Center Manifold Reduction

This technique is based on the following theorem

Theorem: Consider the following system of $n+m$ ordinary differential equations:

$$\begin{cases} \dot{x} = Ax + f(x,y) \\ \dot{y} = By + g(x,y) \end{cases}, \text{with} \begin{cases} f(\mathbf{0}) = \mathbf{0} \wedge Df(\mathbf{0}) = \mathbf{0} \\ g(\mathbf{0}) = \mathbf{0} \wedge Dg(\mathbf{0}) = \mathbf{0} \end{cases}$$

with $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, $f: \mathbb{R}^{n+m} \to \mathbb{R}^n$, $g: \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$.
Here, A is an $n \times n$ matrix, B is an $m \times m$ matrix, with

$$\begin{cases} \forall \lambda \in \lambda(A) : \operatorname{Re}(\lambda) = 0 \\ \forall \lambda \in \lambda(B) : \operatorname{Re}(\lambda) < 0 \end{cases}$$

Then there exists a center-manifold W^c given by

$$W^c = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = h(x)\}$$

with $h: \mathbb{R}^n \to \mathbb{R}^m$, and $h(\mathbf{0}) = \mathbf{0}$, and $Dh(\mathbf{0}) = \mathbf{0}$ such that the solution of the nonlinear system converges to W^c as $t \to +\infty$ if initialized near enough the fixed point $\mathbf{0}$.

The theorem can be used to analyze non-hyperbolic fixed points where all the eigenvalues of the corresponding Jacobian matrix are either zero or negative. The method cannot be applied if at least one eigenvalue is positive (in the real part $\operatorname{Re}(\lambda)$).

Methodology

Let $\dot{x} = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$ be an autonomous dynamical system. Let $x_0 \in \mathbb{R}^n$ be a fixed point with $f(x_0) = 0$. We assume that x_0 is a non-hyperbolic fixed point such that some of the eigenvalues of $Df(x_0)$ have zero real part and none of the eigenvalues have a strictly positive real part. In other words, we assume that

$$\begin{cases} \exists \lambda \in \lambda(Df(x_0)) : Re(\lambda) = 0 \\ \forall \lambda \in \lambda(Df(x_0)) : Re(\lambda) \leq 0 \end{cases}$$

The center manifold reduction technique consists of the following 3 steps:

1) Reduce system to canonical form
2) Apply the center-manifold theorem.
3) Determine series expansion for mapping h.

Reduction to canonical form

- $_1$ We linearize the autonomous system around x_0 and write:

$$\dot{x} = Df(x_0)x + G(x)$$

Here $G(x)$ captures the nonlinear terms of the system.

- $_2$ Assume that $Df(x_0)$ has distinct eigenvalues

$$\lambda(Df(x_0)) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

with corresponding eigenvectors $v_1, v_2, \dots, v_n$.

We daagonalize $Df(x_0)$ by defining

$$P = [V_1 \ V_2 \ \cdots \ V_n]$$

and writing

$$Df(x_0) = P \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) P^{-1}$$

Note that $Df(x_0)$ can be diagonalized even when the eigenvalues are not distinct, into a block-diagonal matrix.

- 3 Define the change of variables $y = P^{-1}x$.
  It follows that $x = Py$, and therefore

$$\dot{y} = P^{-1}\dot{x} = P^{-1}(Df(x_0)x + G(x)) = P^{-1}(Df(x_0)Py + G(Py))$$
$$= [P^{-1}Df(x_0)P]y + G(Py) =$$
$$= [P^{-1}P\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)P^{-1}P]y + G(Py) =$$
$$= \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)y + G(Py)$$

which reduces to the following system of ODEs:

$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 + g_1(y_1, y_2, \ldots, y_n) \\ \dot{y}_2 = \lambda_2 y_2 + g_2(y_1, y_2, \ldots, y_n) \\ \text{-----} \\ \dot{y}_n = \lambda_n y_n + g_n(y_1, y_2, \ldots, y_n) \end{cases}$$

- Determining the center manifold

Let us now assume that $Re(\lambda_a) = 0, \forall a \in [k]$ and $Re(\lambda_a) < 0, \forall a \in [n] - [k]$. Then, according to the center manifold theorem, the first k equations for $y_1, y_2, \ldots, y_k$

drive the dynamics of the system and the other $n-k$ equations are driven by <u>slaving principles</u> given by

$$\begin{cases} y_{k+1} = h_1(y_1, y_2, \dots, y_k) \\ y_{k+2} = h_2(y_1, y_2, \dots, y_k) \\ \vdots \\ y_n = h_{n-k}(y_1, y_2, \dots, y_k) \end{cases}$$

Let us define $u = (y_1, y_2, \dots, y_k)$ and $v = (y_{k+1}, y_{k+2}, \dots, y_n)$ and rewrite the above system as $v = h(u)$, with $h : \mathbb{R}^k \to \mathbb{R}^{n-k}$.
Also, let $(u_0, v_0) = y_0 = P^{-1} x_0$ be the fixed point.
Then according to the center manifold theorem, h has to satisfy:

$$\begin{cases} h(u_0) = 0 \\ Dh(u_0) = 0 \end{cases}$$

Now let us rewrite the original system of ODEs for $y = (u, v)$ as follows:

$$\dot{u} = Au + G_1(u, v)$$
$$\dot{v} = Bv + G_2(u, v)$$

with $A = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ and $B = \mathrm{diag}(\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n)$ and G_1, G_2 are given by

$$G_1 = (g_1, g_2, \dots, g_k)$$
$$G_2 = (g_{k+1}, g_{k+2}, \dots, g_n)$$

To write governing PDEs for h, we note that

$$v = h(u) \Rightarrow \dot{v} = Dh(u)\,\dot{u} = Dh(u)[Au + G_1(u, v)] =$$
$$= Dh(u)[Au + G_1(u, h(u))]$$

and

$\dot{v} = Bv + G_2(u,v) = Bh(u) + G_2(u,h(u))$

and it follows that

$$Dh(u)[Au + G_1(u,h(u))] = Bh(u) + G_2(u,h(u))$$

Now, let us define the operator

$$\boxed{(Nh)(u) = Dh(u)[Au + G_1(u,h(u))] - [Bh(u) + G_2(u,h(u))]}$$

Then, h is given by the solution of the following initial value problem:

$$\boxed{\begin{cases} (Nh)(u) = 0 \\ h(u_0) = 0 \\ Dh(u_0) = 0 \end{cases}}$$

Note that in terms of components, $(Nh)(u)$ is given by:

$$\boxed{(Nh)_a(u) = \sum_{b=1}^{k} \left[(\lambda_b y_b + g_b(y)) \frac{\partial h_a}{\partial y_b} \right] - (\lambda_{k+a} h_a + g_{k+a}(y))}$$

● The spectral gap theorem

The mapping h can be determined via a power-series technique based on the following spectral gap theorem.

Theorem : Let an arbitrary $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ be given with $\psi(u_0) = \mathbf{0}$ and $D\psi(u_0) = \mathbf{0}$. Then, under the limit $u \to u_0$, we can show that

$$\exists q > 1 : (N\psi)(u) = O(\|u - u_0\|^q) \Rightarrow \|h(u) - \psi(u)\| = O(\|u - u_0\|^q)$$

It follows that power-series techniques can be used to approximate the center manifold to any degree of approximation by solving the equation $(N\psi)(u) = \mathbf{0}$ to the same degree of approximation, as shown in the examples below.

EXAMPLES

a) $\begin{cases} \dot{x} = x^2 y - x^5 \\ \dot{y} = -y + x^2 \end{cases}$ $\leftarrow$ Analyze fixed points.

Solution

- Fixed points and Jacobian.

Let $f(x,y) = x^2 y - x^5$ and $g(x,y) = -y + x^2$.

(x,y) fixed point $\Leftrightarrow \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 y - x^5 = 0 \\ -y + x^2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x^2(y - x^3) = 0 \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x^2 = 0 \\ y = x^2 \end{cases} \vee \begin{cases} y = x^3 \\ y = x^2 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^3 - x^2 = 0 \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^2(x - 1) = 0 \\ y = x^2 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = 1 \end{cases} \Leftrightarrow (x,y) \in \{(0,0), (1,1)\}$.

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2xy - 5x^4 & x^2 \\ 2x & -1 \end{bmatrix}$$

- For $(x,y) = (1,1)$:

$$DF(1,1) = \begin{bmatrix} 2-5 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(1,1) - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} =$$

$$= (-3-\lambda)(-1-\lambda) - 2 = (\lambda+3)(\lambda+1) - 2 = \lambda^2 + 4\lambda + 3 - 2$$

$= \lambda^2 + 4\lambda + 1$

$\Delta = b^2 - 4ac = 4^2 - 4\cdot 1\cdot 1 = 16 - 4 = 12 = 4\cdot 3 \Rightarrow$

$\Rightarrow \lambda_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$ (both negative)

thus $\lambda(DF(1,1)) = \{-2-\sqrt{3}, -2+\sqrt{3}\} \Rightarrow (1,1)$ is a stable sink.

- For $(x,y) = (0,0)$:

$DF(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(DF(0,0)) = \{0, -1\} \Rightarrow$

$\Rightarrow (0,0)$ is a non-hyperbolic fixed point.

- Center-Manifold analysis: We note that $\dot{x}$ is the master equation and $\dot{y}$ is the slave equation, the system being already written in canonical form. So, let us consider $y = h(x)$ with $h(0) = 0$ and $h'(0) = 0$. It follows that

$$\left.\begin{aligned} \dot{y} &= h'(x)\dot{x} = h'(x)[x^2y - x^5] = h'(x)[x^2h(x) - x^5] \\ \dot{y} &= -y + x^2 = -h(x) + x^2 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow (Nh)(x) = h'(x)[x^2h(x) - x^5] + h(x) - x^2$

Let $h(x) = ax^2 + bx^3 + O(x^4) \Rightarrow h'(x) = 2ax + 3bx^2 + O(x^3)$

and it follows that

$$\begin{aligned} (Nh)(x) &= [2ax + 3bx^2 + O(x^3)][x^2(ax^2 + bx^3 + O(x^4)) - x^5] \\ &\quad + ax^2 + bx^3 + O(x^4) - x^2 = \\ &= x^2(2ax + 3bx^2)(ax^2 + bx^3) + O(x^9) - x^5[2ax + 3bx^2 + O(x^3)] \\ &\quad + ax^2 + bx^3 + O(x^4) - x^2 = \end{aligned}$$

$$= x^2(2a^2x^3 + 2abx^4 + 3abx^4 + 3b^2x^5) - 2ax^6 - 3bx^7 \underline{+ax^2} + \underline{\underline{bx^3}} \underline{-x^2} + O(x^4)$$
$$= (a-1)x^2 + bx^3 + O(x^4).$$

and therefore:

$$(Nh)(x) = 0 + O(x^4) \Leftrightarrow (a-1)x^2 + bx^3 + O(x^4) = O(x^4) \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} a-1=0 \\ b=0 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=0 \end{cases}$$

It follows that $h(x) = x^2 + O(x^4)$.

Then $\dot{x} = x^2y - x^5 = x^2h(x) - x^5 = x^2[x^2 + O(x^4)] - x^5$
$= x^4 + O(x^5)$

and the center-manifold reduction is:

$$\begin{cases} \dot{x} = x^4 + O(x^5) \leftarrow \text{master equation} \\ y = x^2 + O(x^4) \leftarrow \text{slave equation} \end{cases}$$

From the master equation we see that the $(0,0)$ fixed point is unstable.

$\hookrightarrow$ Note that $\lambda(DF(0,0)) = \{0,-1\}$, thus linear stability analysis might suggest that $(0,0)$ is Lyapunov stable, and in the absense of positive eigenvalues there is no hint of instability. On the other hand, because $(0,0)$ is not hyperbolic, so linear stability analysis is not guaranteed to be accurate, and center manifold analysis shows that the $(0,0)$ fixed point is in fact unstable.

Note that the center manifold ensures local convergence: if the initial condition is close to the center manifold, it will converge to the center manifold. We ~~c~~ can also investigate global convergence, i.e. whether or not ALL initial conditions converge to the center manifold via the following argument:

Since the center manifold is $y = x^2 + O(x^4)$, we define $U(x,y) = (y-x^2)^2$.

It is sufficient to show that $\dot{U}(x,y) < 0$.

$$\dot{U}(x,y) = (d/dt)\left[(y-x^2)^2\right] = 2(y-x^2)(\dot{y} - 2x\dot{x}) =$$
$$= 2(y-x^2)(-y+x^2-2x(x^2y-x^5)) =$$
$$= -2(y-x^2)^2 - 4x(y-x^2)(x^2y-x^5)$$
$$= -2(y-x^2)^2 - 4(y-x^2)(x^3y-x^6)$$
$$= -2(y-x^2)^2 - 4(x^3y^2 - x^6y - x^5y + x^8) =$$
$$= -2(y-x^2)^2 - 4x^8 + 4x^3y(-y+x^3+x^2)$$

First two terms are negative, third term is unclear (negative or positive). Let us assume that $y = x^2+\varepsilon$ with ε small.

Then, it follows that

$$4x^3y(-y+x^2+x^3) = 4x^3(x^2+\varepsilon)(-x^2-\varepsilon+x^2+x^3) =$$
$$= 4x^3(x^2+\varepsilon)(x^3-\varepsilon)$$
$$= 4x^3(x^5 - \varepsilon x^2 + \varepsilon x^3 - \varepsilon^2)$$
$$= 4x^8 + 4\varepsilon x^3(-x^2+x^3-\varepsilon) \Rightarrow$$

$$\Rightarrow \dot{U}(x,y) = -2(y-x^2)^2 - 4x^8 + 4x^8 + 4\varepsilon x^3(x^3-x^2-\varepsilon) =$$
$$= -2(y-x^2)^2 + 4\varepsilon x^3(x^3-x^2-\varepsilon)$$

$$= -2(x^2+\varepsilon-x^2)^2 + 4\varepsilon x^3(x^3-x^2-\varepsilon)$$
$$= -2\varepsilon^2 + 4\varepsilon x^6 - 4\varepsilon x^5 - 4\varepsilon^2 x^3$$
$$= -2\varepsilon^2 - 4\varepsilon^2 x^3 + O(x^4) = -2\varepsilon^2(1+2x^3) + O(x^4). < 0$$

in the limit $x \to 0$, since for small x, $1+2x^3 > 0$.
It follows that we do not have global convergence towards the center manifold.

b) $\begin{cases} \dot{x} = xy \\ \dot{y} = -y - x^2 \end{cases}$ $\longleftarrow$ Find all fixed points and classify with respect to stability

Solution

▸ Fixed points.

Let $f(x,y) = xy \wedge g(x,y) = -y - x^2$.

(x,y) fixed point $\Leftrightarrow \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} xy = 0 \\ -y - x^2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x(-x^2) = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Leftrightarrow (x,y) = (0,0).$

▸ Jacobian

Let $F(x,y) = (f(x,y), g(x,y))$.

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} y & x \\ -2x & -1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(DF(0,0)) = \{0, -1\} \Rightarrow$$

$\Rightarrow (0,0)$ non-hyperbolic fixed point.

▸ Center-Manifold reduction.

System is already in canonical form with $\dot{x} = xy$ the master equation and $\dot{y} = -y - x^2$ the slave equation. Thus, in the limit $x \to 0$, let us define

$$y = h(x) = ax^2 + bx^3 + cx^4 + dx^5 + O(x^6)$$

Then:

$$\dot{y} = h'(x)\dot{x} = h'(x)xy = h'(x)xh(x) = xh(x)h'(x)$$

$$\dot{y} = -y - x^2 = -h(x) - x^2$$

therefore, let us define

$$N(x) = xh(x)h'(x) - [-h(x) - x^2] = xh(x)h'(x) + h(x) + x^2 =$$
$$= x(ax^2 + bx^3 + cx^4 + dx^5)(2ax + 3bx^2 + 4cx^3 + 5dx^4) + O(x^{12})$$
$$+ (ax^2 + bx^3 + cx^4 + dx^5) + O(x^6) + x^2 =$$
$$= (ax^3 + bx^4 + cx^5 + dx^6)(2ax + 3bx^2 + 4cx^3 + 5dx^4) +$$
$$+ ax^2 + bx^3 + cx^4 + dx^5 + x^2 + O(x^6) =$$
$$= 2a^2x^4 + 3abx^5 + 2abx^5 + O(x^6) + ax^2 + bx^3 + cx^4 + dx^5 + x^2 + O(x^6)$$
$$= (a+1)x^2 + bx^3 + (c + 2a^2)x^4 + (5ab + d)x^5 + O(x^6)$$

It follows that

$$N(x) = O(x^6) \Leftrightarrow (a+1)x^2 + bx^3 + (c+2a^2)x^4 + (5ab+d)x^5 + O(x^6) = O(x^6)$$

$$\Leftrightarrow \begin{cases} a+1=0 \\ b=0 \\ c+2a^2=0 \\ 5ab+d=0 \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=0 \\ c=-2a^2 \\ d=-5ab \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=0 \\ c=-2 \\ d=0 \end{cases}$$

and therefore $h(x) = -x^2 - 2x^4 + O(x^6)$.

Thus, $\dot{x} = xy = xh(x) = x(-x^2 - 2x^4 + O(x^6)) =$
$$= -x^3 - 2x^5 + O(x^7)$$
$$y = -x^2 - 2x^4 + O(x^6)$$

and the centermanifold reduction reads:

$$\begin{cases} \dot{x} = -x^3 - 2x^5 + O(x^7) \\ y = -x^2 - 2x^4 + O(x^6) \end{cases}$$

It follows that $(0,0)$ is stable since the fixed point $x=0$ of $\dot{x} = -x^3 - 2x^5 + O(x^7)$ is stable.

↳ Local vs. global convergence

Let us consider the 1st-order approximation

$$y = -x^2 + O(x^4)$$

of the center manifold and therefore define

$$U(x,y) = (y+x^2)^2$$

It follows that

$$\dot{U}(x,y) = 2(y+x^2)(\dot{y}+2x\dot{x}) = 2(y+x^2)(-y-x^2+2x(xy)) =$$
$$= 2(y+x^2)(-y-x^2+2x^2y)$$
$$= -2(y+x^2)^2 + 4x^2y(y+x^2)$$

Note that the 1st term is negative but the 2nd term can be positive or negative. Choose $y = -x^2 + \varepsilon$ with ε small. Then

$$\dot{U}(x,y) = -2(-x^2+\varepsilon+x^2)^2 + 4x^2(-x^2+\varepsilon)(-x^2+\varepsilon+x^2)$$
$$= -2\varepsilon^2 + 4\varepsilon x^2(-x^2+\varepsilon) = -2\varepsilon^2 - 4\varepsilon x^{+4} + 4\varepsilon^2 x^2$$
$$= -4\varepsilon x^4 + 2\varepsilon^2(2x^2-1) = 2\varepsilon^2(2x^2-1) + O(x^4)$$

For small enough ε, $\dot{U}(x,y) < 0$, thus we have local but not global convergence

Inclusion of Linearly Unstable Directions

- The center manifold analysis is still applicable even if in the canonical formulation of the original ODE system some eigenvalues have $Re(\lambda) > 0$.
- Consider the system, in canonical form

$$\begin{cases} \dot{x} = Ax + f(x,y,z) \\ \dot{y} = By + g(x,y,z) \\ \dot{z} = Cz + h(x,y,z) \end{cases}$$

with $(x,y,z) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$ and

$$\begin{cases} \forall \lambda \in \lambda(A) : Re(\lambda) = 0 \\ \forall \lambda \in \lambda(B) : Re(\lambda) < 0 \\ \forall \lambda \in \lambda(C) : Re(\lambda) > 0 \end{cases}$$

and

$$\begin{cases} f(\mathbf{0}) = \mathbf{0} \land g(\mathbf{0}) = \mathbf{0} \land h(\mathbf{0}) = \mathbf{0} \\ Df(\mathbf{0}) = \mathbf{0} \land Dg(\mathbf{0}) = \mathbf{0} \land Dh(\mathbf{0}) = \mathbf{0} \end{cases}$$

(i.e. $\mathbf{0}$ is a fixed point and f, g, h capture only the nonlinear terms).

Then the center-manifold is given by

$$W^c = \{(x,y,z) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c \mid y = h_1(x) \land z = h_2(x)\}$$

with $h_1(\mathbf{0}) = \mathbf{0} \land h_2(\mathbf{0}) = \mathbf{0} \land Dh_1(\mathbf{0}) = \mathbf{0} \land Dh_2(\mathbf{0}) = \mathbf{0}$

- To determine h_1 and h_2, we note that

$$\dot{y} = Dh_1(x)\dot{x} = Dh_1(x)[Ax + f(x,y,z)] =$$
$$= Dh_1(x)[Ax + f(x, h_1(x), h_2(x))]$$

$$\dot{y} = By + g(x,y,z) = Bh_1(x) + g(x,h_1(x),h_2(x))$$

$$\dot{z} = Dh_2(x)\dot{x} = Dh_2(x)[Ax + f(x,y,z)] =$$

$$= Dh_2(x)[Ax + f(x,h_1(x),h_2(x))]$$

$$\dot{z} = Cz + h(x,y,z) = Ch_2(x) + h(x,h_1(x),h_2(x))$$

It follows that if we define

$$(N_1(h_1,h_2))(x) = Dh_1(x)[Ax + f(x,h_1(x),h_2(x))] - Bh_1(x) - g(x,h_1(x),h_2(x))$$

$$(N_2(h_1,h_2))(x) = Dh_2(x)[Ax + f(x,h_1(x),h_2(x))] - Ch_2(x) - h(x,h_1(x),h_2(x))$$

then $(N_1(h_1,h_2))(x) = \mathbf{0} \wedge (N_2(h_1,h_2))(x) = \mathbf{0}$.

Consequently, h_1 and h_2 are the solutions of the following initial value problem:

$$\begin{cases} (N_1(h_1,h_2))(x) = \mathbf{0} \\ (N_2(h_1,h_2))(x) = \mathbf{0} \\ h_1(x) = \mathbf{0} \wedge Dh_1(x) = \mathbf{0} \\ h_2(x) = \mathbf{0} \wedge Dh_2(x) = \mathbf{0} \end{cases}$$

which can still be solved via power-series methods.

EXAMPLES

a) Analyze the stability of the fixed point $(x,y,z) = (0,0,0)$ for the system

$$\begin{cases} \dot{x} = xz \\ \dot{y} = -y + x^2 \\ \dot{z} = z - xy \end{cases}$$

using center-manifold reduction.

Solution

Define $f(x,y,z) = xz \wedge g(x,y,z) = -y + x^2 \wedge h(x,y,z) = z - xy$.
and $F = (f,g,h)$. It follows that

$$DF(x,y,z) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} z & 0 & x \\ 2x & -1 & 0 \\ -y & -x & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda(DF(0,0,0)) = \{0, -1, 1\}$$

$\Rightarrow (0,0,0)$ is a non-hyperbolic fixed point.

▸ Center Manifold analysis.

We note that $\dot{x} = xz$ is the master equation. Thus let $y = h_1(x)$ and $z = h_2(x)$. It follows that

$$\dot{y} = h_1'(x)\dot{x} = h_1'(x)\,xz = x h_1'(x) h_2(x)$$

$$\dot{y} = -y + x^2 = -h_1(x) + x^2$$

therefore, we define $N_1(x) = x h_1'(x) h_2(x) - [-h_1(x) + x^2]$

Likewise,

$\dot{z} = h_2'(x)\dot{x} = h_2'(x)\, xz = x h_2'(x) h_2(x)$

$\dot{z} = z - xy = h_2(x) - x h_1(x)$

therefore, we define

$N_2(x) = x h_2'(x) h_2(x) - [h_2(x) - x h_1(x)]$.

For $h_1(x) = ax^2 + bx^3 + O(x^4)$

$\quad h_2(x) = cx^2 + dx^3 + O(x^4)$

we have

$$N_1(x) = x h_1'(x) h_2(x) + h_1(x) - x^2 =$$
$$= x(2ax + 3bx^2)(cx^2 + dx^3) + O(x^8) + ax^2 + bx^3 - x^2 + O(x^4)$$
$$= (a-1)x^2 + bx^3 + O(x^4)$$

$$N_2(x) = x h_2'(x) h_2(x) - h_2(x) + x h_1(x)$$
$$= x(2cx + 3dx^2)(cx^2 + dx^3) + O(x^8) - cx^2 - dx^3 + O(x^4)$$
$$+ x(ax^2 + bx^3) + O(x^5)$$
$$= -cx^2 - dx^3 + ax^3 + O(x^4) = -cx^2 + (a-d)x^3 + O(x^4)$$

It follows that

$$\begin{cases} N_1(x) = (a-1)x^2 + bx^3 + O(x^4) = O(x^4) \\ N_2(x) = -cx^2 + (a-d)x^3 + O(x^4) = O(x^4) \end{cases} \iff$$

$$\iff \begin{cases} a - 1 = 0 \\ b = 0 \\ -c = 0 \\ a - d = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = a \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = 1 \end{cases}$$

and therefore $h_1(x) = x^2 + O(x^4)$ and $h_2(x) = x^3 + O(x^4)$

It follows that the master equation reads

$\dot{x} = xz = x h_2(x) = x(x^3 + O(x^4)) = x^4 + O(x^5)$.

The center-manifold reduction near the $(0,0,0)$ fixed point is given by:

$$\begin{cases} \dot{x} = x^4 + O(x^5) \\ y = x^2 + O(x^4) \\ z = x^3 + O(x^4) \end{cases}$$

$\leftarrow$ thus $(0,0,0)$ is unstable source.

EXERCISES

(1) Study the dynamics of the following systems near the origin $(x,y)=(0,0)$ via center-manifold analysis for the following autonomous dynamical systems.

a) $\begin{cases} \dot{x} = -x+y^2 \\ \dot{y} = -\sin x \end{cases}$ b) $\begin{cases} \dot{x} = x-2y \\ \dot{y} = x+y+x^4 \end{cases}$ c) $\begin{cases} \dot{x} = -2x+3y+y^3 \\ \dot{y} = 2x-3y+x^3 \end{cases}$

d) $\begin{cases} \dot{x} = y+x^2 \\ \dot{y} = -y-x^2 \end{cases}$ e) $\begin{cases} \dot{x} = -x+y \\ \dot{y} = -e^x+e^{-x}+2x \end{cases}$

f) $\begin{cases} \dot{x} = -x-y+z^2 \\ \dot{y} = 2x+y-z^2 \\ \dot{z} = x+2y-z \end{cases}$

Application of Center Manifold to Local Bifurcations

The center manifold technique can be used to investigate local bifurcations for multidimensional autonomous dynamical systems without an explicit determination of the local fixed point, as in the following example:

EXAMPLE

a) Investigate the possible bifurcation at $\mu=0$ for the following system, using center-manifold reduction.

$$\begin{cases} \dot{x} = \mu x - x^3 + xy \\ \dot{y} = -y + y^2 - x^2 \end{cases}$$

Solution

- Fixed point: There is an obvious fixed point at $(x,y)=(0,0)$.
- Linearization

Define: $f(x,y) = \mu x - x^3 + xy$ and $g(x,y) = -y + y^2 - x^2$ and $F(x,y) = (f(x,y), g(x,y))$. Then:

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} \mu - 3x^2 + y & x \\ -2x & -1+2y \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0) = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \cancel{(0,0)}\ \lambda(DF(0,0)) = \{\mu, -1\} \Rightarrow$$

$\Rightarrow (0,0)$ stable for $\mu<0$, unstable for $\mu>0$.

Note that stability is unknown for $\mu=0$.

▸ Center Manifold analysis: Note that center manifold reduction cannot be applied to the given dynamical system in the absense of zero-eigenvalues. However, we can "cheat" by turning μ into a variable and rewriting the system as:

$$\begin{cases} \dot{x} = \mu x - x^3 + xy \\ \dot{y} = -y + y^2 - x^2 \\ \dot{\mu} = 0 \end{cases}$$

Define $f(x,y,\mu) = \mu x - x^3 + xy$, $g(x,y,\mu) = -y + y^2 - x^2$, and $h(x,y,\mu) = 0$, and also define

$$F(x,y,\mu) = (f(x,y,\mu), g(x,y,\mu), h(x,y,\mu))$$

It follows that

$$DF(x,y,\mu) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial \mu \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial \mu \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial \mu \end{bmatrix} =$$

$$= \begin{bmatrix} \mu - 3x^2 + y & x & x \\ -2x & -1+2y & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \lambda(DF(0,0,0)) = \{0, -1\}$$

$\Rightarrow (0,0,0)$ is a non-hyperbolic fixed point.

Write the linearized equations around $(x,y,\mu) = (0,0,0)$ as follows:

$$\begin{cases} \dot{x} = 0x + (\mu x - x^3 + xy) \\ \dot{y} = -y + (y^2 - x^2) \\ \dot{\mu} = 0\mu \end{cases}$$

Note that $\dot{x}$ and $\dot{\mu}$ are the master equations and $\dot{y}$ is the slave equation. Therefore, let us write $y = H(x,\mu)$. It follows that

$$\dot{y} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \mu}\dot{\mu} = \frac{\partial H}{\partial x}\dot{x} = (\mu x - x^3 + xy)\frac{\partial H}{\partial x} =$$

$$= (\mu x - x^3 + x H(x,\mu))\frac{\partial H}{\partial x}$$

and

$$\dot{y} = -y + (y^2 - x^2) = -H(x,\mu) + H(x,\mu)^2 - x^2$$

therefore we define

$$N(x,\mu) = (\mu x - x^3 + xH(x,\mu))\frac{\partial H}{\partial x} + H(x,\mu) - H(x,\mu)^2 + x^2 .$$

Under the limit $x \to 0$, consider the expansion

$$H(x,\mu) = a(\mu)x^2 + b(\mu)x^3 + O(x^4) \Rightarrow \frac{\partial H(x,\mu)}{\partial x} = 2a(\mu)x + 3b(\mu)x^2 + O(x^3)$$

It follows that

$$N(x,\mu) = [\mu x - x^3 + x(a(\mu)x^2 + b(\mu)x^3 + O(x^4))][2a(\mu)x + 3b(\mu)x^2 + O(x^3)]$$
$$+ [a(\mu)x^2 + b(\mu)x^3 + O(x^4)] - [a(\mu)x^2 + b(\mu)x^3 + O(x^4)]^2 + x^2 =$$
$$= \underline{2\mu a(\mu)x^2} + \underline{\underline{3\mu b(\mu)x^3}} + O(x^4) + \underline{a(\mu)x^2} + \underline{\underline{b(\mu)x^3}} + O(x^4) - O(x^4) + \underline{x^2} =$$
$$= [2\mu a(\mu) + a(\mu) + 1]x^2 + [3\mu b(\mu) + b(\mu)]x^3 + O(x^4) =$$
$$= [(2\mu+1)a(\mu) + 1]x^2 + b(\mu)(3\mu+1)x^3 + O(x^4)$$

and therefore, if we restrict μ to $\mu \in (-1/3, 1/3)$, we have:

$$N(x,\mu) = O(x^4) \Longleftrightarrow \begin{cases} (2\mu+1)\,a(\mu) + 1 = 0 \\ b(\mu)\,(3\mu+1) = 0 \end{cases} \Longleftrightarrow \begin{cases} a(\mu) = \dfrac{-1}{2\mu+1} \\ b(\mu) = 0 \end{cases}$$

and therefore

$$H(x,\mu) = \frac{-x^2}{2\mu+1} + O(x^4)$$

It follows that the $\dot{x}$ master equation reads:

$$\dot{x} = \mu x - x^3 + xy = \mu x - x^3 + x H(x,\mu) = \mu x - x^3 + x\left[\frac{-x^2}{2\mu+1} + O(x^4)\right]$$

$$= \mu x - x^3 - \frac{1}{2\mu+1} x^3 + O(x^4) = \mu x - \left(1 + \frac{1}{2\mu+1}\right) x^3 + O(x^4)$$

$$= \mu x - \frac{2\mu+2}{2\mu+1} x^3 + O(x^4)$$

and consequently, the center manifold reduction reads:

$$\begin{cases} \left.\begin{aligned} &\dot{x} = \mu x - \frac{2\mu+2}{2\mu+1} x^3 + O(x^4) \\ &\dot{\mu} = 0 \end{aligned}\right\} \text{master equations} \\ y = \dfrac{-x^2}{2\mu+1} + O(x^4). \quad \} \text{ slave equation} \end{cases}$$

► Local Bifurcation at $\mu = 0$ and $(x,y) = (0,0)$

We can analyze the local bifurcation of $(x,y) = (0,0)$ at $\mu = 0$, by studying the master equation $\dot{x}$ instead of the original two-dimensional system.

Define $G(x,\mu) = \mu x - \dfrac{2\mu+2}{2\mu+1} x^3 + O(x^4)$.

We note that

$G(0,0) = 0$

$$G_x(x,\mu) = \mu - \frac{2\mu+2}{2\mu+1}(3x^2) + O(x^3) \Rightarrow G_x(0,0) = 0$$

$$G_\mu(x,\mu) = x - x^3 \frac{\partial}{\partial\mu}\left(\frac{2\mu+2}{2\mu+1}\right) + O(x^4) \Rightarrow G_\mu(0,0) = 0$$

$$G_{xx}(x,\mu) = 0 - \frac{2\mu+2}{2\mu+1}(6x) + O(x^2) \Rightarrow G_{xx}(0,0) = 0$$

$$G_{x\mu}(x,\mu) = \frac{\partial}{\partial\mu}\left[\mu - \frac{2\mu+2}{2\mu+1}(3x^2) + O(x^3)\right] =$$

$$= 1 - (3x^2)\frac{\partial}{\partial\mu}\left(\frac{2\mu+2}{2\mu+1}\right) + O(x^3) \Rightarrow$$

$$\Rightarrow G_{x\mu}(0,0) = 1 - 0 = 1 \neq 0$$

$$G_{xxx}(x,\mu) = \frac{-(2\mu+2)}{2\mu+1} \cdot 6 + O(x) \Rightarrow$$

$$\Rightarrow G_{xxx}(0,0) = \frac{-(0+2)}{0+1} \cdot 6 + 0 = -12 \neq 0.$$

To summarize:

$$\left\{\begin{array}{l} G(0,0) = G_x(0,0) = G_\mu(0,0) = G_{xx}(0,0) \\ G_{x\mu}(0,0) = 1 \neq 0 \\ G_{xxx}(0,0) = -12 \neq 0 \end{array}\right. \Rightarrow$$

$\Rightarrow$ At $\mu = 0$, the $(x,y) = (0,0)$ fixed point undergoes a pitchfork bifurcation.

b) Analyze the bifurcation at the origin for the Lorenz equations, given below, using the center-manifold reduction method

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -bz + xy \end{cases} \quad \text{with } b>0, \sigma>0, \text{ and } \rho>0.$$

Solution

We note that $(x,y,z) = (0,0,0)$ is an obvious fixed point.

▶ Direct linearization

Define $f(x,y,z) = \sigma(y-x)$, $g(x,y,z) = \rho x - y - xz$, and $h(x,y,z) = -bz + xy$. Also define

$$F(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$$

It follows that

$$DF(x,y,z) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -b \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(0,0,0) - \lambda I) = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ \rho & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix} =$$

$$= (-b-\lambda)\begin{vmatrix} -\sigma-\lambda & \sigma \\ \rho & -1-\lambda \end{vmatrix} =$$

$= (-b-\lambda)\left[(-\sigma-\lambda)(-1-\lambda)-\rho\sigma\right] =$
$= (-b-\lambda)\left[(\lambda+\sigma)(\lambda+1)-\rho\sigma\right] =$
$= (-b-\lambda)(\lambda^2+(\sigma+1)\lambda+\sigma-\rho\sigma] =$
$= -(\lambda+b)(\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho))$

$\hookrightarrow$ Note that for $\sigma(1-\rho)\neq 0$, the zeroes λ_1, λ_2 of the quadratic factor $\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho)$ will satisfy $\lambda_1\lambda_2\neq 0$ consequently none of the eigenvalues is zero and therefore we cannot apply the center manifold method. On the other hand for $\rho=1$, we have

$p(\lambda) = -(\lambda+b)(\lambda^2+(\sigma+1)\lambda) = -\lambda(\lambda+b)(\lambda+\sigma+1) \Rightarrow$
$\Rightarrow \lambda(DF(0,0,0)) = \{-b, -\sigma-1, 0\} \Rightarrow$
$\Rightarrow (0,0,0)$ non-hyperbolic fixed point.

▸ Center Manifold reduction

To make center manifold reduction applicable, we turn ρ into a variable governed by $\dot{\rho}=0$ with initial condition $\rho=1$ (at $t=0$). To center the 4D fixed point to the origin, we define $\mu=\rho-1$ and rewrite the Lorenz equations as:

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \mu x + x - y - xz \\ \dot{z} = -bz + xy \\ \dot{\mu} = 0 \end{cases}$$

This extended 4D system has an obvious fixed point at $(x,y,z,\mu) = (0,0,0,0)$.

We define $f(x,y,z,\mu) = \sigma(y-x)$,

$$g(x,y,z,\mu) = \mu x + x - y - xz,$$

$$h(x,y,z,\mu) = -bz + xy$$

and also we define

$$F(x,y,z,\mu) = (f(x,y,z,\mu), g(x,y,z,\mu), h(x,y,z,\mu))$$

$$\mathcal{F}(x,y,z,\mu) = (f(x,y,z,\mu), g(x,y,z,\mu), h(x,y,z,\mu), 0)$$

It follows that

$$DF(x,y,z,\mu) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} =$$

$$= \begin{bmatrix} -\sigma & \sigma & 0 \\ \mu+1-z & -1 & -x \\ y & x & -b \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

which is the same as the previous Jacobian matrix with $\rho = 1$, and therefore:

$$\lambda(DF(0,0,0,0)) = \{0, -\sigma-1, -b\}$$

$\hookrightarrow$ Note that it is not necessary to write the full Jacobian for the 4x4 system explicitly since its Jacobian has a block diagonal structure

$$D\mathcal{F}(0,0,0,0) = \begin{bmatrix} DF(0,0,0,0) & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$$

▸ To diagonalize the system we find the corresponding eigenvectors:

$\lambda_1 = -b$ has eigenvector $v_1 = (0,0,1)$

$\lambda_2 = 0$ has eigenvector $v_2 = (1,1,0)$

$\lambda_3 = -(\sigma+1)$ has eigenvector $v_3 = (-\sigma,1,0)$

Consequently, we define

$$P = [v_1 \; v_2 \; v_3] = \begin{bmatrix} 0 & 1 & -\sigma \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{\sigma+1}\begin{bmatrix} 0 & 0 & \sigma+1 \\ 1 & \sigma & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

and define

$(u,v,w) = P^{-1}(x,y,z) \Leftrightarrow (x,y,z) = P(u,v,w)$

Equivalently, we write:

$$\begin{cases} x = v - \sigma w \\ y = v + w \\ z = u \end{cases} \Leftrightarrow \begin{cases} u = z \\ v = (x+\sigma y)/(\sigma+1) \\ w = (-x+y)/(\sigma+1) \end{cases}$$

Now we rewrite the Lorenz equations in terms of the new variables u,v,w:

$\dot{u} = \dot{z} = -bz + xy = -bu + (v-\sigma w)(v+w)$

$$\dot{v} = \frac{\dot{x} + \sigma\dot{y}}{\sigma+1} = \frac{\sigma(y-x) + \sigma(\mu x + x - y - xz)}{\sigma+1} =$$

$$= \frac{\sigma(y - x + \mu x + x - y - xz)}{\sigma+1} = \frac{\sigma(\mu x - xz)}{\sigma+1} = \frac{\sigma x(\mu - z)}{\sigma+1} =$$

$$= \frac{\sigma(v-\sigma w)(\mu-u)}{\sigma+1}$$

$$\dot{w} = \frac{\dot{y}-\dot{x}}{\sigma+1} = \frac{(\mu x + x - y - xz) - \sigma(y-x)}{\sigma+1} =$$

$$= \frac{(\mu+\sigma+1)x - (\sigma+1)y - xz}{\sigma+1} = \frac{(\sigma+1)(x-y) + (\mu x - xz)}{\sigma+1} =$$

$$= (x-y) + \frac{x(\mu-z)}{\sigma+1} =$$

$$= (v-\sigma w) - (v+w) + \frac{(v-\sigma w)(\mu-u)}{\sigma+1} =$$

$$= -(\sigma+1)w + \frac{(\mu-u)(v-\sigma w)}{\sigma+1}$$

To summarize; the diagonalized equations read:

$$\begin{cases} \dot{u} = -bu + (v-\sigma w)(v+w) \\ \dot{v} = 0v + \sigma(v-\sigma w)(\mu-u)/(\sigma+1) \\ \dot{w} = -(\sigma+1)w + (\mu-u)(v-\sigma w)/(\sigma+1) \\ \dot{\mu} = 0\mu \end{cases}$$

We see that v, μ are the master variables and u, w are the slave variables. Let us write, therefore:

$$u = f(v) \quad \text{and} \quad w = g(v)$$

We note that

$$\dot{u} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial f}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} =$$

$$= \frac{\partial f}{\partial v}\frac{\sigma(v-\sigma w)(\mu-u)}{\sigma+1} = \frac{\partial f}{\partial v}\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1}$$

$$\dot{u} = -bu + (v-\sigma w)(v+w) = -bf(v) + (v-\sigma g(v))(v+g(v))$$

$$\dot{w} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial g}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} =$$

$$= \frac{\partial g}{\partial v}\frac{\sigma(v-\sigma w)(\mu-u)}{\sigma+1} = \frac{\partial g}{\partial v}\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1}$$

$$\dot{w} = -(\sigma+1)w + \frac{(\mu-u)(v-\sigma w)}{\sigma+1} = -(\sigma+1)g(v) + \frac{(\mu-f(v))(v-\sigma g(v))}{\sigma+1}$$

consequently, we define:

$$N_u(v) = \frac{\partial f}{\partial v}\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1} + bf(v) - (v-\sigma g(v))(v+g(v))$$

$$= bf(v) + (v-\sigma g(v))\left[\frac{\partial f}{\partial v}\frac{\sigma(\mu-f(v))}{\sigma+1} - (v+g(v))\right]$$

$$N_w(v) = \frac{\partial g}{\partial v}\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1} + (\sigma+1)g(v) - \frac{(\mu-f(v))(v-\sigma g(v))}{\sigma+1} =$$

$$= (\sigma+1)g(v) + (v-\sigma g(v))\left[\frac{\partial g}{\partial v}\frac{\sigma(\mu-f(v))}{\sigma+1} - \frac{\mu-f(v)}{\sigma+1}\right]$$

$$= (\sigma+1)g(v) + \frac{(v-\sigma g(v))(\mu-f(v))}{\sigma+1}\left[\sigma\frac{\partial g}{\partial v} - 1\right]$$

- Use the expansions

$$f(v) = a_1(\mu)v^2 + a_2(\mu)v^3 + O(v^4) \Rightarrow \partial f/\partial v = 2a_1(\mu)v + 3a_2(\mu)v^2 + O(v^3)$$

$$g(v) = b_1(\mu)v^2 + b_2(\mu)v^3 + O(v^4) \Rightarrow \partial g/\partial v = 2b_1(\mu)v + 3b_2(\mu)v^2 + O(v^3)$$

and it follows that $N_u(v)$ and $N_w(v)$ are given by

$$N_u(v) = N_u^{(1)}(v) + N_u^{(2)}(v) + N_u^{(3)}(v)$$

with

$$N_u^{(1)}(v) = bf(v) = b(a_1 v^2 + a_2 v^3 + O(v^4)) =$$
$$= ba_1 v^2 + ba_2 v^3 + O(v^4)$$
$$N_u^{(2)}(v) = (v - \sigma g(v)) \frac{\partial f}{\partial v} \frac{\sigma(\mu - f(v))}{\sigma+1} =$$
$$= \frac{\sigma}{\sigma+1}(v - \sigma b_1 v^2 - \sigma b_2 v^3)(2a_1 v + 3a_2 v^2)(\mu - a_1 v^2 - a_2 v^3) + O(v^4)$$
$$= \frac{\sigma}{\sigma+1}(2a_1 v^2 + 3a_2 v^3 - 2\sigma a_1 b_1 v^3)(\mu - a_1 v^2 - a_2 v^3) + O(v^4)$$
$$= \frac{\sigma}{\sigma+1}(2a_1 \mu v^2) + O(v^4)$$ [Note: We drop μv^3 which is a 4th order]
$$N_u^{(3)}(v) = -(v - \sigma g(v))(v + g(v)) =$$
$$= -(v - \sigma b_1 v^2 - \sigma b_2 v^3)(v + b_1 v^2 + b_2 v^3) + O(v^4)$$
$$= -(v^2 + b_1 v^3 - \sigma b_1 v^3) + O(v^4) =$$
$$= -v^2 - b_1 v^3 + \sigma b_1 v^3 + O(v^4)$$
and therefore
$$N_u(v) = ba_1 v^2 + ba_2 v^3 + \frac{2\sigma a_1}{\sigma+1}\mu v^2 - v^2 - b_1 v^3 + \sigma b_1 v^3 + O(v^4)$$
$$= \left(ba_1 - 1 + \frac{2\mu\sigma}{\sigma+1} a_1\right) v^2 + (ba_2 - b_1 + \sigma b_1) v^3 + O(v^4)$$
Likewise $N_w(v) = N_w^{(1)}(v) + N_w^{(2)}(v) + N_w^{(3)}(v)$ with
$$N_w^{(1)}(v) = (\sigma+1) g(v) = (\sigma+1) b_1 v^2 + (\sigma+1) b_2 v^3 + O(v^4)$$
$$(v - \sigma g(v))(\mu - f(v)) = (v - \sigma b_1 v^2 - \sigma b_2 v^3)(\mu - a_1 v^2 - a_2 v^3) + O(v^4)$$
$$= \mu v - a_1 v^3 - \sigma b_1 \mu v^2 + O(v^4)$$
$$N_w^{(2)}(v) = \frac{\sigma}{\sigma+1}(v - \sigma g(v))(\mu - f(v)) \frac{\partial g}{\partial v} =$$

$$= \frac{\sigma}{\sigma+1}\left(\mu v - a_1 v^3 - \sigma b_1 \mu v^2\right)\left(2b_1 v + 3b_2 v^2\right) + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}\left(2b_1 \mu v^2\right) + O(v^4)$$ [Drop all μv^3 terms because they are 4th order]

$$N_W^{(3)}(v) = \frac{-1}{\sigma+1}\left(v - \sigma g(v)\right)\left(\mu - f(v)\right) =$$

$$= \frac{-1}{\sigma+1}\left(\mu v - a_1 v^3 - \sigma b_1 \mu v^2\right) + O(v^4)$$

and therefore

$$N_W(v) = (\sigma+1) b_1 v^2 + (\sigma+1) b_2 v^3 + \frac{2\sigma b_1}{\sigma+1}\mu v^2 +$$

$$+ \frac{-1}{\sigma+1}\left(\mu v - a_1 v^3 - \sigma b_1 \mu v^2\right) + O(v^4)$$

$$= \frac{-\mu}{\sigma+1} v + \left[b_1(\sigma+1) + \frac{2\sigma\mu}{\sigma+1} b_1 + \frac{\sigma\mu}{\sigma+1} b_1\right] v^2 + \left[b_2(\sigma+1) + \frac{a_1}{\sigma+1}\right] v^3 + O(v^4)$$

- We disregard the μv term on $N_W^{(3)}(v)$ since it can be paired up with other μv terms that we are not keeping track of. Now, we set the coefficients of v^2 and v^3 equal to zero:

$$\begin{cases} b a_1 - 1 + \dfrac{2\mu\sigma}{\sigma+1} a_1 = 0 \\ b a_2 - b_1 + \sigma b_1 = 0 \end{cases} \wedge \begin{cases} b_1(\sigma+1) + \dfrac{3\sigma\mu}{\sigma+1} = 0 \\ b_2(\sigma+1) + \dfrac{a_1}{\sigma+1} = 0 \end{cases} \Leftrightarrow$$

and it follows that:

$$a_1 = \frac{1}{b + \frac{2\mu\sigma}{\sigma+1}} = \frac{1}{b} - \frac{1}{b^2}\frac{2\mu\sigma}{\sigma+1} + O(\mu^2)$$

$$b_1 = \frac{3\sigma\mu}{(\sigma+1)^2}$$

$$a_2 = (1-\sigma)b_1 = \frac{3\sigma(1-\sigma)\mu}{b(\sigma+1)^2}$$

$$b_2 = \frac{-a_1}{(\sigma+1)^2} = \frac{-1}{b(\sigma+1)^2} + \frac{1}{b^2(\sigma+1)^2}\frac{2\mu\sigma}{\sigma+1} + O(\mu^2)$$

The master equation is given by:

$$\dot{v} = \frac{\sigma}{\sigma+1}(v-\sigma w)(\mu-u) = \frac{\sigma}{\sigma+1}(v-\sigma g(v))(\mu - f(v)) =$$

$$= \frac{\sigma}{\sigma+1}(\mu v - a_1 v^3 - \sigma b_1 \mu v^2) + O(v^4) =$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \left(\frac{1}{b} - \frac{2\mu\sigma}{\sigma+1}\frac{1}{b^2}\right)v^3 - \sigma\frac{3\sigma\mu}{(\sigma+1)^2}\mu v^2\right] + O(v^4)$$

$\mu^2 v^2$ term

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \left(\frac{1}{b} - \underbrace{\frac{2\mu\sigma}{\sigma+1}\frac{1}{b^2}}_{\mu v^3 \text{ term can also be dropped}}\right)v^3\right] + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \frac{v^3}{b}\right]$$

which is the standard form of a $\boxed{\text{pitchfork bifurcation.}}$
For $\mu = 0$: $\dot{v} = -[b\sigma/(\sigma+1)]v^3$ which gives stable fixed point.

EXERCISE

(2) Use center manifold reduction to analyze the local bifurcation near the origin for the following autonomous dynamical systems

a) $\begin{cases} \dot{x} = -x + \mu y + y^2 \\ \dot{y} = -\sin x \end{cases}$

b) $\begin{cases} \dot{x} = -x + y + \mu x^2 \\ \dot{y} = -\sin x \end{cases}$

c) $\begin{cases} \dot{x} = 2x + 2y \\ \dot{y} = x + y + x^4 + \mu y^2 \end{cases}$

d) $\begin{cases} \dot{x} = -2x + 3y + y^3 + \mu x^2 \\ \dot{y} = 2x - 3y + x^3 \end{cases}$

e) $\begin{cases} \dot{x} = -x - y + z^2 \\ \dot{y} = 2x + y + \mu y - z^2 \\ \dot{z} = x + 2y - z \end{cases}$

f) $\begin{cases} \dot{x} = -2x + y + z + \mu x + y^2 z \\ \dot{y} = x - 2y + z + \mu x + x z^2 \\ \dot{z} = x + y - 2z + \mu x + x^2 y \end{cases}$