INTRODUCTION

Autonomous dynamical systems

- An autonomous dynamical system is a system of n differential equations of the form:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3, \dots, x_n) \\ \dot{x}_2 = f_2(x_1, x_2, x_3, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, x_3, \dots, x_n) \end{cases}$$

▸ notation: $\dot{x}_K = dx_K/dt = x_K'(t)$.

- The system can be also rewritten as:

$$\dot{x} = f(x)$$

with $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

- We assume that an initial value condition is given at $t = 0$: $x(0) = x_0$, with $x_0 \in \mathbb{R}^n$.

- Classification of autonomous systems

a) Linear Autonomous systems: These are systems where $f(x) = Ax$ with $A \in GL(n, \mathbb{R})$. Note that

$GL(n,\mathbb{R})$ is the set of all nonsingular $n \times n$ matrices.

b) Nonlinear autonomous systems: These are systems where $f(x)$ is nonlinear.

- <u>Jacobian matrix</u>

The Jacobian matrix of the autonomous system $\dot{x} = f(x)$ is defined as

$$\boxed{[Df]_{ab} = \frac{\partial f_a}{\partial x_b}}$$

Note that for a linear system with $f(x) = Ax$ we have $Df = A$.

- <u>Systems reducible to autonomous</u>

a) <u>High-order ODEs</u> $\longrightarrow$ $\boxed{y^{(n)} = F(y, y', y'', \dots, y^{(n-1)})}$

We let: $x_1 = y$, $x_2 = y'$, $x_3 = y''$, $\dots$, $x_n = y^{(n-1)}$.

<u>EXAMPLE</u>

$\ddot{x} - b\dot{x} + kx = 0$ (linear oscillator)

Let $x_1 = x$ and $x_2 = \dot{x} \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = bx_2 - kx_1 \end{cases}$

b) Time-dependent system

A time-dependent system of the form

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 = f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

can be rewritten as an autonomous system by setting $x_0 = t$. Then:

$$\begin{cases} \dot{x}_0 = 1 \\ \dot{x}_1 = f_1(x_0, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 = f_2(x_0, x_1, x_2, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_0, x_1, x_2, \ldots, x_n) \end{cases}$$

EXERCISES

(1) Rewrite the following ODEs as autonomous systems:

a) $\ddot{y} + a\dot{y} + by = 0$

b) $\dddot{y} - a\dot{y} = b\ddot{y}$

c) $\ddot{y} = y\dot{y}$

d) $\dddot{y} = \dot{y}\,(y + \ddot{y})$

e) $\ddot{y} + (1+t^2)\dot{y} + 2ty = 0$

f) $\ddot{y} - \dot{y} + ty = 0$

g) $\dddot{y} + 3t\dot{y} - t^2 y = 0$

h) $\ddot{y} - ty = 0$.

(2) Evaluate the Jacobian matrix for the following autonomous systems

a) $$\begin{cases} \dot{x}_1 = 3x_1 - 2x_2 \\ \dot{x}_2 = 5x_1 + 3x_2 \end{cases}$$

b) $$\begin{cases} \dot{x}_1 = 2x_1 - x_2(x_1 - x_2) \\ \dot{x}_2 = -x_2 + x_1(2x_2 - x_1) \end{cases}$$

c) $$\begin{cases} \dot{x}_1 = x_1^2(x_1 - x_2)^3 \\ \dot{x}_2 = x_2^3(x_1 - x_2)^2 \end{cases}$$

d) $$\begin{cases} \dot{x}_1 = \cos x_1 \sin x_2 \\ \dot{x}_2 = \cos x_2 \sin x_1 \end{cases}$$

Existence and Uniqueness

Consider the problem

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

with $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and $f: \mathbb{R}^n \to \mathbb{R}^n$

We also define

$$\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

with $x = (x_1, x_2, \dots, x_n)$ a vector in $\mathbb{R}^n$.

- Definition:

f Lipschitz continuous in $A \Leftrightarrow$

$\Leftrightarrow \exists L > 0 : \forall x, y \in A : \|f(x) - f(y)\| \leq L\|x - y\|$

with L = Lipschitz constant of f.

If $L < 1 \Rightarrow f$ is a contraction

- Proposition: Let $f: \mathbb{R}^n \to \mathbb{R}^m$. Then

a) $\left.\begin{array}{l} f \text{ differentiable in } A \\ \nabla f \text{ bounded in } A \end{array}\right\} \Rightarrow f$ Lipschitz continuous in A.

b) $\left.\begin{array}{l} f \text{ differentiable in } A \\ f \text{ Lipschitz continuous in } A \end{array}\right\} \Rightarrow \nabla f$ bounded in A

c) $\left.\begin{array}{l} f \text{ differentiable in } A \\ \nabla f \text{ not bounded in } A \end{array}\right\} \Rightarrow f$ not Lipschitz continuous in A.

↳ Note that (c) is the contrapositive of (b).

• <u>Theorem</u> : Assume that

a) f Lipschitz continuous in $B(x_0, \delta)$ with Lipschitz constant L, where $\delta > 0$ and $B(x_0, \delta) = \{ x \in \mathbb{R}^n \mid \|x - x_0\| < \delta \}$

b) $\forall x \in B(x_0, \delta) : \|f(x)\| \leq M$

Then $\dot{x} = f(x)$ with $x(t_0) = x_0$ has a unique solution for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ as long as $0 < \varepsilon < \min(1/L, \delta/M)$

• <u>Theorem</u> : Assume that

a) f Lipschitz continuous in $B(x_0, \delta)$ with Lipschitz constant L

b) y solution for $x(t_0) = y_0$ and z solution for $x(t_0) = z_0$ defined for $t \in [t_0, t_1]$

Then :

$$\boxed{\forall t \in [t_0, t_1] : \|y(t) - z(t)\| \leq \|y_0 - z_0\| e^{L(t - t_0)}}$$

↳ The divergence between two solutions with nearby initial conditions do not grow apart at a faster than exponential rate.

↳ Note that the existence of the unique solution is not guaranteed for infinite time.

EXAMPLES

a) Show that $f(x) = 2x+3$, $\forall x \in \mathbb{R}$ is Lipschitz continuous in $\mathbb{R}$.

Solution

1st method: By definition.

Let $x, y \in \mathbb{R}$ be given. Then

$$
\begin{aligned}
|f(x) - f(y)| &= |(2x+3) - (2y+3)| = |2x+3-2y-3| = \\
&= |2x-2y| = |2(x-y)| = |2||x-y| \\
&= 2|x-y| \Rightarrow |f(x)-f(y)| \leq 2|x-y|.
\end{aligned}
$$

It follows that

$\forall x, y \in \mathbb{R}: |f(x) - f(y)| \leq 2|x-y| \Rightarrow$

$\Rightarrow f$ Lipschitz continuous in $\mathbb{R}$.

2nd method: By proposition

f differentiable in $\mathbb{R}$ with $f'(x) = (2x+3)' = 2$, $\forall x \in \mathbb{R}$ (1)

Since:

$\forall x \in \mathbb{R}: (|f'(x)| = |2| = 2) \Rightarrow \forall x \in \mathbb{R}: (|f'(x)| \leq 2) \Rightarrow$

$\Rightarrow f'$ bounded in $\mathbb{R}$ (2)

From (1) and (2) it follows that f is Lipschitz continuous in $\mathbb{R}$.

b) Show that $f(x) = x^{2/3}$, $\forall x \in (0, +\infty)$ is not Lipschitz continuous on $(0, +\infty)$

Solution

f differentiable in $(0,+\infty)$ with

$$f'(x) = (x^{2/3})' = (2/3)x^{2/3-1} = (2/3)x^{-1/3} =$$

$$= \frac{2}{3\sqrt[3]{x}}, \quad \forall x \in (0,+\infty) \qquad (1)$$

However, since:

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{2}{3\sqrt[3]{x}} = +\infty \Rightarrow$$

$\Rightarrow f'$ not bounded in $(0,+\infty)$ (2)

From (1) and (2), it follows that f not Lipschitz continuous in $(0,+\infty)$.

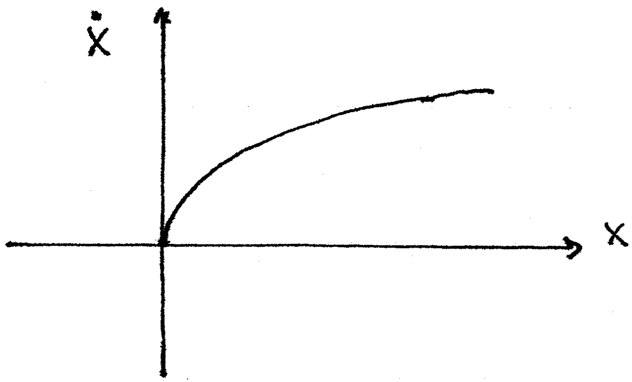
→ Examples on existence and uniqueness

c) $\dot{x} = 1 + x^2$

- We can show, using standard ODE techniques, that $\dot{x} = 1 + x^2 \Leftrightarrow x(t) = \tan(t+c)$ with c dependant on the initial condition. Obviously, a solution <u>exists</u>. Furthermore, the "if and only if" ($\Leftrightarrow$) ensures the <u>uniqueness</u> of this solution. However, we note that the solution has a singularity occuring at finite time when $t+c = k\pi + \pi/2$, consequently the existence and uniqueness holds for finite time only.

d) $\begin{cases} \dot{x} = 3x^{2/3} \\ x(0) = 0 \end{cases}$

has two solutions: $x(t) = 0$ and $x(t) = t^3$ (!!)
thus we have existence but not uniqueness.
Note that $f(x) = 3x^{2/3}$ is continuous but not Lipschitz continuous.



The solution $x(t) = 0$ is VERY unstable because the slope of the function $f(x) = 3x^{2/3}$ is infinite at $x = 0$.
This instability manifests in the existence of a second solution, such as $x(t) = t^3$ (there is in fact an infinite set of such solutions). From a physical standpoint, $x(t) = 0$ is the solution one would expect if we initialize with $x(0) = 0$.
The lack of uniqueness indicates that the system could spontaneously break into the second solution $x(t) = t^3$ at time $t = 0$ if there is even an infinitesimal deviation in the initial condition, thus giving a 3rd solution:

$$x(t) = \begin{cases} t^3 & , \text{ if } t \geq 0 \\ 0 & , \text{ if } t < 0 \end{cases}$$

that combines the previous two solutions. Note that this spontaneous break can just as well occur at any other time t_0. (see homework).

EXERCISES

③ Show that the following functions are Lipschitz continuous in $\mathbb{R}$:

a) $f(x) = ax + b$ with $a \neq 0$.

b) $f(x) = \sin x$

c) $f(x) = |x|$ $\longrightarrow$ Note that it is not differentiable in $\mathbb{R}$. (fails at $x = 0$)

d) $f(x) = \sqrt{x^2 + a}$ with $a > 0$

④ Show that the following functions are not Lipschitz continuous.

a) $f(x) = x^2$ in $\mathbb{R}^2$

b) $f(x) = \sqrt{x}$ in $[0, +\infty)$

c) $f(x) = x^a$ in $[0, +\infty)$ with $0 < a < 1$.

⑤ Consider the system $\dot{x} = x^a$ with $a \in (0,1)$ and with initial condition $x(0) = 0$. Show that this system has an infinite set of solutions.

⑥ a) Consider the system $\dot{x} = x^a$ with $a > 1$ and $x(0) = x_0 > 0$
Show that $x(t)$ becomes infinite at finite time.
What happens when $a = 1$.

b) Show that the solution of $\dot{x} = x^a + b$ with $a > 1$ and $b > 0$ and $x(0) = x_0 > 0$ also becomes infinite at finite time.

Fixed points and stability

- Consider the autonomous system $\dot{x} = f(x)$, with $f: \mathbb{R}^n \to \mathbb{R}^n$. We say that
  x_0 is a <u>fixed point</u> $\Leftrightarrow f(x_0) = 0$

- If x_0 is a fixed point, then $\dot{x} = f(x)$ with initial condition $x(t_0) = x_0$ has solution $x(t) = x_0$. Thus if we start <u>at</u> a fixed point, we will stay at the fixed point. The question of stability concerns what happens when we start with an initial condition <u>near</u> a fixed point.

- <u>Stability of fixed points</u>

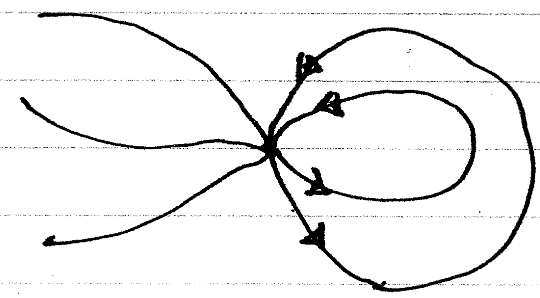
Let $x_0 \in \mathbb{R}^n$ be a fixed point of $\dot{x} = f(x)$.

1. x_0 Lyapunov stable $\Leftrightarrow$
$$\forall \varepsilon > 0 : \exists \delta > 0 : \big(\|x(t_0) - x_0\| < \delta \Rightarrow$$
$$\Rightarrow (\forall t > t_0 : \|x(t) - x_0\| < \varepsilon)\big)$$

2. x_0 attracting $\Leftrightarrow$
$$\exists \delta > 0 : \left(\|x(t_0) - x_0\| < \delta \Rightarrow \lim_{t \to +\infty} x(t) = x_0 \right)$$

In a Lyapunov stable fixed point, solutions that start near the fixed point will stay near the fixed point. In an attracting fixed point, solutions that start near the fixed point will eventually converge into the fixed point.

Note that it is possible for a fixed point to be attractive without being Lyapunov stable, as in the following example:



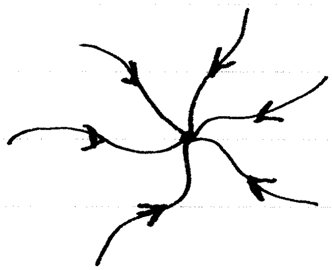
This occurs when there are trajectories that start near the fixed point, then wander far away from the fixed point before returning back to the fixed point for a final approach. This remark motivates the following additional definitions:

③ x_0 asymptotically stable $\Leftrightarrow \begin{cases} x_0 \text{ Lyapunov stable} \\ x_0 \text{ attracting.} \end{cases}$

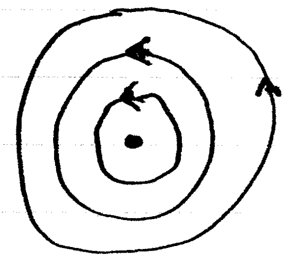
④ x_0 neutrally stable $\Leftrightarrow \begin{cases} x_0 \text{ Lyapunov stable} \\ x_0 \text{ not attracting} \end{cases}$

⑤ x_0 unstable $\Leftrightarrow \begin{cases} x_0 \text{ not Lyapunov stable} \\ x_0 \text{ not attracting.} \end{cases}$

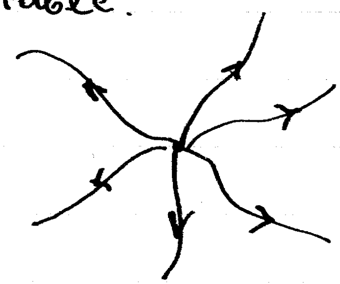
Examples of definitions



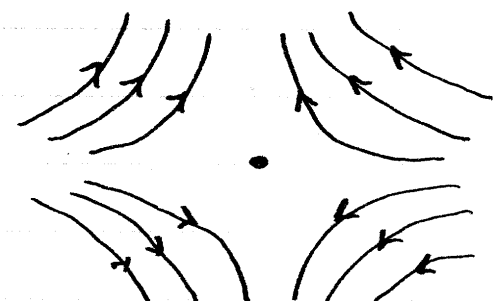
asymptotically stable.



neutrally stable.



unstable (source)



unstable (saddle point)

The distinction between sources and saddle points will be explained later.

⑥ x_0 is <u>exponentially stable</u> if and only if

a) x_0 is asymptotically stable AND

b) $\exists\, a, b, \delta \in (0, +\infty) : \big(\|x(t_0) - x_0\| < \delta \Rightarrow$

$\Rightarrow \big(\forall\, t > t_0 : \|x(t) - x_0\| \leqslant a e^{-bt} \|x(t_0) - x_0\| \big)\big)$

$\blacktriangledown$ Lyapunov functions

Let $\dot{x} = f(x)$ be an autonomous system with $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let x_0 be a fixed point such that $f(x_0) = 0$.

Def: We say that a function $V: A \longrightarrow \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ an open set is a Lyapunov function if it satisfies the following conditions:

a) $V(x_0) = 0$

b) $V(x) > 0, \ \forall x \in A - \{x_0\}$

c) V continuous in A

d) $x(t_0) \in A \Rightarrow \forall t > t_0: V(x(t)) \leq V(x(t_0))$

- The domain A of the Lyapunov function is called a trapping region of the autonomous system.

Thm: (1st Lyapunov Theorem)

If

a) $f(x_0) = 0$

b) There is a Lyapunov function $V: A \to \mathbb{R}$ with $V(x_0) = 0$

Then $x = x_0$ is Lyapunov stable.

Thm: (2nd Lyapunov Theorem)

If:

a) $f(x_0)=0$ with $x_0 \in A$.

b) There is a Lyapunov function $V: A \to \mathbb{R}$ with $V(x_0)=0$

* c) $x(0) \in A \Rightarrow \forall t > t_0 : V(x(t)) < V(x(t_0))$ (!)

Then $x = x_0$ is asymptotically stable.

EXERCISES

⑦ Consider the system

$$\dot{p} = f(p,q)$$
$$\dot{q} = g(p,q)$$

with $f(p,q) = \partial H(p,q)/\partial q$

$g(p,q) = -\partial H(p,q)/\partial p$.

If (p_0,q_0) is a fixed point and if

$H(p,q) > 0$, $\forall (p,q) \in A - \{(p_0,q_0)\}$

with A an open set that contains (p_0,q_0), then show that (p_0,q_0) is Lyapunov stable.

↳ Systems of this type are called Hamiltonian systems.