Linear Multistep Methods

Discretize time: $t_n = n\Delta t$
and define: $u_n = u(t_n)$ and $f_n = f(u(t_n), t_n)$.

The usual suspects

1) Euler's Method (explicit method)

$$u_{n+1} = u_n + \Delta t\, f_n$$

2) Backward Euler Method (implicit method)

$$u_{n+1} = u_n + \Delta t\, f_{n+1}$$

(requires nonlinear equation solver)

3) Crank-Nicholson (implicit method)

$$u_{n+1} = u_n + \Delta t \left(\frac{f_n + f_{n+1}}{2} \right)$$

4) Leap-frog (two-step explicit method)

$$u_{n+1} = u_{n-1} + 2\Delta t\, f_n$$

Naive analysis of LMFs

The ODE $\partial u/\partial t = au$ has exact solution

$$u(t) = u(0)e^{at}$$

This can be compared against the numerical prediction from the LMF.

<u>example</u>: The Euler method

$$u_{n+1} = u_n + \Delta t f_n = u_n + \Delta t\, a u_n = (1 + a\Delta t)u_n \Rightarrow$$

$$\Rightarrow u_n = (1 + a\Delta t)^n u_0 .$$

Since $n\Delta t = t \Rightarrow \Delta t = \frac{t}{n}$. Take the limit $n \to +\infty$:

$$u(t) = \lim_{n\to+\infty}\left(1 + \frac{at}{n}\right)^n u_0 = u_0 e^{at} .$$

▸ This shows that the Euler method is <u>convergent</u> for this particular ODE.

Rigorous Theory of LMFs

Def: An s-step LMF (linear multistep formula) is the difference equation

$$\sum_{k=0}^{s} a_k u_{n+k} = \Delta t \sum_{k=0}^{s} b_k f_{n+k}$$

with $a_s = 1$ and either $a_0 \neq 0$ or $b_0 \neq 0$.

- If $b_s = 0 \Leftrightarrow$ the LMF is explicit
  $b_s \neq 0 \Leftrightarrow$ the LMF is implicit

The numerical implementation of an implicit scheme requires the solution of a nonlinear equation.

Def: With an s-step LMF we associate the characteristic polynomials

$$\rho(z) = \sum_{k=0}^{s} a_k z^k$$

$$\sigma(z) = \sum_{k=0}^{s} b_k z^k$$

example

1) Euler LMF
$s=1$, $\rho(z) = z-1$, $\sigma(z) = 1$
2) Backward Euler
$s=1$, $\rho(z) = z-1$, $\sigma(z) = z$
3) Crank-Nicolson
$s=1$, $\rho(z) = z-1$, $\sigma(z) = (z+1)/2$
4) Leapfrog
$s=2$, $\rho(z) = z^2-1$, $\sigma(z) = 2z$.

→ Consistency and accuracy.

How well does the LMF satisfy the ODE

$$\partial u/\partial t = f(u(t), t).$$

Let $t_n = n\Delta t$, $u_n = u(t_n)$,
$f_n = f(u(t_n), t_n) = \partial u(t_n)/\partial t \equiv u^{(1)}(t_n)$.

▸ notation

$\mathcal{D}$ = the operator $\partial/\partial t$
$u^{(1)} = \mathcal{D}u$, in general $u^{(k)} = \mathcal{D}^k u$

Z is a "time-shift" operator such that

$$Z u(t) = u(t+\Delta t).$$

It follows that $Z u_n = u_{n+1}$.

By Taylor expansion

$$Z u(t) = u(t+\Delta t) = u(t) + \sum_{k=1}^{n} \frac{(\Delta t)^k}{k!} \mathcal{D}^k u(t)$$

$$= \left[\sum_{k=0}^{n} \frac{(\Delta t)^k}{k!} \mathcal{D}^k \right] u(t)$$

$$= \exp(\Delta t \mathcal{D}) u(t)$$

thus: $\underline{Z = \exp(\Delta t \mathcal{D})}$.

▶ <u>solution</u>

Using our notation, the LMF can be rewritten as

$$\boxed{\rho(Z) u_n - \Delta t \, \sigma(Z) f_n = 0}$$

The LMF will approximate the ODE if it is approximately true for $\underline{\underline{f_n = \mathcal{D} u_n}}$!

So, we want to examine the error

$$\varepsilon_n = \rho(Z) u_n - \Delta t\, \sigma(Z) \mathcal{D} u_n$$
$$= [\rho(Z) - \Delta t\, \sigma(Z) \mathcal{D}]\, u_n$$

We define the LMF operator $\mathcal{L}$ as

$$\boxed{\mathcal{L} = \rho(Z) - \Delta t\, \sigma(Z) \mathcal{D}}$$

This operator can be expanded as:

$$\mathcal{L} = C_0 + C_1 (\Delta t \mathcal{D}) + C_2 (\Delta t \mathcal{D})^2 + \cdots$$

and consequently the error is:

$$\varepsilon_n = C_0 u_n + C_1 \Delta t\, u_n^{(1)} + C_2 \Delta t^2 u_n^{(2)} + \cdots$$
$$= C_0 u_n + \sum_{k=1}^{+\infty} C_k (\Delta t)^k u_n^{(k)}.$$

If $C_0 = 0$, $C_1 = 0$, then the error vanishes as $O(\Delta t^2)$! We now give the following defs:

Def : The LMF is

consistent $\Longleftrightarrow$ $C_0 = C_1 = 0$
order p $\Longleftrightarrow$ $C_0 = C_1 = \dots = C_p = 0$

Calculation of C_k

We calculate separately

$$\rho(z) = \rho(e^{\Delta t \lambda}) = \sum_{k=0}^{s} a_k (e^{\Delta t \lambda})^k =$$

$$= \sum_{k=0}^{s} a_k e^{k \Delta t \lambda}$$

$$= \sum_{k=0}^{s} a_k \left[\sum_{m=0}^{+\infty} \frac{(k \Delta t \lambda)^m}{m!} \right]$$

$$= \sum_{m=0}^{+\infty} \left[\sum_{k=0}^{s} \frac{a_k k^m}{m!} \right] (\Delta t \lambda)^m$$

Likewise we find:

$$\Delta t\,\sigma(z)\lambda = \Delta t\,\sigma(e^{\Delta t \lambda})\lambda =$$

$$= \Delta t \left[\sum_{k=0}^{s} b_k (e^{\Delta t \lambda})^k \right] \lambda$$

$$= \Delta t \left[\sum_{k=0}^{s} b_k e^{k \Delta t \lambda} \right] \lambda$$

$$= \Delta t \left[\sum_{k=0}^{s} b_k \left(\sum_{m=0}^{+\infty} \frac{(k \Delta t \lambda)^m}{m!} \right) \right] \lambda$$

$$= \sum_{m=0}^{+\infty} \left[\sum_{k=0}^{s} b_k \frac{k^m}{m!} \right] (\Delta t \lambda)^{m+1}$$

$$= \sum_{m=1}^{+\infty} \left[\sum_{k=0}^{s} b_k \frac{k^{m-1}}{(m-1)!} \right] (\Delta t \lambda)^m$$

It follows that

$$\mathcal{L} = \rho(z) - \Delta t\,\sigma(z)\lambda$$

$$= \sum_{m=0}^{+\infty} \left[\sum_{k=0}^{s} \frac{a_k k^m}{m!} \right] (\Delta t \lambda)^m -$$

$$- \sum_{m=1}^{+\infty} \left[\sum_{k=0}^{s} b_k \frac{k^{m-1}}{(m-1)!} \right] (\Delta t \lambda)^m$$

$$= \underbrace{\left[\sum_{k=0}^{s} a_k\right]}_{C_0} + \sum_{m=1}^{+\infty} \underbrace{\left[\sum_{k=0}^{s} \left(\frac{a_k k^m}{m!} - \frac{b_k k^{m-1}}{(m-1)!} \right)\right]}_{C_m} (\Delta t \lambda)^m$$

thus:

$$C_0 = a_0 + a_1 + \cdots + a_s$$

$$C_m = \sum_{k=0}^{s} \left[\frac{k^m}{m!} a_k - \frac{k^{m-1}}{(m-1)!} b_k \right]$$

Conditions for consistency.

An LMF is consistent $\Leftrightarrow$ $\rho(1) = 0 \wedge \rho'(1) = \sigma(1)$.

Proof

Note that

$$C_0 = a_0 + a_1 + \cdots + a_s = a_0 1^0 + a_1 1^1 + \cdots + a_s 1^s$$
$$= \rho(1)$$

and

$$
\begin{aligned}
C_1 &= (a_1 + 2a_2 + \cdots + s a_s) - (b_0 + \cdots + b_s) \\
&= (a_1 1 + 2a_2 \cdot 1^2 + \cdots + s a_s 1^s) - (b_0 + \cdots + b_s \cdot 1^s) \\
&= \rho'(1) - \sigma(1).
\end{aligned}
$$

LMF consistent $\Leftrightarrow C_0 = C_1 = 0$

$\Leftrightarrow \rho(1) = 0 \land \rho'(1) = \sigma(1).$ □

<u>example</u> : For Crank-Nicolson

$$u_{n+1} = u_n + \frac{\Delta t}{2} (f_n + f_{n+1})$$

we have $s = 1$, $\rho(z) = z - 1$, $\sigma(z) = \frac{1}{2}(z+1)$

Since $\rho(1) = 1 - 1 = 0$

$\left.\begin{array}{l} \rho'(z) = 1 \Rightarrow \rho'(1) = 1 \\ \sigma(1) = \frac{1}{2}(1+1) = 1 \end{array}\right\} \Rightarrow \rho'(1) = \sigma(1)$

the Crank-Nicolson LMF is consistent.

Conditions for accuracy.

The coefficients C_k were defined by expanding

$$\mathcal{L} = \rho(e^{\Delta t \lambda}) - \Delta t \, \sigma(e^{\Delta t \lambda}) \lambda$$

$$= C_0 + C_1 (\Delta t \lambda) + C_2 (\Delta t \lambda)^2 + \cdots$$

Equivalently we have the algebraic statement:

$$\rho(e^x) - x\sigma(e^x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

and it follows that

LMF has order of accuracy p $\iff$ $C_0 = C_1 = C_2 = \cdots = C_p$

$$\iff \rho(e^x) - x\sigma(e^x) = O(x^{p+1}), \quad x \to 0$$

$$\iff \frac{\rho(e^x)}{\sigma(e^x)} - x = O(x^{p+1}), \quad x \to 0$$

$$\iff \frac{\rho(z)}{\sigma(z)} = \log z + O((z-1)^{p+1}), \quad z \to 1$$

$$O((z-1)^p)$$

The main result is

LMF has order of accuracy p $\Leftrightarrow$ $$\boxed{\frac{\rho(z)}{\sigma(z)} = \log z + O((z-1)^{p+1})}$$

To apply this result we use Taylor expansion

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5}$$

<u>example</u> : For the Crank-Nicolson rule we get order 2:

$$\left.\begin{aligned} \rho(z) &= z-1 \\ \sigma(z) &= \frac{1}{2}(z+1) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \frac{\rho(z)}{\sigma(z)} = \frac{z-1}{\frac{1}{2}(z+1)} = \frac{z-1}{1+\frac{1}{2}(z-1)} =$$

$$= (z-1)\left[1 - \frac{z-1}{2} + \frac{(z-1)^2}{4} - \dots\right]$$

$$= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{4} - \dots$$

$$= \log z - \frac{1}{12}(z-1)^2 + O((z-1)^3)$$

- Of course, one may also calculate the coefficients directly.

Order Stars

This result motivates the definition of order stars:

Define $\varphi(z) = \frac{\rho(z)}{\sigma(z)} - \log z$

The order star $A[\rho,\sigma]$ for the corresponding LMF is the subset

$$\boxed{A[\rho,\sigma] = \{ z \in \mathbb{C} \mid \operatorname{Re} \varphi(z) > 0 \}}$$

- For an LMF with order p we have

$$\varphi(z) = C(z-1)^{p+1} + O((z-1)^{p+2})$$

In the neighborhood of $z=1$ we expect $p+1$ "fingers" sticking out. (some bounded, some unbounded).

Stability and Convergence of LMFs

There are two distinct notions of stability

1) If t is fixed, does the numerical evaluation of $u(t)$ remain bounded in the limit $\Delta t \to 0$?
2) If Δt fixed, does the numerical evaluation of $u(t)$ remain bounded in the limit $t \to +\infty$?

Stability and Convergence

Consider the homogeneous part of an LMF:

$$\rho(Z) u_n = 0.$$

The general solution of this equation is

$$u_n = A_1 f_1(n) + A_2 f_2(n) + \dots + A_s f_s(n).$$

where

a) A simple root ρ of $\rho(x) = 0$ contributes
$f_k(n) = \rho^n$

b) A root ρ with multiplicity m contributes
$f_k(n) = \rho^n$, $f_{k+1}(n) = n\rho^n$, $f_{k+2} = n^2 \rho^n$
$f_{k+m-1} = n^{m-1} \rho^m$

If $\rho(z)$ has a root $|z|>1$ or a root $|z|=1$ with multiplicity $m>1$, then the equation

$$\rho(z)\, u_n - \Delta t\, \sigma(z) \mathcal{A} f_n = 0$$

has a "parasitic" mode that is excited by numerical errors and overtakes the approximation of the ODE! Thus,

Def: An LMF is stable iff all the roots z of $\rho(z)=0$ are either

a) $|z|<1$

OR b) $|z|=1$ with multiplicity 1.

Def: An LMF is convergent iff, the numerical evaluation $u_N(t,\Delta t)$ of $u(t)$ with timestep Δt satisfies

$$\lim_{\Delta t \to 0} u_N(t,\Delta t) = u(t)$$

for all initial value problems.

Thm: (Dahlquist Equivalence theorem)
An LMF is convergent iff it is consistent and stable.

Construction of LMFs

① Adams-Bashforth methods

They are LMF that take the form

$$u_{n+s} - u_{n+s-1} = \Delta t \sum_{k=0}^{s-1} b_k f_{n+k}$$

The first 4 AB methods are

1) $u_{n+1} = u_n + \Delta t f_n$

2) $u_{n+2} = u_{n+1} + \Delta t \left(\frac{3}{2} f_{n+1} - \frac{1}{2} f_n \right)$

3) $u_{n+3} = u_{n+2} + \Delta t \left(\frac{23}{12} f_{n+2} - \frac{16}{12} f_{n+1} + \frac{5}{12} f_n \right)$

4) $u_{n+4} = u_{n+3} + \Delta t \left(\frac{55}{24} f_{n+3} - \frac{59}{24} f_{n+2} + \frac{37}{24} f_{n+1} - \frac{9}{24} f_n \right)$

→ General AB formula

Let $\Delta = z - 1$ = forward diff. operator

$\nabla = 1 - z^{-1}$ = backward diff. operator

such that, for example

$$\Delta u_n = u_{n+1} - u_n$$

$$\Delta^2 u_n = u_{n+2} - 2u_{n+1} + u_n$$

$$\nabla u_n = u_n - u_{n-1}$$

The general AB LMF is

$$\boxed{u_{n+s} = u_{n+s-1} + \Delta t \sum_{k=0}^{s-1} \gamma_k \nabla^k f_{n+s-1}}$$

with γ_k given by

$$\gamma(t) = \frac{-t}{(1-t)\log(1-t)} = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \dots$$

$$= 1 - \frac{1}{2} t + \frac{5}{12} t^2 + \frac{3}{8} t^3 + \frac{251}{720} t^4 + \dots$$

It can also be shown that

$$\boxed{\sum_{k=0}^{m} \frac{1}{m+1-k} \gamma_k = 1 \,.,\ \forall\, m \geq 0}$$

Since $\rho(z) = z^{s} - z^{s-1} = z^{s-1}(z-1)$
it follows that Adams-Bashforth schemes are stable
for all $s \geq 1$

② Adams-Moulton formulas

They are LMFs of the form

$$u_{n+s} - u_{n+s-1} = \Delta t \sum_{k=0}^{s} b_k f_{n+k}$$

The first few Adams-Moulton formulas are

1) $u_{n+1} - u_n = \Delta t f_{n+1}$

2) $u_{n+1} - u_n = \Delta t \left(\frac{1}{2} f_{n+1} + \frac{1}{2} f_n \right)$

3) $u_{n+2} - u_{n+1} = \Delta t \left(\frac{5}{12} f_{n+2} + \frac{8}{12} f_{n+1} - \frac{1}{12} f_n \right)$

4) $u_{n+3} - u_{n+2} = \Delta t \left(\frac{9}{24} f_{n+3} + \frac{19}{24} f_{n+2} - \frac{5}{24} f_{n+1} + \frac{1}{24} f_n \right)$

The general Adams-Moulton formula is given by

$$u_{n+s} = u_{n+s-1} + \Delta t \sum_{k=0}^{s} \gamma_k^* \nabla^k f_{n+s}$$

with the coefficients γ_k^* given by

$$\gamma^*(t) = \frac{-t}{\log(1-t)} = \gamma_0^* + \gamma_1^* t + \gamma_2^* t^2 + \cdots$$

$$= 1 - \frac{1}{2} t - \frac{1}{12} t^2 - \frac{1}{24} t^3 - \frac{19}{720} t^4 \cdots$$

or equivalently by the recurrence

$$\begin{cases} \displaystyle\sum_{k=0}^{m} \frac{1}{m+1-k} \gamma_k^* = 0 \,, \quad \forall\ m \geq 1 \\ \gamma_0^* = 1. \end{cases}$$

- The Adams-Moulton LMFs. are stable for all $s > 1$.

③ Backwards Differentiation Formulas

In these formulas we choose $\sigma(z) = z^s$

The general BDF is

$$\boxed{\sum_{k=1}^{s} \frac{1}{k} \nabla^k u_{n+s} = \Delta t\, f_{n+s}}$$

Some special cases are:

1) $u_n - u_{n-1} = \Delta t f_n$

2) $u_n - \frac{4}{3} u_{n-1} + \frac{1}{3} u_{n-2} = \frac{2}{3} \Delta t f_n$

3) $u_n - \frac{18}{11} u_{n-1} + \frac{9}{11} u_{n-2} - \frac{2}{11} u_{n-3} = \frac{6}{11} \Delta t f_n.$

▸ BDF formulas are stable iff $\boxed{1 \leq s \leq 6}$!!!!

BDF formulas with $s \geq 7$ are unstable!

Analytic construction of optimal LMFs

① Explicit methods

Form:

$$\rho(Z)\,u_n = \Delta t \sum_{k=0}^{s-1} \gamma_k \nabla^k f_{n+s-1}$$

with $\deg \rho(Z) = s$

- 1 Let $t = 1 - z^{-1}$ (Canalogous to $\nabla = 1 - Z^{-1}$) and write

$$\left.\begin{aligned} \rho(z) &= z^s R(t) = z^{s-1}\frac{R(t)}{1-t} \\ \sigma(z) &= z^{s-1} S(t) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \frac{\rho(z)}{\sigma(z)} = \frac{R(t)}{(1-t)\,S(t)} \leftarrow \text{ ~~is~~ independent of } s\ !!$$

- 2 Use $\log z = \log \frac{1}{1-t} = -\log(1-t)$

  and solve

$$\frac{R(t)}{(1-t)S(t)} = -\log(1-t)$$

$$\Leftrightarrow \boxed{S(t) = \frac{-R(t)}{(1-t)\log(1-t)}}$$

$$= \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \dots + \gamma_k t^k + \dots$$

<u>example</u> : Adams-Bashforth.

$$\rho(z) = z^s - z^{s-1} = z^s(1-z^{-1}) = z^s t = z^{s-1}\frac{t}{1-t}$$

$\sigma(z) = z^s S(t)$, thus

$$\frac{\rho(z)}{\sigma(z)} = \frac{t/(1-t)}{S(t)} = -\log(1-t)$$

$$\Leftrightarrow S(t) = \frac{-t}{(1-t)\log(1-t)}$$

② <u>Optimal implicit LMFs</u>

Form:

$$\rho(Z)u_n = \Delta t \sum_{k=0}^{s} \gamma_k \nabla^k f_{n+s}$$

with $\deg \rho(Z) = s$.

- 1 Again we let $t = 1 - z^{-1}$ and

$\rho(z) = z^s R(t)$ and $\sigma(z) = z^s S(t)$.

- 2 Then

$$\left.\begin{array}{l} \dfrac{\rho(z)}{\sigma(z)} = \dfrac{R(t)}{S(t)} \\ \log z = -\log(1-t) \end{array}\right\} \Rightarrow \frac{R(t)}{S(t)} = -\log(1-t)$$

$$\Rightarrow \boxed{S(t) = \frac{-R(t)}{\log(1-t)}}$$

<u>example</u> : Adams – Moulton

$$\rho(z) = z^s - z^{s-1} = z^s(1 - z^{-1}) = z^s t \Rightarrow R(t) = t$$

$$\Rightarrow S(t) = \frac{-t}{\log(1-t)} \leftarrow \text{gives } \gamma_n^*$$

③ Derivation of BDFs

To derive the BDF:

$$\sum_{k=1}^{s} \frac{1}{k} \nabla^k u_{n+s} = \Delta t f_{n+s}$$

we set $\sigma(z) = z^s$ and calculate $\rho(z)$:

$$\frac{\rho(z)}{\sigma(z)} = \frac{\rho(z)}{z^s} = \log z = -\log z^{-1} =$$

$$= -\left[(z^{-1}-1) - \frac{1}{2}(z^{-1}-1) + \frac{1}{3}(z^{-1}-1) - \dots \right]$$
$$+ O((z-1)^{p+1})$$

$$= \left[(1-z^{-1}) + \frac{1}{2}(1-z^{-1}) + \frac{1}{3}(1-z^{-1}) + \dots \right]$$
$$+ O((z-1)^{p+1})$$

$$\Rightarrow \rho(z) = z^s \left[t + \frac{1}{2} t^2 + \frac{1}{3} t^3 + \dots \right] + O((z-1)^{p+1})$$

We choose:

$$\rho(z) = \left[t + \frac{1}{2} t^2 + \frac{1}{3} t^3 + \dots + \frac{1}{s} t^s \right] z^s.$$

which gives the BDF.

Dahlquist stability barrier

It is possible for an s-step LMF to have order of accuracy $p = 2s$. However many such LMFs are ruled out because they are unstable.

Thm: If an LMF is stable with s steps and p order of accuracy, then

$$p \leq \begin{cases} s+2, & \text{if } s \text{ even} \\ s+1, & \text{if } s \text{ odd} \\ s, & \text{if LMF explicit} \end{cases}$$

A geometric proof is possible using order-stars.

Stability regions

If Δt is fixed, does LMF remain stable in the limit $t \to +\infty$? ← Eigenvalue stability.

One "estimates" eigenvalue stability by assessing the LMF against the problem

$$\frac{\partial u}{\partial t} = au \quad , \quad \text{with } a \in \mathbb{C}$$

The corresponding LMF is:

$$\rho(z)u_n - \Delta t\, \sigma(z)\, a u_n = 0 \Leftrightarrow$$

$$\Leftrightarrow [\rho(z) - a\Delta t\, \sigma(z)]\, u_n = 0 .$$

Define the stability polynomial

$$\boxed{\pi(z) = \rho(z) - a\Delta t\, \sigma(z)}$$

<u>Def</u> : An LMF is <u>absolutely stable</u> for a given $a\Delta t$ iff all the zeroes of $\pi(z)$ satisfy

a) $|z| < 1$

OR b) $|z| = 1$ with z simple

<u>Def</u> : The stability region S of an LMF is the set of all $a\Delta t$ such that the LMF is absolutely stable.

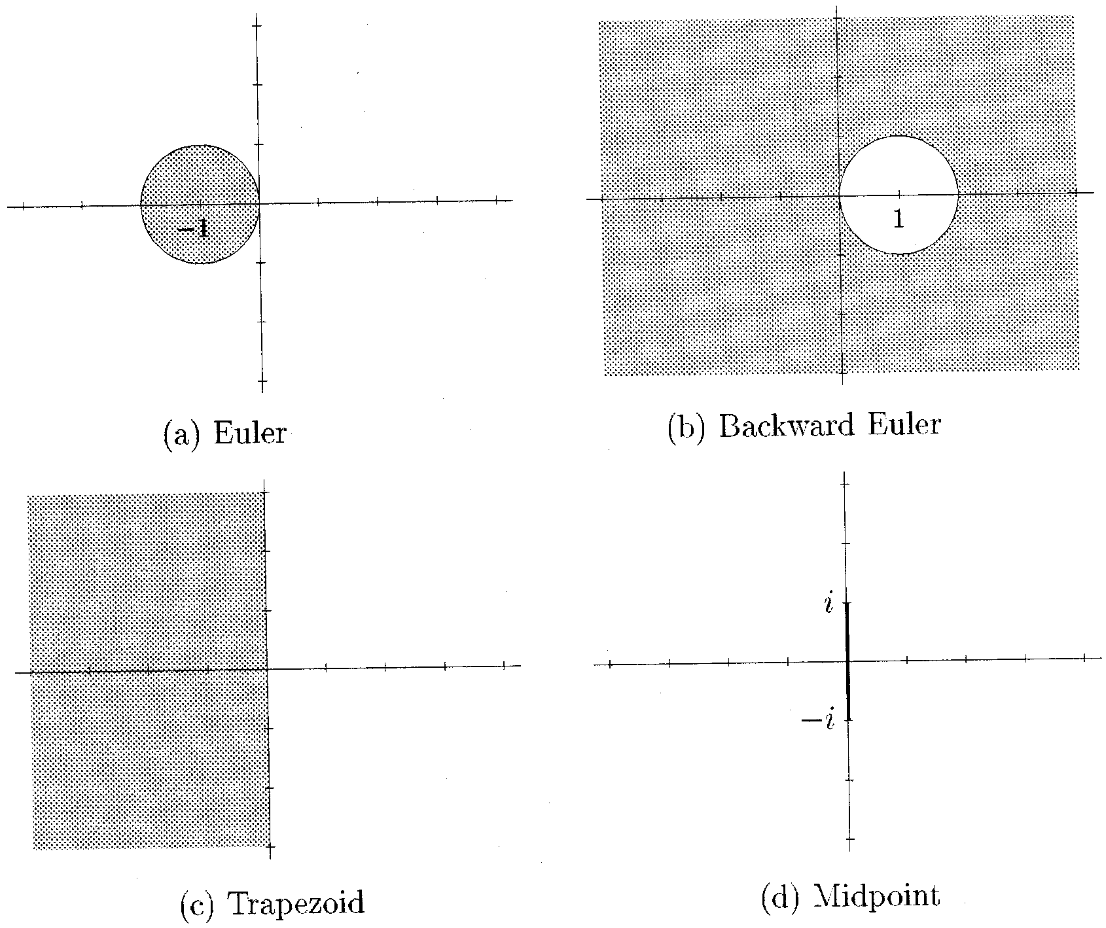

Figure 1.7.1. Stability regions (shaded) for four linear multistep formulas. In case (d) the stability region is the open complex interval $(-i, i)$.

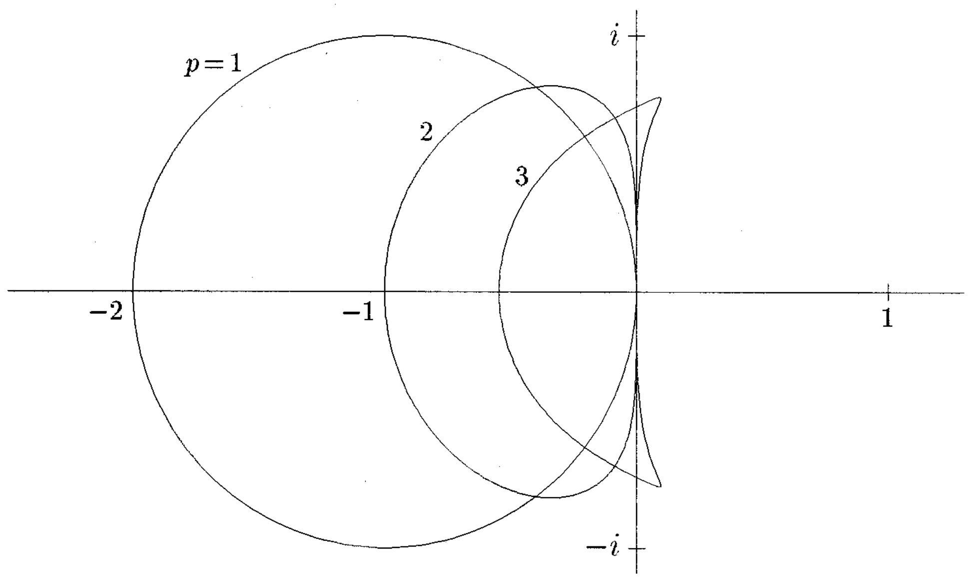

Figure 1.7.2. Boundaries of stability regions for Adams-Bashforth formulas of orders 1–3.

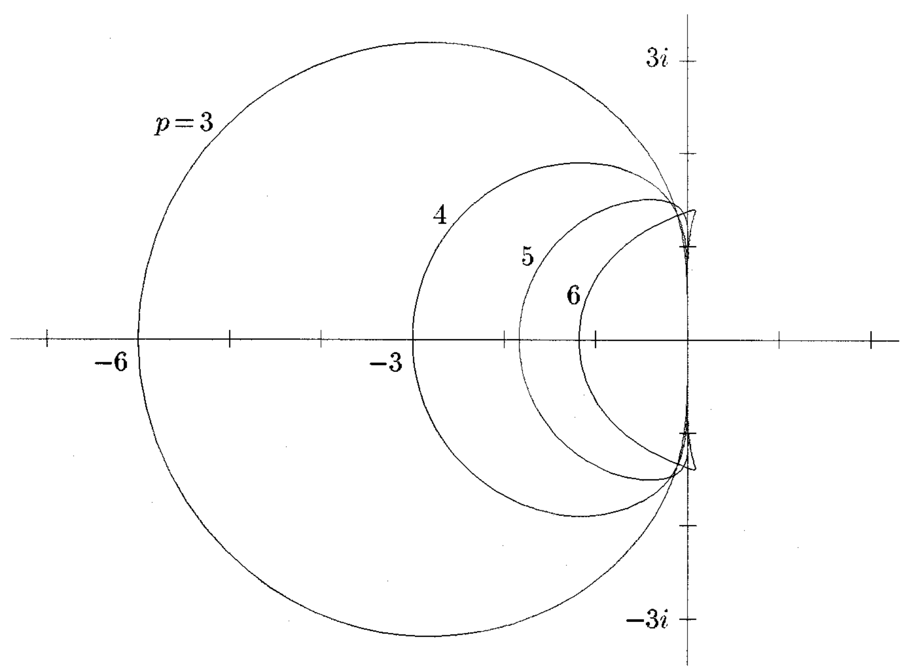

Figure 1.7.3. Boundaries of stability regions for Adams-Moulton formulas of orders 3–6. (Orders 1 and 2 were displayed already in Figure 1.7.1(b,c).) Note that the scale is very different from that of the previous figure.

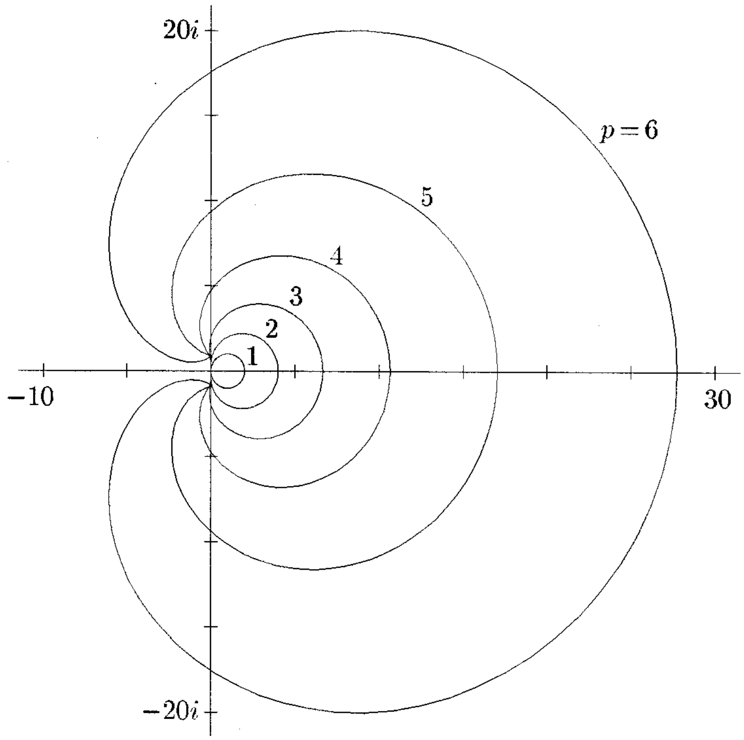

Figure 1.7.4. Boundaries of stability regions for backwards differentiation formulas of orders 1–6 (exteriors of curves shown).

Stiff ODEs

An ODE is stiff if the stepsize required to maintain stability is much smaller than that required to maintain accuracy.

→ Characteristic features of stiff problems

1) The choice of timestep is dictated by stability, not accuracy.
2) The problem is characterized by widely varying timescales.
3) Explicit methods don't work.

example : $\partial u/\partial t = -100(u - \cos t) - \sin t$

$u(0) = 1$

General solution is

$$u(t) = A\exp(-100t) - \sin t$$

two different timescales

- Stiff problems require an LMF with stability region that includes as much of

$$\{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$$

as possible.

Def: An LMF is A-stable iff
$\{z \in \mathbb{C} \mid \operatorname{Re} z < 0\} \subseteq S$

Def: An LMF is A(a)-stable iff
$\{z \in \mathbb{C} \mid \pi - a < \operatorname{Arg} z < \pi + a\} \subseteq S$

Def: An LMF is A(0)-stable iff
$\exists\, a > 0$: LMF A(a)-stable

1) An LMF which is A-stable must satisfy $p < 2$

2) Backward Euler, Crank-Nicholson } A-stable.

3) Backward Diff. Formulas are A(0)-stable for $p \leq 6$

- An A-stable method will perform well on any stiff problem → price: small timesteps.
- An A(0)-stable method will perform well on stiff problems with no oscillatory modes (i.e. $\forall \lambda \in sp(J(t)) : \operatorname{Im} \lambda = 0$)

The eigenvalue stability of Non-linear ODEs is determined with linearization:

For $\partial u/\partial t = f(u,t)$

- 1 Linearize:

$$\partial u/\partial t = J(t)\,u + \cdots$$

with

$$[J(t)]_{ab} = \frac{\partial f_a(u(t),t)}{\partial u_b}$$

- 2 Freeze coefficients
- 3 Find eigenvalues of $sp(J(t))$.

Condition for eigenvalue stability:

$$\boxed{\forall \lambda \in sp(J(t)) : \lambda \Delta t \in S}$$

<u>WARNING</u>: This condition is "rule of thumb", not rigorous.