LINEAR AUTONOMOUS SYSTEMS

- A linear autonomous system is a system of ordinary differential equations of the form
$$\dot{x} = Ax$$
with $x \in \mathbb{R}^n$ a vector and $A \in M_n(\mathbb{R})$ an $n \times n$ matrix. In detail:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ \dot{x}_n = A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n \end{cases}$$

Exact solutions

- An exact solution can be written in terms of the matrix exponential.

Def: $$\boxed{\exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!}}$$ (with $A^0 = I$)

→ Properties:
- $AB = BA \Rightarrow \exp(A+B) = \exp(A)\exp(B)$
- $[\exp(A)]^{-1} = \exp(-A)$
- $\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$

- The solution of $\dot{x} = Ax$ with $x(0) = x_0$ is

$$\boxed{x(t) = \exp(tA)\, x(0)}$$

- <u>Eigenvalues and eigenvectors</u>

<u>Def</u> : $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in M_n(\mathbb{R})$ with eigenvector $x \in \mathbb{C}^n$ if and only if $Ax = \lambda x$.

- <u>notation</u> : $\lambda(A)$ = the set of all eigenvalues of A.

<u>Thm</u> : $\boxed{\lambda \in \lambda(A) \iff \det(A - \lambda I) = 0}$

- We note that $p(\lambda) = \det(A - \lambda I)$ is a polynomial called the <u>characteristic polynomial</u> of A.
- Assume that A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$. Then,

  a) The eigenvalues $v_1, v_2, \dots, v_n$ are linearly independent. Thus any $x \in \mathbb{R}^n$ can be written as:

  $$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

  with $c_1, c_2, \dots, c_n$ constant.

  b) For $P = [v_1 v_2 \cdots v_n]$, A can be written as

  $$A = P \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) P^{-1}.$$

  with

$$\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

c) If the initial condition of $\dot{x} = Ax$ satisfies

$$x(0) = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

then

$$\boxed{x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n}$$

<u>Proof</u>

$$x(t) = \exp(tA)\, x(0) = \exp(tA)\left[c_1 v_1 + c_2 v_2 + \cdots + c_n v_n\right]$$

$$= \sum_{a=1}^{n} c_a \exp(tA)\, v_a = \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{1}{k!} (tA)^k\right] v_a$$

$$= \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{t^k}{k!} (A^k v_a)\right] =$$

$$= \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{t^k}{k!} \lambda_a^k v_a\right] =$$

$$= \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{(\lambda_a t)^k}{k!}\right] v_a = \sum_{a=1}^{n} c_a e^{\lambda_a t} v_a \quad \square$$

We see that when the eigenvalues are all distinct, we can find the exact solution without calculating the matrix exponential.

Matrix Exponential - 2x2 case

Let $A \in M_2(\mathbb{R})$ be a 2x2 matrix with eigenvalues λ_1, λ_2.

a) If $\lambda_1 \neq \lambda_2$, then

$$\boxed{\exp(tA) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} A}$$

b) If $\lambda_1 = \lambda_2 = \lambda$, then

$$\boxed{\exp(tA) = e^{\lambda t}(1 - \lambda t) I + t e^{\lambda t} A}$$

EXERCISES

① Write the exact solution for the following systems

a) $\begin{cases} \dot{x}_1 = 4x_1 + x_2 \\ \dot{x}_2 = -2x_1 + x_2 \end{cases}$

b) $\begin{cases} \dot{x}_1 = -5x_1 - x_2 \\ \dot{x}_2 = x_1 - 3x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_1 + x_2 \end{cases}$

d) $\begin{cases} \dot{x}_1 = 2x_1 + x_2 + 3x_3 \\ \dot{x}_2 = x_1 + 2x_2 + 3x_3 \\ \dot{x}_3 = 3x_1 + 3x_2 + 20x_3 \end{cases}$

② Show that

$$\exp\left(\vartheta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

③ Let $R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$

Show that $R(\vartheta)$ has real-only eigenvalues if and only if $\sin\vartheta = 0$.

④ Let $A \in M_n(\mathbb{R})$ with $A^2 = I$.
Show that: $\lambda \in \lambda(A) \Rightarrow \lambda = 1$ or $\lambda = -1$

⑤ Let $A \in M_n(\mathbb{R})$ with $\det A \neq 0$. Show that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}. Can a non-singular matrix have $\lambda = 0$ as an eigenvalue?

Lyapunov function for $\dot{x} = Ax$

- Consider the linear autonomous system $\dot{x} = Ax$. If $\det A \neq 0$, then $Ax = 0 \Leftrightarrow x = 0$. Thus $x = 0$ is the unique fixed point. It's stability can be investigated by constructing an appropriate Lyapunov function.

Definition of $V(x)$

Let $x, y \in \mathbb{C}^n$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. We define the inner product:

$$\boxed{\langle x | y \rangle = \overline{x_1}\, y_1 + \overline{x_2}\, y_2 + \cdots + \overline{x_n}\, y_n}$$

The bar (e.g. $\bar{x}$) represents the complex conjugate. We note that

$$|\langle x | y \rangle|^2 = \langle x | y \rangle \langle y | x \rangle .$$

For the matrix $A = [A_{ab}]$ we define the Hermitian matrix $A^H = [\overline{A}_{ba}]$. It can then be shown that

$$\langle x | Ay \rangle = \langle A^H x | y \rangle$$
$$\langle Ax | y \rangle = \langle x | A^H y \rangle$$

Let λ_a be the eigenvalues of A with eigenvectors u_a for $a \in \{1, 2, 3, \ldots, n\}$. Also, let $\overline{\lambda_a}$ be the

eigenvalues of A^H with eigenvectors v_a.
We define the Lyapunov function $V(x)$ as:

$$\boxed{V(x) = \sum_a b_a |\langle v_a | x \rangle|^2}$$

Here $b_a > 0$ are arbitrary positive constants.
The sum runs from $a = 1, 2, 3, \ldots, n$.
By definition, it is easy to see that

$$V(0) = 0$$
$$x \neq 0 \Rightarrow V(x) > 0.$$

Stability theorem

$$\boxed{\begin{array}{l} \mathrm{Re}(\lambda_a) \leqslant 0, \forall a \Rightarrow x = 0 \text{ is Lyapunov stable} \\ \mathrm{Re}(\lambda_a) < 0, \forall a \Rightarrow x = 0 \text{ is asymptotically stable} \end{array}}$$

Proof

We note that

$$\langle v_a | Ax \rangle = \langle A^H v_a | x \rangle = \langle \overline{\lambda_a} v_a | x \rangle =$$
$$= \lambda_a \langle v_a | x \rangle$$

and

$$\langle Ax | v_a \rangle = \langle x | A^H v_a \rangle = \langle x | \lambda_a v_a \rangle =$$
$$= \overline{\lambda_a} \langle x | v_a \rangle$$

It follows that:

$$\frac{dV}{dt} = \frac{d}{dt}\sum_a b_a |\langle v_a|x\rangle|^2 =$$

$$= \frac{d}{dt}\sum_a b_a \langle v_a|x\rangle\langle x|v_a\rangle =$$

$$= \sum_a \left[b_a\left(\frac{d}{dt}\langle v_a|x\rangle\right)\langle x|v_a\rangle + b_a\langle v_a|x\rangle\left(\frac{d}{dt}\langle x|v_a\rangle\right)\right]$$

$$= \sum_a b_a\left[\langle v_a|Ax\rangle\langle x|v_a\rangle + \langle v_a|x\rangle\langle Ax|v_a\rangle\right] =$$

$$= \sum_a b_a\left[\lambda_a\langle v_a|x\rangle\langle x|v_a\rangle + \langle v_a|x\rangle(\overline{\lambda_a}\langle x|v_a\rangle)\right]$$

$$= \sum_a b_a(\lambda_a+\overline{\lambda_a})\langle v_a|x\rangle\langle x|v_a\rangle =$$

$$= \sum_a 2b_a \operatorname{Re}(\lambda_a)|\langle v_a|x\rangle|^2$$

For $x \neq 0$, $|\langle v_a|x\rangle|^2 > 0$, and by definition $b_a > 0$ for all a. Recall that $V(0)=0$ and $V(x)>0$ for $x\neq 0$

a) If $\operatorname{Re}(\lambda_a) \leq 0 \Rightarrow dV/dt \leq 0 \Rightarrow$

$\Rightarrow x=0$ Lyapunov stable.

b) If $\operatorname{Re}(\lambda_a) < 0 \Rightarrow dV/dt < 0 \Rightarrow$

$\Rightarrow x=0$ asymptotically stable. □

↳ A matrix A whose eigenvalues satisfy $\operatorname{Re}(\lambda_a)<0$, $\forall a$ is called <u>negative-definite</u>

Assuming $A \in M_n(\mathbb{R})$, it can be shown that

A negative-definite $\Rightarrow \forall x \in \mathbb{R}^n: \langle x|Ax\rangle < 0$

EXERCISES

⑥ a) Show that if $A + A^H$ is negative-definite then $V(x) = \langle x | x \rangle$ is a Lyapunov function of the system $\dot{x} = Ax$.

b) Consider the 2×2 case:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$$

Show that if $a + d < 0$ and $4ad > (b+c)^2$ then $A + A^H$ is negative-definite.

⑦ Consider the system

$$\begin{cases} \dot{x}_1 = \mu x_1 - x_2 \\ \dot{x}_2 = x_1 + (\mu+1) x_2 \end{cases}$$

a) Show that $(x_1, x_2) = (0,0)$ is the unique fixed-point of the system for all $\mu \in \mathbb{R}$.

b) Show that if $2\mu + 1 < 0$, then $(x_1, x_2) = (0,0)$ is an asymptotically stable fixed-point.

c) What happens when $2\mu + 1 = 0$.

▼ The 2×2 linear autonomous system

Consider the 2×2 linear autonomous system:

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The eigenvalues λ_1, λ_2 of A are found by solving the equation:

$$\begin{aligned} \det(A - \lambda I) = 0 &\Leftrightarrow (a-\lambda)(d-\lambda) - bc = 0 \Leftrightarrow \\ &\Leftrightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \\ &\Leftrightarrow \lambda^2 - \tau\lambda + D = 0 \end{aligned}$$

with: $\tau = \operatorname{tr} A = a + d = \lambda_1 + \lambda_2$

$D = \det A = ad - bc = \lambda_1 \lambda_2$

The solution reads:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

The general solution of the system reads

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

with v_1, v_2 the eigenvectors corresponding to the eigenvalues λ_1, λ_2.

We note that an eigenvalue λ that satisfies

a) $Re(\lambda) < 0 \longrightarrow$ Gives a contribution that vanishes thus approaching the fixed-point.

b) $Re(\lambda) > 0 \longrightarrow$ Gives a contribution that diverges away from the fixed-point.

c) $Re(\lambda) = 0 \longrightarrow$ Gives a contribution that neither approaches nor diverges from the fixed-point.

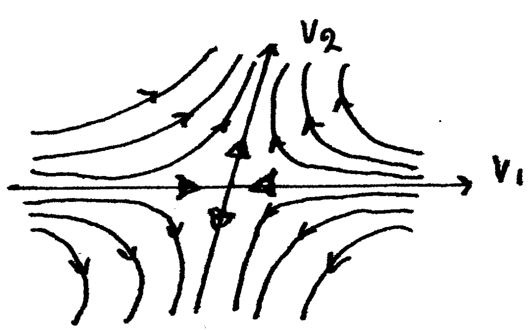
d) $Im(\lambda) \neq 0 \longrightarrow$ Gives a contribution that spirals around the fixed point.

e) $Im(\lambda) = 0 \longrightarrow$ Gives a contribution that does not spiral around the fixed point.

Based on that, we classify the $(0,0)$ fixed point as follows:

Classification of fixed-points in 2d

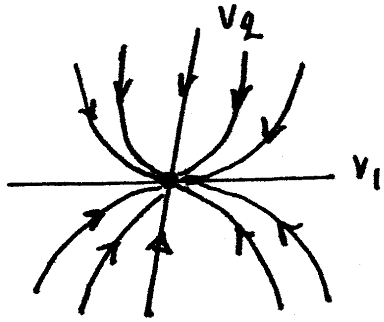
1) Saddle node :



- Eigenvalue condition: $\lambda_1 \lambda_2 < 0$
- τ-D condition: $D < 0$
- Unstable

The shape of the saddle node is controlled by the eigenvectors v_1, v_2.

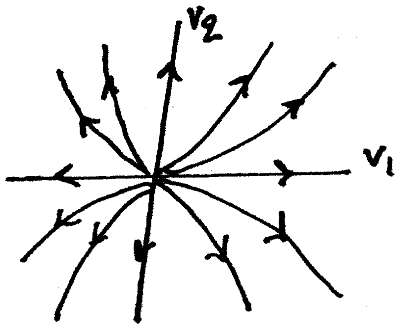
2) <u>Sink</u> :



- Eigenvalue condition
  $\lambda_1, \lambda_2 \in \mathbb{R} \wedge \lambda_1 < 0 \wedge \lambda_2 < 0$
- τ-D condition:
  $D > 0 \wedge \tau^2 - 4D > 0 \wedge \tau < 0$
- Exponentially stable

(v_1 slow, v_2 fast : $|\lambda_1| < |\lambda_2|$)

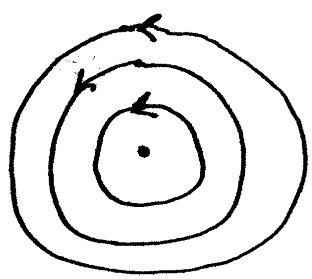
3) <u>Source</u> :



- Eigenvalue condition:
  $\lambda_1, \lambda_2 \in \mathbb{R} \wedge \lambda_1 > 0 \wedge \lambda_2 > 0$
- τ-D condition:
  $D > 0 \wedge \tau^2 - 4D > 0 \wedge \tau > 0$
- Unstable

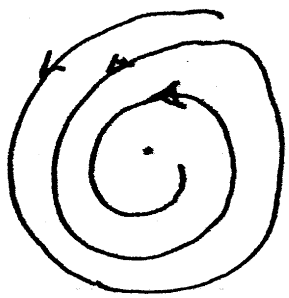
(v_1 fast , v_2 slow : $\lambda_1 > \lambda_2 > 0$)

4) <u>Center</u> :



- Eigenvalue condition
  $$\begin{cases} Re(\lambda_1) = Re(\lambda_2) = 0 \\ Im(\lambda_1) \neq 0 \wedge Im(\lambda_2) \neq 0 \end{cases}$$
- τ-D condition
  $D > 0 \wedge \tau = 0$
- Neutrally stable
  (i.e. Lyapunov stable but not attracting)
- Note that $D > 0 \wedge \tau = 0 \Rightarrow \Delta < 0$

5) <u>Stable spiral</u>:



- Eigenvalue condition

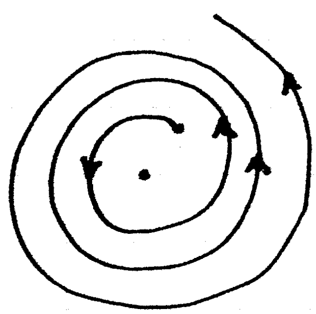
$$\begin{cases} Re(\lambda_1) < 0 \wedge Re(\lambda_2) < 0 \\ Im(\lambda_1) \neq 0 \wedge Im(\lambda_2) \neq 0 \end{cases}$$

- τ-D condition

$D > 0 \wedge \tau^2 - 4D < 0 \wedge \tau < 0$

- Exponentially stable

6) <u>Unstable spiral</u>:



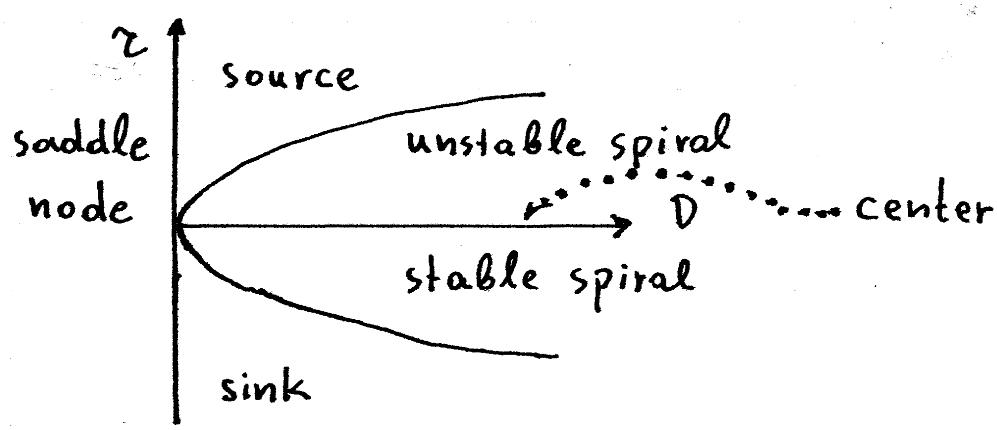
- Eigenvalue condition

$$\begin{cases} Re(\lambda_1) > 0 \wedge Re(\lambda_2) > 0 \\ Im(\lambda_1) \neq 0 \wedge Im(\lambda_2) \neq 0 \end{cases}$$

- τ-D condition

$D > 0 \wedge \tau^2 - 4D < 0 \wedge \tau > 0$

→ <u>Summary of τ-D conditions</u>



$D < 0$: saddle point

$D > 0$: $\tau = 0$: center

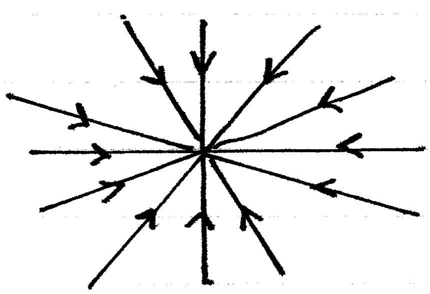
$\tau^2 - 4D > 0$: source $(\tau > 0)$, sink $(\tau < 0)$

$\tau^2 - 4D < 0$: spiral, stable $(\tau < 0)$ or unstable $(\tau > 0)$

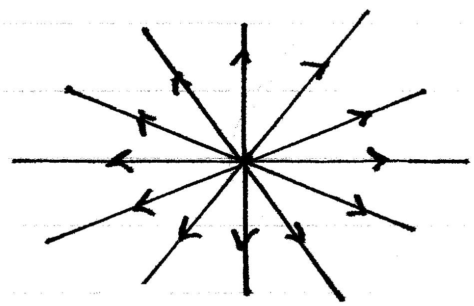
Borderline nodes

Borderline nodes occur when $\lambda_1 = \lambda_2$ which occurs when $\tau^2 - 4D = 0$. Let $E_\lambda = \{v \in \mathbb{R}^2 \mid Av = \lambda v\}$ be the eigenspace associated with the eigenvalue $\lambda = \lambda_1 = \lambda_2$. We distinguish between two cases: $\dim E_\lambda = 1$ or $\dim E_\lambda = 2$.

7) Stars

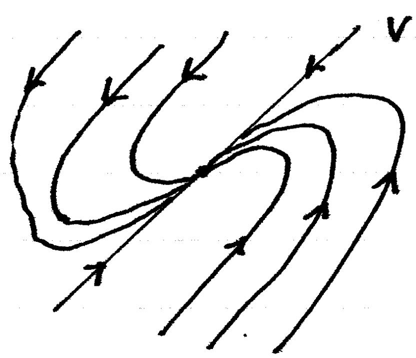


$\lambda_1 = \lambda_2 < 0$
$\dim E_\lambda = 2$

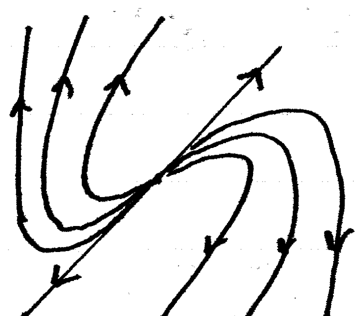


$\lambda_1 = \lambda_2 > 0$
$\dim E_\lambda = 2$

8) Degenerate nodes



$\lambda_1 = \lambda_2 < 0$
$\dim E_\lambda = 1$



$\lambda_1 = \lambda_2 > 0$
$\dim E_\lambda = 1$.

EXAMPLES

a) $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = 4x_1 - 2x_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(-2-\lambda) - 4 = -2 - \lambda + 2\lambda + \lambda^2 - 4 =$$

$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0 \Leftrightarrow \underline{\lambda = -3 \vee \lambda = 2}.$$

Since $\begin{cases} \lambda_1, \lambda_2 \in \mathbb{R} \\ \lambda_1 \lambda_2 < 0 \end{cases} \Rightarrow (0,0)$ is a saddle-node.

- To draw a phase portrait we need the eigenvectors.
  In general, for eigenvalue λ
  $Av = \lambda v \Leftrightarrow (A - \lambda I)v = 0 \Leftrightarrow \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  For $\lambda_1 = 2$:
  $\begin{bmatrix} 1-2 & 1 \\ 4 & -2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} -x + y = 0 \\ 4x - 4y = 0 \end{cases} \Leftrightarrow y = x$

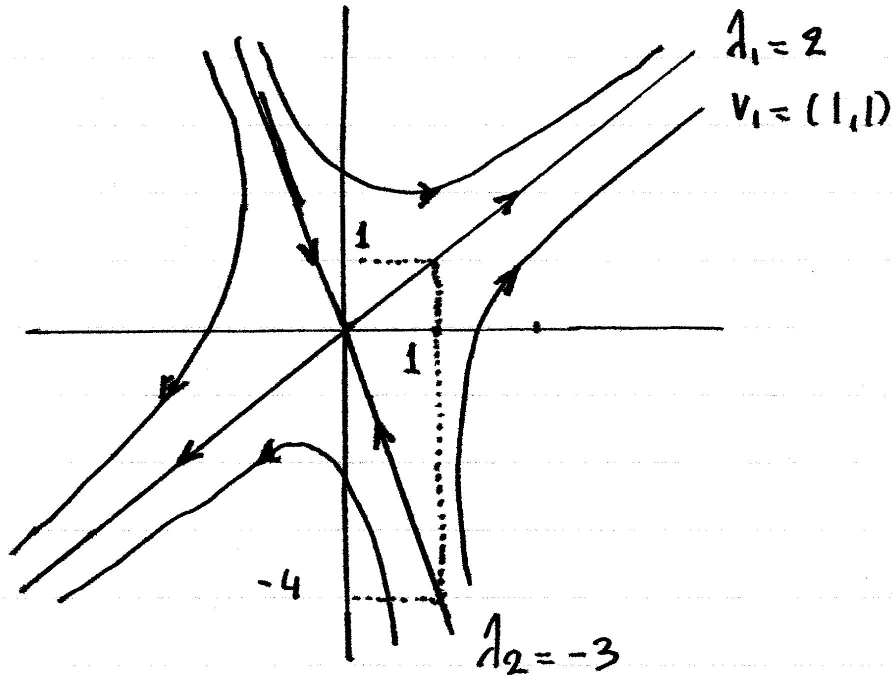
  $\Leftrightarrow (x,y) = (x,x) = x(1,1)$

  thus $v_1 = (1,1)$.

  For $\lambda_2 = -3$:
  $\begin{bmatrix} 1-(-3) & 1 \\ 4 & -2-(-3) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 4x + y = 0 \\ 4x + y = 0 \end{cases} \Leftrightarrow$

  $\Leftrightarrow y = -4x \Leftrightarrow (x,y) = (x,-4x) = x(1,-4)$

  thus $v_2 = (1,-4)$



"saddle
node"

b) $\begin{cases} \dot{x}_1 = x_1 - 2x_2 \\ \dot{x}_2 = 2x_1 - x_2 \end{cases} \leftarrow A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(-1-\lambda) + 4 = -1 - \lambda + \lambda + \lambda^2 + 4 =$$

$$= \lambda^2 + 3 = 0 \Leftrightarrow \lambda = \pm i\sqrt{3}$$

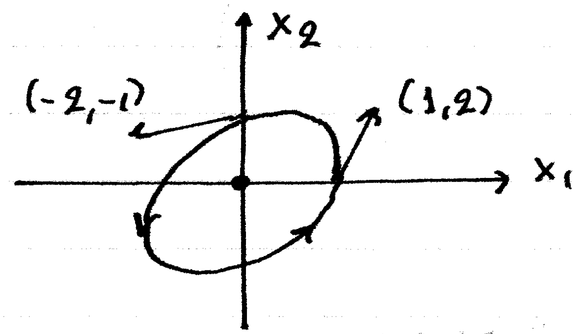
It follows that $(0,0)$ is a center.

- Clockwise or counterclockwise?

The direction of the orbits can be determined by calculating Ax with x a unit vector:

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



When a linear system has a center, the shape of the orbits can be derived by noting that

$$V(x) = ax_1^2 + bx_1x_2 + cx_2^2$$

with appropriate choice of a, b, c remains constant along the orbits around $(0,0)$.
This $V(x)$ is a Lyapunov function.

For this problem:

$V(x) = ax_1^2 + bx_1x_2 + cx_2^2 \Rightarrow$

$\Rightarrow dV(x)/dt = 2ax_1\dot{x}_1 + b(\dot{x}_1x_2 + x_1\dot{x}_2) + 2cx_2\dot{x}_2 =$

$= 2ax_1(x_1 - 2x_2) + b[(x_1-2x_2)x_2 + x_1(2x_1 - x_2)] + 2cx_2(2x_1 - x_2)$

$= 2ax_1^2 - 4ax_1x_2 + bx_1x_2 - 2bx_2^2 + 2bx_1^2 - bx_1x_2 + 4cx_1x_2 - 2cx_2^2$

$= (2a+2b)x_1^2 + (-4a+b-b+4c)x_1x_2 - 2(b+c)x_2^2 =$

$= 2(a+b)x_1^2 + 4(c-a)x_1x_2 - 2(b+c)x_2^2$

Require:

$$\begin{cases} a+b=0 \\ c-a=0 \\ b+c=0 \end{cases} \Leftrightarrow \begin{cases} c-c=0 \\ a=c \\ b=-c \end{cases} \Leftrightarrow \begin{cases} a=c \\ b=-c \end{cases} \Leftrightarrow (a,b,c) = c(1,-1,1)$$

Choose: $(a,b,c) = (1,-1,1)$, thus

$V(x) = x_1^2 - x_1x_2 + x_2^2$.

Center orbits have equation:

$$\boxed{(C):\ x_1^2 - x_1x_2 + x_2^2 = C_1}$$

c) $\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = 3x_1 \end{cases} \longleftarrow A = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - (-1)\cdot 3$$

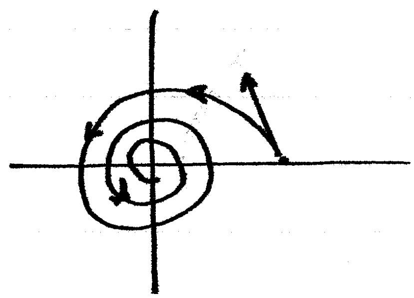
$$\left.\begin{aligned} &= \lambda + \lambda^2 + 3 = \lambda^2 + \lambda + 3 = 0 \\ &\Delta = 1 - 12 = -11 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{11}}{2}$$

Since $\lambda_{1,2}$ are complex and $Re(\lambda_1) = Re(\lambda_2) < 0$ it follows that $(0,0)$ is <u>stable spiral.</u>

Since

$$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

the direction is counterclockwise.



d) $\begin{cases} \dot{x}_1 = 4x_1 - x_2 \\ \dot{x}_2 = -x_1 + 4x_2 \end{cases} \longleftarrow A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1$$

$$\left.\begin{aligned} &= 16 - 8\lambda + \lambda^2 - 1 = \lambda^2 - 8\lambda + 15 \\ &\Delta = 64 - 4\cdot 15 = 64 - 60 = 4 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{8 \pm 2}{2} = \begin{cases} 5 \\ 3 \end{cases}$$

thus $(0,0)$ is a source.

- Eigenvalues:

For $\lambda_1 = 3$:

$Ax = 3x \Leftrightarrow \begin{cases} 4x_1 - x_2 = 3x_1 \\ -x_1 + 4x_2 = 3x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2 \Leftrightarrow (x_1, x_2) = (1,1)x_1$

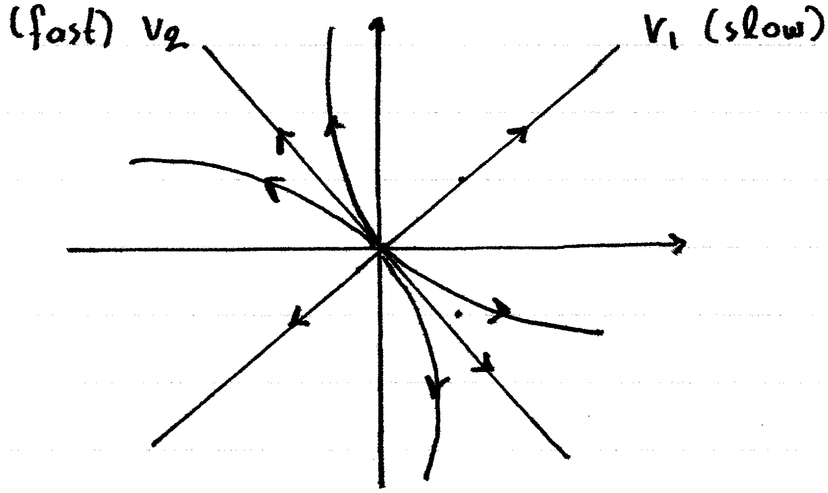
Choose $v_1 = (1,1)$.

For $\lambda_2 = 5$:

$Ax = 5x \Leftrightarrow \begin{cases} 4x_1 - x_2 = 5x_1 \\ -x_1 + 4x_2 = 5x_2 \end{cases} \Leftrightarrow \begin{cases} -x_1 - x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow x_1 = -x_2 \Leftrightarrow (x_1, x_2) = (-1,1)x_2$

Choose $v_2 = (-1,1)$.



EXERCISES

⑧ Classify the fixed-point of the following systems and draw the phase-portrait.

a) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 3x_2 \end{cases}$

b) $\begin{cases} \dot{x}_1 = 5x_1 + 10x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = 3x_1 - 4x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$

d) $\begin{cases} \dot{x}_1 = -3x_1 + 2x_2 \\ \dot{x}_2 = x_1 - 2x_2 \end{cases}$

e) $\begin{cases} \dot{x}_1 = 5x_1 + 2x_2 \\ \dot{x}_2 = -17x_1 - 5x_2 \end{cases}$

f) $\begin{cases} \dot{x}_1 = -3x_1 + 4x_2 \\ \dot{x}_2 = -2x_1 + 3x_2 \end{cases}$

g) $\begin{cases} \dot{x}_1 = 4x_1 - 3x_2 \\ \dot{x}_2 = 8x_1 - 6x_2 \end{cases}$

h) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 2x_2 \end{cases}$

⑨ Consider the system

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = ax_2 \end{cases}$$

with $a \neq 0$ and $b \neq 0$. Show that $(0,0)$ is a degenerate node.