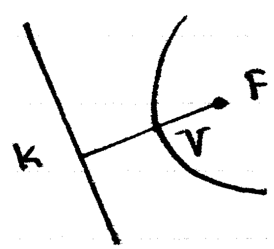
INTRODUCTION TO ANALYTICAL GEOMETRY

▼ Parabola

- Let (ℓ) be a line and let a point $F \notin (\ell)$.
  Then (c) is a parabola with
  a) Focus F
  b) Directrix (ℓ)
  if and only if $\boxed{M \in (c) \Leftrightarrow MF = d(M,(\ell))}$

- Let $FK \perp (\ell)$ with $K \in (\ell)$. Let V be the midpoint of FK. We claim that $V \in (c)$.

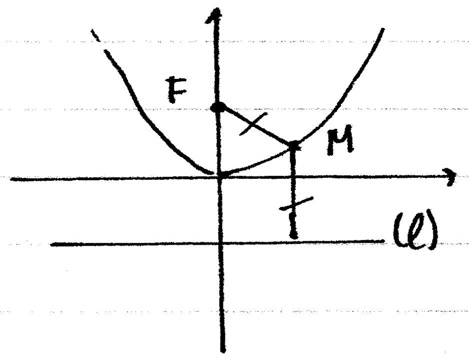


Proof: $VF = VK = d(V,(\ell)) \Rightarrow V \in (c)$. □

Thm:

$$\boxed{\left.\begin{array}{l}(c) \text{ parabola with} \\ \text{focus } F(0,p) \\ \text{directrix } (\ell): y=-p\end{array}\right\} \Rightarrow (c): x^2 = 4py}$$

Proof



Let $M \in (c)$. Then

$$MF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

$$d(M,(\ell)) = |y-(-p)| = |y+p|.$$

It follows that

$$M \in (c) \Leftrightarrow MF = d(M,(\ell)) \Leftrightarrow \sqrt{x^2+(y-p)^2} = |y+p|^2$$

$$\Leftrightarrow x^2 + (y-p)^2 = (y+p)^2 \Leftrightarrow$$

$$\Leftrightarrow x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \Leftrightarrow$$

$$\Leftrightarrow x^2 - 2py = 2py \Leftrightarrow \underline{x^2 = 4py}. \quad \square$$

In general:

$F(x_0+p, y_0)$ $(\ell): x = x_0 - p$ $\Downarrow$ $(c): (y-y_0)^2 = 4p(x-x_0)$	$F(x_0, y_0+p)$ $(\ell): y = y_0 - p$ $\Downarrow$ $(c): (x-x_0)^2 = 4p(y-y_0)$

EXAMPLES

a) Find the parabola (c) with focus $F(1,3)$ and directrix $(\ell): y=1$

Solution

In general, for focus $F(x_0, y_0+p)$ and directrix $(\ell): y = y_0 - p$ the corresponding parabola is $(c): (x-x_0)^2 = 4p(y-y_0)$.

Then,

$$\begin{cases} \text{Focus } F(1,3) \\ \text{Directrix } (\ell): y=1 \end{cases} \Leftrightarrow \begin{cases} x_0 = 1 \\ y_0 + p = 3 \\ y_0 - p = 1 \end{cases} + \Leftrightarrow \begin{cases} x_0 = 1 \\ 2y_0 = 4 \\ y_0 + p = 3 \end{cases}$$

$$2y_0 = 4$$

$$\Leftrightarrow \begin{cases} x_0 = 1 \\ y_0 = 2 \\ 2+p = 3 \end{cases} \Leftrightarrow \begin{cases} x_0 = 1 \\ y_0 = 2 \\ p = 1 \end{cases}$$

and it follows that

$$(c): (x-1)^2 = 4 \cdot 1 \cdot (y-2) \Leftrightarrow x^2 - 2x + 1 = 4y - 8 \Leftrightarrow$$

$$\Leftrightarrow x^2 - 2x + 1 - 4y + 8 = 0 \Leftrightarrow$$

$$\Leftrightarrow x^2 - 2x - 4y + (1+8) = 0$$

$$\Leftrightarrow x^2 - 2x - 4y + 9 = 0.$$

Thus:

$(c): x^2 - 2x - 4y + 9 = 0$

B) Find the focus and directrix of the parabola

$(c): y^2-2x-6y+7=0$

Solution

Since,

$(c): y^2-2x-6y+7=0 \Leftrightarrow (y^2-6y+9)-2x+7-9=0 \Leftrightarrow$

$\Leftrightarrow (y-3)^2-2x-2=0 \Leftrightarrow (y-3)^2 = 2x+2 \Leftrightarrow$

$\Leftrightarrow (y-3)^2 = 2(x+1) \Leftrightarrow (y-3)^2 = 4\cdot(1/2)(x-(-1))$

In general, $(c): (y-y_0)^2 = 4p(x-x_0)$ has focus $F(x_0+p, y_0)$ and $(\ell): x=x_0-p$. It follows that

$x_0=-1 \wedge y_0=3 \wedge p=1/2 \Rightarrow \begin{cases} \text{Focus } F(-1+1/2, 3) \\ \text{Directrix } (\ell): x=(-1)-1/2 \end{cases}$

$\Rightarrow \begin{cases} \text{Focus } F(-1/2, 3) \\ \text{Directrix } (\ell): x=-3/2 \end{cases}$

$\hookrightarrow$ Curves with equations of the form

$(c): x^2+Ax+By+C=0$ or

$(c): y^2+Ax+By+C=0$

are sometimes parabolas. To rewrite in standard form

$(c): (y-y_0)^2=4p(x-x_0)$ or

$(c): (x-x_0)^2=4p(y-y_0)$

we complete the square as shown in the example above.

EXERCISES

① Find the equation for the parabola with focus F and directrix (ℓ) with

a) $F(1,2)$, $(\ell): x=-1$
b) $F(-1,3)$, $(\ell): x=2$
c) $F(0,0)$, $(\ell): x=-2$
d) $F(2,5)$, $(\ell): y=1$
e) $F(-2,3)$, $(\ell): y=3$
f) $F(-1,-3)$, $(\ell): y=-2$

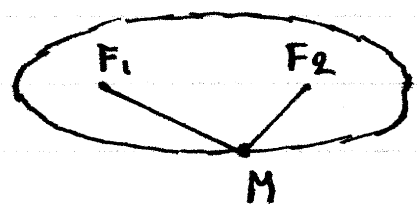
② Find the focus and directrix of the following parabolas:

a) $x^2+4x+2y+1=0$
b) $x^2+6x+3y-1=0$
c) $y^2+2x+8y+3=0$
d) $y^2+3x-4y+2=0$
e) $x^2-x+y-1=0$
f) $x^2+3x-2y+5=0$

Ellipse

Let F_1, F_2 be two points. An ellipse (c) with foci F_1 and F_2 is any curve such that

$$\boxed{M \in (c) \Leftrightarrow MF_1 + MF_2 = 2a}$$



Here $a \in (0, +\infty)$ is a constant.
We also define:
(a) Focal distance: $F_1F_2 = 2c$
(b) Eccentricity: $e = c/a$

Prop: $\boxed{0 < c < a}$

Proof

We apply the triangle inequality to $\overset{\Delta}{MF_1F_2}$:

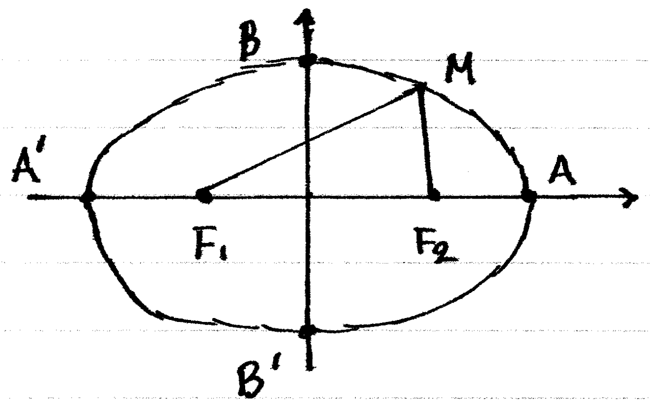
$$\begin{aligned} 2c &= F_1F_2 && [\text{def}] \\ &< MF_1 + MF_2 && [\text{triangle ineq.}] \\ &= 2a && [M \in (c)] \end{aligned}$$

$\Rightarrow c < a \Rightarrow 0 < c < a$ $\square$

Equation of the ellipse

Consider an ellipse (c) with foci $F_1(-c,0)$ and $F_2(c,0)$. Then, for $M(x,y)$:

$$\boxed{M \in (c) \Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad \text{with} \quad \boxed{a^2 = b^2 + c^2}$$



Note that
$A(a,0)$
$A'(-a,0)$
$B(0,b)$
$B'(0,-b)$

Terminology:

a) Vertices: A, A', B, B'

b) Major axis: AA'

c) Minor axis: BB'

d) Focal radii:
$r_1 = F_1M$
$r_2 = F_2M$

It can also be shown that for $M(x,y)$:

$r_1 = MF_1 = a + \frac{cx}{a}$	$r_2 = MF_2 = a - \frac{cx}{a}$

We now prove the above statements:

Thm : $\boxed{M(x,y) \in (c) \Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_2 = a - cx/a \end{cases}}$

Proof

$(\Rightarrow)$

Assume $M(x,y) \in (c)$.

It follows that

$$r_1 + r_2 = MF_1 + MF_2 = 2a \quad (1)$$

Also note that:

$$\left.\begin{array}{l} r_1^2 = MF_1^2 = (x+c)^2 + y^2 \\ r_2^2 = MF_2^2 = (x-c)^2 + y^2 \end{array}\right\} \Rightarrow$$

$$\begin{aligned} \Rightarrow r_1^2 - r_2^2 &= [(x+c)^2+y^2] - [(x-c)^2+y^2] = \\ &= (x+c)^2 - (x-c)^2 = \\ &= x^2 + 2cx + c^2 - (x^2 - 2cx + c^2) = \\ &= 2cx + 2cx = 4cx \Rightarrow \end{aligned}$$

$$\Rightarrow (r_1 - r_2)(r_1 + r_2) = 4cx \Rightarrow (r_1 - r_2)2a = 4cx \Rightarrow$$

$$\Rightarrow r_1 - r_2 = \frac{2cx}{a} \quad (2).$$

From (1) and (2):

$$\begin{cases} r_1 + r_2 = 2a \\ r_1 - r_2 = 2cx/a \end{cases} \Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_1 + r_2 = 2a \end{cases} \Leftrightarrow$$

$$2r_1 = 2a + 2cx/a$$

$$\Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_2 = 2a - r_1 = 2a - (a + cx/a) = a - cx/a \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r_1 = a + cx/a \\ r_2 = a - cx/a \end{cases}$$

$(\Leftarrow)$: Assume that $\begin{cases} r_1 = a + cx/a \\ r_2 = a - cx/a \end{cases}$

Then:

$$MF_1 + MF_2 = r_1 + r_2 = (a + cx/a) + (a - cx/a)$$
$$= 2a \Rightarrow M \in (c). \qquad \square$$

Thm : $\boxed{M(x,y) \in (c) \Leftrightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1}$

Proof

$(\Rightarrow)$: Assume that $M(x,y) \in (c) \Rightarrow \underline{r_1 = a + cx/a}$ (1)

Recall that $\underline{r_1^2 = (x+c)^2 + y^2}$. (2)

From (1) and (2):

$$(a + cx/a)^2 = (x+c)^2 + y^2 \Leftrightarrow$$
$$\Leftrightarrow a^2 + 2 \cdot cx + (cx/a)^2 = x^2 + 2cx + c^2 + y^2 \Leftrightarrow$$
$$\Leftrightarrow x^2 + 2cx + c^2 + y^2 - a^2 - 2\ cx - (c^2/a^2)x^2 = 0 \Leftrightarrow$$
$$\Leftrightarrow (1 - c^2/a^2)x^2 + y^2 = a^2 - c^2 \Leftrightarrow$$
$$\Leftrightarrow \frac{a^2 - c^2}{a^2} x^2 + y^2 = a^2 - c^2 \Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad (3)$$

$(\Leftarrow)$: Assume that $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} \underset{\uparrow (3)}{=} 1 \Rightarrow$

$$\left.\begin{aligned} &\Rightarrow (a + cx/a)^2 = (x+c)^2 + y^2 \quad (4) \\ &\quad r_1^2 = (x+c)^2 + y^2 \end{aligned}\right\} \Rightarrow r_1^2 = (a + cx/a)^2$$

$\Rightarrow \underline{r_1 = a + cx/a}$. (5).

From (4), replace c with $-c$:

$$\left.\begin{array}{l}(a-cx/a)^2=(x-c)^2+y^2\\ r_2^2=(x-c)^2+y^2\end{array}\right\}\Rightarrow r_2^2=(a-cx/a)^2\Rightarrow$$

$$\Rightarrow \underline{r_2=a-cx/a} \quad (6)$$

From (5) and (6): $M(x,y)\in(\mathcal{C})$. □

- <u>General equation of the ellipse</u>

$$\boxed{M(x,y)\in(\mathcal{C})\Leftrightarrow \frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1}$$

Vertices: $A(x_0+a,y_0)$, $A'(x_0-a,y_0)$
$B(x_0,y_0+b)$, $B'(x_0,y_0-b)$

$a>b$	$a<b$
Foci: $F_1(x_0-c,y_0)$ $F_2(x_0+c,y_0)$ with $c^2=a^2-b^2$	Foci: $F_1(x_0,y_0-c)$ $F_2(x_0,y_0+c)$ with $c^2=b^2-a^2$

EXAMPLES

a) Find the foci and eccentricity of the ellipse
$(c): x^2+3y^2+4x+6y+3=0$

Solutions

$(c): x^2+3y^2+4x+6y+3=0 \Leftrightarrow$

$\Leftrightarrow (x^2+4x+4)+(3y^2+6y+3)-4=0$

$\Leftrightarrow (x+2)^2+3(y^2+2y+1)=4$

$\Leftrightarrow (x+2)^2+3(y+1)^2=4 \Leftrightarrow$

$$\Leftrightarrow \frac{(x+2)^2}{4}+\frac{3(y+1)^2}{4}=1 \Leftrightarrow$$

$$\Leftrightarrow \frac{(x-(-2))^2}{2^2}+\frac{(y-(-1))}{(2/\sqrt{3})^2}=1.$$

It follows that:

$x_0=-2,\ y_0=-1,\ a=2,\ b=2/\sqrt{3}.$

Since $a>b \Rightarrow$

$\Rightarrow$ Foci : $F_1(x_0-c, y_0)$ and $F_2(x_0+c, y_0)$

with

$c^2=a^2-b^2=2^2-(2/\sqrt{3})^2=4-4/3=4\cdot 2/3 \Rightarrow$

$$\Rightarrow c=\frac{2\sqrt{2}}{\sqrt{3}}=\frac{2\sqrt{6}}{3}$$

It follows that $F_1(-2-2\sqrt{6}/3, -1)$ and $F_2(-2+2\sqrt{6}/3, -1)$.

Eccentricity: $e=\dfrac{c}{a}=\dfrac{2\sqrt{6}/3}{2}=\dfrac{\sqrt{6}}{3}.$

b) Find the equation of the ellipse with foci $F_1(2,1)$ and $F_2(2,5)$ and major axis $AA' = 12$.

Solution

Let $(c): \dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$.

$2a = AA' = 12 \Rightarrow a = 6$.

$2c = F_1F_2 = |y_{F_1} - y_{F_2}| = |1-5| = 4 \Rightarrow c = 2$.

Since $F_1F_2 \parallel y\text{-axis} \Rightarrow a < b \Rightarrow$

$\Rightarrow b^2 = a^2 + c^2 = 6^2 + 2^2 = 36 + 4 = 40 \Rightarrow b = 2\sqrt{10}$.

Since O midpoint of F_1F_2:

$x_0 = x_{F_1} = 2$

$y_0 = \dfrac{1}{2}(y_{F_1} + y_{F_2}) = \dfrac{1}{2}(1+5) = \dfrac{6}{2} = 3$

Thus:

$(c): \dfrac{(x-2)^2}{36} + \dfrac{(y-3)^2}{40} = 1$

c) Find the equation of the ellipse with foci $F_1(2,2)$ and $F_2(5,2)$ and eccentricity $e = 1/2$.

Solution

Let $(c): \dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$

$2c = F_1F_2 = |x_{F_1} - x_{F_2}| = |2-5| = 3 \Rightarrow c = 3/2$

$e = c/a \Rightarrow a = \dfrac{c}{e} = \dfrac{3/2}{1/2} = 3$

Since $F_1F_2 \parallel x\text{-axis} \Rightarrow a > b \Rightarrow a^2 = b^2 + c^2 \Rightarrow$

$\Rightarrow b^2 = a^2 - c^2 = 3^2 - (3/2)^2 = 9[1 - 1/4] = 9 \cdot 3/4 = 27/4$

$\Rightarrow b = \dfrac{3\sqrt{3}}{2}$.

O midpoint of F_1F_2, thus:

$x_0 = \dfrac{1}{2}(x_{F_1} + x_{F_2}) = \dfrac{1}{2}(2+5) = \dfrac{7}{2}$

$y_0 = y_{F_1} = 2$.

It follows that:

$(c): \dfrac{(x-7/2)^2}{3^2} + \dfrac{(y-2)^2}{27/4} = 1 \Leftrightarrow$

$\Leftrightarrow \dfrac{(2x-7)^2}{4 \cdot 3^2} + \dfrac{4(y-2)^2}{27} = 1.$

$\Leftrightarrow \dfrac{(2x-7)^2}{36} + \dfrac{4(y-2)^2}{27} = 1.$

EXERCISES

③ Find the foci and eccentricity of the following ellipses:

a) $x^2+2y^2+6x+8y+1=0$
b) $2x^2+3y^2-6x+6y-2=0$
c) $5x^2+2y^2-10x+12y+3=0$
d) $3x^2+y^2+30x+12y+2=0$
e) $3x^2+4y^2-6x-16y+3=0$
f) $2x^2+y^2+12x+4y-1=0$.

④ Find an equation for the ellipse with focus F_1, F_2 and eccentricity e; with

a) $F_1(-1,0)$, $F_2(2,0)$, $e=1/2$
b) $F_1(2,2)$, $F_2(2,6)$, $e=2/3$
c) $F_1(1,-\sqrt{2})$, $F_2(1,+\sqrt{2})$, $e=1/\sqrt{2}$
d) $F_1(1-\sqrt{2},3)$, $F_2(1+\sqrt{2},3)$, $e=1/3$
e) $F_1(-1,\sqrt{2})$, $F_2(-1,2\sqrt{2})$, $e=\sqrt{2}-1$.

▼ Hyperbola

Let F_1, F_2 be two points. A hyperbola (c) with focus F_1 and F_2 is a set of points such that

$$\boxed{M \in (c) \Leftrightarrow |MF_1 - MF_2| = 2a}$$

with $a \in (0, +\infty)$ a constant. We also define:

(a) Focal distance: $F_1F_2 = 2c$

(b) Eccentricity: $e = c/a$

Prop : $\boxed{e > 1}$

Proof

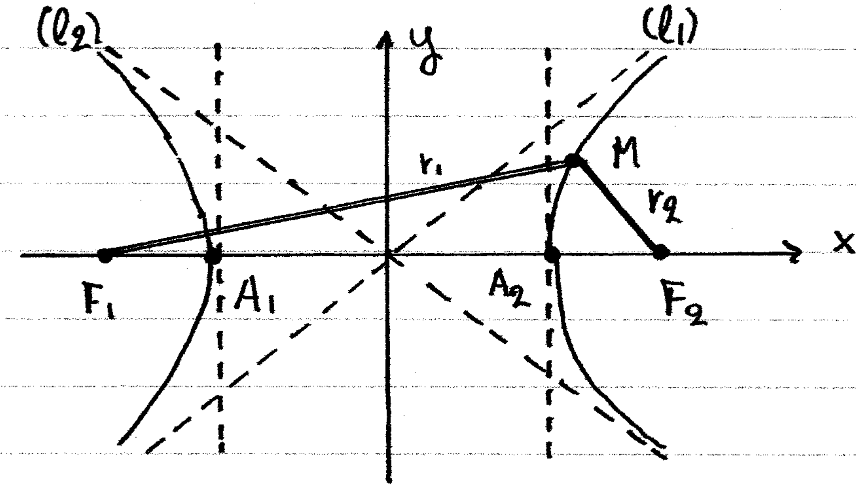
Apply triangle inequality to $M\hat{F_1}F_2$:

$$\begin{aligned} 2a = |MF_1 - MF_2| &< && [\text{def}] \\ &< F_1F_2 && [\text{triangle inequality}] \\ &= 2c \Rightarrow && [\text{def}] \\ \Rightarrow a < c \Rightarrow e = \frac{c}{a} &> 1 && \square \end{aligned}$$

❶ Equation of hyperbola

Consider a hyperbola (c) with $F_1(-c,0)$ and $F_2(c,0)$. Then:

$\boxed{M(x,y) \in (c) \Leftrightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ with $\boxed{c^2 = b^2 + a^2}$



Terminology:

a) Vertices ⟶ $A_1(-a,0)$ and $A_2(a,0)$

b) Asymptotes

Focus – Asymptote distance

$(\ell_1):\ y = \dfrac{b}{a}x$	$(\ell_2):\ y = -\dfrac{b}{a}x$

(!) $\boxed{d(F_1,(\ell_1)) = d(F_1,(\ell_2)) = d(F_2,(\ell_1)) = d(F_2,(\ell_2)) = b}$

c) Focal radii

$r_1 = MF_1 = \left\lvert \dfrac{cx}{a} + a \right\rvert$	$r_2 = MF_2 = \left\lvert \dfrac{cx}{a} - a \right\rvert$

• Proof of equation of the ellipse

Let (c) be a hyperbola with focii $F_1(-c,0)$ and $F_2(c,0)$ with $c>0$ such that

$$M(x,y)\in(c)\Leftrightarrow |MF_1-MF_2|=2a \quad,\text{ with } a>0$$

First we show that:

Prop : $\boxed{M(x,y)\in(c)\Leftrightarrow \begin{cases} r_1=MF_1=|cx/a+a| \\ r_2=MF_2=|cx/a-a| \end{cases}}$

Proof

($\Rightarrow$): Assume that $M(x,y)\in(c)$. Then:

$$r_1^2=MF_1^2=(x_M-x_{F_1})^2+(y_M-y_{F_1})^2=$$
$$=(x-(-c))^2+(y-0)^2=(x+c)^2+y^2$$
$$r_2^2=MF_2^2=(x_M-x_{F_2})^2+(y_M-y_{F_2})^2=$$
$$=(x-c)^2+(y-0)^2=(x-c)^2+y^2.$$

It follows that

$$(r_1-r_2)(r_1+r_2)=r_1^2-r_2^2=[(x+c)^2+y^2]-[(x-c)^2+y^2]$$
$$=(x+c)^2-(x-c)^2=$$
$$=x^2+2cx+c^2-x^2+2cx-c^2=$$
$$=2cx+2cx=4cx$$

thus:

$$\begin{cases}(r_1-r_2)(r_1+r_2)=4cx & (1)\\ |r_1-r_2|=2a\end{cases}$$

since $M(x,y)\in(c)\Rightarrow|MF_1-MF_2|=2a\Rightarrow|r_1-r_2|=2a.$

Case 1: Assume $x=0 \Rightarrow 4cx=0 \Rightarrow r_1^2-r_2^2=0 \Rightarrow$
$\Rightarrow r_1=r_2 \Rightarrow MF_1=MF_2 \Rightarrow$
$\Rightarrow |MF_1-MF_2|=0 \neq 2a \leftarrow$ contradiction.

Case 2: Assume $x>0 \Rightarrow 4cx>0 \Rightarrow (r_1-r_2)(r_1+r_2)>0$
$\Rightarrow r_1-r_2>0$, therefore

$$(1) \Leftrightarrow \begin{cases} (r_1-r_2)(r_1+r_2)=4cx \\ r_1-r_2=2a \end{cases} \Leftrightarrow \begin{cases} 2a(r_1+r_2)=4cx \\ r_1-r_2=2a \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r_1+r_2=4cx/2a=2cx/a \\ r_1-r_2=2a \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} r_1=cx/a+a \\ r_2=cx/a-a. \end{cases}$$

Case 3: Assume $x<0 \Rightarrow 4cx<0 \Rightarrow (r_1-r_2)(r_1+r_2)<0$
$\Rightarrow r_1-r_2<0$, therefore

$$(1) \Leftrightarrow \begin{cases} (r_1-r_2)(r_1+r_2)=4cx \\ r_1-r_2=-2a \end{cases} \Leftrightarrow \begin{cases} -2a(r_1+r_2)=4cx \\ r_1-r_2=-2a \end{cases}$$

$$\Leftrightarrow \begin{cases} r_1+r_2=-2cx/a \\ r_1-r_2=-2a \end{cases} \Leftrightarrow \begin{cases} r_1=-cx/a-a \\ r_2=-cx/a+a \end{cases}$$

From cases 1,2,3 above:

$$(1) \Leftrightarrow \begin{cases} r_1=|cx/a+a| \\ r_2=|cx/a-a| \end{cases}$$

$(\Leftarrow)$: Assume that

$$r_1 = MF_1 = |cx/a + a| \quad (1)$$
$$r_2 = MF_2 = |cx/a - a|$$

Without making any assumptions, we show again that

$$(r_1 - r_2)(r_1 + r_2) = 4cx \quad (2)$$

Case 1: Assume that $x = 0 \overset{(1)}{\Longrightarrow} r_1 = r_2 = |a| = a$

Also:

$$\left.\begin{matrix} r_1^2 = (x+c)^2 + y^2 = c^2 + y^2 \\ r_1^2 = a^2 \end{matrix}\right\} \Rightarrow c^2 + y^2 = a^2 \Rightarrow$$

$$\Rightarrow y^2 = a^2 - c^2 < 0 \quad (\text{since } a < c) \Rightarrow$$

$$\Rightarrow y^2 < 0 \leftarrow \text{Contradiction.}$$

Case 2: Assume that $x \neq 0$. Under this assumption, from cases 2, 3 of the $(\Rightarrow)$ argument above, we can show, without making any further assumptions, that:

$$\begin{cases} (r_1 - r_2)(r_1 + r_2) = 4cx \\ |r_1 - r_2| = 2a \end{cases} \Leftrightarrow \begin{cases} r_1 = |cx/a + a| \\ r_2 = |cx/a - a| \end{cases}$$

It follows that

$$(1) \Rightarrow |r_1 - r_2| = 2a \Rightarrow |MF_1 - MF_2| = 2a \Rightarrow$$
$$\Rightarrow M(x, y) \in (C). \qquad \square$$

We will now show that

<u>Thm</u> : $\boxed{M(x,y) \in (c) \Leftrightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1}$

<u>Proof</u>

$(\Rightarrow)$: Assume that $M(x,y) \in (c)$.
We show again that $r_1^2 = MF_1^2 = (x+c)^2 + y^2$. Then
$M(x,y) \in (c) \Rightarrow r_1 = \left| \dfrac{cx}{a} + a \right| \Rightarrow r_1^2 = \left(\dfrac{cx}{a} + a \right)^2$

$$\Rightarrow (x+c)^2 + y^2 = \left(\frac{cx}{a} + a \right)^2. \quad (1).$$

Note that:

$$(1) \Leftrightarrow x^2 + 2cx + c^2 + y^2 = \frac{c^2x^2}{a^2} + 2cx + a^2 \Leftrightarrow$$

$$\Leftrightarrow x^2 + c^2 + y^2 = \frac{c^2x^2}{a^2} + a^2 \Leftrightarrow$$

$$\Leftrightarrow \left[1 - \frac{c^2}{a^2} \right] x^2 + y^2 = a^2 - c^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{a^2-c^2}{a^2} x^2 + y^2 = a^2 - c^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1 \Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1. \quad (2)$$

Thus $(1) \Rightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1.$

$(\Leftarrow)$: Assume that $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2-a^2} = 1$ (3)

Using (2) we have

$$(3) \Rightarrow (x+c)^2 + y^2 = \left(\frac{cx}{a} + a\right)^2 \Rightarrow r_1^2 = \left(\frac{cx}{a} + a\right)^2$$

$$\Rightarrow r_1 = \left|\frac{cx}{a} + a\right| \quad (4)$$

Furthermore:

$$r_2^2 = (x-c)^2 + y^2 = (x-c)^2 + \left[\left(\frac{cx}{a} + a\right)^2 - (x+c)^2\right] =$$

$$= \left(\frac{cx}{a} + a\right)^2 + (x^2 - 2cx + c^2) - (x^2 + 2cx + c^2) =$$

$$= \left(\frac{cx}{a} + a\right)^2 - 4cx = \left(\frac{cx}{a} - a\right)^2 \Rightarrow$$

$$\Rightarrow r_2 = \left|\frac{cx}{a} - a\right| \quad (5).$$

From (4) and (5) it follows that $M(x,y) \in (c)$. $\square$

We now show that:

- $\boxed{\text{At } x \to \pm\infty, \ (c) \text{ approaches the lines } (\ell_{1,2}): y = \pm \frac{b}{a} x}$

Proof

$$M(x,y) \in (c) \Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Leftrightarrow b^2x^2 - a^2y = a^2b^2$$

$$\Leftrightarrow a^2y^2 = b^2x^2 - a^2b^2 \Leftrightarrow$$

$$\Leftrightarrow y^2 = \frac{b^2x^2 - a^2b^2}{a^2} = \frac{b^2(x^2-a^2)}{a^2} =$$

$$= \frac{b^2x^2}{a^2}\left[1 - \frac{a^2}{x^2}\right] \Leftrightarrow$$

$$\Leftrightarrow y = \pm \frac{bx}{a}\sqrt{1 - \frac{a^2}{x^2}}$$

For $x \to \pm\infty$: $\sqrt{1 - a^2/x^2} \longrightarrow 1$

thus $y/x \sim \pm b/a$.

By symmetry the two lines have to intersect at the origin, thus:

$$(\ell_{1,2}): y = \pm \frac{b}{a}x. \qquad \square$$

- $\boxed{d(F_1,(\ell_1)) = d(F_1,(\ell_2)) = d(F_2,(\ell_1)) = d(F_2,(\ell_2)) = b}$

Proof

Recall that in general, the distance of the point $M(x_0,y_0)$ from the line $(\ell): Ax+By+C=0$ is given by:

$$d(M,(\ell)) = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2+B^2}}$$

For $F_1(-c,0)$ and (ℓ_1): $y = \frac{b}{a}x \Leftrightarrow bx - ay = 0$
we have:

$$d(F_1,(\ell_1)) = \frac{|bx_{F_1} - ay_{F_1}|}{\sqrt{b^2+(-a)^2}} = \frac{|b\cdot(-c) - a\cdot 0|}{\sqrt{a^2+b^2}} =$$
$$= \frac{|-bc|}{\sqrt{c^2}} = \frac{|b||c|}{|c|} = |b| = b$$

Similar argument gives:

$$d(F_2,(\ell_1)) = d(F_1,(\ell_2)) = d(F_2,(\ell_2)) = b \quad \square$$

General Equation of the hyperbola

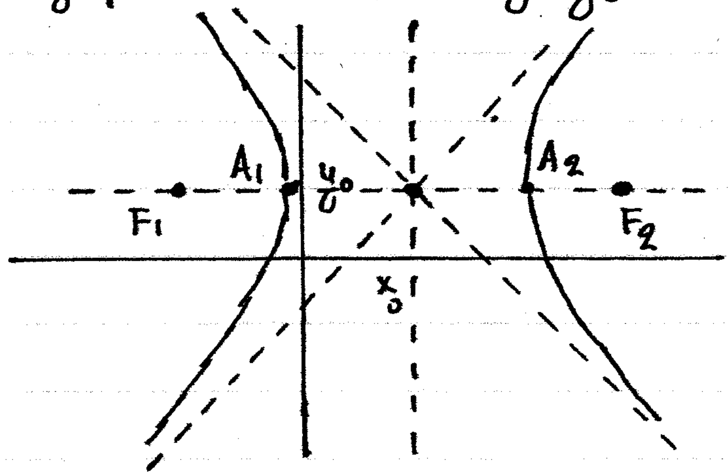
1) $$\boxed{M(x,y) \in (C) \Leftrightarrow \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1}$$

$c^2 = a^2 + b^2$

Focus: $F_1(x_0 - c, y_0)$, $F_2(x_0 + c, y_0)$

Vertices: $A_1(x_0 - a, y_0)$, $A_2(x_0 + a, y_0)$

Asymptotes: $(l_{1,2})$: $y - y_0 = \pm \frac{b}{a}(x - x_0)$.



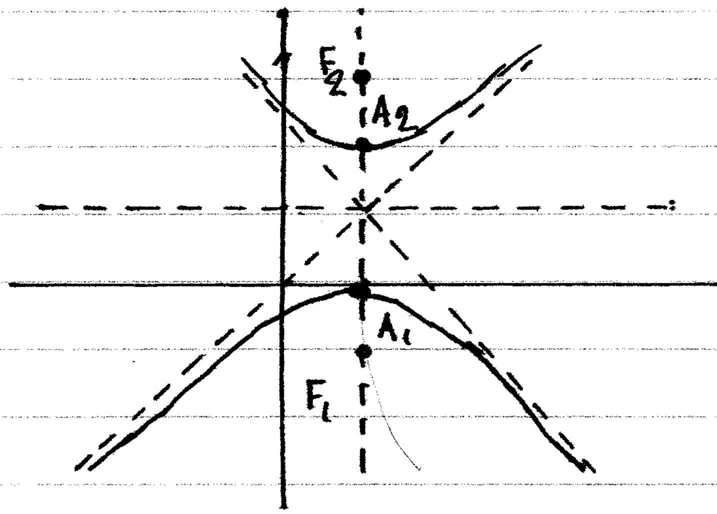
2) $$\boxed{M(x,y) \in (C) \Leftrightarrow \frac{(y-y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1}$$

$c^2 = a^2 + b^2$

Focus: $F_1(x_0, y_0 - c)$, $F_2(x_0, y_0 + c)$

Vertices: $A_1(x_0, y_0 - a)$, $A_2(x_0, y_0 + a)$

Asymptotes: $(l_{1,2})$: $x - x_0 = \pm \frac{b}{a}(y - y_0)$.



F2
A2
A1
F1

EXAMPLES

a) Find the foci, vertices, and asymptotes of the hyperbola

$$(c): x^2 - 2y^2 + 4x - 12y - 20 = 0.$$

Solution

$(c): x^2 - 2y^2 + 4x - 12y - 20 = 0 \Leftrightarrow$

$\Leftrightarrow (x^2 + 4x + 4) - 2(y^2 + 6y + 9) - 20 - 4 + 18 = 0 \Leftrightarrow$

$\Leftrightarrow (x+2)^2 - 2(y+3)^2 - 6 = 0 \Leftrightarrow$

$\Leftrightarrow (x+2)^2 - 2(y+3)^2 = 6 \Leftrightarrow$

$$\Leftrightarrow \frac{(x+2)^2}{6} - \frac{(y+3)^2}{3} = 1$$

$$\Leftrightarrow \frac{(x-(-2))^2}{(\sqrt{6})^2} - \frac{(y-(-3))^2}{(\sqrt{3})^2} = 1$$

For $a = \sqrt{6}$ and $b = \sqrt{3}$:

$c^2 = a^2 + b^2 = (\sqrt{6})^2 + (\sqrt{3})^2 = 6 + 3 = 9 \Rightarrow$

$\Rightarrow c = 3$

It follows that

a) Focus: $F_1(-2-3, -3) = F_1(-5, -3)$

$F_2(-2+3, -3) = F_2(1, -3)$

b) Vertices: $A_1(-2+\sqrt{6}, -3)$

$A_2(-2-\sqrt{6}, -3)$

c) Asymptotes:

$$(\ell_{1,2}):\ y-(-3)=\pm\frac{\sqrt{3}}{\sqrt{6}}\,(x-(-2)) \Leftrightarrow$$

$$\Leftrightarrow y+3=\pm\frac{1}{\sqrt{2}}\,(x+2) \Leftrightarrow \sqrt{2}\,(y+3)=\pm(x+2) \Leftrightarrow$$

$$\Leftrightarrow \mp(x+2)+\sqrt{2}\,(y+3)=0\ .$$

thus:

$$(\ell_1):\ (x+2)+\sqrt{2}\,(y+3)=0 \Leftrightarrow$$

$$\Leftrightarrow \underline{x+\sqrt{2}\,y+(2+3\sqrt{2})=0}$$

and

$$(\ell_2):\ -(x+2)+\sqrt{2}\,(y+3)=0 \Leftrightarrow$$

$$\Leftrightarrow \underline{-x+\sqrt{2}\,y+(3\sqrt{2}-2)=0}\ .$$

b) Find the hyperbola with $F_1(1,2)$, $F_2(6,2)$ focis and vertices $A_1(2,2)$ and $A_2(5,2)$.

Solution

$2a = A_1A_2 = |x_{A_2} - x_{A_1}| = |5-2| = 3 \Rightarrow \underline{a = 3/2}$
$2c = F_1F_2 = |x_{F_2} - x_{F_1}| = |6-1| = 5 \Rightarrow \underline{c = 5/2}$
$c^2 = a^2 + b^2 \Rightarrow$

$$\Rightarrow b^2 = c^2 - a^2 = (5/2)^2 - (3/2)^2 = \frac{25-9}{4} = \frac{16}{4} = 4 \Rightarrow \underline{b = 2.}$$

Origin O midpoint of F_1F_2 thus:

$$x_0 = \frac{x_{F_1} + x_{F_2}}{2} = \frac{1+6}{2} = \frac{7}{2}$$

$$y_0 = \frac{y_{F_1} + y_{F_2}}{2} = \frac{2+2}{2} = 2$$

It follows that:

$$(c): \frac{(x-7/2)^2}{(3/2)^2} - \frac{(y-2)^2}{2^2} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{4(x-7/2)^2}{9} - \frac{(y-2)^2}{4} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{(2x-7)^2}{9} - \frac{(y-2)^2}{4} = 1 \Leftrightarrow$$

$$\Leftrightarrow 4(2x-7)^2 - 9(y-2)^2 = 36 \Leftrightarrow$$

$$\Leftrightarrow 4(4x^2 - 28x + 49) - 9(y^2 - 4y + 4) = 36$$

$\Leftrightarrow 16x^2 - 112x + 196 - 9y^2 + 36y - 36 = 36$

$\Leftrightarrow 16x^2 - 9y^2 - 112x + 36y + (196 - 36 - 36) = 0$

$\Leftrightarrow 16x^2 - 9y^2 - 112x + 36y + 124 = 0$

thus:

$(C): 16x^2 - 9y^2 - 112x + 36y + 124 = 0.$

c) Find the hyperbola with focus $F_1(1-\sqrt{2},1)$ and $F_2(1+\sqrt{2},1)$ and asymptotes $(l_{1,2}): y-1=\pm 2(x-1)$.

<u>Solution</u>

$$2c = F_1F_2 = |x_{F_2}-x_{F_1}| = |(1+\sqrt{2})-(1-\sqrt{2})| =$$
$$= |1+\sqrt{2}-1+\sqrt{2}| = |2\sqrt{2}| = 2\sqrt{2} \Rightarrow c=\sqrt{2}$$

$(l_{1,2}): y-1=\pm 2(x-1)$ asymptotes $\Rightarrow$

$$\Rightarrow \frac{b}{a} = 2 \Rightarrow \underline{b=2a}.$$

It follows that

$$\begin{cases} b=2a \\ c^2=a^2+b^2 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a^2+(2a)^2=(\sqrt{2})^2 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a^2+4a^2=2 \end{cases}$$

$$\Leftrightarrow \begin{cases} b=2a \\ 5a^2=2 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a^2=2/5 \end{cases} \Leftrightarrow \begin{cases} b=2\sqrt{10}/5 \\ a=\dfrac{\sqrt{2}}{\sqrt{5}}=\dfrac{\sqrt{10}}{5} \end{cases}$$

We also note that

$$x_0 = \frac{x_{F_1}+x_{F_2}}{2} = \frac{(1-\sqrt{2})+(1+\sqrt{2})}{2} =$$
$$= \frac{2}{2} = 1 \text{ and}$$

$$y_0 = 1.$$

and therefore:

$$(c): \frac{(x-1)^2}{2/5} - \frac{(y-1)^2}{4/5} = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{5(x-1)^2}{2} - \frac{5(y-1)^2}{4} = 1 \Leftrightarrow$$

$$\Leftrightarrow 10(x-1)^2 - 5(y-1)^2 = 4 \Leftrightarrow$$

$$\Leftrightarrow 10(x^2-2x+1) - 5(y^2-2y+1) = 4 \Leftrightarrow$$

$$\Leftrightarrow 10x^2 - 20x + 10 - 5y^2 + 10y - 5 = 4 \Leftrightarrow$$

$$\Leftrightarrow 10x^2 - 5y^2 - 20x + 10y + (10-5-4) = 0$$

$$\Leftrightarrow 10x^2 - 5y^2 - 20x + 10y + 1 = 0$$

Thus:

$$(c): \underline{10x^2 - 5y^2 - 20x + 10y + 1 = 0}$$

EXERCISES

⑤ Find the focii, vertices, and asymptotes of the following hyperbolas

a) $2x^2-y^2+4x-2y+3=0$
b) $x^2-3y^2+6x+6y+1=0$
c) $x^2-5y^2+10x-20y+10=0$
d) $3x^2-4y^2+12x+40y-3=0$
e) $2x^2-2y^2+24x+28y-7=0$

⑥ Find an equation of the hyperbola with
a) Focus $F_1(3,3)$, $F_2(7,3)$;
Vertices $A_1(4,3)$, $A_2(6,3)$
b) Focus $F_1(2,1-\sqrt{3})$, $F_2(2,1+\sqrt{3})$;
Vertices $A_1(2,0)$, $A_2(2,2)$
c) Focus $F_1(1,-1)$, $F_2(3,-1)$;
asymptote $(\ell): y+1=3(x-2)$
d) Focus $F_1(2,1)$, $F_2(2,7)$;
asymptote $(\ell): y-4=2(x-2)$
(careful with this one!)