LIMIT CYCLES

- A limit cycle is a closed orbit that functions as a stable or unstable fixed point.
- Recall that $x(t)$ is a closed orbit if and only if:

$$\boxed{\exists T > 0 : \forall t \in \mathbb{R} : x(t+T) = x(t).}$$

Ruling out closed orbits

We use the following techniques to rule out the existence of closed orbits.

(1) → Index theory.

We have already shown previously that index theory can be used to rule out the existence of closed orbits.

(2) → Gradient systems

- A gradient system is a system of the form $\dot{x} = -\nabla V$ with $x(t) \in \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$.
- A gradient system has no closed orbits.

Proof

Assume $x(t)$ is a closed orbit with $x(T)=x(0)$.
Then $V(x(T))=V(x(0)) \Rightarrow$

$$\Rightarrow V(x(T))-V(x(0)) = \int_0^T \frac{dV}{dt}\, dt =$$

$$= \int_0^T (\nabla V)\cdot(\dot{x}(t))\, dt = \int_0^T (-\dot{x}(t))\cdot(\dot{x}(t))\, dt =$$

$$= \int_0^T -\|\dot{x}(t)\|^2 dt = 0 \Rightarrow$$

$\Rightarrow \forall t \in (0,T): \dot{x}(t)=0 \Rightarrow \forall t\in(0,T): x(t)=x(0)$
$\Rightarrow x(0)$ fixed point. $\Rightarrow x(t)$ not an orbit. $\square$

EXAMPLE

Show that
$$\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = x_1 \cos x_2 \end{cases}$$
has no closed orbits.

Solution

Let $f_1(x_1,x_2)=\sin x_2$ and $f_2(x_1,x_2)=x_1\cos x_2$
It follows that:

$$\int f_1(x_1,x_2)dx_1 = \int \sin x_2 \, dx_1 = x_1 \sin x_2 + c$$

$$\int f_2(x_1,x_2)\, dx_2 = \int x_1 \cos x_2 \, dx_2 = x_1 \int \cos x_2 \, dx_2 =$$

$$= x_1 \sin x_2 + c$$

Choose $c=0$. For $V(x_1,x_2) = -x_1 \sin x_2 \Rightarrow$

$\Rightarrow \nabla V(x_1,x_2) = -(f_1(x_1,x_2), f_2(x_1,x_2))$

$\Rightarrow$ the system is a gradient system

$\Rightarrow$ there are no closed orbits.

EXERCISES

① Show that the following systems have no closed orbits by showing that they are gradient systems.

a) $\begin{cases} \dot{x}_1 = x_2^2 + x_2 \cos x_1 \\ \dot{x}_2 = 2x_1x_2 + \sin x_1 \end{cases}$

b) $\begin{cases} \dot{x}_1 = 3x_1^2 - 1 - e^{x_2} \\ \dot{x}_2 = -2x_1 e^{2x_2} \end{cases}$

c) $\begin{cases} \dot{x}_1 = -2x_1 e^{x_1^2+x_2^2} \\ \dot{x}_2 = -2x_2 e^{x_1^2+x_2^2} \end{cases}$

3 → Lyapunov functions

- Consider the system $\dot{x} = f(x)$. Assume that there is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

a) $V(x_0) = 0$

b) $V(x) > 0, \forall x \in \mathbb{R}^n - \{x_0\}$

c) $x(t) \neq x_0 \Rightarrow (d/dt) V(x(t)) < 0$.

Then the system has no closed orbits.

Proof

Assume that $x(t)$ is a periodic orbit with $x(0) \neq x_0$. It follows that

$$\exists T > 0 : \forall n \in \mathbb{N} : x(nT) = x(0) \neq x_0. \quad (1)$$

From (a), (b), (c) it follows that x_0 is an asymptotically stable fixed-point with attracting basin $\mathbb{R}^n$, and therefore

$$\lim_{t \to +\infty} x(t) = x_0 \quad (2)$$

From (1) and (2):

$$x(0) = \lim_{n \in \mathbb{N}} x(nT) = \lim_{t \to +\infty} x(t) = x_0 \Rightarrow$$

$$\Rightarrow x(0) = x_0 \leftarrow \text{contradiction.}$$

EXAMPLES

a) Show that the system

$$\begin{cases} \dot{x}_1 = -x_1 + 4x_2 \\ \dot{x}_2 = -x_1 - x_2^3 \end{cases}$$

has no closed orbits.

Solution

We try $V(x_1, x_2) = x_1^2 + a x_2^2$. Then

a) $V(0,0) = 0$ and

b) $V(x_1, x_2) > 0$, $\forall (x_1, x_2) \in \mathbb{R}^2 - \{(0,0)\}$ and

c) $dV/dt = 2x_1\dot{x}_1 + 2ax_2\dot{x}_2 =$

$$= 2x_1(-x_1 + 4x_2) + 2ax_2(-x_1 - x_2^3) =$$
$$= -2x_1^2 + 8x_1x_2 - 2ax_1x_2 - 2ax_2^4 =$$
$$= -2x_1^2 - 2ax_2^4 + 2(4-a)x_1x_2.$$

For $a = 4$: $V(x_1, x_2) = x_1^2 + 4x_2^2$ and

$dV/dt = -2x_1^2 - 2ax_2^4 < 0$, $\forall (x_1, x_2) \in \mathbb{R}^2 - \{(0,0)\}$.

From (a), (b), (c) it follows that the system has no closed orbits.

b) Show that the system $\ddot{x} + (\dot{x})^3 + x = 0$ has no closed orbits.

Solution

We try $V(x, \dot{x}) = x^2 + \dot{x}^2$. We note that

a) $V(0,0) = 0$

b) $V(x,\dot{x}) > 0$ if $(x,\dot{x}) \neq 0$

c) $dV/dt = 2x\dot{x} + 2\dot{x}\ddot{x} = 2\dot{x}(x+\ddot{x}) =$
$= 2\dot{x}(-(\dot{x})^3) = -2(\dot{x})^4 < 0$
for $\dot{x} \neq 0$.

From (a), (b), (c) it follows that the system has no periodic solutions.

EXERCISES

(2) Show that the system

$$\begin{cases} \dot{x}_1 = x_2 - x_1^3 \\ \dot{x}_2 = -x_1 - x_2^3 \end{cases}$$

has no closed orbits.
(Hint: Use $V(x_1, x_2) = ax_1 + bx_2$)

(3) Show that the system

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_2^3 - 2x_2^4 \\ \dot{x}_2 = -x_1 - x_2 + x_1 x_2 \end{cases}$$

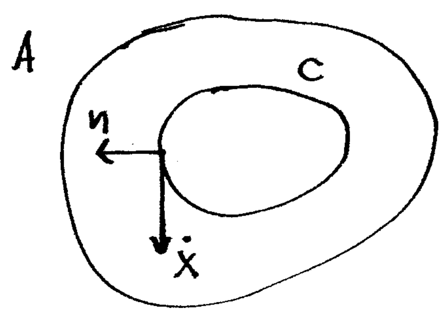
has no closed orbits.
(Hint: Use $V(x_1, x_2) = x_1^m + ax_2^n$)

④ → Dulac's Criterion

Consider the system $\dot{x} = f(x)$ with $f: \mathbb{R}^2 \to \mathbb{R}^2$.
We assume that
a) f continously differentiable in $A \subseteq \mathbb{R}^2$
b) A simply connected
c) $g: A \to \mathbb{R}$ continuously differentiable in A
* d) $\nabla \cdot (g(x) f(x))$ does not change sign in A
(i.e. $\nabla \cdot (g(x) f(x)) > 0, \forall x \in A$ OR
$\nabla \cdot (g(x) f(x)) < 0, \forall x \in A$)
Then, the system has no closed orbits in A.

Proof

Assume that there is a closed orbit C within A.



Since $\dot{x}$ is always tangent to C, then if n is a vector normal to C, then $\dot{x} \cdot n = 0$.
Let B be the interior of C.
Then from (d):

$$I = \iint_B \nabla \cdot (g(x) f(x))\, dx \neq 0. \qquad (1)$$

On the other hand:

$$I = \iint_B \nabla \cdot (f(x) g(x))\, dx = \iint_B \nabla \cdot (g(x)\dot{x})\, dx =$$

$$= \oint_C [g(x)\dot{x}] \cdot n\, dl = \oint_C g(x)\,(\dot{x} \cdot n)\, dl = 0 \quad (2)$$

since everywhere $\dot{x} \cdot n = 0$. However, (1) and (2) contradict thus there are no closed orbits.

→ There is no general method for finding the function $g(x_1, x_2)$. We usually try out the following possibilities:

$g(x_1, x_2) = 1$

$g(x_1, x_2) = \dfrac{1}{x_1^{a} x_2^{b}}$

$g(x_1, x_2) = \exp(a x_1)$

$g(x_1, x_2) = \exp(b x_2)$

EXAMPLES

a) Show that the system

$$\begin{cases} \dot{x}_1 = x_1 (2 - x_1 - x_2) \\ \dot{x}_2 = x_2 (4x_1 - x_1^2 - 3) \end{cases}$$

has no closed orbits in the first quadrant.

Solution

Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0 \wedge x_2 > 0\}$

Choose $g(x) = \dfrac{1}{x^a y^b}$. Then

$$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial x_1}\left[\frac{x_1(2 - x_1 - x_2)}{x_1^a x_2^b}\right] + \frac{\partial}{\partial x_2}\left[\frac{x_2(4x_1 - x_1^2 - 3)}{x_1^a x_2^b}\right]$$

$$= \frac{1}{x_2^b}\frac{\partial}{\partial x_1}\left[2x_1^{1-a} - x_1^{2-a} - x_2 x_1^{1-a}\right] + \frac{\partial}{\partial x_2}\left[x_2^{1-b}\,\frac{4x_1 - x_1^2 - 3}{x_1^a}\right]$$

$$= \frac{1}{x_2^b}\left[(2 - x_2)(1-a)x_1^{-a} - (2-a)x_1^{1-a}\right] + \frac{4x_1 - x_1^2 - 3}{x_1^a}(1-b)x^{-b}$$

For $b = 1$: the 2nd term vanishes.

For $a = 1$: the 1st term loses x_1 dependence.

Then:

$$\nabla \cdot (g\dot{x}) = \frac{1}{x_2}\left[0 - (2-1)x_1^{1-1}\right] = \frac{-1}{x_2} < 0, \ \forall (x_1, x_2) \in A.$$

b) Show that the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + x_1^2 + x_2^2 \end{cases}$$

has no closed orbits.

<u>Solution</u>

For $g(x) = e^{a x_1}$:

$$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial x_1}\left[e^{ax_1} x_2\right] + \frac{\partial}{\partial x_2}\left[e^{ax_1}(-x_1 - x_2 + x_1^2 + x_2^2)\right]$$

$$= a x_2 e^{ax_1} + e^{ax_1}(-0 - 1 + 0 + 2x_2) =$$

$$= e^{ax_1}\left[a x_2 - 1 + 2x_2\right] =$$

$$= e^{ax_1}\left[(a+2)x_2 - 1\right]$$

For $a = -2$: $\nabla \cdot (g\dot{x}) = e^{-2x_1} \cdot (0 - 1) =$

$$= -e^{-2x_1} < 0 \,, \; \forall (x_1, x_2) \in \mathbb{R}^2$$

$\Rightarrow$ no closed orbits. $\square$

EXERCISES

④ Show that the system

$$\begin{cases} \dot{x}_1 = a_1 x_1 - b_1 x_1^2 - c_1 x_1 x_2 \\ \dot{x}_2 = a_2 x_2 - b_2 x_2^2 - c_2 x_1 x_2 \end{cases}$$

with $a_1, a_2, b_1, b_2, c_1, c_2 \in (0, +\infty)$ has no closed orbits in the first quadrant.

(Use: $g(x_1, x_2) = \dfrac{1}{x_1 x_2}$)

⑤ Show that the system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 - 1 \\ \dot{x}_2 = x_2 (x_1 - 1) \end{cases}$$

has no closed orbits in the first quadrant.

(Use: $g(x_1, x_2) = \dfrac{1}{x_1^2 x_2}$)

Definition of orbital stability

- Limit cycles are isolated periodic orbits that act as a global fixed point that may be stable or unstable. We are therefore interested in defining the concept of orbital stability for limit cycles.
- Consider the system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$. Let $x(t|x_0, t_0)$ be the solution with initial condition $x(t_0) = x_0$. We define the orbit:
$$\Gamma_+(x_0, t_0) = \{x(t|x_0, t_0) \mid t > t_0\}.$$
- Point-set distance.
Let $p \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$. We define the distance $d(p, A)$ of p from A as:
$$d(p, A) = \inf_{x \in A} \|x - p\|.$$

Stability definitions

① $\Gamma_+(x_0, t_0)$ orbitally stable $\Leftrightarrow$
$$\boxed{(\forall \varepsilon > 0 : \exists \delta > 0 : \forall y_0 \in B(x_0, \delta) : \forall t > t_0 : d(x(t|y_0, t_0), \Gamma_+(x_0, t_0)) < \varepsilon)}$$

(2) $\Gamma_+(x_0,t_0)$ <u>orbitally attracting</u> $\Leftrightarrow$
$(\exists \delta > 0 : \forall y_0 \in B(x_0,\delta) :$
$: \lim_{t \to +\infty} d(x(t|y_0,t_0), \Gamma_+(x_0,t_0)) = 0$

(3) $\Gamma_+(x_0,t_0)$ <u>asymptotically orbitally stable</u> $\Leftrightarrow$

$$\begin{cases} \Gamma_+(x_0,t_0) \text{ orbitally stable} \\ \Gamma_+(x_0,t_0) \text{ orbitally attracting.} \end{cases}$$

(4) $\Gamma_+(x_0,t_0)$ <u>neutrally orbitally stable</u> $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} \Gamma_+(x_0,t_0) \text{ orbitally stable} \\ \Gamma_+(x_0,t_0) \text{ NOT orbitally attracting.} \end{cases}$$

(5) $\Gamma_+(x_0,t_0)$ <u>orbitally unstable</u> $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} \Gamma_+(x_0,t_0) \text{ NOT orbitally stable} \\ \Gamma_+(x_0,t_0) \text{ NOT orbitally attracting.} \end{cases}$$

- <u>Stability regions</u>

- Let C be an orbitally attracting limit cycle. We define the <u>basin of attraction</u> $\mathcal{B}(C)$ of C as:

$$\boxed{\mathcal{B}(C) = \{y_0 \in \mathbb{R}^n \mid \lim_{t \to +\infty} d(x(t|y_0,0), C) = 0\}}$$

- Let $A \subseteq \mathbb{R}^n$ be a simple bounded region. We say that

$$\boxed{A \text{ trapping region} \Leftrightarrow \forall y_0 \in A : \forall t > 0 : x(t \mid y_0, 0) \in A}$$

The Poincare - Bendixson theorem

Consider the two-dimensional autonomous system:

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$

and let $x(t) = (x_1(t), x_2(t))$.

Now we introduce the following notation:

- Trajectories through $x(0)$:

  $\Gamma(x_0, S) = \{x(t) \mid x(0) = x_0 \wedge t \in S\}$ with $S \subseteq \mathbb{R}$.

  $\Gamma(x_0) = \Gamma(x_0, \mathbb{R})$

  $\Gamma_+(x_0, t) = \Gamma(x_0, (t, +\infty))$, $\Gamma_+(x_0) = \Gamma(x_0, (0, +\infty))$

- Closure of a set.

  Let $A \subseteq \mathbb{R}^2$ be an open set and let ∂A be it's boundary set. We define the closure of A as:

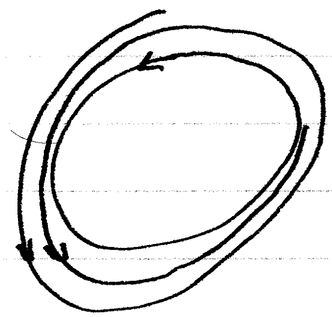
  $$cl(A) = A \cup \partial A.$$

- ω limit set

  Let $x_0 \in \mathbb{R}^2$ be given. The ω limit set of x_0 is defined as:

$$\boxed{\omega(x_0) = \bigcap_{t > 0} cl(\Gamma_+(x_0, t))}$$

▸ <u>interpretation</u> : Consider a periodic orbit C that tends to attract neighbohring orbits. Then $\Gamma_+(x_0,t)$ looks like a thick ring around C that becomes increasingly thinner when $t \to +\infty$. Thus, for x_0 being the starting point of a trajectory attracted to C, $\omega(x_0)$ is equal to the limit cycle C itself.



Note that if $\Gamma(x_0)$ is itself a periodic orbit, then $\Gamma(x_0) = \omega(x_0)$.
Furthermore, if $\Gamma(x_0)$ approaches a fixed point y, then $\omega(x_0) = \{y\}$.

<u>Thm</u> : (<u>Poincare - Bendixson</u>)

> Assume that
> a) $A \subseteq \mathbb{R}^2$ is closed and bounded
> b) A contains no fixed points
> c) $\exists\, x_0 \in A : \Gamma_+(x_0) \subseteq A$
> Then : $\omega(x_0)$ is a periodic orbit.

↳ It follows that either $\Gamma_+(x_0)$ itself is a periodic orbit, in which case $\Gamma_+(x_0) = \omega(x_0)$ or $\Gamma_+(x_0)$ approaches the periodic limit cycle $\omega(x_0)$.

- The immediate consequence of the Poincare-Bendixson theorem is that the only "structures" that can attract orbits in two-dimensional systems are fixed points or periodic limit cycles. In higher dimensional systems, orbits can also be attracted by strange attractors.
- The main difficulty with applying this theorem is in establishing the condition

  $$\exists x_0 \in A : \Gamma_+(x_0) \subseteq A$$

  One way is by locating a <u>trapping region</u>:

<u>Def</u>: Let $A \subseteq \mathbb{R}^2$ be a bounded set. with $N(x,y) \in \mathbb{R}^2$ an outward normal vector defined on ∂A (the boundary of A). We say that A is a <u>trapping region</u> if and only if:

$$N(x,y) \cdot (f(x,y), g(x,y)) < 0, \ \forall (x,y) \in \partial A$$

<u>Prop</u>: If A is a trapping region then:

$$\forall x_0 \in A : \Gamma_+(x_0) \subseteq A$$

↳ In other words, in a trapping region <u>all</u> trajectories inside the region do <u>not</u> leave the region.

EXAMPLE

a) Show that the system

$$\begin{cases} \dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 + x_2 - x_2(x_1^2 + x_2^2). \end{cases}$$

has a limit cycle.

Solution

▸ Fixed points.

We note that $(x_1, x_2) = (0,0)$ is an obvious fixed point. We will now show it is unique. Define:

$f_1(x_1, x_2) = x_1 + x_2 - x_1(x_1^2 + x_2^2)$

$f_2(x_1, x_2) = -x_1 + x_2 - x_2(x_1^2 + x_2^2)$

Using polar coordinates: $x_1 = R\cos\vartheta \wedge x_2 = R\sin\vartheta$ it follows that $x_1^2 + x_2^2 = R^2(\cos^2\vartheta + \sin^2\vartheta) = R^2$ and therefore

$$f_1(x_1, x_2) = R\cos\vartheta + R\sin\vartheta - R\cos\vartheta \cdot R^2 =$$
$$= (R - R^3)\cos\vartheta + R\sin\vartheta = R[(1-R^2)\cos\vartheta + \sin\vartheta]$$

$$f_2(x_1, x_2) = -R\cos\vartheta + R\sin\vartheta - R\sin\vartheta \cdot R^2 = -R\cos\vartheta + (R - R^3)\sin\vartheta$$
$$= R[-\cos\vartheta + (1-R^2)\sin\vartheta].$$

Since $R = 0 \Leftrightarrow (x_1, x_2) = (0,0)$, we assume with no loss of generality that $R \neq 0$. Then:

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases} \Leftrightarrow \begin{cases} R[(1-R^2)\cos\vartheta + \sin\vartheta] = 0 \\ R[-\cos\vartheta + (1-R^2)\sin\vartheta] = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1-R^2)\cos\vartheta + \sin\vartheta = 0 \\ -\cos\vartheta + (1-R^2)\sin\vartheta = 0 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} 1-R^2 & 1 \\ -1 & 1-R^2 \end{bmatrix} \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (1)$$

Since $D = \begin{vmatrix} 1-R^2 & 1 \\ -1 & 1-R^2 \end{vmatrix} = (1-R^2)^2 - (-1) = (1-R^2)^2 + 1 \geq 1 > 0$

it follows that

$(1) \Leftrightarrow \begin{cases} \cos\vartheta = 0 \\ \sin\vartheta = 0 \end{cases} \Leftrightarrow \cos^2\vartheta + \sin^2\vartheta = 0 \leftarrow$ contradiction

Thus, the fixed point $(x_1, x_2) = (0,0)$ is unique

▶ Trapping region

We construct a trapping region bound by two concentric circles centered at $(0,0)$ with radius R_1 and R_2 with $R_1 < R_2$. For a general circle with radius R and a normal vector $N(x_1, x_2)$ oriented outward, we define

$N(x_1, x_2) = (x_1, x_2)$ with $x_1^2 + x_2^2 = R^2$.

It follows that

$N(x_1, x_2) \cdot (f_1(x_1, x_2), f_2(x_1, x_2)) = (x_1, x_2) \cdot (f_1(x_1, x_2), f_2(x_1, x_2))$

$= x_1 f_1(x_1, x_2) + x_2 f_2(x_1, x_2) =$

$= x_1[x_1 + x_2 - x_1(x_1^2 + x_2^2)] + x_2[-x_1 + x_2 - x_2(x_1^2 + x_2^2)]$

$= x_1^2 + x_1 x_2 - x_1^2 R^2 - x_1 x_2 + x_2^2 - x_2^2 R^2 =$

$= (x_1^2 + x_2^2) - R^2(x_1^2 + x_2^2) = R^2 - R^2 R^2 = R^2(1-R^2)$

For $R_2 > 1$, we have $N(x_1, x_2) \cdot (f_1(x_1, x_2), f_2(x_1, x_2)) < 0$, thus the normal vector $N(x_1, x_2)$ points inward to the big circle, and therefore the trapping region.

For $R_1 < 1$, we have $N(x_1, x_2) \cdot (f_1(x_1, x_2), f_2(x_1, x_2)) > 0$, thus the normal vector points outward of the small circle, and

therefore inwards to the trapping region.
Consequently for $R_1 = \sqrt{1/2}$ and $R_2 = \sqrt{2}$, we define
$A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid +1/2 \leq x_1^2 + x_2^2 \leq 2 \}$ and
we have:

$\begin{cases} A \text{ closed and bounded} \\ A \text{ contains no fixed points} \\ A \text{ is a trapping region.} \end{cases}$

It follows from the Poincare-Bendixson theorem that the system has a limit cycle.

Constructing a trapping region

When polor coordinates are not helpful, we can use the nullclines to obtain evidence that there may be a limit cycle and manually construct a trapping region A using the necessary and sufficient condition

$$N(x,y)\cdot(f(x,y),g(x,y))<0\,,\ \forall (x,y)\in\partial A.$$

EXAMPLE

Construct a trapping region for:

$$\begin{cases}\dot{x}_1=-x_1+ax_2+x_1^2x_2 & (1)\\ \dot{x}_2=b-ax_2-x_1^2x_2 & (2)\end{cases}$$

with $a>0$ and $b>0$. Show that the system has limit cycle

Solution

- Nullcline analysis

For (1):

$$f(x_1,x_2)\geqslant 0 \Leftrightarrow -x_1+ax_2+x_1^2x_2\geqslant 0 \Leftrightarrow$$

$$\Leftrightarrow -x_1+x_2(a+x_1^2)\geqslant 0 \Leftrightarrow x_2(a+x_1^2)\geqslant x_1 \Leftrightarrow$$

$$\Leftrightarrow x_2\geqslant \frac{x_1}{a+x_1^2}$$

thus $\dot{x}_1>0$ above curve (1)

$\dot{x}_1<0$ below curve (1)

For (2):

$$g(x_1,x_2) \geq 0 \Leftrightarrow b - a x_2 - x_1^2 x_2 \geq 0 \Leftrightarrow$$

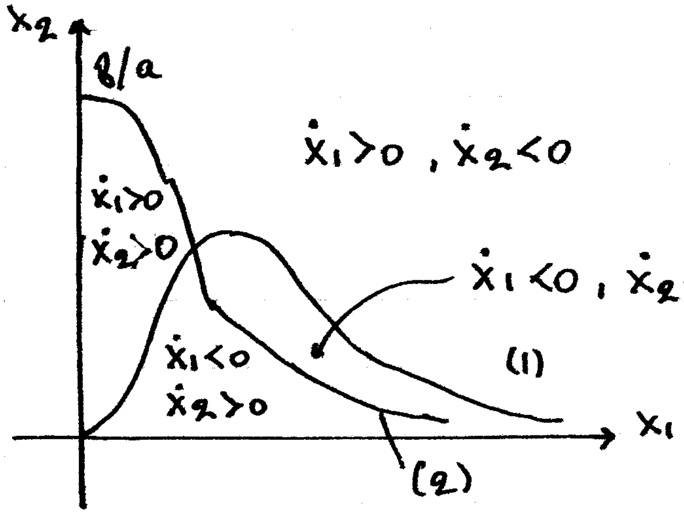
$$\Leftrightarrow b - x_2(a + x_1^2) \geq 0 \Leftrightarrow x_2(a+x_1^2) \leq b \Leftrightarrow$$

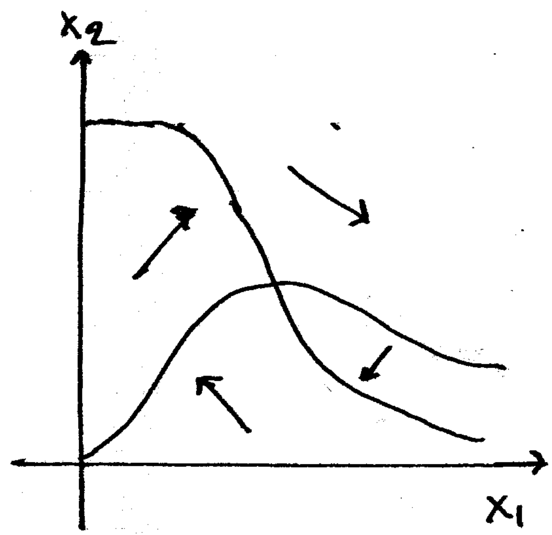
$$\Leftrightarrow x_2 \leq \frac{b}{a+x_1^2}$$

thus $\dot{x}_2 > 0$ below curve (2)

$\dot{x}_2 < 0$ above curve (2).

It follows that:





Thus a limit cycle is indicated.

- <u>Trapping Region</u>

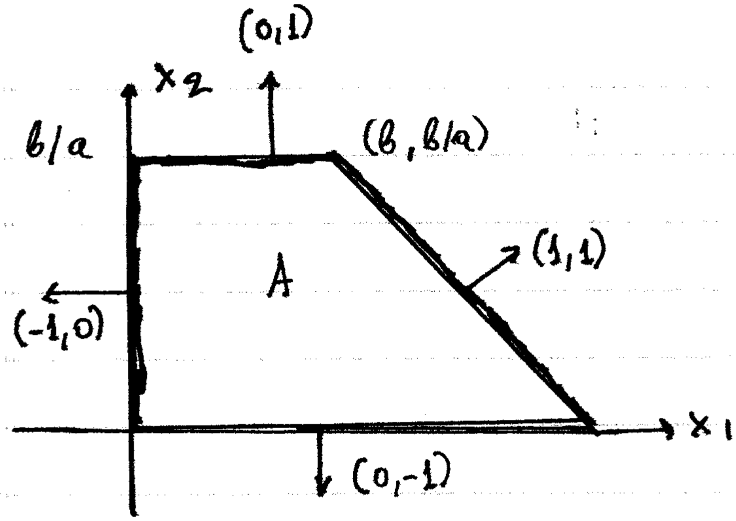
a) x_1-axis with $x_1 > 0$.

At $x_2 = 0$, normal vector is $(0,-1)$ and

$$(0,-1)\cdot(f(x_1,0), g(x_1,0)) = -g(x_1,0) =$$

$$= -(b - a\cdot 0 - x_1^2 \cdot 0) = -b < 0$$

thus x_1-axis is trapping.



The claimed trapping region with the autward normal vectors indicated.

b) At x_2-axis with $x_2>0$
At $x_1=0$, normal vector is $(-1,0)$ and
$(-1,0)\cdot(f(0,x_2), g(0,x_2)) = -f(0,x_2) =$
$= -(-0+ax_2+0x_2) = -ax_2 < 0 \Rightarrow$
$\Rightarrow$ x_2-axis is trapping.

c) At infinity: $\begin{cases} \dot{x}_1 \sim x_1^2 x_2 \\ \dot{x}_2 \sim -x_1^2 x_2 \end{cases} \Rightarrow \dfrac{\dot{x}_1}{\dot{x}_2} \sim -1$

Consider any line with slope -1. Thus $n=(1,1)$.
We note that
$(1,1)\cdot(f(x_1,x_2), g(x_1,x_2)) = f(x_1,x_2)+g(x_1,x_2) =$
$= (-x_1+\underline{\underline{ax_2}}+\underline{x_1^2x_2}) + (b-\underline{\underline{ax_2}}-\underline{x_1^2x_2}) =$
$= b-x_1 < 0$ for $x_1 > b$.

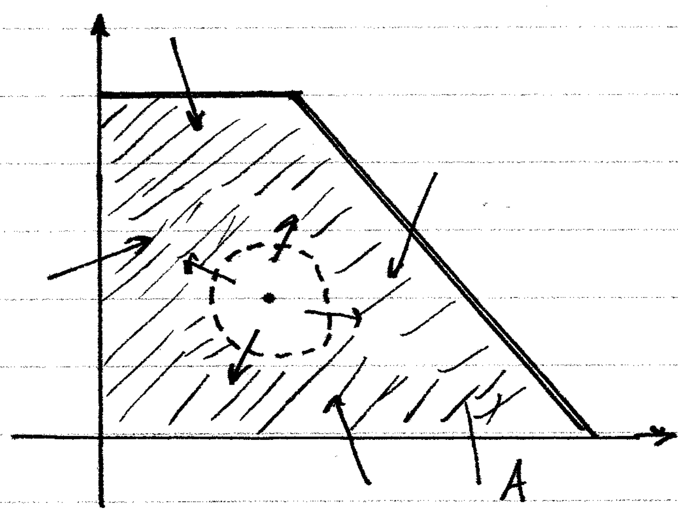
Thus construct a line from $(0,b/a)$ to $(b,b/a)$ and from $(b,b/a)$ to the x_1-axis with slope -1. The latter segment has been shown to be trapping. Now consider the line from $(0,b/a)$ to $(b,b/a)$:

At $x_2 = b/a$ with $0 < x_1 < b$, using $n = (0,1)$

$$(0,1)\cdot(f(x_1, b/a), g(x_1, b/a)) = g(x_1, b/a) =$$
$$= b - a\cdot(b/a) - x_1^2 (b/a) =$$
$$= b - b - x_1^2 b/a = \frac{-b x_1^2}{a} < 0 \Rightarrow \text{also trapping.}$$

Thus the claimed region A is a trapping region.

→ Because the region A contains a fixed point where the two nullclines intersect, we cannot apply the Poincare – Bendixson theorem. However if we show that the fixed point is <u>repelling</u> (i.e. a source, unstable spiral/star, unstable degenerate node), then we can remove a ball around the fixed point and redefine the trapping region as follows:



Thus we will now derive the necessary and sufficient condition for the fixed point to be repelling.

- Fixed point location

$$\begin{cases} f(x_1,x_2)=0 \\ g(x_1,x_2)=0 \end{cases} \Leftrightarrow \begin{cases} -x_1+ax_2+x_1^2x_2=0 \\ b-ax_2-x_1^2x_2=0 \end{cases} \Leftrightarrow$$

Add: $b-x_1=0$

$$\Leftrightarrow \begin{cases} b-x_1=0 \\ b-ax_2-x_1^2x_2=0 \end{cases} \Leftrightarrow \begin{cases} x_1=b \\ b-ax_2-b^2x_2=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1=b \\ b-(a+b^2)x_2=0 \end{cases} \Leftrightarrow \begin{cases} x_1=b \\ x_2=\dfrac{b}{a+b^2} \end{cases}$$

- Jacobian

$$DF(x_1,x_2)=\begin{bmatrix} -1+2x_1x_2 & a+x_1^2 \\ -2x_1x_2 & -(a+x_1^2) \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(b, b/(a+b^2)) = \begin{bmatrix} -1+2b\cdot\dfrac{b}{a+b^2} & a+b^2 \\ -2b\dfrac{b}{a+b^2} & -(a+b^2) \end{bmatrix} =$$

$$= \frac{1}{a+b^2}\begin{bmatrix} -(a+b^2)+2b^2 & (a+b^2)^2 \\ -2b^2 & -(a+b^2)^2 \end{bmatrix} =$$

$$= \frac{1}{a+b^2}\begin{bmatrix} b^2-a & (a+b^2)^2 \\ -2b^2 & -(a+b^2)^2 \end{bmatrix}$$

- <u>Conditions</u>

The fixed point $(b, b/(a+b^2))$ is repelling if and only if $\tau = \lambda_1 + \lambda_2 > 0$ and $D = \lambda_1 \lambda_2 > 0$.

$$\tau = tr(DF(b, b/a)) = \frac{1}{a+b^2}\left[(b^2-a) - (a+b^2)^2\right] =$$

$$= \frac{(b^2-a) - (a^2+2ab^2+b^4)}{(a+b^2)^2} =$$

$$= \frac{b^2 - a - a^2 - 2ab^2 - b^4}{(a+b^2)^2} = \frac{-b^4 + (1-2a)b^2 - (a+a^2)}{(a+b^2)^2}$$

$$D = \det(DF(b, b/(a+b^2))) =$$

$$= \frac{1}{(a+b^2)^2}\left[(b^2-a)\left[-(a+b^2)^2\right] - (-2b^2)(a+b^2)^2\right] =$$

$$= \frac{-(b^2-a)(a+b^2)^2 + 2b^2(a+b^2)^2}{(a+b^2)^2} =$$

$$= \frac{(a+b^2)^2\left[-(b^2-a)+2b^2\right]}{(a+b^2)^2} = -(b^2-a)+2b^2 =$$

$$= b^2 + a.$$

Note that $a > 0 \wedge b > 0 \Rightarrow D = a + b^2 > 0$

thus the necessary and sufficient condition is:

$$\tau > 0 \Leftrightarrow -b^4 + (1-2a)b^2 - (a+a^2) > 0$$

$$\Leftrightarrow b^4 + (2a-1)b^2 + (a+a^2) < 0 \Leftrightarrow p(b^2) < 0$$

with $p(x) \equiv x^2 + (2a-1)x + (a+a^2)$.

Discriminant:

$$\Delta = (2a-1)^2 - 4\cdot 1\cdot (a^2+a) = 4a^2 - 4a + 1 - 4a^2 - 4a = 1 - 8a$$

For $\Delta \leq 0 \Rightarrow p(b^2) \geq 0, \forall b \in \mathbb{R} \Rightarrow \tau \leq 0$

For $\Delta > 0 \Leftrightarrow 1 - 8a > 0 \Leftrightarrow a < 1/8 \Leftrightarrow 0 < a < 1/8$

we have two zeroes:

$$b_{1,2}^2 = \frac{(1-2a) \pm \sqrt{1-8a}}{2}$$

thus: $\tau > 0 \Leftrightarrow p(b^2) < 0 \Leftrightarrow b_1^2 < b^2 < b_2^2 \Leftrightarrow$

$$\Leftrightarrow \frac{1-2a-\sqrt{1-8a}}{2} < b^2 < \frac{1-2a+\sqrt{1-8a}}{2}$$

It follows that the fixed point is repelling when

$$\boxed{\begin{array}{l} 0 < a < 1/8 \\ \dfrac{1-2a-\sqrt{1-8a}}{2} < b^2 < \dfrac{1-2a+\sqrt{1-8a}}{2} \end{array}}$$

EXERCISES

⑥ Use the Poincare - Bendixson theorem to show that the following systems have a limit cycles.

a) $\begin{cases} \dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + 5x_2^2) \\ \dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{cases}$

b) $\begin{cases} \dot{x}_1 = x_1 - x_2 - x_1^3 \\ \dot{x}_2 = x_1 + x_2 - x_2^3 \end{cases}$

c) $\begin{cases} \dot{x}_1 = -x_1 - x_2 + x_1(x_1^2 + 2x_2^2) \\ \dot{x}_2 = x_1 - x_2 + x_2(x_1^2 + 2x_2^2) \end{cases}$

⑦ Consider the system

$$\begin{cases} \dot{x}_1 = x_1(1 - 4x_1^2 - x_2^2) - (1/2)x_2(1 + x_1) \\ \dot{x}_2 = x_2(1 - 4x_1^2 - x_2^2) + 2x_1(1 + x_1) \end{cases}$$

Use the function $V(x_1, x_2) = (1 - 4x_1^2 - x_2^2)^2$ to show that all trajectories converge to $(c): 4x_1^2 + x_2^2 = 1$ as $t \to +\infty$.

⑧ Consider the nonlinear oscillator

$$\ddot{x} + F(x, \dot{x})\dot{x} + x = 0$$

We assume that $F(x, \dot{x}) < 0$ when $z \leq a$, and $F(x,\dot{x}) > 0$ when $z \geq b$, with $z = x^2 + \dot{x}^2$. Show that the system has a limit cycle with $a < z < b$.

⑨ Consider the system

$$\begin{cases} \dot{x}_1 = x_2 + a x_1 (1 - 2b - x_1^2 - x_2^2) \\ \dot{x}_2 = -x_1 + a x_2 (1 - x_1^2 - x_2^2) \end{cases}$$

a) Write the system in polar coordinates.

b) Determine how many limit cycles exist in this system.

Lienard systems

Consider the system governed by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \quad (1)$$

Equivalently, we may rewrite (1) as an autonomaus system as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1)x_2 - g(x_1) \end{cases}$$

Thm: (Lienard's theorem)

Assume that

a) f, g are continuously differentiable in $\mathbb{R}$.

b) f even $(\forall x \in \mathbb{R} : f(-x) = f(x))$

c) g odd $(\forall x \in \mathbb{R} : g(-x) = -g(x))$

d) $g(x) > 0, \ \forall x \in (0, +\infty)$

e) For $F(x)$ given by:

$$F(x) = \int_0^x f(t)dt$$

there is an $a \in (0, +\infty)$ such that:

$$\begin{cases} F(x) < 0, & \forall x \in (0, a) \\ F(a) = 0 \\ F(x) > 0, & \forall x \in (a, +\infty) \end{cases}$$

Then there is a stable limit cycle around the origin $(0,0)$. Furthermore, the limit cycle is <u>unique</u>.

EXAMPLE

a) Show that

$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$ (van der Pol oscillator)

has a limit cycle, for $\mu > 0$.

Solution

Let $f(x) = \mu(x^2-1)$ and $g(x) = x$. We note that:

a) f, g are continuously differentiable

b) $f(-x) = \mu((-x)^2-1) = \mu(x^2-1)$
$= f(x)$, $\forall x \in \mathbb{R} \Rightarrow f$ even

c) $g(-x) = -x = -g(x)$, $\forall x \in \mathbb{R} \Rightarrow g$ odd.

d) $g(x) = x > 0$, $\forall x \in (0, +\infty)$

e) Let

$$F(x) = \int_0^x f(t)dt = \int_0^x \mu(t^2-1)\,dt =$$

$$= \mu\left[\frac{x^3}{3} - x\right]_0^x = \mu\left[\frac{x^3}{3} - x\right] =$$

$$= \frac{\mu}{3}(x^3-3x) = \frac{1}{3}\mu x(x^2-3)$$

For $a = \sqrt{3}$:

$\forall x \in (0, \sqrt{3})$: $x^2-3 < 0 \wedge x > 0 \Rightarrow F(x) < 0$

$$F(\sqrt{3}) = \frac{1}{3}\mu\sqrt{3}\left((\sqrt{3})^2 - 3\right) = 0$$

$$\forall x \in (\sqrt{3}, +\infty): \; x^2 - 3 > 0 \wedge x > 0 \Rightarrow F(x) > 0.$$

From (a), (b), (c), (d), (e) it follows that the system has a stable limit cycle.

EXERCISES

(10) Show that the system

$$\ddot{x} + \mu (x^2 - 1)\dot{x} + \tanh x = 0$$

has a unique stable limit cycle when $\mu > 0$.

(11) Consider the system

$$\ddot{x} + \mu (x^4 - 1)\dot{x} + x = 0$$

a) Show that it has a unique stable limit cycle when $\mu > 0$.

b) If $\mu < 0$, does the system still have a stable limit cycle?

Hopf bifurcation

- Consider the two-dimensional system

$$\begin{cases} \dot{x} = f(x,y,\mu) \\ \dot{y} = g(x,y,\mu) \end{cases} \quad (1)$$

and define

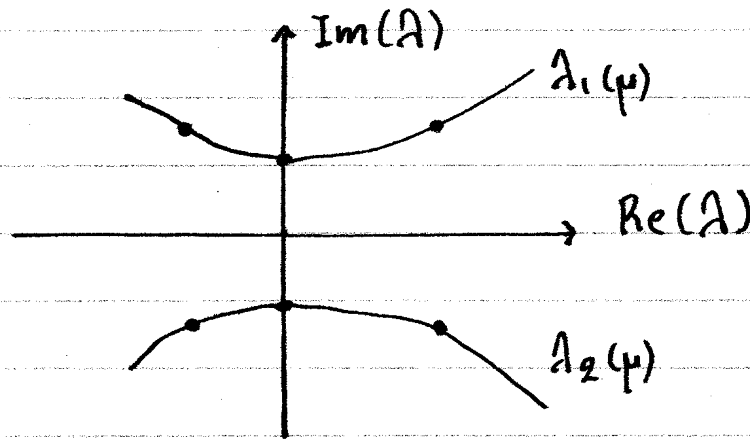
$$F(x,y,\mu) = (f(x,y,\mu), g(x,y,\mu))$$

Let $(x_0(\mu), y_0(\mu))$ be the location of a fixed point, dependant on the parameter μ.
Let $\lambda_1(\mu), \lambda_2(\mu)$ be the eigenvalues of the Jacobian of the above-referenced fixed point:

$$\lambda(DF(x_0(\mu), y_0(\mu))) = \{\lambda_1(\mu), \lambda_2(\mu)\}.$$

- Def: We say that (1) undergoes a Hopf bifurcation at $\mu = \mu_0$ when the following conditions are satisfied:

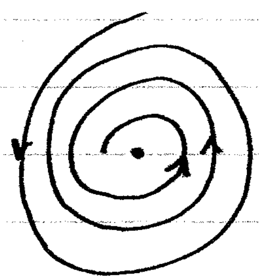
a) $\forall \mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon) : \lambda_{1,2}(\mu) = \gamma(\mu) \pm i\omega(\mu)$
b) $\gamma(\mu_0) = 0$
c) $\gamma(\mu) < 0, \ \forall \mu < \mu_0$
d) $\gamma(\mu) > 0, \ \forall \mu > \mu_0$



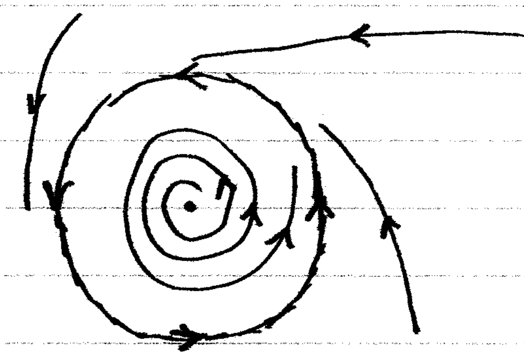
Eigenvalue diagram.

Phenomenology of the Hopf bifurcation

① Supercritical Hopf Bifurcation



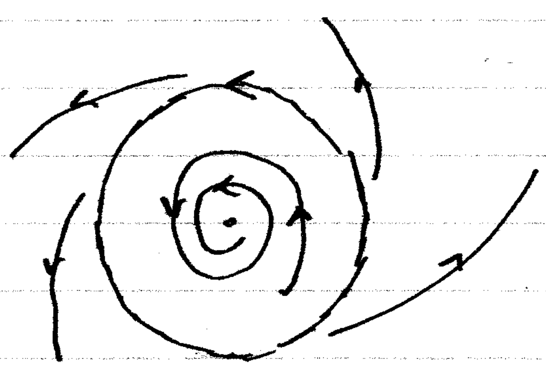
$\mu<\mu_0$



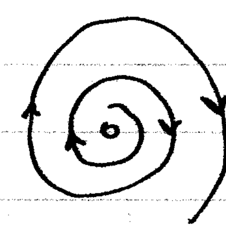
$\mu>\mu_0$

A stable spiral fixed point for $\mu<\mu_0$ becomes an unstable spiral for $\mu>\mu_0$ surrounded by a stable limit cycle which expands with increasing μ.

② Subcritical Hopf Bifurcation



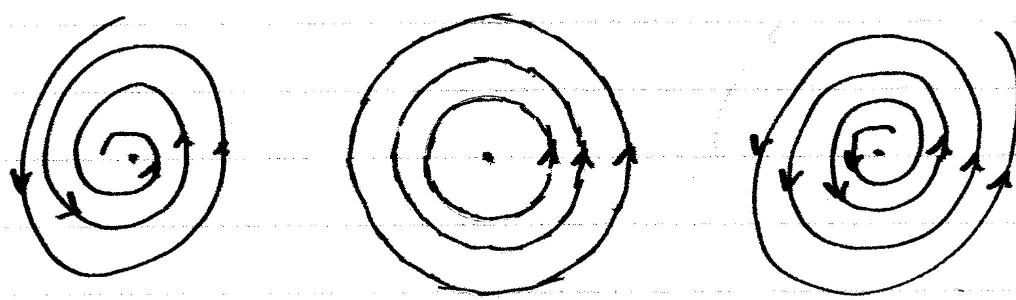
$\mu<\mu_0$



$\mu>\mu_0$

At $\mu < \mu_0$, a stable spiral is bounded by an unstable limit cycle. With increasing μ, the unstable limit cycle becomes smaller. At $\mu = \mu_0$, the cycle collapses onto the fixed point. At $\mu > \mu_0$, the fixed point becomes an unstable spiral.

③ Degenerate Hopf Bifurcation



$\mu < \mu_0$ $\qquad\qquad$ $\mu = \mu_0$ $\qquad\qquad$ $\mu > \mu_0$

A stable spiral at $\mu < \mu_0$ becomes a nonlinear center at $\mu = \mu_0$ and then an unstable spiral at $\mu > \mu_0$. No limit cycle occurs at either $\mu < \mu_0$ or $\mu > \mu_0$.

Hopf bifurcation prototype

Consider the system:

$$\boxed{\begin{cases} \dot{x}_1 = \mu x_1 - x_2 + \sigma x_1 (x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 + \mu x_2 + \sigma x_2 (x_1^2 + x_2^2) \end{cases}}$$

This system undergoes a Hopf bifurcation at $(0,0)$ when $\mu = 0$.

a) $\sigma = +1 \Rightarrow$ subcritical Hopf bifurcation

b) $\sigma = -1 \Rightarrow$ supercritical Hopf bifurcation

c) $\sigma = 0 \Rightarrow$ degenerate Hopf bifurcation.

Proof

$$DF(0,0) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(0,0) - \lambda I) = \begin{vmatrix} \mu - \lambda & -1 \\ 1 & \mu - \lambda \end{vmatrix} =$$

$$= (\mu - \lambda)(\mu - \lambda) - (-1)\cdot 1 = (-\mu + \lambda)^2 + 1 =$$

$$= (-\mu + \lambda + i)(-\mu + \lambda - i) \Rightarrow$$

$$\Rightarrow \lambda(DF(0,0)) = \{ \mu - i, \mu + i \}$$

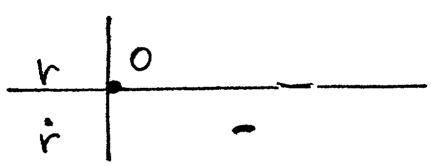
Thus, there is a Hopf bifurcation at $\mu = 0$.

- <u>Polar representation</u>

$$r^2 = x_1^2 + x_2^2 \Rightarrow$$
$$\Rightarrow 2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 =$$
$$= 2x_1\left[\mu x_1 - x_2 + \sigma x_1(x_1^2+x_2^2)\right] + 2x_2\left[x_1 + \mu x_2 + \sigma x_2(x_1^2+x_2^2)\right]$$
$$= 2\mu x_1^2 - \underline{2x_1x_2} + 2\sigma x_1^2 r^2 + \underline{2x_1x_2} + 2\mu x_2^2 + 2\sigma x_2^2 r^2$$
$$= 2\mu(x_1^2+x_2^2) + 2\sigma r^2(x_1^2+x_2^2) = 2\mu r^2 + 2\sigma r^4 \Rightarrow$$
$$\Rightarrow \dot{r} = \sigma r^3 + \mu r \Rightarrow \dot{r} = r(\sigma r^2 + \mu)$$

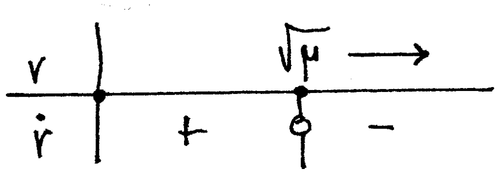
- <u>Analysis</u>

a) For $\sigma = -1$: $\dot{r} = r(-r^2+\mu)$



stable spiral

$\mu < 0$

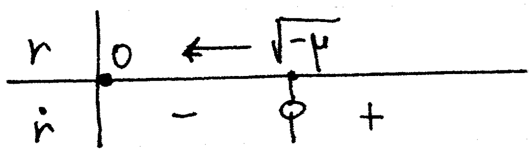


unstable spiral
+ stable limit cycle

$\mu > 0$

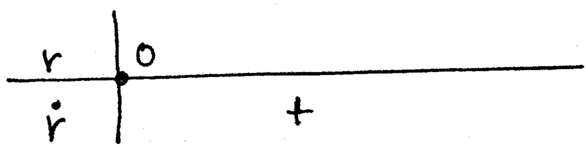
Thus: supercritical Hopf bifurcation

b) For $\sigma = 1$: $\dot{r} = r(r^2+\mu)$



stable spiral + unstable limit cycle.

$\mu < 0$



unstable spiral

$\mu > 0$

Thus: subcritical Hopf bifurcation

c) For $\sigma = 0$: $\boxed{\dot{r} = +\mu r}$

Note that:

$\mu < 0 \Rightarrow \dot{r} < 0 \Rightarrow$ stable spiral

$\mu = 0 \Rightarrow \dot{r} = 0 \Rightarrow$ nonlinear center

$\mu > 0 \Rightarrow \dot{r} > 0 \Rightarrow$ unstable spiral

Thus: degenerate Hopf bifurcation.

• Classifying Hopf Bifurcations

It is rather easy to show that a system undergoes a Hopf bifurcation, simply by examining the Jacobian's eigenvalues as a function of μ. What is more challenging is to determine whether the Hopf bifurcation is supercritical, subcritical, or degenerate. To do that, we work as follows:

•1 Write the system in the form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \begin{bmatrix} f(x, y, \mu) \\ g(x, y, \mu) \end{bmatrix}$$

with f, g nonlinear functions such that

$f(x_0, y_0, \mu_0) = 0$ and $g(x_0, y_0, \mu_0) = 0$

•2 Evaluate the 1st Lyapunov coefficient

$$\Lambda_1 = [f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{w}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right]$$

Here the subscripts denote partial derivatives and all terms are evaluated at $(x, y, \mu) = (x_0, y_0, \mu_0)$.

•3 $\Lambda_1 < 0 \Rightarrow$ supercritical Hopf bifurcation

$\Lambda_1 = 0 \Rightarrow$ degenerate Hopf bifurcation

$\Lambda_1 > 0 \Rightarrow$ subcritical Hopf bifurcation

EXAMPLE

a) Show that

$\dot{x} = \mu x - y + xy^2$

$\dot{y} = x + \mu y + y^3$

undergoes a subcritical Hopf bifurcation at $(x,y) = (0,0)$ and $\mu_0 = 0$

Solution

Let $f(x,y) = \mu x - y + xy^2$ and $g(x,y) = x + \mu y + y^3$ and $F(x,y) = (f(x,y), g(x,y))$. It follows that

$$DF(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} = \begin{bmatrix} \mu + y^2 & -1 + 2xy \\ 1 & \mu + 3y^2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(0,0) - \lambda I) = \begin{vmatrix} \mu - \lambda & -1 \\ 1 & \mu - \lambda \end{vmatrix} =$$

$$= (\mu - \lambda)^2 - 1 \cdot (-1) = (\mu - \lambda)^2 + 1 = (\mu - \lambda - i)(\mu - \lambda + i) \Rightarrow$$

$\Rightarrow \lambda(DF(0,0)) = \{\mu + i, \mu - i\} \Rightarrow$ Hopf bifurcation at $(0,0)$ at $\mu = 0$.

- Classification.

We note that

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mu x + xy^2 \\ \mu y + y^3 \end{bmatrix}$$

with $f(x,y) = \mu x + xy^2$

$g(x,y) = \mu y + y^3$.

At $(x,y)=0$:

$f_x(x,y) = \mu + y^2 \Rightarrow f_{xy}(x,y) = 0 \Rightarrow f_{xy}(0,0) = 0$

$f_{xx}(x,y) = (\partial/\partial x) f_x(x,y) = (\partial/\partial x)(\mu + y^2) = 0 \Rightarrow f_{xx}(0,0) = 0$

$f_{xxx}(x,y) = (\partial/\partial x) f_{xx}(x,y) = 0 \Rightarrow f_{xxx}(0,0) = 0$

$f_y(x,y) = (\partial/\partial y)(\mu x + xy^2) = 2xy \Rightarrow f.$

$\Rightarrow f_{yy}(x,y) = (\partial/\partial y) f_y(x,y) = (\partial/\partial y)(2xy) = 2x \Rightarrow$

$\Rightarrow f_{yy}(0,0) = 0$

$f_{xyy}(x,y) = (\partial/\partial x) f_{yy}(x,y) = (\partial/\partial x) 2x = 2 \Rightarrow f_{xyy}(0,0) = 2.$

$g_x(x,y) = (\partial/\partial x) g(x,y) = (\partial/\partial x)(\mu y + y^3) = 0$

$\Rightarrow g_{xx}(x,y) = 0 \Rightarrow g_{xxx}(x,y) = 0$

It follows that $g_{xx}(0,0) = 0$ and $g_{xxx}(0,0) = 0$

Also $g_{xy}(x,y) = (\partial/\partial y) g_x(x,y) = 0 \Rightarrow$

$\Rightarrow g_{xxy}(x,y) = (\partial/\partial x) g_{xy}(x,y) = 0$

and therefore: $g_{xy}(0,0) = 0$ and $g_{xxy}(0,0) = 0$

$g_y(x,y) = (\partial/\partial y)(\mu y + y^3) = \mu + 3y^2 \Rightarrow g_{yy}(x,y) = 6xy \Rightarrow$

$\Rightarrow g_{yyy}(x,y) = 6$ and therefore:

$g_{yy}(0,0) = 0$ and $g_{yyy}(0,0) = 6.$

From the above, using $\omega = 1$, we have

$$\Lambda_1 = [f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx} g_{xx} + f_{yy} g_{yy}] =$$

$$= [0 + 2 + 0 + 6] + (1/1)[0(0+0) - 0(0+0) - 0 \cdot 0 + 0 \cdot 0] = 8 > 0 \Rightarrow$$

$\Rightarrow$ subcritical Hopf bifurcation.

EXERCISES

(13) Show that the following systems undergo a Hopf bifurcation at $\mu = 0$ and classify it as supercritical, subcritical, or degenerate.

a) $$\begin{cases} \dot{x} = \mu x + y \\ \dot{y} = -x + \mu y - x^2 y \end{cases}$$

b) $$\begin{cases} \dot{x} = \mu x + y - x^3 \\ \dot{y} = -x + \mu y + 2y^3 \end{cases}$$

c) $$\begin{cases} \dot{x} = \mu x + y - x^2 \\ \dot{y} = -x + \mu y + 2x^2 \end{cases}$$