi}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{2k\pi}{3} - \frac{\pi}{18} \vee x = \frac{(2k+1)\pi}{3} - \frac{\pi}{6}$$

2) Forms:

$$\boxed{\begin{array}{l} \sin(f(x)) = \cos(g(x)) \\ \tan(f(x)) = \cot(g(x)) \end{array}}$$

We use the cofunction identities to reduce to the previous form:

$$\boxed{\begin{array}{l} \sin(x) = \cos(\pi/2 - x) \\ \cos(x) = \sin(\pi/2 - x) \end{array}}$$

EXAMPLE

$$\sin(\pi - 2x) - \cos(x + \pi/4) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sin(\pi - 2x) = \cos(x + \pi/4) \Leftrightarrow$$

$$\Leftrightarrow \cos(\pi/2 - \pi + 2x) = \cos(x + \pi/4) \Leftrightarrow$$

$$\Leftrightarrow \cos(2x - \pi/2) = \cos(x + \pi/4) \Leftrightarrow$$

$$\Leftrightarrow 2x - \pi/2 = 2k\pi + x + \pi/4 \vee 2x - \pi/2 = 2k\pi - x - \pi/4 \Leftrightarrow$$

$$\Leftrightarrow 8x - 2\pi = 8k\pi + 4x + \pi \vee 8x - 2\pi = 8k\pi - 4x - \pi$$

$$\Leftrightarrow 4x = 8k\pi + 3\pi \vee 12x = 8k\pi + \pi$$

$$\Leftrightarrow x = 2k\pi + \frac{3\pi}{4} \vee x = \frac{2k\pi}{3} + \frac{\pi}{12}$$

3) <u>Form</u> :

$$\boxed{\begin{aligned} \sin(f(x)) &= -\sin(g(x)) \\ \tan(f(x)) &= -\tan(g(x)) \\ \cot(f(x)) &= -\cot(g(x)) \end{aligned}}$$

We use the fact that sin, tan, cot are odd functions.
i.e. $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$ and $\cot(-x) = \cot x$.

▸ <u>Remark</u>

For equations containing terms of the form $\tan(f(x))$ or $\cot(f(x))$ we introduce the following restrictions and need to reject solutions that violate these restrictions

For $\tan(g(x)) \longleftrightarrow$ Require $g(x) \neq k\pi + \pi/2$ with $k \in \mathbb{Z}$.

For $\cot(g(x)) \longleftrightarrow$ Require $g(x) \neq k\pi$ with $k \in \mathbb{Z}$

The process for enforcing such restrictions is as follows:

- •1 Solving the original equation gives
  $x = f_1(k) \vee x_2 = f_2(k) \vee \cdots \vee x = f_n(k)$ with $k \in \mathbb{Z}$.
  which may include solutions that need to be rejected.
  With no loss of generality, consider the case $n=1$ where we have
  $x = f(k)$ with $k \in \mathbb{Z}$.
- •2 Given a restriction $g(x) \neq k\pi + \pi/2$, solve:
  $g(x) = k\pi + \pi/2 \Leftrightarrow x = G(k)$
  To accept $x = f(k)$ with $k \in \mathbb{Z}$, we require
  $\forall \lambda \in \mathbb{Z} : f(k) \neq G(\lambda)$
  therefore, we solve:
  $f(k) = G(\lambda) \Leftrightarrow \cdots \Leftrightarrow \lambda = \Lambda(k)$

We reject all solutions $x = f(k)$ for which $\lambda(k) \in \mathbb{Z}$
We accept all solutions $x = f(k)$ for which $\lambda(k) \notin \mathbb{Z}$.

- 3 We work similarly for any restrictions of the form $g(x) \neq k\pi$ and process all restrictions, rejecting solutions as needed.

EXAMPLE

$\tan(3x) + \tan x = 0$

Solution

Require $\begin{cases} x \neq k\pi + \pi/2 \\ 3x \neq k\pi + \pi/2 \end{cases} \iff \begin{cases} x \neq k\pi + \pi/2 \\ x \neq k\pi/3 + \pi/6 \end{cases}$

We have:

$\tan(3x) + \tan(x) = 0 \iff \tan(3x) = -\tan x \iff \tan(3x) = \tan(-x)$

$\iff 3x = k\pi - x \iff 3x + x = k\pi \iff 4x = k\pi \iff x = k\pi/4.$

a) We apply $x \neq k\pi + \pi/2$

Solve $\dfrac{k\pi}{4} = \lambda\pi + \dfrac{\pi}{2} \iff \dfrac{k}{4} = \lambda + \dfrac{1}{2} \iff$

$$\iff \lambda = \frac{k}{4} - \frac{1}{2} = \frac{k-2}{4}$$

We reject $x = k\pi/4$ for $k \in \mathbb{Z}$ such that $k-2$ is multiple of 4

Thus we remove: $S_1 = \{k\pi/4 \mid \lambda \in \mathbb{Z} \wedge k = 4\lambda + 2\}$.

b) We apply $x \neq k\pi/3 + \pi/6$

Solve: $\dfrac{k\pi}{4} = \dfrac{\lambda\pi}{3} + \dfrac{\pi}{6} \iff 12\,\dfrac{k\pi}{4} = 12\left(\dfrac{\lambda\pi}{3} + \dfrac{\pi}{6}\right) \iff$

$3k\pi = 4\lambda\pi + 2\pi \Leftrightarrow 3k = 4\lambda + 2 \Leftrightarrow 4\lambda = 3k - 2 \Leftrightarrow$

$\Leftrightarrow \lambda = \dfrac{3k-2}{4}$

We thus reject solutions $x = k\pi/4$ when $3k-2$ is a multiple of 4.

► Consider the possibilities $k = 4\lambda$, $k = 4\lambda+1$, $k = 4\lambda+2$, $k = 4\lambda+3$ with $\lambda \in \mathbb{Z}$. Note that $k = 4\lambda+2$ with $\lambda \in \mathbb{Z}$ solutions are already rejected.

For $k = 4\lambda$:

$3k-2 = 3(4\lambda) - 2 = 4(3\lambda) - 4 + 2 = 4(3\lambda - 1) + 2 \Rightarrow$

$\Rightarrow 3k-2$ not multiple of 4.

For $k = 4\lambda+1$:

$3k-2 = 3(4\lambda+1) - 2 = 4(3\lambda) + 3 - 2 = 4(3\lambda) + 1 \Rightarrow$

$\Rightarrow 3k-2$ not multiple of 4.

For $k = 4\lambda+3$:

$3k-2 = 3(4\lambda+3) - 2 = 4(3\lambda) + 9 - 2 = 4(3\lambda) + 7 = 4(3\lambda+1) + 3 \Rightarrow$

$\Rightarrow 3k-2$ not multiple of 4.

It follows that no additional solutions need to be rejected

The solutions that are accepted are:

$S = \{k\pi/4 \mid \lambda \in \mathbb{Z} \wedge (k = 4\lambda \vee k = 4\lambda+1 \vee k = 4\lambda+3)\}.$

4) Form: $\boxed{\cos(f(x)) = -\cos(g(x))} \Leftrightarrow$

$$\Leftrightarrow \cos(f(x)) = \cos(\pi + g(x)) \Leftrightarrow \dots \text{ etc.}$$

EXAMPLE

$$\cos(3x - \pi/4) + \cos(2\pi/3 - 2x) = 0 \Leftrightarrow \cos(3x - \pi/4) = -\cos(2\pi/3 - 2x)$$

$$\Leftrightarrow \cos(3x - \pi/4) = \cos(\pi + 2\pi/3 - 2x) \Leftrightarrow$$

$$\Leftrightarrow \cos(3x - \pi/4) = \cos(5\pi/3 - 2x) \Leftrightarrow$$

$$\Leftrightarrow 3x - \pi/4 = 2k\pi + (5\pi/3 - 2x) \vee 3x - \pi/4 = 2k\pi - (5\pi/3 - 2x) \Leftrightarrow$$

$$\Leftrightarrow 3x + 2x = 2k\pi + 5\pi/3 + \pi/4 \vee 3x - \pi/4 = 2k\pi - 5\pi/3 + 2x \Leftrightarrow$$

$$\Leftrightarrow 5x = 2k\pi + \frac{(5\cdot 4 + 3\cdot 1)\pi}{12} \vee 3x - 2x = 2k\pi + \pi/4 - 5\pi/3 \Leftrightarrow$$

$$\Leftrightarrow 5x = 2k\pi + \frac{23\pi}{12} \vee x = 2k\pi + \frac{3\pi - 4\cdot 5\pi}{12} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{2k\pi}{5} + \frac{23\pi}{60} \vee x = 2k\pi - \frac{17\pi}{12}$$

It is possible to have equations that require a combination of techniques from the forms above.

EXAMPLE

$\tan(3x) + \cot(2x) = 0 \quad (1)$

Require: $\begin{cases} 3x \neq k\pi + \pi/2 \\ 2x \neq k\pi \end{cases} \Leftrightarrow \begin{cases} x \neq \frac{k\pi}{3} + \frac{\pi}{2} & (2) \\ x \neq \frac{k\pi}{2} & (3) \end{cases}$

$$(1) \Leftrightarrow \tan(3x) = -\cot(2x) \Leftrightarrow \tan(3x) = \cot(-2x) \Leftrightarrow$$

$$\Leftrightarrow \tan(3x) = \tan\left(\frac{\pi}{2} - (-2x)\right) \Leftrightarrow \tan(3x) = \tan\left(\frac{\pi}{2} + 2x\right)$$

$$\Leftrightarrow 3x = k\pi + \frac{\pi}{2} + 2x \Leftrightarrow x = k\pi + \frac{\pi}{2} = \frac{(2k+1)\pi}{2}$$

This violates condition (3) thus the equation does not have any solution.

EXERCISES

① Solve the following equations

a) $\sin\left(\frac{x}{3}+\frac{\pi}{4}\right)=\sin\left(x-\frac{\pi}{4}\right)$

b) $\tan 3x=\tan\left(7x+\frac{\pi}{8}\right)$

c) $\cos 2x-\cos(x/2)=0$

d) $\tan\left(2x+\frac{\pi}{3}\right)=\cot(\pi-3x)$

e) $\sin(\pi-2x)-\cos(x+\pi/4)=0$

f) $\cos(\pi/6+5x)+\sin(-3x)=0$

g) $\cos(3x-\pi/4)+\cos(2\pi/3-2x)=0$

h) $\tan(x-\pi/3)=\cot(2x)$

i) $\sin 3x+\sin 2x=0$.

▼ Trigonometric Equations – 1 unknown

• These are equations of the form

$$f(\sin x)=0 \qquad f(\tan x)=0$$
$$f(\cos x)=0 \qquad f(\cot x)=0$$

and they can be solved by auxiliary substitution.

EXAMPLES

a) $(3+\cot x)^2 = 5(3+\cot x)$. (1)

Require $x \neq k\pi$.

Let $y = 3+\cot x$. Then

(1) $\Leftrightarrow y^2 = 5y \Leftrightarrow y^2-5y=0 \Leftrightarrow y(y-5)=0 \Leftrightarrow$

$\Leftrightarrow y=0 \vee y=5$ (2)

Note that

$y=0 \Leftrightarrow 3+\cot x=0 \Leftrightarrow \cot x=-3 \Leftrightarrow x=k\pi+\mathrm{Arccot}(-3)$

$y=5 \Leftrightarrow 3+\cot x=5 \Leftrightarrow \cot x=2 \Leftrightarrow x=k\pi+\mathrm{Arccot}(2)$.

Both solutions are accepted. Thus

(2) $\Leftrightarrow x=k\pi+\mathrm{Arccot}(-3) \vee x=k\pi+\mathrm{Arccot}(2)$.

b) $\sin^3 x - 4\sin x = 0$ (1).

Let $y=\sin x$. Then

(1) $\Leftrightarrow y^3-4y=0 \Leftrightarrow y(y^2-4)=0 \Leftrightarrow y(y-2)(y+2)=0$

$\Leftrightarrow y=0 \vee y=2 \vee y=-2$. (2)

We note that:

$y=0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = k\pi$
$y=2 \Leftrightarrow \sin x = 2 \leftarrow$ no solutions
$y=-2 \Leftrightarrow \sin x = -2 \leftarrow$ no solutions.

Thus

$(2) \Leftrightarrow x = k\pi.$

$\hookrightarrow$ Note that since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ $\sin x = a$ has no solution when $a>1$ or $a<-1$. Likewise, $\cos x = a$ has no solution when $a>1$ or $a<-1$.

EXERCISE

(2) Solve the equations:

a) $3\tan^2 x - 4\tan x + 1 = 0$

b) $2\cos^2 x = \sqrt{2}\cos x + 2$

c) $2\sin^2 x + \sqrt{3} = (2+\sqrt{3})\sin x$

d) $\tan^2 x - (1+\sqrt{3})\tan x + \sqrt{3} = 0$

e) $4\cos^4 x - 37\cos^2 x + 9 = 0$

▼ Trigonometric Equations – Multiple unknowns

- If possible, we use trigonometric identities to convert all terms into the same angle and the same trigonometric function.

EXAMPLE

a) $\cos 2x - \sin 3x = 1 \Leftrightarrow$

$\Leftrightarrow (1 - 2\sin^2 x) - (-4\sin^3 x + 3\sin x) = 1 \Leftrightarrow$

$\Leftrightarrow -2\sin^2 x + 4\sin^3 x - 3\sin x = 0 \qquad (1)$

Let $y = \sin x$. Then

$(1) \Leftrightarrow -2y^2 + 4y^3 - 3y = 0 \Leftrightarrow 4y^3 - 2y^2 - 3y = 0 \Leftrightarrow$

$\Leftrightarrow y(4y^2 - 2y - 3) = 0 \Leftrightarrow$

$\Leftrightarrow y = 0 \vee 4y^2 - 2y - 3 = 0 \qquad (2)$

Solve $4y^2 - 2y - 3 = 0$:

$\Delta = (-2)^2 - 4 \cdot 4 \cdot (-3) = 4 + 48 = 52 = 4 \cdot 13 \Rightarrow$

$$\Rightarrow x_{1,2} = \frac{-(-2) \pm 2\sqrt{13}}{2 \cdot 4} = \frac{1 \pm \sqrt{13}}{4}$$

Thus

$$(2) \Leftrightarrow y = 0 \vee y = \frac{1+\sqrt{13}}{4} \vee y = \frac{1-\sqrt{13}}{4} \qquad (3)$$

Note that

$y = 0 \Leftrightarrow \sin x = 0 \Leftrightarrow x = k\pi$

$y = \frac{1+\sqrt{13}}{4} \Leftrightarrow \sin x = \frac{1+\sqrt{13}}{4} > 1 \leftarrow$ no solutions.

$$y = \frac{1-\sqrt{13}}{4} = \sin x \Leftrightarrow x = 2k\pi + \text{Arcsin}\left(\frac{1-\sqrt{13}}{4}\right) \vee x = (2k+1)\pi - \text{Arcsin}\left(\frac{1-\sqrt{13}}{4}\right)$$

b) $2\sin x + \tan x = 0$ (1)

Require: $x \neq k\pi + \pi/2$

(1) $\Leftrightarrow 2\sin x + \frac{\sin x}{\cos x} = 0 \Leftrightarrow \sin x\left(2 + \frac{1}{\cos x}\right) = 0$

$\Leftrightarrow \sin x \cdot \frac{2\cos x + 1}{\cos x} = 0 \Leftrightarrow \sin x\,(2\cos x + 1) = 0$

$\Leftrightarrow \sin x = 0 \lor 2\cos x + 1 = 0$ (2)

We note that:

$\sin x = 0 \Leftrightarrow x = k\pi \leftarrow$ accepted

$2\cos x + 1 = 0 \Leftrightarrow \cos x = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right)$

$\Leftrightarrow \cos x = \cos\left(\frac{2\pi}{3}\right) \Leftrightarrow x = 2k\pi \pm \frac{2\pi}{3}$

accepted.

therefore:

(2) $\Leftrightarrow x = k\pi \lor x = 2k\pi \pm \frac{\pi}{3}$.

→ Turning sums to products

c) $\sin 5x - \sin 3x = \sin x \Leftrightarrow 2\sin\left(\frac{5x-3x}{2}\right)\cos\left(\frac{5x+3x}{2}\right) = \sin x$

$\Leftrightarrow 2\sin x\cos 4x - \sin x = 0 \Leftrightarrow \sin x\,(2\cos 4x - 1) = 0 \Leftrightarrow$

$\Leftrightarrow \sin x = 0 \lor \cos 4x = \frac{1}{2} = \cos\frac{\pi}{3} \Leftrightarrow$

$\Leftrightarrow x = k\pi \lor 4x = 2k\pi \pm \frac{\pi}{3} \Leftrightarrow x = k\pi \lor x = \frac{k\pi}{2} \pm \frac{\pi}{12}$.

Turn products to sums

d) $\sin(3x)\sin x = \frac{1}{2} \Leftrightarrow$

$$\Leftrightarrow \frac{1}{2}\left[\cos(3x-x) - \cos(3x+x)\right] = \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \cos 2x - \cos 4x = 1 \Leftrightarrow \cos 2x - (2\cos^2 2x - 1) = 1$$

$$\Leftrightarrow \cos 2x - 2\cos^2 2x = 0 \Leftrightarrow \cos 2x\,(1 - 2\cos 2x) = 0$$

$$\Leftrightarrow \cos 2x = 0 \lor \cos 2x = \frac{1}{2} = \cos\frac{\pi}{3} \Leftrightarrow$$

$$\Leftrightarrow 2x = k\pi + \frac{\pi}{2} \lor 2x = 2k\pi \pm \frac{\pi}{3} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{k\pi}{2} \pm \frac{\pi}{4} \lor x = k\pi \pm \frac{\pi}{6}$$

EXERCISES

③ Solve the following equations

a) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$

b) $\sin^2 2x - \sin^2 x = 1/2$

c) $\sin 2x = \sin^3 x$

d) $\cos 4x + 2\cos^2 x = 0$

e) $\sin 3x - \cos 2x = 0$

f) $\tan\left(\frac{\pi}{4} + x\right) + \tan x - 2 = 0$

g) $\sqrt{3}\tan x = 2\sin x$

④ Solve the following equations:
(Hint: turn sums to products or vice versa)

a) $\cos 2x + \cos x = \sin x + \sin 2x$

b) $\sin x + \sin 2x + \sin 3x = 0$

c) $2\cos x + \cos 3x + \cos 5x = 0$

d) $\cos 6x + \sin 5x = \sin 3x - \cos 2x$

e) $\cos x \cdot \cos 7x = \cos 3x \cos 5x$

f) $2\sin x \sin 3x = 1$

V Special types of trigonometric equations

① → $\boxed{a\sin x + b\cos x = c}$ (Linear Trigonometric)

These equations have solutions when $a^2+b^2 \geq c^2$ which can be obtained as follows:

$$a\sin x + b\cos x = c \Leftrightarrow \sin x + \frac{b}{a}\cos x = \frac{c}{a} \quad (1)$$

Let $\tan w = \dfrac{b}{a}$. Then

$$(1) \Leftrightarrow \sin x + \tan w \cos x = \frac{c}{a} \Leftrightarrow$$

$$\Leftrightarrow \sin x + \frac{\sin w}{\cos w}\cos x = \frac{c}{a} \Leftrightarrow$$

$$\Leftrightarrow \sin x \cos w + \sin w \cos x = \frac{c}{a}\cos w \Leftrightarrow$$

$$\Leftrightarrow \sin(x+w) = \frac{c}{a}\cos w \quad (2).$$

Let $\sin\vartheta = \dfrac{c}{a}\cos w$. Then $(2) \Leftrightarrow \sin(x+w) = \sin\vartheta$
$\Leftrightarrow \ldots$ etc.

To define ϑ we require $|(c/a)\cos w| \leq 1$.

Note that:

$$\left|\frac{c}{a}\cos w\right|^2 = \frac{c^2}{a^2}\cos^2 w = \frac{c^2}{a^2}\,\frac{1}{1+\tan^2 w} =$$

$$= \frac{c^2}{a^2}\,\frac{1}{1+(b/a)^2} = \frac{c^2}{a^2+b^2} \leq 1 \Leftrightarrow$$

$$\Leftrightarrow a^2+b^2 \geq c^2.$$

EXAMPLE

$$\sin 4x + \sqrt{3}\cos 4x = \sqrt{2} \Leftrightarrow \sin 4x + \tan(\pi/3)\cos 4x = \sqrt{2}$$

$$\Leftrightarrow \sin 4x \cos(\pi/3) + \sin(\pi/3)\cos 4x = \sqrt{2}\cos(\pi/3) \Leftrightarrow$$

$$\Leftrightarrow \sin(4x + \pi/3) = \sqrt{2}\cdot(1/2) = \frac{\sqrt{2}}{2} = \sin\left(\frac{\pi}{4}\right) \Leftrightarrow$$

$$\Leftrightarrow 4x + \pi/3 = 2k\pi + \frac{\pi}{4} \vee 4x + \pi/3 = (2k+1)\pi - \frac{\pi}{4} \Leftrightarrow$$

$$\Leftrightarrow 4x = 2k\pi - \frac{\pi}{12} \vee 4x = (2k+1)\pi - \frac{7\pi}{12}$$

$$\Leftrightarrow x = \frac{k\pi}{2} - \frac{\pi}{48} \vee x = \frac{(2k+1)\pi}{4} - \frac{7\pi}{48}.$$

(2) $\rightarrow$ $\boxed{a\sin^2 x + b\sin x\cos x + c\cos^2 x = 0}$ (Homogeneous)

If $\cos x = 0$, then the equation gives:

$a\sin^2 x = 0 \Leftrightarrow \sin x = 0$

which implies that $\sin^2 x + \cos^2 x = 0 \neq 1 \leftarrow$ Contradiction.
We may therefore assume that $\cos x \neq 0$ and divide the equation with $\cos^2 x$:

$$a\frac{\sin^2 x}{\cos^2 x} + b\frac{\sin x\cos x}{\cos^2 x} + c\frac{\cos^2 x}{\cos^2 x} = 0 \Leftrightarrow$$

$\Leftrightarrow a\tan^2 x + b\tan x + c = 0 \Leftrightarrow \ldots$ etc.

(3) $\rightarrow$ $\boxed{a\sin^2 x + b\sin x\cos x + c\cos^2 x = d}$ (Pseudohomogeneous)

Can be reduced to homogeneous as follows:

$a\sin^2 x + b\sin x\cos x + c\cos^2 x = d(\sin^2 x + \cos^2 x) \Leftrightarrow$

$\Leftrightarrow (a-d)\sin^2 x + b\sin x\cos x + (c-d)\cos^2 x = 0 \Leftrightarrow$

$\Leftrightarrow \ldots$ etc.

EXAMPLE

$\sin^2 x + \sin 2x + 2\cos^2 x = \frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \sin^2 x + 2\sin x\cos x + 2\cos^2 x = (1/2)(\sin^2 x + \cos^2 x)$

$\Leftrightarrow 2\sin^2 x + 4\sin x\cos x + 4\cos^2 x = \sin^2 x + \cos^2 x$

$\Leftrightarrow \sin^2 x + 4\sin x\cos x + 3\cos^2 x = 0 \Leftrightarrow$

$$\Leftrightarrow \tan^2 x + 4\tan x + 3 = 0$$
$$\Delta = 16 - 4\cdot 3 = 16 - 12 = 4$$
$$\Rightarrow \tan x = \frac{-4 \pm 2}{2} = \begin{cases} -3 \\ -1 = -\tan(\pi/4) = \tan(-\pi/4) \end{cases} \Leftrightarrow$$
$$\Leftrightarrow x = k\pi + \operatorname{Arctan}(-3) \vee x = k\pi - \pi/4.$$

(4) → $\boxed{F(\sin x+\cos x,\ \sin x\cos x)=0}$

Let $y=\sin x+\cos x$. Then

$$y^2=\sin^2 x+2\sin x\cos x+\cos^2 x=$$
$$=1+2\sin x\cos x \Rightarrow \sin x\cos x=\frac{y^2-1}{2}$$

It follows that $F\left(y,\frac{y^2-1}{2}\right)=0 \Leftrightarrow \dots$ etc.

EXAMPLE

$\sin x+\cos x=\sin x\cos x+1$ (1).

Let $y=\sin x+\cos x$. Then

$$y^2=(\sin x+\cos x)^2=\sin^2 x+2\sin x\cos x+\cos^2 x=$$
$$=1+2\sin x\cos x\Rightarrow \sin x\cos x=\frac{y^2-1}{2}.$$

$$(1)\Leftrightarrow y=\frac{y^2-1}{2}+1\Leftrightarrow 2y=y^2-1+2\Leftrightarrow$$

$$\Leftrightarrow y^2-2y+1=0\Leftrightarrow (y-1)^2=0\Leftrightarrow y-1=0\Leftrightarrow y=1$$

$$\Leftrightarrow \sin x+\cos x=1\Leftrightarrow \sin x+\tan(\pi/4)\cos x=1\Leftrightarrow$$
$$\Leftrightarrow \sin x\cos(\pi/4)+\sin(\pi/4)\cos x=\cos(\pi/4)$$
$$\Leftrightarrow \sin(x+\pi/4)=\sin(\pi/2-\pi/4)\Leftrightarrow \sin(x+\pi/4)=\sin(\pi/4)$$
$$\Leftrightarrow x+\pi/4=2k\pi+\pi/4 \vee x+\pi/4=(2k+1)\pi-\pi/4$$
$$\Leftrightarrow x=2k\pi \vee x=(2k+1)\pi-\pi/2 .$$

EXERCISES

⑤ Solve the following equations:

a) $3\sin x - \sqrt{3}\cos x = 3$

b) $\sin 4x + \sqrt{3}\cos 4x = \sqrt{2}$

c) $\sin x + \cos x = 1$

d) $2\sin x + 3\cos x = 1$

e) $5\sin^2 x - 3\sin x\cos x - 2\cos^2 x = 0$

f) $\cos^2 x + 4\sin 2x + 3 = 0$

g) $\sin^2 x + \sin 2x - 2\cos^2 x = 1/2$

h) $\sin x + \cos x = 1 + \sin x\cos x$

i) $2\sin x + 2\cos x - 4\sin x\cos x = 1$

j) $\dfrac{1}{\sin x} + \dfrac{1}{\cos x} = 2\sqrt{2}$

k) $\sin x - \cos x + \sin x\cos x = 1.$

▼ Solving trigonometric equations in an interval

To solve a trigonometric equation in an interval (a,b) or $(a,b]$ or $[a,b)$ or $[a,b]$, we work as follows:

- 1 Find the general solutions in terms of $k \in \mathbb{Z}$.
- 2 Require that x belongs to the interval and derive a corresponding inequality for k
- 3 List the solutions that satisfy the inequality for k.

EXAMPLE

$$\tan\left(\frac{\pi}{4}+x\right)-\tan\left(\frac{\pi}{4}-x\right)=2\sqrt{3} \qquad (1)$$

Find all solutions in the interval $[0,\pi]$.

Solution

We require

$$\begin{cases} \frac{\pi}{4}+x \neq k\pi+\frac{\pi}{2} \\ \frac{\pi}{4}-x \neq k\pi+\frac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} x \neq k\pi+\pi/4 \\ x \neq k\pi-\pi/4 \end{cases}$$

Let $y=\tan x$. We note that

$$\tan\left(\frac{\pi}{4}+x\right)=\frac{\tan(\pi/4)+\tan x}{1-\tan(\pi/4)\tan x}=\frac{1+\tan x}{1-\tan x}=\frac{1+y}{1-y}$$

$$\tan\left(\frac{\pi}{4}-x\right)=\frac{\tan(\pi/4)-\tan x}{1+\tan(\pi/4)\tan x}=\frac{1-\tan x}{1+\tan x}=\frac{1-y}{1+y}$$

$$(1)\Leftrightarrow \frac{1+y}{1-y}-\frac{1-y}{1+y}=2\sqrt{3}\Leftrightarrow$$

$$\Leftrightarrow (1+y)^2-(1-y)^2=2\sqrt{3}\,(1-y)(1+y)$$

$$\Leftrightarrow 1+2y+y^2-1+2y-y^2=2\sqrt{3}-y^2\cdot 2\sqrt{3}$$

$$\Leftrightarrow 4y=2\sqrt{3}-(2\sqrt{3})y^2\Leftrightarrow$$

$$\Leftrightarrow (2\sqrt{3})y^2+4y-2\sqrt{3}=0\Leftrightarrow$$

$$\left.\begin{array}{l}\Leftrightarrow \sqrt{3}\,y^2+2y-\sqrt{3}=0\\ \Delta=4-4\sqrt{3}\cdot(-\sqrt{3})=\\ =4+12=16=4^2\end{array}\right\}\Rightarrow y_{1,2}=\frac{-2\pm 4}{2\sqrt{3}}\Rightarrow$$

$$\Rightarrow y_1=\frac{-6}{2\sqrt{3}}=\frac{-3}{\sqrt{3}}=-\sqrt{3} \text{ or}$$

$$y_2=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

We note that

$$y=-\sqrt{3}\Leftrightarrow \tan x=-\sqrt{3}=-\tan\left(\frac{\pi}{3}\right)=\tan\left(\frac{-\pi}{3}\right)\Leftrightarrow$$

$$\Leftrightarrow x=k\pi-\frac{\pi}{3} \leftarrow \text{accepted}$$

and

$$y=\frac{\sqrt{3}}{3}\Leftrightarrow \tan x=\frac{\sqrt{3}}{3}=\tan\left(\frac{\pi}{6}\right)\Leftrightarrow x=k\pi+\frac{\pi}{6}$$

accepted.

Now we require that $x \in [0, \pi]$:

a) For $x = k\pi + \pi/3$:

$$0 \leq k\pi - \pi/3 \leq \pi \Leftrightarrow 0 \leq k - 1/3 \leq 1 \Leftrightarrow$$

$$\Leftrightarrow 1/3 \leq k \leq 4/3 \Leftrightarrow k = 1$$

$*$ (k is an integer).

Thus: $x = \pi - \pi/3 = 2\pi/3$

b) For $K = k\pi + \pi/6$

$$0 \leq k\pi + \pi/6 \leq \pi \Leftrightarrow 0 \leq k + 1/6 \leq 1 \Leftrightarrow$$

$$\Leftrightarrow -1/6 \leq k \leq 5/6 \Leftrightarrow k = 0$$

Thus $x = 0\pi + \pi/6 = \pi/6$.

Thus solution set in $[0, \pi]$ is: $S = \{\pi/6, 2\pi/3\}$

EXERCISES

(6) Solve the following equation in $[-\pi, \pi]$

$$\cos(2x) + 3\cos x = 0$$

(7) Solve $\sin(3x) + \sin(5x) = \sin(8x)$ in $[0, 2\pi)$.

(8) Solve $4\cos^4 x - 37\cos^2 x + 9 = 0$ in $(\pi/2, 3\pi/2]$.

(9) Solve $\cos^2 x + 4\sin 2x + 3 = 0$ in $(2\pi, 3\pi)$

(10) Solve $\sqrt{3}\cos x - 3\sin x = 3$ in $[\pi, 3\pi]$

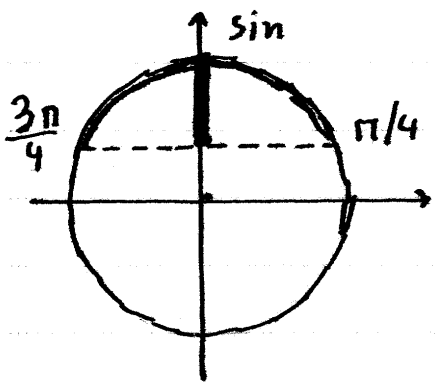
Trigonometric Inequalities

The solution of trigonometric inequalities in the $[0, 2\pi]$ interval can be visualized on the trigonometric circle. These solutions can then be generalized by adding $2k\pi$ for sin or cos and $k\pi$ for tan or cot.

EXAMPLES

a) $2\sin(3x-1) - \sqrt{2} \geq 0 \Leftrightarrow \sin(3x-1) \geq \frac{\sqrt{2}}{2} \Leftrightarrow$

$\Leftrightarrow \sin(3x-1) \geq \sin\left(\frac{\pi}{4}\right) \Leftrightarrow$



$\Leftrightarrow 2k\pi + \pi/4 \leq 3x - 1 \leq 2k\pi + 3\pi/4$

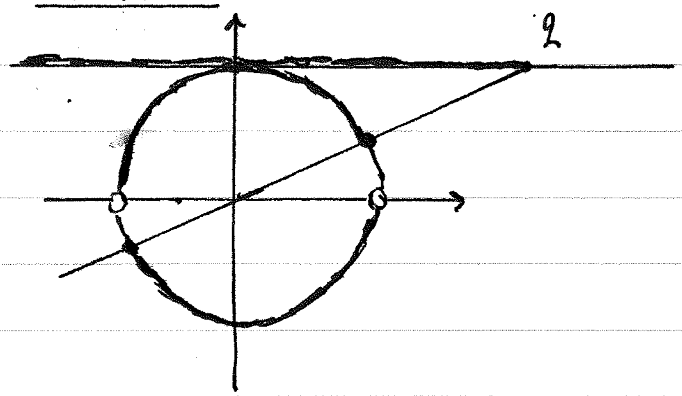
$\Leftrightarrow 2k\pi + \pi/4 + 1 \leq 3x \leq 2k\pi + 3\pi/4 + 1$

$\Leftrightarrow \frac{2k\pi}{3} + \frac{\pi}{12} + \frac{1}{3} \leq x \leq \frac{2k\pi}{3} + \frac{3\pi}{12} + \frac{1}{3}$

$\Leftrightarrow \frac{2k\pi}{3} + \frac{\pi+4}{12} \leq x \leq \frac{2k\pi}{3} + \frac{3\pi+4}{12}$.

B) $\cot(x+\pi/3) \leq 2$

Solution



$\cot(x+\pi/3) \leq 2 \Leftrightarrow \cot(x+\pi/3) \leq \cot(\mathrm{Arccot}(2))$

$\Leftrightarrow \mathrm{Arccot}(2) + k\pi \leq x + \frac{\pi}{3} < \pi + k\pi \Leftrightarrow$

$\Leftrightarrow \mathrm{Arccot}(2) + k\pi - \frac{\pi}{3} \leq x < \pi + k\pi - \frac{\pi}{3} \Leftrightarrow$

$\Leftrightarrow (\mathrm{Arccot}(2) - \pi/3) + k\pi \leq x < \frac{2\pi}{3} + k\pi$

EXERCISES

(11) Solve the following inequalities

a) $\sin(3x) > \sqrt{2}/2$

b) $\cos(2x) \leq \sqrt{3}/2$

c) $\tan x \geq \sqrt{3}/3$

d) $\cot 4x < \sqrt{3}$

e) $\cos 4x \leq -\sqrt{3}/2$

f) $\sqrt{2}\cos x \leq 1$

(12) Solve the following inequalities

a) $\sin(x-\pi/6) > 0$

b) $\cos(2x+\pi/3) \leq 1/2$

c) $\tan(3x-\pi/4) < 0$

d) $\cot(x+\pi/3) \leq 1$

e) $\tan(x/3) > \sqrt{3}/3$

f) $\sin(x+2\pi/3) > -1/2$

g) $-1/2 < \sin 3x < \sqrt{2}/2$

h) $-\sqrt{2}/2 < \cos 2x < 1/2$

i) $2\sin x + 1 > 0$

j) $2\cos(2x/5) < 1$

k) $2\cos(x+\pi/6) - \sqrt{2} > 0$

l) $2\cos(x-\pi/3) > -\sqrt{3}$

m) $3\tan 2x - \sqrt{3} \leq 0$

n) $\cot(3x) + \sqrt{3} > 0$

o) $\tan(x-\pi/4) - 1 > 0$

p) $-\sqrt{3}/3 < \tan 5x < 1$

q) $-\sqrt{3} < \cot 3x < 1$