INTRODUCTION TO SERIES

Sequences and series

Recall that:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$
$$\mathbb{N}^* = \{1, 2, 3, \ldots\}$$

Definition : Any function $a : \mathbb{N} \to \mathbb{R}$ or $a : \mathbb{N}^* \to \mathbb{R}$ is called a real sequence (or just sequence) and we write:

$$a_n = a(n) \ , \ \forall n \in \mathbb{N}$$

Defining a sequence

There are two methods for defining a sequence (a_n) :

1) Directly $\to$ We provide a formula for directly calculating a_n.

e.g. $a_n = \dfrac{(-1)^n}{2n} \ , \ \forall n \in \mathbb{N}$.

2) Recursively $\to$ We define the first few terms of the sequence and a recursive formula give the next term in terms of previous terms.

e.g.: $(a_n): \begin{cases} a_1 = 2 \\ a_{n+1} = 3a_n - 1 \end{cases}$

e.g.: $(a_n): \begin{cases} a_1 = 1 \wedge a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$ ← Fibonacci sequence.

Series

A series is a sequence S_n defined via a partial sum of the terms of a sequence a_n. For example:

$$S_n = a_1 + a_2 + \cdots + a_n.$$

▶ Notation : $\boxed{\sum_{n=p}^{q} a_n = a_p + a_{p+1} + \cdots + a_q}$

We note that:

$$\sum_{n=p}^{q} (a_n + b_n) = \sum_{n=p}^{q} a_n + \sum_{n=p}^{q} b_n$$

$$\sum_{n=p}^{q} (a_n - b_n) = \sum_{n=p}^{q} a_n - \sum_{n=p}^{q} b_n$$

$$\sum_{n=p}^{q} c a_n = c \sum_{n=p}^{q} a_n$$

Basic Sums

$S_1(n) = \sum_{k=1}^{n} k = 1+2+\cdots+n = \frac{n(n+1)}{2}$
$S_2(n) = \sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$
$S_3(n) = \sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2$

Proof

▶ For $S_1(n)$

We note that $(x+1)^2 = x^2+2x+1$.

For $x=1: \quad 2^2 = 1^2+2\cdot 1+1$

$x=2: \quad 3^2 = 2^2+2\cdot 2+1$

$\vdots$

$x=n: \quad (n+1)^2 = n^2+2n+1$

Add the equations above:

$$[2^2+3^2+\cdots+(n+1)^2] = [1^2+2^2+\cdots+n^2]+2S_1(n)+n \Leftrightarrow$$

$$\Leftrightarrow (n+1)^2 = 1+2S_1(n)+n \Leftrightarrow$$

$$\Leftrightarrow 2S_1(n) = (n+1)^2-1-n = n^2+2n+1-1-n =$$

$$= n^2+n = n(n+1) \Leftrightarrow$$

$$\Leftrightarrow S_1(n) = \frac{n(n+1)}{2}.$$

- **For $S_2(n)$**

We note that $(x+1)^3 = x^3 + 3x^2 + 3x + 1$

For $x=1$: $2^3 = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1$

$x=2$: $3^3 = 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1$

$\vdots$

$x=n$: $(n+1)^3 = n^3 + 3n^2 + 3n + 1$

Add the equations above:

$$[2^3 + \cdots + (n+1)^3] = [1^3 + \cdots + n^3] + 3S_2(n) + 3S_1(n) + n \Leftrightarrow$$

$$\Leftrightarrow (n+1)^3 = 1 + 3S_2(n) + 3S_1(n) + n \Leftrightarrow$$

$$\Leftrightarrow 3S_2(n) = (n+1)^3 - 1 - 3S_1(n) - n$$

$$= (n+1)^3 - 3\frac{n(n+1)}{2} - (n+1) =$$

$$= (n+1)\left[(n+1)^2 - \frac{3n}{2} - 1\right] =$$

$$= (n+1)\left[n^2 + 2n + 1 - \frac{3n}{2} - 1\right] =$$

$$= (n+1)\left(n^2 + \frac{n}{2}\right) = n(n+1)\left(n + \frac{1}{2}\right) =$$

$$= \frac{1}{2} n(n+1)(2n+1) \Leftrightarrow$$

$$\Leftrightarrow S_2(n) = \frac{n(n+1)(2n+1)}{6}.$$

▶ <u>For $S_3(n)$</u>

We note that $(x+1)^4 = x^4+4x^3+6x^2+4x+1$

$x=1: \; 2^4 = 1^4+4\cdot 1^3+6\cdot 1^2+4\cdot 1+1$

$x=2: \; 3^4 = 2^4+4\cdot 2^3+6\cdot 2^2+4\cdot 2+1$

$\vdots$

$x=n: \; (n+1)^4 = n^4+4n^3+6n^2+4n+1$

Adding the above equations:

$$2^4+\dots+(n+1)^4 = [1^4+\dots+n^4]+4S_3(n)+6S_2(n)+4S_1(n)+n$$

$$\Leftrightarrow (n+1)^4 = 1+4S_3(n)+6S_2(n)+4S_1(n)+n$$

$$\Leftrightarrow 4S_3(n) = (n+1)^4-(n+1)-6S_2(n)-4S_1(n) =$$

$$= (n+1)^4-(n+1)-6\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} =$$

$$= (n+1)^4-(n+1)-n(n+1)(2n+1)-2n(n+1)$$

$$= (n+1)[(n+1)^3-1-n(2n+1)-2n] =$$

$$= (n+1)[(n+1)^3-n(2n+1)-(2n+1)] =$$

$$= (n+1)[(n+1)^3-(n+1)(2n+1)] =$$

$$= (n+1)(n+1)[(n+1)^2-(2n+1)]$$

$$= (n+1)^2[n^2+2n+1-2n-1] = n^2(n+1)^2 \Leftrightarrow$$

$$\Leftrightarrow S_3(n) = \frac{n^2(n+1)^2}{4} = [S_1(n)]^2. \qquad \square$$

EXAMPLES

a) $S_n = 1\cdot 3 + 2\cdot 5 + 3\cdot 7 + \cdots + n(2n+1)$

Solution

$$S_n = 1\cdot 3 + 2\cdot 5 + 3\cdot 7 + \cdots + n(2n+1) = \sum_{k=1}^{n} k(2k+1) =$$

$$= \sum_{k=1}^{n} (2k^2 + k) = 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k =$$

$$= 2S_2(n) + S_1(n) = 2\,\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} =$$

$$= n(n+1)\left[\frac{2n+1}{3} + \frac{1}{2}\right] = \frac{1}{6}\, n(n+1)[2(2n+1)+3]$$

$$= \frac{n(n+1)(4n+2+3)}{6} = \frac{n(n+1)(4n+5)}{6}.$$

b) $S_n = 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3$

Solution

$$S_n = 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = \sum_{k=1}^{n} (2k-1)^3 =$$

$$= \sum_{k=1}^{n} (8k^3 - 3(2k)^2 + 3(2k) - 1) =$$

$$= \sum_{k=1}^{n} (8k^3 - 12k^2 + 6k - 1) =$$

$$= 8S_3(n) - 12S_2(n) + 6S_1(n) - n =$$

$$= 8\,\frac{n^2(n+1)^2}{4} - 12\,\frac{n(n+1)(2n+1)}{6} + 6\,\frac{n(n+1)}{2} - n =$$

$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n =$$

$$= n(n+1)[2n(n+1) - 2(2n+1) + 3] - n =$$

$$= n(n+1)[2n^2 + 2n - 4n - 2 + 3] - n$$

$$= n(n+1)(2n^2 - 2n + 1) - n$$

Application to arithmetic series

Def : $\boxed{\left.\begin{array}{l}(a_n) \text{ arithmetic} \\ \text{sequence}\end{array}\right\} \Leftrightarrow \forall n \in \mathbb{N} : a_{n+1} = a_n + c}$

- It is easy to see that if (a_n) is an arithmetic sequence, then

$$\boxed{a_n = a_1 + (n-1)c \,, \forall n \in \mathbb{N}}$$

Thm : $\boxed{\left.\begin{array}{l}(a_n) \text{ arithmetic} \\ \text{sequence}\end{array}\right\} \Rightarrow \sum_{k=1}^{n} a_k = \frac{n(a_1 + a_n)}{2}}$

Proof

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} [a_1 + (k-1)c] = a_1 n + c \sum_{k=1}^{n} (k-1)$$

$$= a_1 n + c \sum_{k=0}^{n-1} k = a_1 n + c\, S_1(n-1) =$$

$$= a_1 n + \frac{(n-1)[(n-1)+1]}{2} \cdot c = a_1 n + \frac{cn(n-1)}{2} =$$

$$= \frac{a_1 n}{2} + \frac{a_1 n}{2} + \frac{cn(n-1)}{2} =$$

$$= \frac{a_1 n}{2} + \frac{n}{2}[a_1 + c(n-1)] = \frac{a_1 n}{2} + \frac{n}{2} \cdot a_n =$$

$$= \frac{n(a_1 + a_n)}{2}. \quad \square$$

EXERCISES

① Show that:

a) $1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = (1/3)\, n(n+1)(n+2)$
b) $1\cdot 2 + 2\cdot 5 + \cdots + n(3n-1) = n^2(n+1)$
c) $1^2 + 3^2 + \cdots + (2n-1)^2 = (1/3)\, n(2n-1)(2n+1)$
d) $1^3 + 3^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$
e) $1\cdot 2^2 + 2\cdot 3^2 + \cdots + n(n+1)^2 = (1/12)\, n(n+1)(n+2)(3n+5)$
f) $1^2\cdot 2 + 2^2\cdot 3 + \cdots + n^2(n+1) = (1/12)\, n(n+1)(n+2)(3n+1)$
g) $1^2\cdot 3 + 2^2\cdot 5 + \cdots + n^2(2n+1) = (1/6)\, n(n+1)(3n^2+5n+1)$
h) $1\cdot 3^2 + 2\cdot 5^2 + \cdots + n(2n+1)^2 = (1/6)\, n(n+1)(6n^2+14n+7)$

Geometric sums

$$\boxed{G_a(n) = 1 + a + a^2 + \dots + a^n = \frac{1-a^{n+1}}{1-a}}$$

Proofs

We note that

$$G_a(n) = 1 + a + a^2 + \dots + a^n \qquad (1)$$

$$aG_a(n) = a + a^2 + a^3 + \dots + a^{n+1} \qquad (2)$$

Subtract (2) from (1):

$$G_a(n) - aG_a(n) = (1 + a + \dots + a^n) - (a + a^2 + \dots + a^{n+1})$$

$$= 1 - a^{n+1} \Rightarrow$$

$$\Rightarrow (1-a)\,G_a(n) = 1 - a^{n+1} \Rightarrow$$

$$\Rightarrow G_a(n) = \frac{1-a^{n+1}}{1-a} \qquad \square$$

Application to geometric sequences

Def : (a_n) geometric sequence $\Leftrightarrow \forall n \in \mathbb{N}^*: a_{n+1} = \lambda a_n$

It follows that

$$\boxed{a_n = a_1 \lambda^{n-1}, \ \forall n \in \mathbb{N}^*}$$

Thm: $\boxed{(a_n) \text{ geometric sequence} \Rightarrow S_n = a_1 + \cdots + a_n = \dfrac{a_1(1-\lambda^n)}{1-\lambda}}$

Proof

$$S_n = a_1 + \cdots + a_n = \sum_{k=1}^{n} a_1 \lambda^{k-1} = a_1 \sum_{k=1}^{n} \lambda^{k-1} =$$

$$= a_1 \sum_{k=0}^{n-1} \lambda^k = a_1 G_\lambda(n-1) = a_1 \frac{1-\lambda^n}{1-\lambda} =$$

$$= \frac{a_1(1-\lambda^n)}{1-\lambda} \quad \square$$

EXAMPLES

a) $\displaystyle\sum_{k=0}^{n} \left(\frac{2}{3}\right)^k$.

Solution

$$S_n = \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = \frac{1-(2/3)^{n+1}}{1-(2/3)} = \frac{1-(2/3)^{n+1}}{1/3} =$$

$$= 3\left[1-(2/3)^{n+1}\right] = \frac{3\left[3^{n+1}-2^{n+1}\right]}{3^{n+1}} =$$

$$= \frac{3^{n+1}-2^{n+1}}{3^n}.$$

b) $\displaystyle\sum_{k=0}^{n} (-1)^k \left(\frac{1}{3}\right)^{2k}$.

Solution

$$S_n = \sum_{k=0}^{n} (-1)^k \left(\frac{1}{3}\right)^{2k} = \sum_{k=0}^{n} \left[-\left(\frac{1}{3}\right)^2\right]^k =$$

$$= \sum_{k=0}^{n} \left(-\frac{1}{9}\right)^k = \frac{1-(-1/9)^{n+1}}{1-(-1/9)} = \frac{1-(-1)^{n+1}(1/9)^{n+1}}{1+1/9}$$

$$= \frac{1+(-1)^n(1/9)^{n+1}}{10/9} = \frac{9}{10}\,\frac{1}{9^{n+1}}\left[9^{n+1}+(-1)^n\right]$$

$$= \frac{9^{n+1}+(-1)^n}{9^n \cdot 10}.$$

c) $\displaystyle\sum_{k=n}^{2n} \left(\frac{\sqrt{2}}{2}\right)^{k+2}$

Solution

$$S_n = \sum_{k=n}^{2n} \left(\frac{\sqrt{2}}{2}\right)^{k+2} = \left(\frac{\sqrt{2}}{2}\right)^2 \sum_{k=n}^{2n} \left(\frac{\sqrt{2}}{2}\right)^k =$$

$$= \frac{1}{2} \sum_{k=0}^{n} \left(\frac{\sqrt{2}}{2}\right)^{k+n} = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^n \sum_{k=0}^{n} \left(\frac{\sqrt{2}}{2}\right)^k =$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^n \frac{1-(\sqrt{2}/2)^{n+1}}{1-(\sqrt{2}/2)} = \left(\frac{\sqrt{2}}{2}\right)^n \frac{1-(\sqrt{2}/2)^{n+1}}{2-\sqrt{2}} =$$

$$= \left(\frac{\sqrt{2}}{2}\right)^n \frac{\left[1-(\sqrt{2}/2)^{n+1}\right](2+\sqrt{2})}{2^2-(\sqrt{2})^2} =$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^n (2+\sqrt{2})\left[1-(\sqrt{2}/2)^{n+1}\right] =$$

$$= \frac{2+\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)^n\left[1-\left(\frac{\sqrt{2}}{2}\right)^{n+1}\right]$$

- Infinite geometric series

$$\boxed{-1 < a < 1 \Rightarrow \sum_{k=0}^{+\infty} a^k = \frac{1}{1-a}}$$

EXAMPLES

a) $\sum_{k=0}^{+\infty} (\sqrt{3}-1)^k$

Solution

Since $1 < \sqrt{3} < 2 \Rightarrow 0 < \sqrt{3}-1 < 1 \Rightarrow$

$$\Rightarrow s = \sum_{k=0}^{+\infty} (\sqrt{3}-1)^k = \frac{1}{1-(\sqrt{3}-1)} = \frac{1}{1-\sqrt{3}+1} =$$

$$= \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2+\sqrt{3}}{2^2-(\sqrt{3})^2} =$$

$$= \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}.$$

b) $\sum_{k=2}^{+\infty} (\sqrt{2}-1)^k$

Solution

$$S = \sum_{k=2}^{+\infty} (\sqrt{2}-1)^k = \sum_{k=0}^{+\infty} (\sqrt{2}-1)^k - (\sqrt{2}-1)^0 - (\sqrt{2}-1)^1 =$$

$$= \frac{1}{1-(\sqrt{2}-1)} - 1 - (\sqrt{2}-1) = \frac{1}{1-\sqrt{2}+1} - 1 - \sqrt{2} + 1 =$$

$$= \frac{1}{2-\sqrt{2}} - \sqrt{2} = \frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})} - \sqrt{2} =$$

$$= \frac{2+\sqrt{2}}{2^2-(\sqrt{2})^2} - \sqrt{2} = \frac{2+\sqrt{2}}{2} - \sqrt{2} = \frac{2+\sqrt{2}-2\sqrt{2}}{2} =$$

$$= \frac{2-\sqrt{2}}{2}.$$

c) $\displaystyle\sum_{k=n}^{+\infty} \left(\frac{1}{3}\right)^{k-1}$

Solution

$$S = \sum_{k=n}^{+\infty} \left(\frac{1}{3}\right)^{k-1} = \left(\frac{1}{3}\right)^{-1} \sum_{k=n}^{+\infty} \left(\frac{1}{3}\right)^k =$$

$$= 3\left[\sum_{k=0}^{+\infty} \left(\frac{1}{3}\right)^k - \sum_{k=0}^{n-1} \left(\frac{1}{3}\right)^k\right] =$$

$$= 3\left[\frac{1}{1-1/3} - \frac{1-(1/3)^n}{1-1/3}\right] =$$

$$= 3.\left[\frac{1-(1-(1/3)^n)}{2/3}\right] = 3.\frac{3}{2}.\left[1-1+(1/3)^n\right] =$$

$$= \frac{9}{2}\left(\frac{1}{3}\right)^n.$$

EXERCISES

2. Evaluate the following sums:

a) $\displaystyle\sum_{k=0}^{n} \left(\frac{1}{3}\right)^{k}$

b) $\displaystyle\sum_{k=0}^{n} (-1)^{k} \cdot \left(\frac{1}{2}\right)^{2k+1}$

c) $\displaystyle\sum_{k=0}^{n} \left(\frac{1}{2}\right)^{2k-1}$

d) $\displaystyle\sum_{k=0}^{n} 2^{k/2}$

e) $\displaystyle\sum_{k=0}^{n} (\sqrt{2})^{2k-1}$

f) $\displaystyle\sum_{k=0}^{n} (-1)^{k+1} (\sqrt{3})^{k-1}$

$\mapsto$ Try $n=1$ or $n=2$ to check your answer.

3. Similarly, evaluate the following sums:

a) $\displaystyle\sum_{k=3}^{n+3} 2^{k}$

b) $\displaystyle\sum_{k=3}^{2n+1} (\sqrt{3})^{k}$

c) $\displaystyle\sum_{k=n}^{2n-1} (-1)^{k} \left(\frac{1}{3}\right)^{k+1}$

d) $\sum_{k=n+1}^{2n} (\sqrt{2})^k$

e) $\sum_{k=2n}^{3n+1} \left(\frac{2}{3}\right)^k$

f) $\sum_{k=n}^{2n} (-1)^k (1+\sqrt{2})^{2k}$

(4) Similarly, evaluate the following infinite sums:

a) $\sum_{k=0}^{+\infty} \left(\frac{1}{3}\right)^k$

b) $\sum_{k=0}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{2}}\right)^{k+1}$

c) $\sum_{k=2}^{+\infty} (-1)^k \left(\frac{1}{\sqrt{3}}\right)^k$

d) $\sum_{k=n}^{+\infty} \left(\frac{2}{5}\right)^k$

e) $\sum_{k=n+1}^{+\infty} \left(\frac{2}{\sqrt{3}}\right)^k$

f) $\sum_{k=2n+1}^{+\infty} (\sqrt{2}-1)^k$