NONLINEAR AUTONOMOUS SYSTEMS

▼ Local analysis of fixed points

Consider the nonlinear autonomous systems $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.
Let $x_0 \in \mathbb{R}^n$ be a fixed point with $f(x_0) = 0$.
Let $x_0(t) = x_0$ be a solution with the fixed point as initial condition.

To examine the stability of x_0, we consider the following perturbation around x_0:

$$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$$

with $0 < \varepsilon \ll 1$. It follows that:

$$\dot{x}(t) = \dot{x}_0 + \varepsilon \dot{x}_1(t) + O(\varepsilon^2) = \varepsilon \dot{x}_1(t) + O(\varepsilon^2)$$

$$f(x) = f(x_0 + \varepsilon x_1) = f(x_0) + (\varepsilon)\, Df(x_0)\, x_1 + O(\varepsilon^2)$$
$$= \varepsilon\, Df(x_0)\, x_1 + O(\varepsilon^2)$$

Equating the ε terms gives the linearization

$$\boxed{\dot{x}_1(t) = Df(x_0)\, x_1(t)}$$

Here Df is the Jacobian matrix given by

$$\boxed{[Df]_{ab} = \frac{\partial f_a}{\partial x_b}}$$

Def: We say that the fixed point x_0 is a <u>hyperbolic point</u> if and only if

$$\boxed{\forall \lambda \in \lambda(Df(x_0)) : \mathrm{Re}(\lambda) \neq 0}$$

- It can be shown that if x_0 is a hyperbolic fixed-point, then the local behavior of the nonlinear systems is topologically equivalent to the local behaviour of the linearized equation $\dot{x} = Df(x_0)\,x$.
- It follows that hyperbolic fixed-points can be classified according to the eigenvalues of the Jacobian matrix $Df(x_0)$.

EXAMPLE

$$\begin{cases} \dot{x}_1 = x_1(3 - x_1 - x_2) \\ \dot{x}_2 = x_2(x_1 - 1) \end{cases}$$

- <u>Fixed points:</u>

$$\begin{cases} x_1(3-x_1-x_2) = 0 \\ x_2(x_1-1) = 0 \end{cases} \Leftrightarrow \begin{cases} x_1(3-x_1) = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} 1\cdot(3-1-x_2) = 0 \\ x_1 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_1 = 3 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_2 = 2 \\ x_1 = 1 \end{cases}$$

thus set of fixed points: $\{(0,0), (3,0), (1,2)\}$.

- <u>Jacobian</u>

$$\frac{\partial f_1}{\partial x_1} = 1\cdot(3-x_1-x_2)+x_1(-1) = 3-2x_1-x_2$$

$$\frac{\partial f_1}{\partial x_2} = -x_1$$

$$\frac{\partial f_2}{\partial x_1} = x_2$$

$$\frac{\partial f_2}{\partial x_2} = x_1-1$$

thus $Df(x_1,x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$

$$= \begin{bmatrix} 3-2x_1-x_2 & -x_1 \\ x_2 & x_1-1 \end{bmatrix}$$

- <u>At (0,0)</u>

$$Df(0,0) = \begin{bmatrix} 3-0-0 & 0 \\ 0 & 0-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 3$ with $v_1 = (1,0)$

$\lambda_2 = -1$ with $v_2 = (0,1)$

thus $(0,0)$ is a saddle point.

- <u>At (1,2)</u>

$$Df(1,2) = \begin{bmatrix} 3-2\cdot 1-2 & -1 \\ 2 & 1-1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$p(\lambda) = \det(Df(1,2) - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} =$$

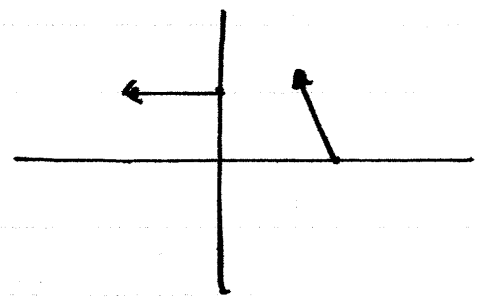
$$= (-1-\lambda)(-\lambda) - (-1)\cdot 2 = \lambda(\lambda+1) + 2 =$$

$$\left.\begin{aligned} &= \lambda^2 + \lambda + 2 \\ &\Delta = 1^2 - 4\cdot 1 \cdot 2 = -7 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{2}$$

Note that

$$\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



Thus $(1,2)$ is a counterclockwise stable spiral.

• <u>At $(3,0)$</u>

$$Df(3,0) = \begin{bmatrix} 3-2\cdot 3-0 & -3 \\ 0 & 3-1 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix}$$

$$p(\lambda) = \det(Df(3,0) - \lambda I) = \begin{vmatrix} -3-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} =$$

$$= (-3-\lambda)(2-\lambda) - (-3)\cdot 0 = (\lambda+3)(\lambda-2) = 0 \Leftrightarrow$$

$\Leftrightarrow \lambda_1 = -3$ or $\lambda_2 = 2. \leftarrow (3,0)$ is a saddle point.

Eigenvectors:

a) For $\underline{\lambda_1 = -3}$:

$$Ax = -3x \Leftrightarrow \begin{cases} -3x_1 - 3x_2 = -3x_1 \\ 2x_2 = -3x_2 \end{cases} \Leftrightarrow \begin{cases} -3x_2 = 0 \\ 5x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x_2 = 0 \Leftrightarrow (x_1, x_2) = (1,0)\, x_1 \leftarrow \underline{v_1 = (1,0)}$$

b) For $\underline{\lambda_2 = 2}$:

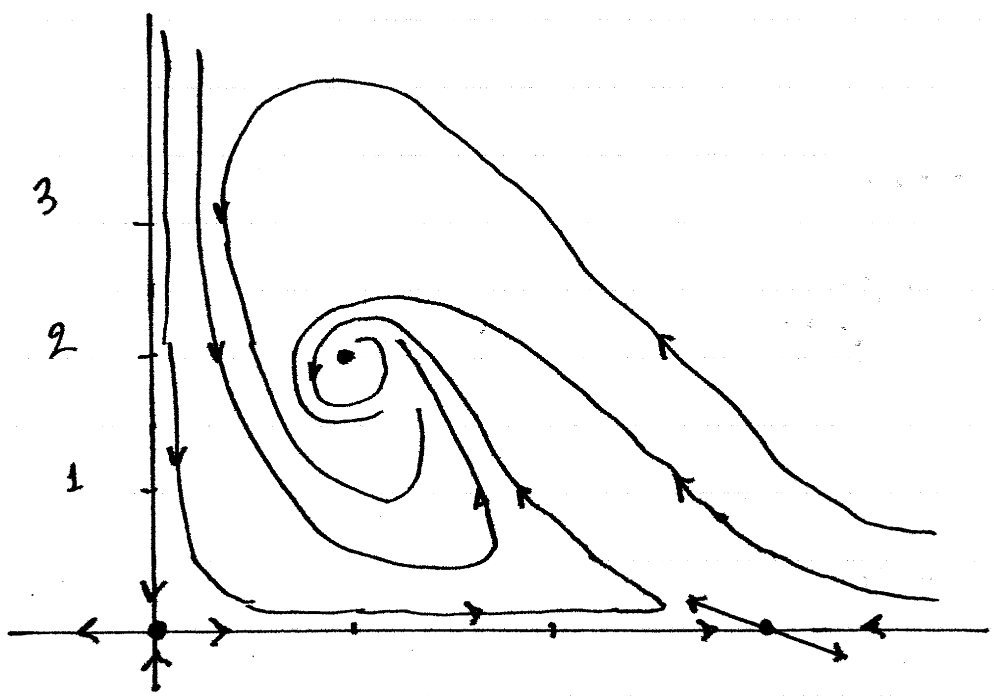
$$Ax = 2x \Leftrightarrow \begin{cases} -3x_1 - 3x_2 = 2x_1 \\ 2x_2 = 2x_2 \end{cases} \Leftrightarrow -5x_1 - 3x_2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x_2 = \frac{-5}{3} x_1 \Leftrightarrow (x_1, x_2) = \left(1, \frac{-5}{3}\right) x_1$$

thus choose $v_2 = (3, -5)$.

- <u>Phase Portrait</u>

$(0,0)$ saddle point with $\lambda_1 = 3$, $v_1 = (1,0)$, $\lambda_2 = -1$, $v_2 = (0,1)$
$(1,2)$ counterclockwise stable spiral
$(3,0)$ saddle point with $\lambda_1 = -3$, $v_1 = (1,0)$, $\lambda_2 = 2$, $v_2 = (3,-5)$



EXERCISES

① Classify the fixed points for the following systems and attempt to draw the phase portrait.

a) $\begin{cases} \dot{x}_1 = x_1 - x_2 \\ \dot{x}_2 = x_1^2 - 4 \end{cases}$

b) $\begin{cases} \dot{x}_1 = 1 + x_2 - e^{-x_1} \\ \dot{x}_2 = x_1^3 - x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = x_2 + x_1 - x_1^3 \\ \dot{x}_2 = -x_2 \end{cases}$

d) $\begin{cases} \dot{x}_1 = x_1 x_2 - 1 \\ \dot{x}_2 = x_1 - x_2^3 \end{cases}$

e) $\begin{cases} \dot{x}_1 = x_1(3 - 2x_1 - 2x_2) \\ \dot{x}_2 = x_2(2 - x_1 - x_2) \end{cases}$

f) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = x_1 - x_1^3 \end{cases}$

g) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = \cos x_1 \end{cases}$

Nonlinear Centers

- Fixed points which, according to local linear analysis, appear to be centers are NOT hyperbolic. It follows that the original nonlinear system may or may not be a center. To determine whether a fixed point with
$$\exists \lambda \in \lambda(Df(x_0)) : Re(\lambda) = 0$$
is or is not a center, we rely on the following methods:

① → Conversion to polar coordinates

A two-dimensional autonomous system of the form
$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$
can be rewritten in polar coordinates (r, ϑ) with $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ using the following identities:

$$\boxed{\dot{r} = \frac{x_1 \dot{x}_1 + x_2 \dot{x}_2}{r}} \quad \boxed{\dot{\vartheta} = \frac{x_1 \dot{x}_2 - \dot{x}_1 x_2}{r^2}}$$

Proof

$$x_1^2+x_2^2 = r^2\cos^2\vartheta + r^2\sin^2\vartheta = r^2(\cos^2\vartheta+\sin^2\vartheta) = r^2 \Rightarrow$$

$$\Rightarrow 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2r\dot{r} \Rightarrow \dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$$

Since $\begin{cases} x_1 = r\cos\vartheta \\ x_2 = r\sin\vartheta \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta \\ \dot{x}_2 = \dot{r}\sin\vartheta + r\dot{\vartheta}\cos\vartheta \end{cases} \Rightarrow$

$$\Rightarrow x_1\dot{x}_2 - \dot{x}_1x_2 = (r\cos\vartheta)(\dot{r}\sin\vartheta + r\dot{\vartheta}\cos\vartheta) - (\dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta)(r\sin\vartheta)$$

$$= r\dot{r}\cos\vartheta\sin\vartheta + r^2\dot{\vartheta}\cos^2\vartheta - r\dot{r}\cos\vartheta\sin\vartheta + r^2\dot{\vartheta}\sin^2\vartheta =$$

$$= r^2\dot{\vartheta}\cos^2\vartheta + r^2\dot{\vartheta}\sin^2\vartheta = r^2\dot{\vartheta}(\cos^2\vartheta+\sin^2\vartheta) = r^2\dot{\vartheta} \Rightarrow$$

$$\Rightarrow \dot{\vartheta} = \frac{x_1\dot{x}_2 - \dot{x}_1x_2}{r^2}$$

EXAMPLE

$$\begin{cases} \dot{x}_1 = -x_2 + ax_1(x_1^2+x_2^2) = f_1(x_1,x_2) \\ \dot{x}_2 = x_1 + ax_2(x_1^2+x_2^2) = f_2(x_1,x_2) \end{cases}$$

Solution

Obvious fixed point at $(x_1,x_2) = (0,0)$

Jacobian

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 3ax_1^2 + ax_2^2 \\ \frac{\partial f_1}{\partial x_2} &= -1 + 2ax_1x_2 \\ \frac{\partial f_2}{\partial x_1} &= 1 + 2ax_1x_2 \\ \frac{\partial f_2}{\partial x_2} &= ax_1^2 + 3ax_2^2 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow Df(x_1,x_2) = \begin{bmatrix} 3ax_1^2 + ax_2^2 & -1 + 2ax_1x_2 \\ 1 + 2ax_1x_2 & ax_1^2 + 3ax_2^2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow Df(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - (-1)\cdot 1 = \lambda^2 + 1 \Rightarrow$$

$\Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \longleftarrow$ a center ?

- Convert to polar coordinates:

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2 =$$
$$= x_1[-x_2 + ax_1(x_1^2+x_2^2)] + x_2[x_1 + ax_2(x_1^2+x_2^2)]$$
$$= -x_1x_2 + ax_1^2r^2 + x_1x_2 + ax_2^2r^2 =$$
$$= ax_1^2r^2 + ax_2^2r^2 = ar^2(x_1^2+x_2^2) = ar^4 \Rightarrow$$

$\Rightarrow \underline{\dot{r} = ar^3}$, and

$$r^2\dot{\theta} = x_1\dot{x}_2 - \dot{x}_1x_2 =$$
$$= x_1[x_1 + ax_2(x_1^2+x_2^2)] - [-x_2 + ax_1(x_1^2+x_2^2)]x_2 =$$
$$= x_1^2 + ax_1x_2r^2 + x_2^2 - ax_1x_2r^2 =$$
$$= x_1^2 + x_2^2 = r^2 \Rightarrow \underline{\dot{\theta} = 1}$$

Thus $\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases}$.

For $a = 0$: $\dot{r} = 0$ and $\dot{\theta} = 1 \Rightarrow (0,0)$ is a center.

For $a > 0$: $\dot{r} > 0$ and $\dot{\theta} = 1 \Rightarrow$

$\Rightarrow (0,0)$ is unstable counterclockwise spiral.

For $a<0$: $\dot{r}<0$ and $\dot{\vartheta}=1 \Rightarrow$

$\Rightarrow (0,0)$ is a counterclockwise stable spiral.

② → Conservative systems

Consider a general autonomous system $\dot{x}=f(x)$ with $x \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$.

Def: We say that the system is conservative if and only if

a) There is a function $V: \mathbb{R}^n \to \mathbb{R}$ such that $(d/dt) V(x(t)) = 0$

b) $\forall A \subset \mathbb{R}^n : (A \text{ open set} \Rightarrow V \text{ non-constant in } A)$.

Def: $x_0 \in \mathbb{R}^n$ is an isolated fixed-point if and only if

a) $f(x_0) = 0$

b) $\exists \varepsilon > 0 : \forall x \in \mathbb{R}^n : (0 < \|x - x_0\| < \varepsilon \Rightarrow f(x) \neq 0)$.

Def: $x(t)$ is a closed orbit if and only if $\forall t > 0 : \exists \tau > 0 : x(t+\tau) = x(t)$.

Thm: Assume that

a) $x_0 \in \mathbb{R}^n$ is an isolated fixed-point

b) f is continuously differentiable in $\mathbb{R}^n$

c) the system is conservative with $(d/dt) V(x(t)) = 0$

d) x_0 is a local min or max of $V(x)$.

Then,

$\exists \varepsilon > 0 : (\|x(0) - x_0\| < \varepsilon \Rightarrow x(t)$ is a closed orbit$)$.

<u>Prop</u> : If $\dot{x} = f(x)$ is conservative then it has no attracting fixed points.

<u>Proof</u>

Let $x_0 \in \mathbb{R}^n$ be an attracting fixed point. Let A be the basin of attraction of x_0 such that

$$\forall y \in A : (x(0) = y \Rightarrow \lim_{t \to +\infty} x(t) = x_0).$$

Let $y \in A$ be given and choose $x(0) = y$. Then

$V(y) = V(x(0)) = V(x(t)), \forall t > 0 \Rightarrow$

$$\Rightarrow V(y) = \lim_{t \to +\infty} V(x(t)) = V(\lim_{t \to +\infty} x(t)) = V(x_0), \forall y \in A$$

$\Rightarrow V$ constant in $A \leftarrow$ contradiction.

Thus x_0 cannot be an attracting fixed point. □

↳ Thus to show that a system is NOT conservative it is sufficient to show that it has an attracting fixed point.

- Constructing $V(x)$ is easy for systems of the following forms:

Form 1 : $\boxed{\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1) \end{cases}} \longleftarrow \boxed{\ddot{x} = f(x)}$

Consider:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\dot{x}_1 + x_2 f(x_1) = x_1\dot{x}_1 + \dot{x}_1 f(x_1) =$$
$$= \dot{x}_1 (x_1 + f(x_1)) \Rightarrow$$
$$\Rightarrow -f(x_1)\dot{x}_1 + x_2\dot{x}_2 = 0 \longleftarrow \text{easily integrated to yield } V(x).$$

EXAMPLE

$$\begin{cases} \dot{x}_1 = -x_2 - x_2^3 \\ \dot{x}_2 = x_1 \end{cases}$$

- Fixed points

$$\begin{cases} -x_2 - x_2^3 = 0 \\ x_1 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2(1+x_2^2) = 0 \\ x_1 = 0 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} x_2 = 0 \\ x_1 = 0 \end{cases} \vee \begin{cases} 1 + x_2^2 = 0 \\ x_1 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$
$$\Leftrightarrow (x_1, x_2) = (0,0)$$

- Local linear analysis

$$Df(x_1,x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -1-3x_2^2 \\ 1 & 0 \end{bmatrix} \Rightarrow Df(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - (-1)\cdot 1 = \lambda^2 + 1 \Rightarrow$$

$\Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \Rightarrow (0,0)$ is a <u>linear center.</u>

▶ However we still have to prove that it is a nonlinear center.

- <u>Nonlinear center.</u>

Let:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_2 - x_2^3) + x_2x_1 =$$

$$= -x_1x_2 - x_1x_2^3 + x_1x_2 = -x_1x_2^3 =$$

$$= -x_2^3\dot{x}_2 \Rightarrow$$

$$\Rightarrow x_1\dot{x}_1 + (x_2 + x_2^3)\dot{x}_2 = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left[\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_2^4}{4}\right] = 0$$

$$\Rightarrow \underline{2x_1^2 + 2x_2^2 + x_2^4 = c} \qquad (1)$$

For $V(x_1,x_2) = 2x_1^2 + 2x_2^2 + x_2^4$ we have $V(0,0) = 0$ and $V(x_1,x_2) > 0$, $\forall (x_1,x_2) \in \mathbb{R}^2 - \{(0,0)\}$, thus $(0,0)$ is a local minimum. It follows that $(0,0)$ is a nonlinear center. The closed trajectories are given by (1).

Form 2 : $\boxed{\begin{aligned}\dot{x}_1 &= f(x_1)\,g_1(x_2)\\ \dot{x}_2 &= f(x_2)\,g_2(x_1)\end{aligned}}$

Consider:

$$\frac{dV}{dt} = \frac{g_2(x_1)}{f(x_1)}\dot{x}_1 - \frac{g_1(x_2)}{f(x_2)}\dot{x}_2 =$$

$$= \frac{g_2(x_1)}{f(x_1)} f(x_1) g_1(x_2) - \frac{g_1(x_2)}{f(x_2)} f(x_2) g_2(x_1) =$$

$$= g_2(x_1) g_1(x_2) - g_1(x_2) g_2(x_1) = 0$$

which can then be easily integrated to yield the Lyapunov function.

EXAMPLE

$$\begin{cases} \dot{x}_1 = x_1 - x_1 x_2 \\ \dot{x}_2 = -x_2 + x_1 x_2 \end{cases}$$

- Fixed points

$$\begin{cases} x_1 - x_1 x_2 = 0 \\ -x_2 + x_1 x_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1(1-x_2) = 0 \\ x_2(x_1 - 1) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2(0-1) = 0 \end{cases} \vee \begin{cases} x_2 = 1 \\ 1\cdot(x_1 - 1) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases} \leftarrow \text{Fixed points: } \{(0,0),(1,1)\}.$$

- Local linear analysis

$$Df(x_1,x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1-x_2 & -x_1 \\ x_2 & x_1-1 \end{bmatrix}$$

- At $(0,0)$:

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(Df(0,0)) = \{+1,-1\} \Rightarrow$$

$\Rightarrow (0,0)$ is a saddle point.

- At $(1,1)$

$$Df(1,1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(1,1) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - 1\cdot(-1) = \lambda^2 + 1 \Rightarrow$$

$\Rightarrow \lambda(Df(1,1)) = \{+i, -i\} \Rightarrow (1,1)$ is a linear center.

- Nonlinear center.

We now show that $(1,1)$ is a nonlinear center.

Construct a Lyapunov function:

Note that:

$$\dot{x}_1 = x_1(1-x_2)$$

$$\dot{x}_2 = x_2(x_1-1)$$

so we define:

$$\frac{dV}{dt} = \frac{x_1 - 1}{x_1}\dot{x}_1 - \frac{1-x_2}{x_2}\dot{x}_2 =$$

$$= \frac{x_1-1}{x_1} x_1(1-x_2) - \frac{1-x_2}{x_2} x_2(x_1-1) =$$

$$= (x_1-1)(1-x_2) - (1-x_2)(x_1-1) = 0 \Rightarrow$$

$$\Rightarrow \left(1 - \frac{1}{x_1}\right)\dot{x}_1 + \left(1 - \frac{1}{x_2}\right)\dot{x}_2 = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left[x_1 - \ln x_1 + x_2 - \ln x_2\right] = 0 \Rightarrow$$

$$\Rightarrow \underline{x_1 + x_2 - \ln(x_1 x_2) = c} \leftarrow \text{shape of trajectories.}$$

We now show that $(1,1)$ is an extremum by calculating the Hessian:

Let $f(x_1,x_2) = x_1 + x_2 - \ln(x_1x_2)$. Then

$$\nabla f(x_1,x_2) = (\partial f/\partial x_1, \partial f/\partial x_2) =$$
$$= (1 - 1/x_1, 1 - 1/x_2) \Rightarrow$$

$$\Rightarrow \nabla f(1,1) = (1-1, 1-1) = (0,0).$$

Since:

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(1 - \frac{1}{x_1}\right) = \frac{1}{x_1^2}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(1 - \frac{1}{x_2}\right) = \frac{1}{x_2^2}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(1 - \frac{1}{x_2}\right) = 0$$

the Hessian reads:

$$\Delta(x_1,x_2) = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 =$$

$$= \frac{1}{x_1^2}\frac{1}{x_2^2} - 0^2 = \left(\frac{1}{x_1 x_2}\right)^2 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \Delta(1,1) = 1 > 0 \\ \dfrac{\partial^2 f(1,1)}{\partial x_1^2} = \dfrac{1}{1^2} = 1 > 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ $(1,1)$ is a local min of
$f(x_1,x_2) = x_1 + x_2 - \ln(x_1 x_2)$

It follows that $(1,1)$ is a nonlinear center.

Recall that for

$$\Delta = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2$$

we have the following sufficient conditions:

a) $\left.\begin{array}{l} \Delta(x_0) > 0 \\ \partial^2 f(x_0)/\partial x_1^2 > 0 \end{array}\right\} \Rightarrow x_0 \in \mathbb{R}^2$ is a local min

b) $\left.\begin{array}{l} \Delta(x_0) > 0 \\ \partial^2 f(x_0)/\partial x_1^2 < 0 \end{array}\right\} \Rightarrow x_0 \in \mathbb{R}^2$ is a local max.

EXERCISES

② Show that the following systems are conservative, locate and classify all fixed points. Draw a phase portrait.

a) $\ddot{x} = x^3 - x$ c) $\ddot{x} = 1 - e^x$

b) $\ddot{x} = x - x^2$ d) $\ddot{x} = (x-a)(x^2-a)$

③ Similarly for the following systems:

a) $\begin{cases} \dot{x} = -kxy \\ \dot{y} = kxy - ly \end{cases}$ b) $\dot{x} = x - xy$, $\dot{y} = \mu xy - \mu y$

with $k > 0$, $l > 0$

④ Consider the system

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -x^2 \end{cases}$$

a) Show that $V(x,y) = x^2 + y^2$ is conserved.

b) Show that $(x,y) = (0,0)$ is a fixed point but not an isolated fixed point.

c) Show that V has a minimum at $(0,0)$ but $(0,0)$ is not a nonlinear center.

(3) → Reversible systems

- Consider the system $\dot{x} = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$

Def: We say that a mapping $P: \mathbb{R}^n \to \mathbb{R}^n$ is an involution if and only if
$\forall x \in \mathbb{R}^n : P(P(x)) = x$.

Def: We say that the system $\dot{x} = f(x)$ is reversible if and only if there is an involution P such that

$$\frac{d}{dt} P(x) = -f(P(x))$$

- A reversible system is invariant under the transformation

$$t \to -t$$
$$x \to P(x)$$

- We define the symmetry section of the involution P as:

$$\mathrm{Fix}(P) = \{x \in \mathbb{R}^n \mid P(x) = x\}$$

Thm: Assume that the system $\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$ is reversible under the involution P. Then, if

$$\left.\begin{array}{l} x_0 \in \mathrm{Fix}(P) \\ x_0 \text{ linear center} \end{array}\right\} \Rightarrow x_0 \text{ nonlinear center.}$$

We confine our attention to the two-dimensional system

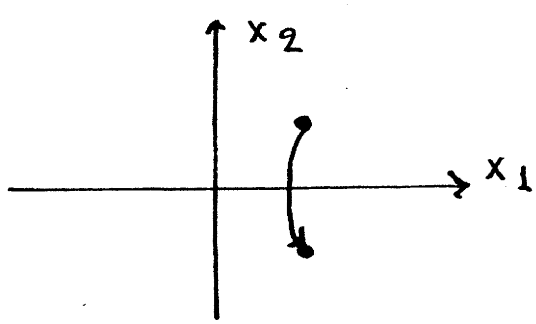
$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases} \qquad (1)$$

► <u>Reflection around x-axis</u>

Assume that

$$f(x_1, -x_2) = -f(x_1, +x_2) \ , \ \forall x_1, x_2 \in \mathbb{R}$$
$$g(x_1, -x_2) = g(x_1, x_2) \ , \ \forall x_1, x_2 \in \mathbb{R}$$

Then, the system (1) is reversible under the involution $(x_1, -x_2) = P(x_1, x_2)$



We note that the symmetry section is $Fix(P) = \{(x, 0) \mid x \in \mathbb{R}\}$.

<u>Proof</u>

Let $x = (x_1, x_2)$ and $F(x) = (f(x_1, x_2), g(x_1, x_2))$.
Then:

$$\frac{d}{dt} P(x) = \frac{d}{dt}(x_1, -x_2) = (\dot{x}_1, -\dot{x}_2) =$$
$$= (f(x_1, x_2), -g(x_1, x_2)) =$$
$$= (-f(x_1, -x_2), -g(x_1, -x_2)) =$$
$$= -(f(x_1, -x_2), g(x_1, -x_2)) = -F(P(x)) \ \square$$

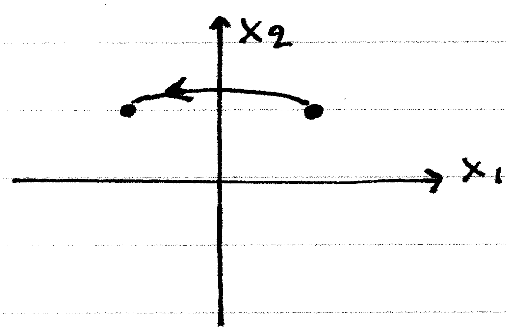
Reflection around y-axis

Assume that

$$f(-x_1,x_2)=f(x_1,x_2) \quad , \forall x_1,x_2\in\mathbb{R}$$
$$g(-x_1,x_2)=-g(x_1,x_2), \quad \forall x_1,x_2\in\mathbb{R}$$

then (1) is reversible under the involution

$$P(x_1,x_2)=(-x_1,x_2)$$



We note that the symmetry section is

$$\mathrm{Fix}(P)=\{(0,y)\mid y\in\mathbb{R}\}$$

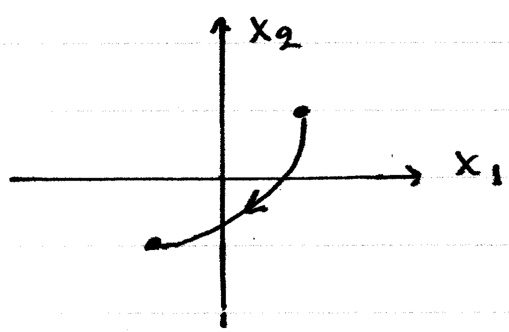
Reflection around x-axis and y-axis

Assume that

$$f(-x_1,-x_2)=f(x_1,x_2) \quad , \forall x_1,x_2\in\mathbb{R}$$
$$g(-x_1,-x_2)=g(x_1,x_2) \quad , \forall x_1,x_2\in\mathbb{R}$$

then (1) is reversible under the involution

$$P(x_1,x_2)=(-x_1,-x_2)$$



We note that the symmetry section is:

$$\mathrm{Fix}(P)=\{(0,0)\}.$$

EXAMPLE

$$\begin{cases} \dot{x}_1 = x_2 - x_2^3 \\ \dot{x}_2 = -x_1 - x_2^2 \end{cases} \leftarrow \text{Classification of fixed points.}$$

Proof

Let $f(x_1,x_2) = x_2 - x_2^3$ and $g(x_1,x_2) = -x_1 - x_2^2$.

- Fixed points:

$$\begin{cases} f(x_1,x_2) = 0 \\ g(x_1,x_2) = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 - x_2^3 = 0 \\ -x_1 - x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2(1-x_2)(1+x_2) = 0 \\ x_1 = -x_2^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_1 = -1 \\ x_2 = 1 \end{cases} \vee \begin{cases} x_1 = -1 \\ x_2 = -1 \end{cases}$$

- Jacobian

$$Df(x_1,x_2) = \begin{bmatrix} 0 & 1-3x_2^2 \\ -1 & -2x_2 \end{bmatrix}$$

- At $(x_1,x_2) = (0,0)$

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - (-1) =$$

$$= \lambda^2 + 1 \Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \Rightarrow$$

$\Rightarrow (0,0)$ is a linear center.

Since $f(x_1,-x_2) = (-x_2) - (-x_2)^3 = -(x_2 - x_2^3) =$

$$= -f(x_1,x_2)$$

and

$g(x_1,-x_2) = -x_1-(-x_2)^2 = -x_1 - x_2^2 = g(x_1,x_2)$
it follows that the system is reversible. Thus, since

$$\left.\begin{array}{l}(0,0) \text{ is a linear center} \\ (0,0)\in \mathrm{Fix}(P) = \{(x,0) \mid x\in\mathbb{R}\}\end{array}\right\} \Rightarrow (0,0) \text{ is a nonlinear center.}$$

- At $(x_1,x_2) = (-1,1)$

$$Df(-1,1) = \begin{bmatrix} 0 & 1-3\cdot 1^2 \\ -1 & -2\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(-1,1)-\lambda I) = \begin{vmatrix} -\lambda & -2 \\ -1 & -\lambda-2 \end{vmatrix} =$$

$$= (-\lambda)(-\lambda-2)-(-1)(-2) = \lambda(\lambda+2)-2 = \lambda^2+2\lambda-2.$$

$$\Delta = 2^2-4\cdot 1\cdot(-2) = 4+8 = 12 = 4\cdot 3 \Rightarrow \lambda_{1,2} = \frac{-2\pm 2\sqrt{3}}{2} = -1\pm\sqrt{3}$$

$$\Rightarrow \lambda(Df(-1,1)) = \{-1-\sqrt{3}, -1+\sqrt{3}\} \Rightarrow (-1,1) \text{ is a saddle point.}$$

- At $(x_1,x_2) = (-1,-1)$

$$Df(-1,-1) = \begin{bmatrix} 0 & 1-3(-1)^2 \\ -1 & -2(-1) \end{bmatrix} = \begin{bmatrix} 0 & 1-3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(-1,-1)-\lambda I) = \begin{vmatrix} -\lambda & -2 \\ -1 & -\lambda+2 \end{vmatrix} = (-\lambda)(-\lambda+2)-(-1)(-2)$$

$$= \lambda^2-2\lambda-2$$

$$\Delta = (-2)^2-4\cdot 1\cdot(-2) = 4+8 = 12 = 4\cdot 3 \Rightarrow \lambda_{1,2} = \frac{-(-2)\pm 2\sqrt{3}}{2} = 1\pm\sqrt{3}$$

$$\Rightarrow \lambda(Df(-1,-1)) = \{1+\sqrt{3}, 1-\sqrt{3}\} \Rightarrow$$

$\Rightarrow (-1,-1)$ is a saddle point.

EXAMPLE

$$\begin{cases} \dot{x}_1 = 2\cos x_1 + \cos x_2 \\ \dot{x}_2 = 2\cos x_2 + \cos x_1 \end{cases} \leftarrow$$ Show that system is reversible but not conservative.

- **Reversability.**

Let $f(x_1, x_2) = 2\cos x_1 + \cos x_2$ and
$g(x_1, x_2) = 2\cos x_2 + \cos x_1$.

Since:

$$f(-x_1, -x_2) = 2\cos(-x_1) + \cos(-x_2) =$$
$$= 2\cos x_1 + \cos x_2 = f(x_1, x_2)$$

and

$$g(-x_1, -x_2) = 2\cos(-x_2) + \cos(-x_1) =$$
$$= 2\cos x_2 + \cos x_1 = g(x_1, x_2)$$

thus the system is reversible with respect to the involution $P(x_1, x_2) = (-x_1, -x_2)$

- **Not conservative**

It is sufficient to show that the system has an attracting fixed point.
We first find the fixed points of the system:

$$\begin{cases} 2\cos x_1 + \cos x_2 = 0 \\ 2\cos x_2 + \cos x_1 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \cos x_1 \\ \cos x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cos x_1 = 0 \\ \cos x_2 = 0 \end{cases} \Leftrightarrow \exists k, \lambda \in \mathbb{Z} : \begin{cases} x_1 = k\pi + \pi/2 \\ x_2 = \lambda\pi + \pi/2 \end{cases}$$

The Jacobian of the system reads:

$$Df(x_1, x_2) = \begin{bmatrix} -2\sin x_1 & -\sin x_2 \\ -\sin x_1 & -2\sin x_2 \end{bmatrix}$$

At $(x_1, x_2) = (\pi/2, \pi/2)$:

$$Df(\pi/2, \pi/2) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(\pi/2, \pi/2) - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)^2 - (-1)^2 = (\lambda+2)^2 - 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow (\lambda+2)^2 = 1 \Leftrightarrow \lambda + 2 = \pm 1 \Leftrightarrow \lambda = -2 \pm 1 = \{-3, -1\}$$

thus $\lambda(Df(\pi/2, \pi/2)) = \{-3, -1\} \Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is a sink $\Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is asymptotically stable $\Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is attracting $\Rightarrow$

$\Rightarrow$ the system is not conservative.

EXERCISES

(5) Show that the following systems are reversible and then find and classify all fixed points.

a) $\begin{cases} \dot{x}_1 = x_2(1-x_1^2) \\ \dot{x}_2 = 1-x_2^2 \end{cases}$

b) $\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \cos x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = \sin x_1 \end{cases}$

d) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = x_2^2 - x_1 \end{cases}$

e) $\begin{cases} \dot{x}_1 = x_2 - x_2^3 \\ \dot{x}_2 = x_1 \cos x_2 \end{cases}$

f) $\ddot{x} + (\dot{x})^2 + x = 3$

g) $\ddot{x} + x\dot{x} + x = 0$

Index theory

Index theory is a global method that provides global information about the phase portrait of a two-dimensional autonomous system.

Definition of the index

Consider the two-dimensional autonomous system

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$

We note that at (x_1, x_2), the angle φ of the vector $(\dot{x}_1, \dot{x}_2)$ is given by

$$\varphi(x_1, x_2) = \operatorname{Arctan}\left(\frac{g(x_1, x_2)}{f(x_1, x_2)}\right)$$

Let C be a simple closed curve. We define the index $I(C)$ of C as:

$$\boxed{I(C) = \oint_C \frac{d\varphi(x_1, x_2)}{2\pi}}$$

Explicit form of the index integral

We note that:

$$d\varphi = d\left(\mathrm{Arctan}\,(g/f)\right) = \frac{1}{1+(g/f)^2}\, d\left(\frac{g}{f}\right) =$$

$$= \frac{1}{1+(g/f)^2}\,\frac{f\,dg - g\,df}{f^2} =$$

$$= \frac{f\,dg - g\,df}{f^2+g^2} \Rightarrow$$

$$\Rightarrow I(C) = \oint_C \frac{d\varphi}{2\pi} = \oint_C \frac{f\,dg - g\,df}{2\pi(f^2+g^2)}$$

Let $C: \rho(t) \in \mathbb{R}^2$, $t \in [0,1]$ be a parameterization of the curve C. Then, the differentials df and dg are given by:

$$df = [\dot{\rho}(t)\cdot \nabla f(\rho(t))]\,dt$$
$$dg = [\dot{\rho}(t)\cdot \nabla g(\rho(t))]\,dt$$

It follows that:

$$I(C) = \oint_C \frac{f\,dg - g\,df}{f^2+g^2} =$$

$$= \int_0^1 dt\, \frac{f(\rho(t))\nabla g(\rho(t))\cdot\dot{\rho}(t) - g(\rho(t))\nabla f(\rho(t))\cdot\dot{\rho}(t)}{2\pi\left[f^2(\rho(t)) + g^2(\rho(t))\right]}$$

$$= \int_0^1 dt\, \dot{\rho}(t)\cdot\left[\frac{f(\rho(t))\nabla g(\rho(t)) - g(\rho(t))\nabla f(\rho(t))}{2\pi\left[f^2(\rho(t)) + g^2(\rho(t))\right]}\right]$$

- Properties of the index

① $\boxed{I(C) \in \mathbb{Z}}$ (i.e. $I(C)$ is an integer).

Proof

Going around the curve C, both initial and final value of φ point in the same direction, therefore the variation $\Delta\varphi$ of the angle must be a multiple of 2π. It follows that

$$\Delta\varphi = \oint_C d\varphi = 2k\pi \ , \text{ with } k \in \mathbb{Z} \Rightarrow$$

$$\Rightarrow I(C) = \frac{1}{2\pi} \oint_C d\varphi = \frac{1}{2\pi} \cdot (2k\pi) = k \in \mathbb{Z} \quad \square$$

② $\boxed{\text{Assume that there are no fixed points in the interior of a simple closed curve } C. \text{ Then } I(C) = 0.}$

Proof

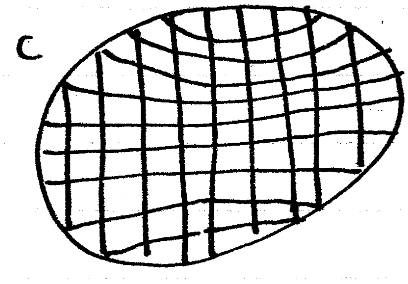


Fig. 1

We divide the interior of the curve C into a mesh of N closed simple curves γ_k with $k \in [N]$. We assume that the loops γ_k are small enough so that the maximum angle variation around γ_K does not exceed $\pi/2$. This is possible only because there are no fixed points in the interior of any γ_K (see fig. 2)

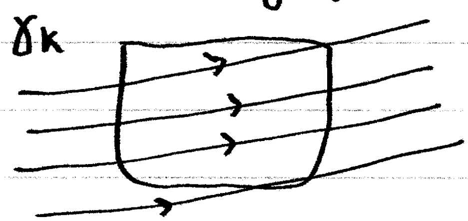


Fig. 2

It follows that

$$\forall K \in [N] : \oint_{\gamma_K} d\varphi = 0$$

and therefore

$$I(C) = \frac{1}{2\pi} \oint_C d\varphi = \frac{1}{2\pi} \left[\sum_{K=1}^{N} \oint_{\gamma_K} d\varphi \right] = 0 \quad \square$$

③ Invariance with contour deformation

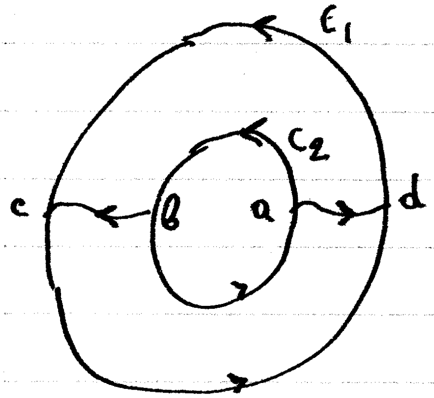
Def : Let C_1, C_2 be two simple closed curves with
$C_1 : \rho_1(t) \in \mathbb{R}^2,\ t \in [0,1]$ and
$C_2 : \rho_2(t) \in \mathbb{R}^2,\ t \in [0,1]$.
We say that $C_1 \sim C_2$ if and only if there is a mapping $\rho : [0,1]^2 \to \mathbb{R}^2$ such that
a) $\forall t \in [0,1] : (\rho(t,0) = \rho_1(t) \wedge \rho(t,1) = \rho_2(t))$
b) ρ continuous at $[0,1]^2$

c) $\forall (t,a) \in [0,1]^2$: $\rho(t,a)$ not a fixed point.

$\hookrightarrow$ $C_1 \sim C_2$ means that C_1 can be continuously deformed into C_2 without crossing any fixed points.

- $\boxed{C_1 \sim C_2 \Rightarrow I(C_1) = I(C_2)}$

<u>Proof</u>



Define:

C_{bc} : path from b to c

C_{ad} : path from a to d

C_{ab} : counterclockwise path from a to b

C_{ba} : counterclockwise path from b to a

C_{cd} : counterclockwise path from c to d.

C_{dc} : counterclockwise path from d to c.

We also let $-C$ represent the path C with its direction reversed. (e.g. $-C_{ab}$ vs. C_{ba}).

Now consider the paths Γ_1 and Γ_2 defined as:

$\Gamma_1 = C_{ad} \cup C_{dc} \cup (-C_{bc}) \cup (-C_{ab})$

$\Gamma_2 = C_{bc} \cup C_{cd} \cup (-C_{ad}) \cup (-C_{ba})$

There are no fixed points in the interiors of Γ_1 and Γ_2, therefore $I(\Gamma_1) = 0$ and $I(\Gamma_2) = 0$.

We note that

$$2\pi I(\Gamma_1) = \int_{C_{ad}} d\varphi + \int_{C_{dc}} d\varphi + \int_{-C_{bc}} d\varphi + \int_{-C_{ab}} d\varphi =$$

$$= \int_{C_{ad}} d\varphi + \int_{C_{dc}} d\varphi - \int_{C_{bc}} d\varphi - \int_{C_{ab}} d\varphi \qquad (1)$$

and

$$2\pi I(\Gamma_2) = \int_{C_{bc}} d\varphi + \int_{C_{cd}} d\varphi + \int_{-C_{ad}} d\varphi + \int_{-C_{ba}} d\varphi =$$

$$= \int_{C_{bc}} d\varphi + \int_{C_{cd}} d\varphi - \int_{C_{ad}} d\varphi - \int_{C_{ba}} d\varphi \qquad (2)$$

Adding (1) and (2) gives: the cancellations: C_{bc}, C_{ad}

$$2\pi [I(\Gamma_1) + I(\Gamma_2)] = \int_{C_{cd}} d\varphi + \int_{C_{dc}} d\varphi - \int_{C_{ab}} d\varphi - \int_{C_{ba}} d\varphi =$$

$$= \oint_{C_1} d\varphi - \oint_{C_2} d\varphi = 2\pi [I(C_1) - I(C_2)]$$

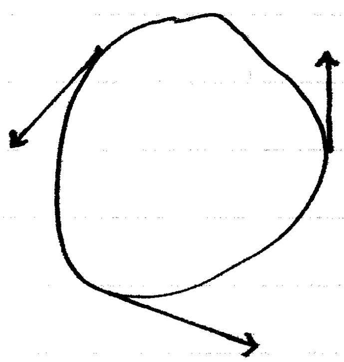
$\Rightarrow I(C_1) - I(C_2) = I(\Gamma_1) + I(\Gamma_2) = 0 + 0 = 0 \Rightarrow$

$\Rightarrow I(C_1) = I(C_2)$. $\square$

④ Index of closed orbits

- $\boxed{\text{If } C \text{ is a closed orbit of the system then } I(C) = 1}$

Proof: If C is a closed orbit of the system, then it is easy to see that the vector $(\dot{x}_1, \dot{x}_2)$ is tangent to C for all points of C. Thus, the total change in the angle φ is $\Delta\varphi = 2\pi$. It follows that



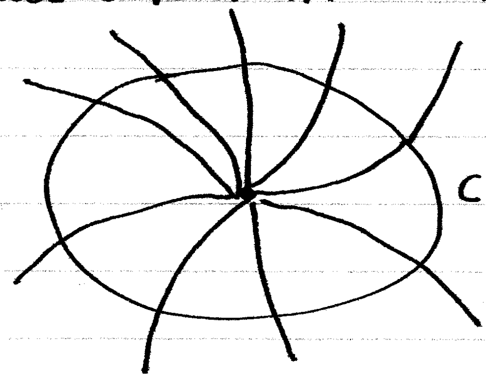
$$I(C) = \frac{1}{2\pi} \oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{2\pi}{2\pi} = 1 \quad \square$$

- Index of a fixed point

Def: Let $x_0 \in \mathbb{R}^2$ be a fixed point. Let C be a counterclockwise curve whose interior contains the fixed point x_0, but no other fixed points. We define the index $I(x_0)$ of the fixed point x_0 as $I(x_0) = I(C)$

↳ We note that from property 3 above, $I(x_0)$ is independent of our choice of C, subject to the stated constraints.

<u>Thm</u> : Let $x_0 \in \mathbb{R}^2$ be a fixed point such that trajectories radiate from or towards x_0 in all directions. Then $I(x_0) = +1$.



<u>Proof</u>

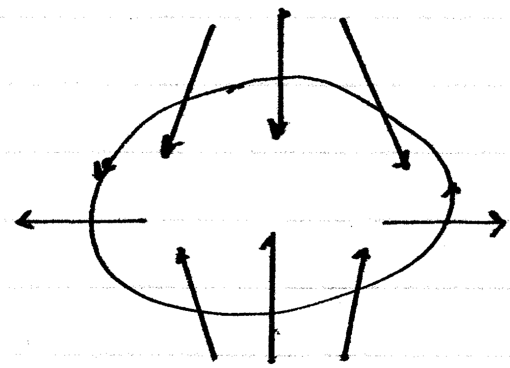
Consider a small enough loop C around x_0 constructed so that it is perpendicular to every trajectory it intersects. Then the total change in the angle around C is $\Delta\varphi = 2\pi$. It follows that:

$$I(x_0) = I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{2\pi}{2\pi} = 1. \quad \square$$

↳ It follows that the following fixed points have $I(x_0) = 1$:

a) sources c) stable spirals e) degenerate nodes
b) sinks d) unstable spirals f) stars.

Thm : $\boxed{\text{Let } x_0 \in \mathbb{R}^2 \text{ be a } \underline{\text{saddle node}}\text{. Then } I(x_0) = -1.}$



Proof

Let C be a small loop around the saddle node x_0. The angle φ varies clockwise around C with $\Delta\varphi = -2\pi$ (see fig.) It follows that

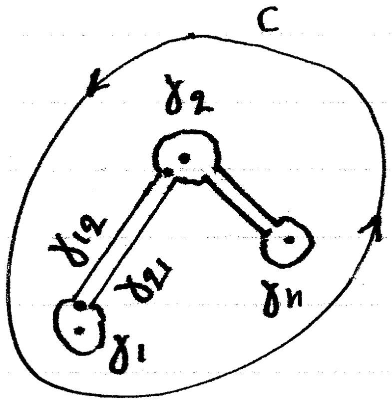
$$I(x_0) = I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{-2\pi}{2\pi} = -1 \quad \square$$

- <u>Index of a curve surrounding fixed points.</u>

Thm : Let C be a simple closed curve containing the fixed points $x_1, x_2, \dots, x_n$. Then:

$$\boxed{I(C) = \sum_{a=1}^{n} I(x_a)}$$

Proof



We deform continuously C into a curve $\Gamma \sim C$ such that Γ consists of small loops γ_a around the fixed points x_a and connecting bridges γ_{ab} connecting x_a to x_b as shown in the figure. We further assume that the gap between γ_{ab} to γ_{ba} tends to zero. That implies that $\gamma_{ab} = -\gamma_{ba}$ and γ_a are closed. It follows that

$$I(C) = I(\Gamma) = \frac{1}{2\pi} \oint_\Gamma d\varphi =$$

$$= \frac{1}{2\pi}\left[\sum_{a=1}^{n} \oint_{\gamma_a} d\varphi + \sum_{a=1}^{n-1} \int_{\gamma_{a,a+1}} d\varphi + \sum_{a=1}^{n} \int_{\gamma_{a+1,a}} d\varphi\right]$$

$$= \frac{1}{2\pi}\left[\sum_{a=1}^{n} \oint_{\gamma_a} d\varphi + \sum_{a=1}^{n-1} \int_{\gamma_{a,a+1}} d\varphi - \sum_{a=1}^{n-1} \int_{\gamma_{a,a+1}} d\varphi\right]$$

$$= \frac{1}{2\pi}\left[\sum_{a=1}^{n} \oint_{\gamma_a} d\varphi\right] = \sum_{a=1}^{n}\left[\frac{1}{2\pi} \oint_{\gamma_a} d\varphi\right] =$$

$$= \sum_{a=1}^{n} I(x_a) \qquad \square$$

Corollary: Let C be a closed trajectory enclosing the fixed points $x_1, x_2, \ldots, x_n$. Then

$$\sum_{a=1}^{n} I(x_a) = +1$$

Proof

Since C is a closed trajectory, from property 4, we have $I(C) = +1$. Thus, from the theorem:

$$\sum_{a=1}^{n} I(x_a) = I(C) = +1 \qquad \square$$

EXAMPLES

a) Show that the system

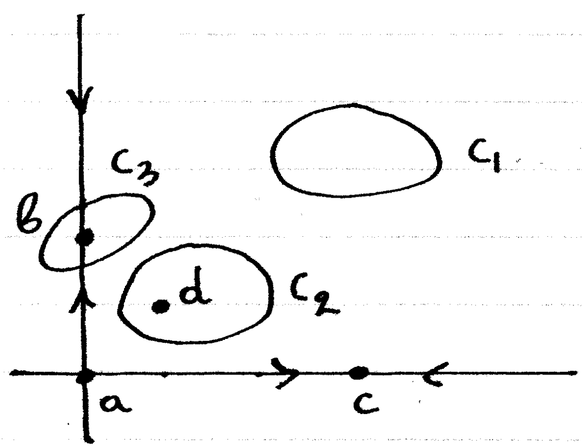
$$\begin{cases} \dot{x}_1 = x_1(3 - x_1 - 2x_2) \\ \dot{x}_2 = x_2(2 - x_1 - x_2) \end{cases}$$

does not have any closed trajectories.

Solution

It can be shown that this system has the following fixed points:

$a = (0,0)$ unstable node $\Rightarrow I(a) = 1$
$b = (0,2)$ stable node $\Rightarrow I(b) = 1$
$c = (3,0)$ stable node $\Rightarrow I(c) = 1$
$d = (1,1)$ saddle node $\Rightarrow I(d) = -1$.



- Let C_1 be a curve enclosing no fixed points. Then
  $I(C_1) = 0 \neq 1 \Rightarrow$
  $\Rightarrow C_1$ not a trajectory.

- Let C_2 be any curve that encloses only the fixed point $d = (1,1)$. Then
  $I(C_2) = I(d) = -1 \neq +1 \Rightarrow C_2$ not a trajectory.

- Let C_3 be any curve enclosing a or b or c or any combination of these three fixed points. Then C_3 will intersect at least the x_1-axis or the x_2-axis (or both). Since both the x_1-axis and the x_2-axis are trajectories, and trajectories cannot intersect, it follows that C_3 is not a trajectory. □

↳ We see that trajectories that cannot be ruled out by index theory, can be eliminated, sometimes, by the constraint that two trajectories cannot intersect.

b) Show that the system

$$\begin{cases} \dot{x}_1 = x_1 e^{-x_1} \\ \dot{x}_2 = 1 + x_1 + x_2^2 \end{cases}$$

does not have any closed trajectories.

Solution

Fixed points:

$$\begin{cases} x_1 e^{-x_1} = 0 \\ 1 + x_1 + x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ 1 + x_1 + x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ 1 + x_2^2 = 0 \end{cases}$$

Since $1 + x_2^2 = 0$ is inconsistent, there are no

fixed points.
Now, let C be any closed curve. Then
$I(C) = 0 \neq 1 \Rightarrow C$ not a trajectory.

EXERCISES

(10) Show that the following systems do not have any closed trajectories.

a) $\begin{cases} \dot{x}_1 = x_1(4 - x_2 - x_1^2) \\ \dot{x}_2 = x_2(x_1 - 1) \end{cases}$

b) $\begin{cases} \dot{x}_1 = x_1^2 + x_2^2 \\ \dot{x}_2 = x_1 - 2 \end{cases}$

(11) A system has three closed trajectories C_1, C_2, C_3, all counterclockwise, with C_2, C_3 enclosed by C_1. We also know that C_2 is not enclosed by C_3 and vice versa. Show that there is at least one fixed point enclosed by C_1, but not enclosed by C_2 and C_3.

(12) Consider the parameterized system

$\begin{cases} \dot{x}_1 = f(x_1, x_2, a) \\ \dot{x}_2 = g(x_1, x_2, a) \end{cases}$

As we vary a from a_0 to a_1 this system undergoes one or more local bifurcations. Show that the sum $J(a)$ of all indices of all fixed points is constant with respect to a.