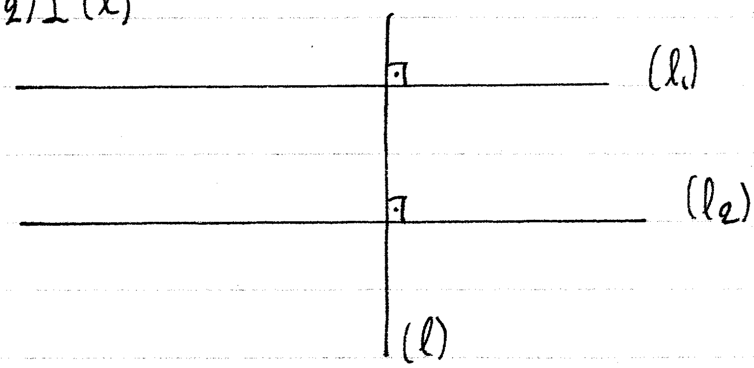
REVIEW OF GEOMETRY

Parallel and perpendicular lines

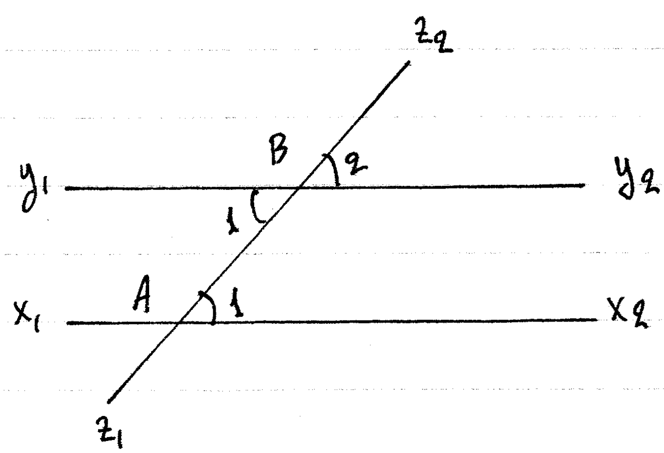
We give the following two results without proof.

1) Given 3 lines $(\ell_1), (\ell_2), (\ell)$

$$\begin{cases} (\ell_1) \perp (\ell) \\ (\ell_2) \perp (\ell) \end{cases} \Rightarrow (\ell_1) \parallel (\ell_2)$$



2) Given two lines $(\ell_1), (\ell_2)$ and a line (ℓ) such that $(\ell_1) \parallel (\ell_2)$ and $(\ell) \cap (\ell_1) = \{A\}$ and $(\ell) \cap (\ell_2) = \{B\}$, we



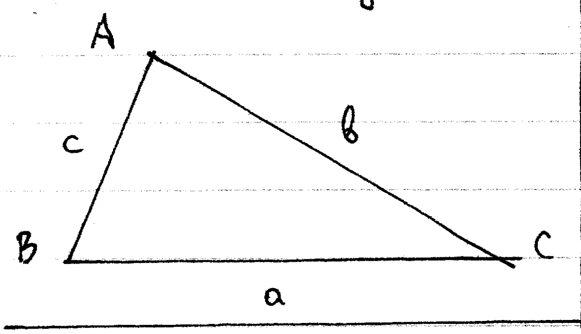
define the angles $\hat{A}_1 = x_2\hat{A}B$ and $\hat{B}_1 = y_1\hat{B}A$ and $\hat{B}_2 = y_2\hat{B}z_2$

Then: $\boxed{\hat{A}_1 = \hat{B}_1 = \hat{B}_2}$

Terminology: A_1, B_1 : interior alternate angles
A_1, B_2 : interior-exterior corresponding angles.
B_1, B_2 : vertical angles.

Basic properties of triangles

Consider a triangle $\overset{\Delta}{ABC}$. We define:



1) Three angles

$\hat{A} = B\hat{A}C$

$\hat{B} = C\hat{B}A$

$\hat{C} = A\hat{C}B$

2) Three sides

$a = BC$

$b = CA$

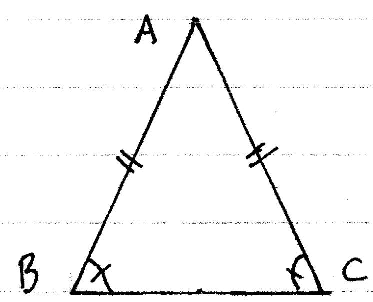
$c = AB$

(1) → Isosceles property

$$\boxed{\begin{array}{l} a = b \iff \hat{A} = \hat{B} \\ b = c \iff \hat{B} = \hat{C} \\ c = a \iff \hat{C} = \hat{A} \end{array}}$$

This property can be shown via equality of triangles. We omit the proof.

example



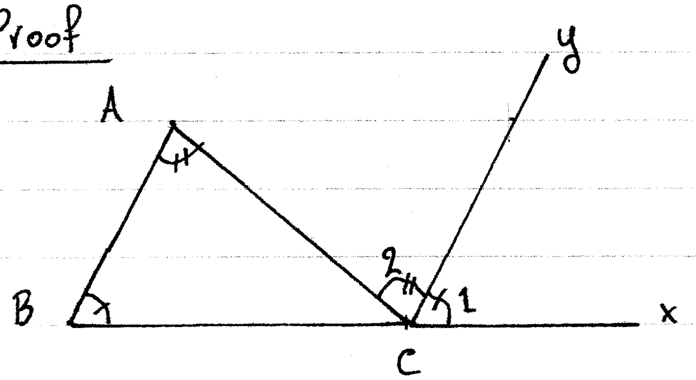
The case:
$b = c \iff \hat{B} = \hat{C}$

We say that
$\overset{\Delta}{ABC}$ is isosceles $\iff a = b \vee b = c \vee c = a$

(2) → Angle sum

$$\boxed{\hat{A} + \hat{B} + \hat{C} = 180^\circ}$$

Proof

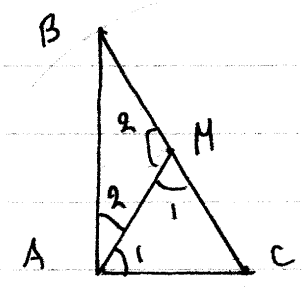


Extend BC to the side of C with the half line Cx such that $B\hat{C}x = 180^\circ$. Bring $Cy \parallel AB$ on the same half-plane as A.

Define $\hat{C}_1 = x\hat{C}y$ and $\hat{C}_2 = A\hat{C}y$. Then
$\hat{C}_1 = \hat{B}$, as interior-exterior corresponding angles
$\hat{C}_2 = \hat{A}$, as interior-alternate angles
It follows that
$\hat{A} + \hat{B} + \hat{C} = \hat{C}_2 + \hat{C}_1 + \hat{C} = B\hat{C}x = 180^\circ$

(3) → 30-60 triangle

$$\boxed{\left.\begin{array}{l}\hat{A} = 90^\circ \\ \hat{B} = 30^\circ\end{array}\right\} \Rightarrow b = \frac{a}{2}}$$



Proof

Choose M on BC such that $M\hat{A}C = 60^\circ$. Define $\hat{A}_1 = M\hat{A}C$ and $\hat{A}_2 = B\hat{A}M$ and $\hat{M}_1 = A\hat{M}C$ and $\hat{M}_2 = A\hat{M}B$. Note that given $\hat{A}_1 = M\hat{A}C = 60^\circ$ we have:

$\hat{C} = 180° - \hat{A} - \hat{B} = 180° - 90° - 30° = 60°$

$\hat{M}_1 = 180° - \hat{C} - \hat{A}_1 = 180° - 60° - 60° = 60°$

It follows that $\hat{A}_1 = \hat{C} = \hat{M}_1 = 60° \Longrightarrow CM = AM = AC$ (1)

We also have

$\hat{A}_2 = \hat{A} - \hat{A}_1 = 90° - 60° = 30° = \hat{B} \Longrightarrow AM = BM$ (2)

and therefore:

$$\begin{aligned} a = BC &= BM + MC \\ &= AM + AC && [\text{via } BM = AM \text{ and } MC = AC] \\ &= AC + AC && [\text{via } AM = AC] \\ &= 2AC = 2b \Longrightarrow b = a/2. \end{aligned}$$

□

Similar triangles and the Pythagorean theorem

Def : Consider two triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$.
We define:

$$\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \iff \begin{cases} \hat{A}_1 = \hat{A}_2 \wedge \hat{B}_1 = \hat{B}_2 \wedge \hat{C}_1 = \hat{C}_2 \\ \dfrac{A_1B_1}{A_2B_2} = \dfrac{B_1C_1}{B_2C_2} = \dfrac{C_1A_1}{C_2A_2} \end{cases}$$

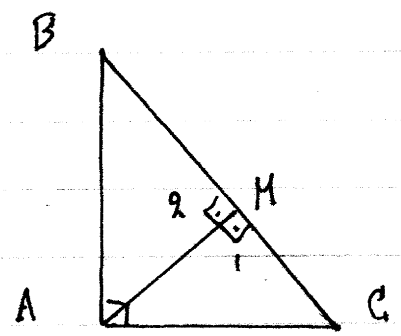
- If $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$, then we say that the triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are <u>similar</u>.
- We can show (proof omitted) that

$$\boxed{\begin{cases} \hat{A}_1 = \hat{A}_2 \\ \hat{B}_1 = \hat{B}_2 \end{cases} \Rightarrow \triangle A_1B_1C_1 \sim \triangle A_2B_2C_2}$$

- This result can be used to establish the Pythagorean theorem:

$$\boxed{\hat{A} = 90° \Rightarrow a^2 = b^2 + c^2}$$

Proof



Choose M on BC such that $AM \perp BC$.
Define $\hat{M}_1 = A\hat{M}C$ and $\hat{M}_2 = A\hat{M}B$
and note that
$AM \perp BC \Rightarrow \hat{M}_1 = \hat{M}_2 = 90°$
Compare $\triangle ABC$ with $\triangle AMC$. Both share $\hat{C}$. Also $\hat{A} = \hat{M}_1$. It follows that

$\overset{\Delta}{ABC} \sim \overset{\Delta}{MAC} \Rightarrow \dfrac{CM}{AC} = \dfrac{AC}{BC} \Rightarrow \dfrac{CM}{b} = \dfrac{b}{a} \Rightarrow CM = \dfrac{b^2}{a}$ (1)

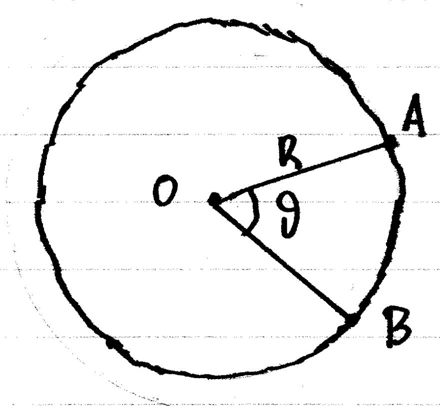
Compare $\overset{\Delta}{ABC}$ with $\overset{\Delta}{AMB}$. Both share $\hat{B}$ and also $\hat{A} = \hat{M}_2$.
It follows that

$\overset{\Delta}{ABC} \sim \overset{\Delta}{MBA} \Rightarrow \dfrac{BM}{AB} = \dfrac{AB}{BC} \Rightarrow \dfrac{BM}{c} = \dfrac{c}{a} \Rightarrow BM = \dfrac{c^2}{a}$ (2)

From Eq.(1) and Eq.(2):

$BM + CM = BC \Rightarrow \dfrac{c^2}{a} + \dfrac{b^2}{a} = a \Rightarrow \underline{a^2 = b^2 + c^2}$ $\square$

Circles



Circumference:

$l = 2\pi R$

Area:

$A = \pi R^2$

- Consider the arc $\widehat{AB}$.
- We say that the angle $A\hat{O}B$ subtends the arc $\widehat{AB}$.
- Length of arc:

$$\boxed{l(\widehat{AB}) = 2\pi R \cdot \frac{(A\hat{O}B)}{360}}$$

with $(A\hat{O}B)$ given in degrees.

- Angles in radians

The measure ϑ of the angle $A\hat{O}B$ is defineds as

$$\boxed{\vartheta = \frac{2\pi}{360}(A\hat{O}B)} \Rightarrow \boxed{l(AB) = R\vartheta}$$

Some commonly used angles in degrees and in radians:

$A\hat{O}B$	30°	45°	60°	90°	180°	360°
ϑ	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	2π

To convert:

$$\boxed{\begin{array}{l} (\text{radians}) = \dfrac{2\pi}{360}\,(\text{degrees}) \\ (\text{degrees}) = \dfrac{360}{2\pi}\,(\text{radians}) \end{array}}$$

- <u>Area of a sector</u>

The area (OAB) of the sector defined by the arc $\widehat{AB}$ is:

$$\boxed{(OAB) = \frac{1}{2} R^2 \vartheta}$$

For $\vartheta = 2\pi$, this gives the area of of the whole circle $A = \pi R^2$.