1D AUTONOMOUS SYSTEMS

Stability analysis for 1d systems

- We recall from analysis strong differentiability:

Def: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$. We say that f is strongly differentiable at $x_0 \in A$ if and only if there is a function $g: A \to \mathbb{R}$ such that

$$\begin{cases} \forall x \in A: f(x) = f(x_0) + (x - x_0) f'(x_0) + |x| g(x) \\ \lim_{x \to x_0} g(x) = 0 \end{cases}$$

Prop: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $x_0 \in A$.

$$\left.\begin{array}{l} f \text{ differentiable at } x_0 \\ f' \text{ continuous at } x_0 \end{array}\right\} \Rightarrow f \text{ strongly differentiable at } x_0$$

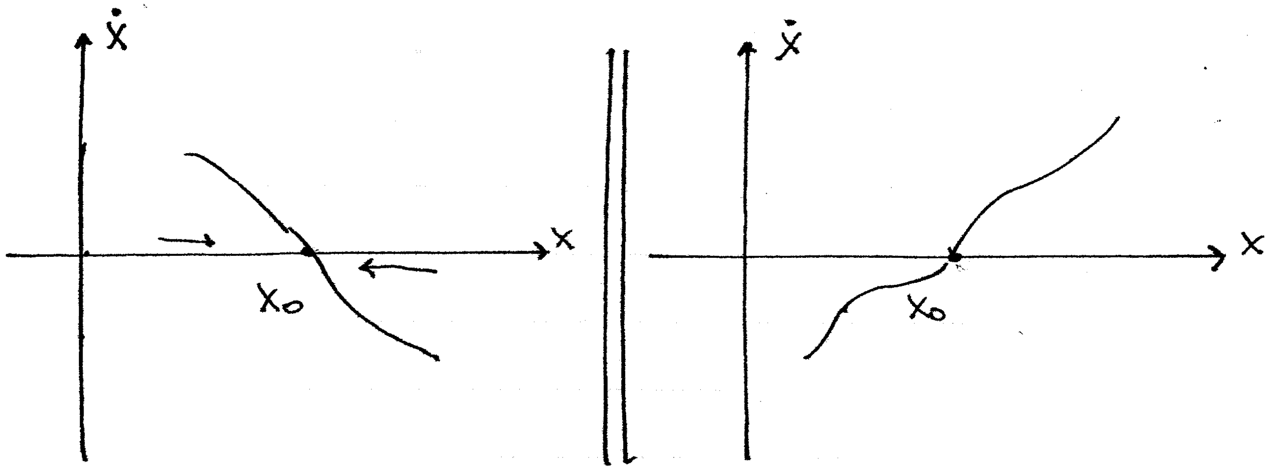
- The stability of 1d autonomous dynamical systems is determined via the following theorem.

Thm: Consider the system $\dot{x} = f(x)$ with $f: \mathbb{R} \to \mathbb{R}$ a function which is strongly differentiable on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a fixed point with $f(x_0) = 0$. Then:

a) $f'(x_0) < 0 \Rightarrow x_0$ asymptotically stable

b) $f'(x_0) > 0 \Rightarrow x_0$ unstable.

$\mapsto$ The theorem is interpreted according to the following phase portraits:



x_0 asymptotically stable $\qquad\qquad$ x_0 unstable

<u>Proof</u>

Define $y(t) = x(t) - x_0 \Rightarrow x(t) = y(t) + x_0 \Rightarrow$

$$\Rightarrow \dot{y} = \dot{x} = f(x) = f(x_0 + y) = f(x_0) + yf'(x_0) + |y|g(y) =$$
$$= yf'(x_0) + |y|g(y)$$

with $\lim_{y\to 0} g(y) = 0$, since f is strongly differentiable in x_0. It follows that

$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in (-\delta, 0) \cup (0, \delta) : |g(y)| < \varepsilon$

Let $\varepsilon = (1/2)|f'(x_0)|$ and let $\delta > 0$ be the corresponding δ such that $\forall y \in (-\delta, 0) \cup (0, \delta) : |g(y)| < \varepsilon$. We see that:

$$|\dot{y} - yf'(x_0)| = ||y|g(y)| = |y||g(y)| < |y|\,\varepsilon = |y|(1/2)|f'(x_0)|$$
$$= (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow |\dot{y} - yf'(x_0)| < (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow -(1/2)|yf'(x_0)| < \dot{y} - yf'(x_0) < (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow yf'(x_0) - (1/2)|yf'(x_0)| < \dot{y} < yf'(x_0) + (1/2)|yf'(x_0)|$$

First, we note that:

a) If $y f'(x_0) > 0 \Rightarrow \dot{y} > y f'(x_0) - (1/2)|y f'(x_0)| =$
$= y f'(x_0) - (1/2) y f'(x_0) =$
$= (1/2) y f'(x_0) \Rightarrow \dot{y} > (1/2) y f'(x_0)$

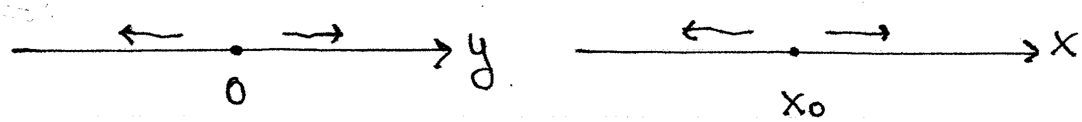
b) If $y f'(x_0) < 0 \Rightarrow$
$\Rightarrow \dot{y} < y f'(x_0) + (1/2)|y f'(x_0)| = y f'(x_0) - (1/2) y f'(x_0)$
$= (1/2) y f'(x_0) \Rightarrow \dot{y} < (1/2) y f'(x_0)$.

We have thus shown that for $y \in (-\delta, 0) \cup (0, \delta)$:
$y f'(x_0) > 0 \Rightarrow \dot{y} > (1/2) y f'(x_0)$
$y f'(x_0) < 0 \Rightarrow \dot{y} < (1/2) y f'(x_0)$

We now distinguish between the following cases:

<u>Case 1</u>: Assume that $f'(x_0) > 0$. Then for
$y \in (0, \delta) \Rightarrow y f'(x_0) > 0 \Rightarrow \dot{y} > (1/2) y f'(x_0) > 0 \Rightarrow$
$\Rightarrow y(t)$ increasing.
$y \in (-\delta, 0) \Rightarrow y f'(x_0) < 0 \Rightarrow \dot{y} < (1/2) y f'(x_0) < 0 \Rightarrow$
$\Rightarrow y(t)$ decreasing.

It follows that the fixed point x_0 is unstable:



<u>Case 2</u>: Assume that $f'(x_0) < 0$. Then for
$y \in (0, \delta) \Rightarrow y f'(x_0) < 0 \Rightarrow \dot{y} < (1/2) y f'(x_0) < 0 \Rightarrow$
$\Rightarrow y(t)$ decreasing $\Rightarrow$ Lyapunov stability.

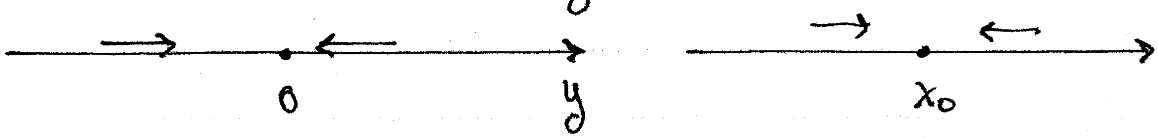
Since $y = 0$ is a fixed point, it follows that if we initialize at $\cancel{y(0) \in (-\delta, 0)}$ $y(0) \in (0, \delta)$, then $y(t) \geq 0$ and furthermore $y(0) \exp((1/2) f'(x_0) t) \geq y(t) \geq 0 \Rightarrow$
$\Rightarrow \lim_{t \to +\infty} y(t) = 0 \Rightarrow$ fixed point is attracting

Likewise, for

$y \in (-\delta, 0) \Rightarrow y f'(x_0) > 0 \Rightarrow \dot{y} > (1/2) y f'(x_0) > 0 \Rightarrow$
$\Rightarrow y(t)$ increasing $\Rightarrow$ Lyapunov stability

and similarly we can show that

$y(0) \exp\left((1/2) f'(x_0) t\right) \leqslant y(t) \leqslant 0 \Rightarrow \lim_{t \to +\infty} y(t) = 0 \Rightarrow$
$\Rightarrow$ fixed point is attracting.



In both cases, initializing at $y(0) \in (-\delta, 0) \cup (0, \delta)$ yields both Lyapunov stability and the attracting property, therefore the fixed point x_0 is asymptotically stable.

EXAMPLES

a) $\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases}$ with $a>0$ (Exponential growth model)

- Exact solution $x(t) = x_0 \exp(at)$
- Fixed points.

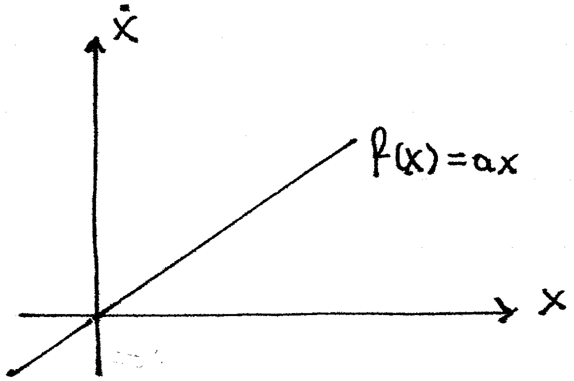
Let $f(x) = ax$. Then

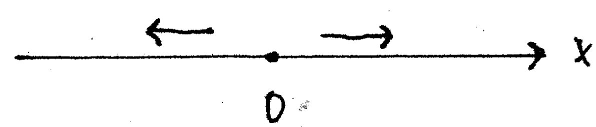
x fixed point $\Leftrightarrow f(x)=0 \Leftrightarrow ax=0 \Leftrightarrow \underline{x=0}$

- Stability.

$f'(x) = (ax)' = a$

At $x=0$: $f'(0) = a>0 \Rightarrow x=0$ is an unstable fixed point.





b) $\begin{cases} \dot{x} = (a/b)x(b-x) \\ x(0) = x_0 \end{cases}$ with $a>0$ and $b>0$ (Logistic Model)

Here $a =$ growth rate
$b =$ carrying capacity.

- Fixed points.

Let $f(x) = (a/b)\,x\,(b-x)$.

x fixed point $\Leftrightarrow f(x) = 0 \Leftrightarrow (a/b)x(b-x) = 0 \Leftrightarrow$
$\Leftrightarrow x(b-x) = 0 \Leftrightarrow x = 0 \vee x = b$

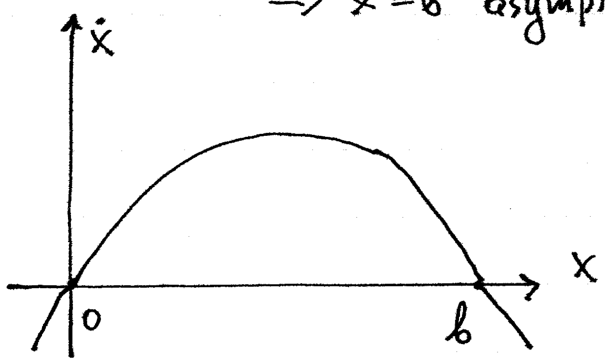
- Stability:

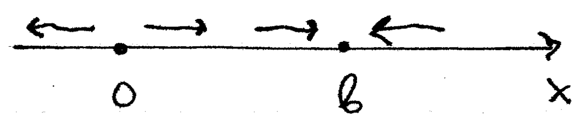
$$f'(x) = \frac{a}{b}\frac{d}{dx}x(b-x) = \frac{a}{b}\frac{d}{dx}(bx - x^2) =$$

$$= \frac{a}{b}(b - 2x) = a - \frac{2ax}{b}$$

For $x=0$: $f'(0) = a > 0 \Rightarrow x = 0$ unstable.
For $x=b$: $f'(b) = a - \dfrac{2ab}{b} = a - 2a = -a < 0 \Rightarrow$

$\Rightarrow x = b$ asymptotically stable.





The stability theorem given above may fail if at a fixed point $x = x_0$ we have $f'(x_0) = 0$. An alternative method for determining fixed point stability useful in such situations is the construction of a sign table.

c) $\dot{x} = 2x(x-1)^2(x-2)^3$

▸ Fixed points

Let $f(x) = 2x(x-1)^2(x-2)^3$. Then

x fixed point $\Leftrightarrow f(x)=0 \Leftrightarrow 2x(x-1)^2(x-2)^3 = 0 \Leftrightarrow$

$\Leftrightarrow 2x = 0 \vee (x-1)^2 = 0 \vee (x-2)^3 = 0 \Leftrightarrow$

$\Leftrightarrow x = 0 \vee x = 1 \vee x = 2$.

▸ Stability

x		0		1		2	
$2x$	$-$	0	$+$		$+$		$+$
$(x-1)^2$	$+$		$+$	0	$+$		$+$
$(x-2)^3$	$-$		$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$-$	0	$+$
	$\rightarrow$		$\leftarrow$		$\leftarrow$		$\rightarrow$
		stab.		unst.		unst.	

Thus $x=0$ is asymptotically stable and $x=1$ and $x=2$ are unstable.

EXERCISES

① Find the fixed-points and determine their stability for the following systems:

a) $\dot{x} = x^2 - 9$

b) $\dot{x} = x - x^3$

c) $\dot{x} = x^2(6-x)$

d) $\dot{x} = x(1-x)(2-x)$

e) $\dot{x} = (x+1)^3(x-3)^2$

f) $\dot{x} = (2x-1)^2(3x+1)^4$

g) $\dot{x} = 1 - 2\cos x$

h) $\dot{x} = e^{-x}\sin x$

i) $\dot{x} = 2\sin x + \sin 2x$

j) $\dot{x} = \cos x + 2\sin x$

k) $\dot{x} = -2x\ln(3x)$

▼ Potential and 1d systems

Consider a 1d autonomous system $\dot{x} = f(x)$ with f continuous in $\mathbb{R}$. Then we may define a potential function

$$V(x) = \int_x^c f(t)\,dt \Rightarrow f(x) = -\frac{dV(x)}{dx} = -V'(x).$$

It follows that

$$\frac{dx}{dt} = -V'(x).$$

- Let $x(t)$ be a solution of the autonomous system. We will show that $V(x(t))$ decreases with time, that is the system evolves towards lower potentials. Formally:

$$\boxed{t_1 < t_2 \Rightarrow V(x(t_1)) > V(x(t_2))}$$

Proof

$$\frac{d}{dt} V(x(t)) = V'(x(t)) \frac{dx(t)}{dt} = V'(x(t)) f(x(t)) =$$

$$= V'(x(t))\left[-V'(x(t))\right] = -\left[V'(x(t))\right]^2 \Rightarrow$$

$$\Rightarrow V(x(t_2)) - V(x(t_1)) = \int_{t_1}^{t_2} \left[\frac{d}{dt} V(x(t))\right] dt =$$

$$= \int_{t_1}^{t_2} -[V'(x(t))]^2 dt \leq 0 \Rightarrow$$

$$\Rightarrow V(x(t_1)) \geq V(x(t_2)). \qquad \square$$

Remarks

a) Fixed points occur at the min/max points of the potential function $V(x)$.

b) Stable fixed points occur at the min points of $V(x)$.

c) Unstable fixed points occur at the max points of $V(x)$.

$\mapsto$ No periodic solutions

- A 1d autonomous system $\dot{x} = f(x)$ never has any periodic solution that is not constant for all time.

Proof

Let $x(t)$ be a solution of $\dot{x} = f(x)$ such that
$x(t) = x(t+T)$, $\forall t \in \mathbb{R}$. (1)
Let $V(x)$ be the potential function. Then
$(1) \Rightarrow V(x(t)) = V(x(t+T))$, $\forall t \in \mathbb{R} \Rightarrow$

$$\Rightarrow \int_t^{t+T} [V'(x(\tau))]^2 d\tau = 0, \; \forall t \in \mathbb{R} \Rightarrow$$

$\Rightarrow V'(x(\tau)) = 0 \quad , \forall \tau \in [t, t+T] , \forall t \in \mathbb{R} \Rightarrow$

$\Rightarrow V'(x(t)) = 0 \quad , \forall t \in \mathbb{R}$

$\Rightarrow dx(t)/dt = 0 \quad , \forall t \in \mathbb{R} \Rightarrow x(t)$ constant $\square$.

EXERCISES

(2) Show that the system $\dot{x} = x - \sin t$ with $x(0) = 1/2$ admits the periodic solution $x(t) = (1/2)(\sin t + \cos t)$. How does this reconcile with our claim that 1d autonomous systems do not admit periodic solutions?

(3) Likewise show that the system $\ddot{x} = -\omega^2 x$ with $x(0) = 0$ admits the periodic solution $x(t) = \quad \sin(\omega t)$. Again, how do we reconcile this paradox?

(4) Find the fixed-points and determine their stability for 1d autonomous systems with potentials given by

a) $V(x) = x^4 - 3x^3$

b) $V(x) = (x^2 - 1)^2$

c) $V(x) = x + 4\sin x$

d) $V(x) = e^{-x}\cos x$

e) $V(x) = (x^2 - 2)e^{-x}$

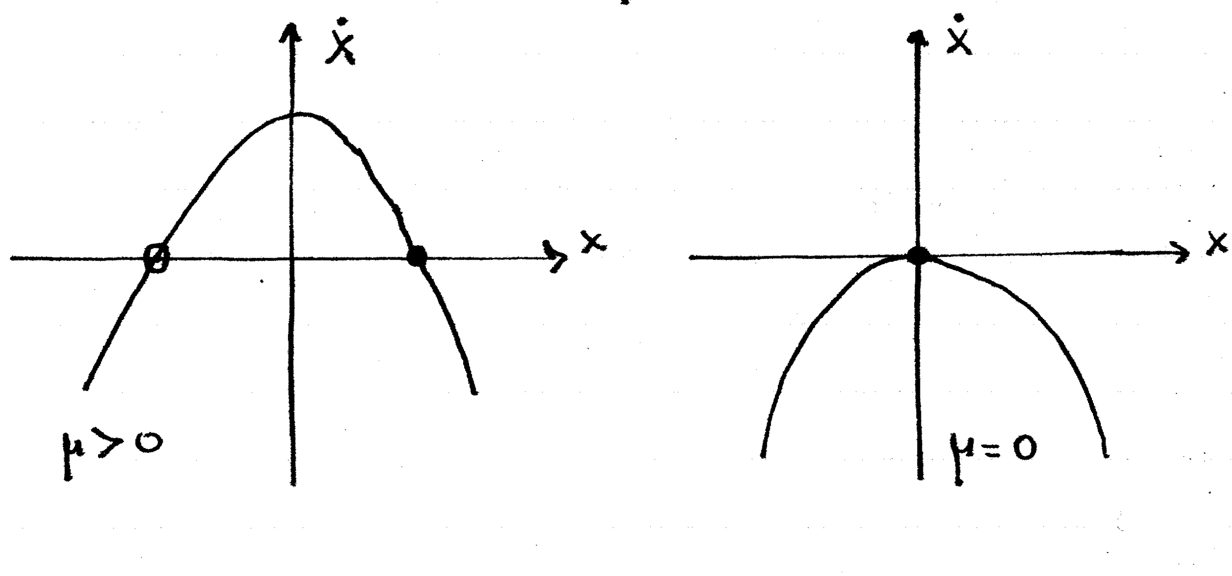
Local Bifurcations with 1d systems

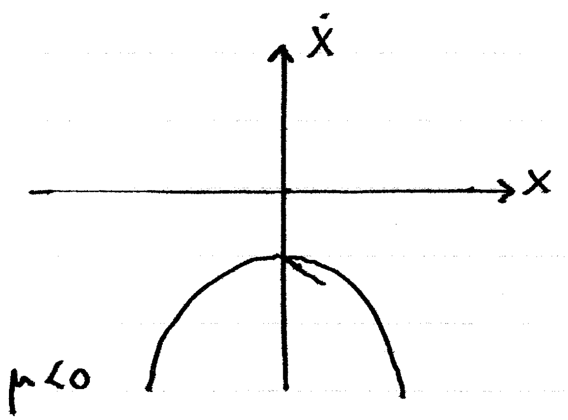
- In general, bifurcations fall under two general categories
  (a) Local Bifurcations
  (b) Global Bifurcations
  However 1d systems admit only local bifurcations.

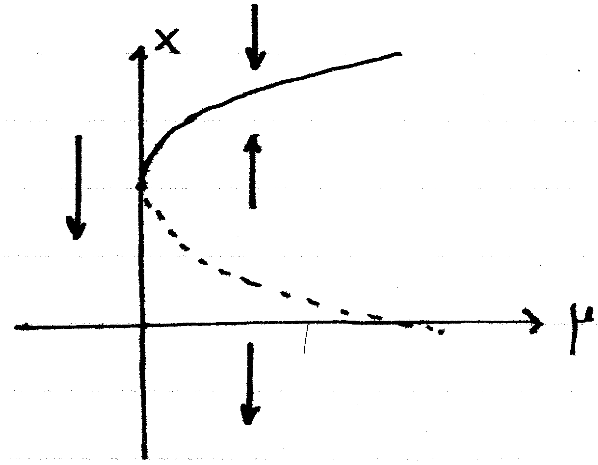
- Consider the 1d autonomous system $\dot{x} = f(x,\mu)$ with $\mu \in \mathbb{R}$ a parameter. A <u>local bifurcation</u> occurs when the number of fixed points changes as we vary the value of the parameter μ. The three most common types of local bifurcations are:

① <u>Saddle-node bifurcation</u> $\longrightarrow$ $\boxed{\dot{x} = \mu - x^2}$

Two fixed-points with opposite stability properties collide into a saddle point which then vanishes:





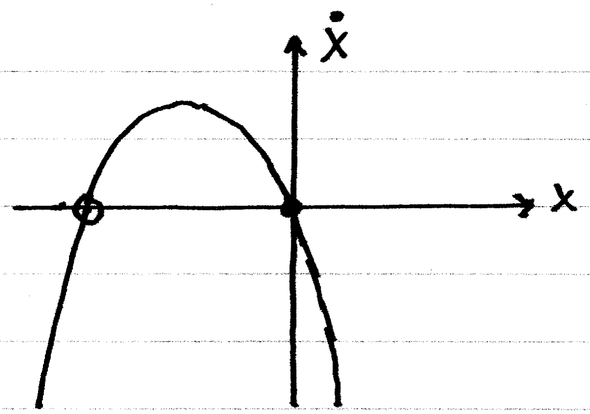


(Bifurcation diagram)

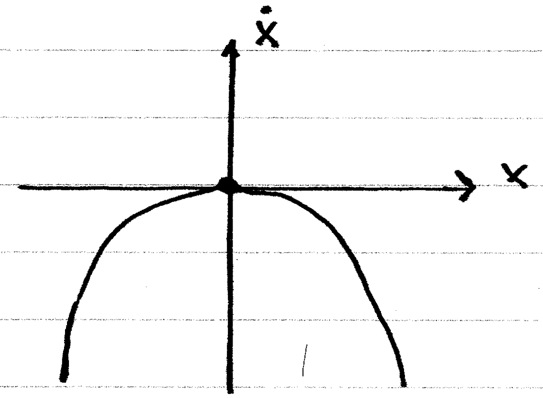
↳ A bifurcation diagram shows the motion of the fixed-points on the x-axis as a function of the parameter μ. We use a solid line to denote the motion of a stable fixed-point and a dotted-line to show the motion of an unstable fixed-point.

(2) → Transcritical bifurcation → $\boxed{\dot{x} = \mu x - x^2}$

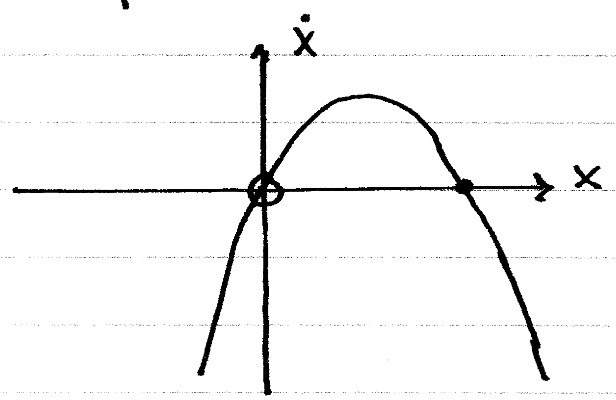
Two fixed points with opposite stability properties collide into a saddle-point which breaks up into two fixed points again with opposite stability properties but also with their stability properties exchanged. One fixed point is independent of μ.



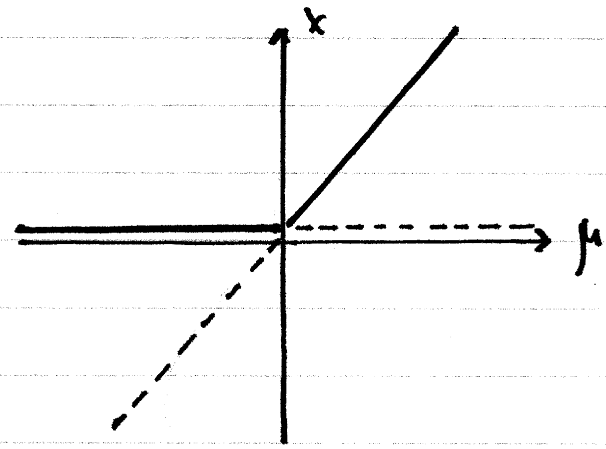
$\mu < 0$



$\mu = 0$



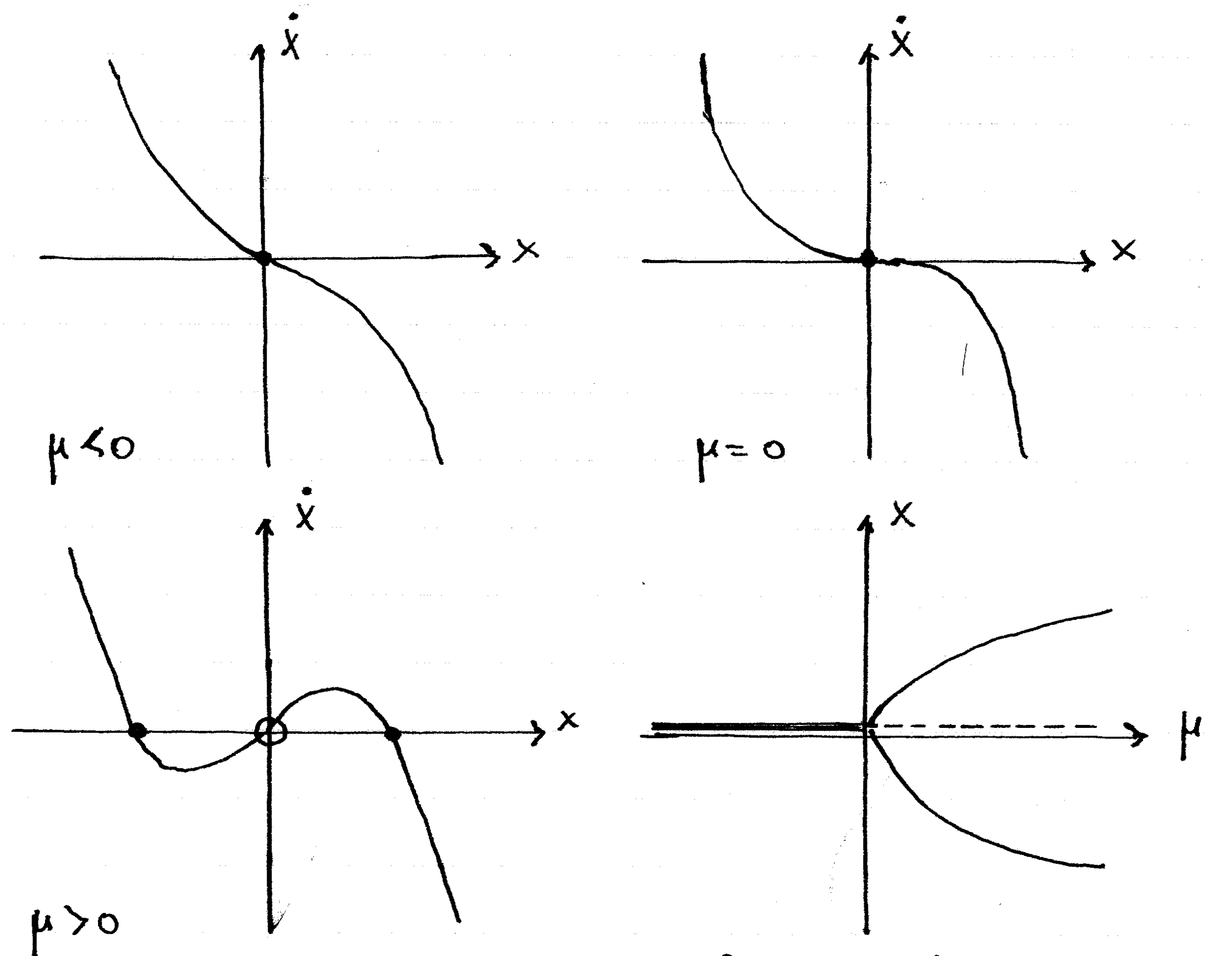
$\mu > 0$



Bifurcation diagram.

③ → Pitchfork Bifurcation → $\boxed{\dot{x} = \mu x - x^3}$

In a pitchfork bifurcation, a fixed-point breaks into 3 fixed-points. The inner fixed-point has opposite stability property with respect to the original fixed-point. The 2 outer fixed-points have the same stability property as the original fixed-point. We call this bifurcation a pitchfork bifurcation because the bifurcation diagram resembles a pitchfork. The inner fixed-point is independant of the parameter μ.



Bifurcation diagram

Tangency condition

From the examples above we see that bifurcations occur always when the graph of $f(x)$ is tangent to the x-axis. It follows that candidate points (x_0, μ_0) for bifurcation events can be located by solving the system of equations:

$$\boxed{\begin{cases} f(x_0, \mu_0) = 0 \\ f_x(x_0, \mu_0) = 0 \end{cases}}$$

Here, the subscripts represent partial derivatives, thus $f_x = \dfrac{\partial f}{\partial x}$.

Sufficient conditions

Once we identify a candidate for a bifurcation event at (x_0, μ_0) it can be classified by confirming the corresponding sufficient conditions. The sufficient conditions for the bifurcations considered above are:

Saddle-node bifurcation	Transcritical bifurcation	Pitchfork bifurcation.
$f(x_0, \mu_0) = 0$	$f(x_0, \mu_0) = 0$	$f(x_0, \mu_0) = 0$
$f_x(x_0, \mu_0) = 0$	$f_x(x_0, \mu_0) = 0$	$f_x(x_0, \mu_0) = 0$
$f_\mu(x_0, \mu_0) \neq 0$	$f_\mu(x_0, \mu_0) = 0$	$f_\mu(x_0, \mu_0) = 0$
$f_{xx}(x_0, \mu_0) \neq 0$	$f_{xx}(x_0, \mu_0) \neq 0$	$f_{xx}(x_0, \mu_0) = 0$
	$f_{x\mu}(x_0, \mu_0) \neq 0$	$f_{x\mu}(x_0, \mu_0) \neq 0$
		$f_{xxx}(x_0, \mu_0) \neq 0$

Procedure

To identify and classify bifurcation events (x_0, μ_0) we work as follows:

- 1 Solve the equations

$$f(x,\mu) = 0$$
$$f_x(x,\mu) = 0$$

to identify candidates (x_0, μ_0).

- 2 Calculate $f_\mu(x_0, \mu_0)$.

1) If $f_\mu(x_0,\mu_0) \neq 0$, then check that $f_{xx}(x_0,\mu_0) \neq 0$. If so, then $(x_0,\mu_0) \leftarrow$ saddle-node bifurcation

2) If $f_\mu(x_0,\mu_0) = 0$, then check that $f_{x\mu}(x_0,\mu_0) \neq 0$. Then, if:

a) $f_{xx}(x_0,\mu_0) \neq 0 \leftarrow$ trancritical bifurcation

b) $f_{xx}(x_0,\mu_0) = 0 \leftarrow$, then check that $f_{xxx}(x_0,\mu_0) \neq 0 \leftarrow$ pitchfork bifurcation.

- 3 For a saddle-node bifurcation we have 2 fixed points

a) For $\mu > \mu_0$, if $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) < 0$

b) For $\mu < \mu_0$, if $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) > 0$

(see exercise 9)

- 4 For a pitchfork bifurcation we have 3 fixed points

a) For $\mu > \mu_0$, if $f_{xxx}(x_0,\mu_0) f_{x\mu}(x_0,\mu_0) < 0$

b) For $\mu < \mu_0$, if $f_{xxx}(x_0,\mu_0) f_{x\mu}(x_0,\mu_0) > 0$

(see exercise 11).

EXAMPLES

a) <u>Saddle-Node Bifurcation</u>: $\dot{x} = \mu - x - e^{-x}$

Let $f(x,\mu) = \mu - x - e^{-x} \Rightarrow f_x(x,\mu) = -1 + e^{-x}$.

$$\begin{cases} f(x,\mu) = 0 \\ f_x(x,\mu) = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - x - e^{-x} = 0 \\ -1 + e^{-x} = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - x - e^{-x} = 0 \\ e^{-x} = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu - 0 - e^{-0} = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - 1 = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu = 1 \\ x = 0 \end{cases}$$

thus possible bifurcation at $(x_0, \mu_0) = (0, 1)$

$f_\mu(x,\mu) = 1 \Rightarrow f_\mu(x_0,\mu_0) = 1 \neq 0$ (1)

$f_{xx}(x,\mu) = -e^{-x} \Rightarrow f_{xx}(x_0,\mu_0) = -e^{-0} = -1 \neq 0$ (2)

From (1) and (2): saddle-node bifurcation at $(x_0,\mu_0) = (0,1)$

Since $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) = 1 \cdot (-1) = -1 < 0 \Rightarrow$
$\Rightarrow$ two fixed points for $\mu > 1$ and no fixed points for $\mu < 1$.

b) <u>Transcritical Bifurcation</u>: $\dot{x} = \mu \ln x + x - 1$

Let $f(x,\mu) = \mu \ln x + x - 1 \Rightarrow f_x(x,\mu) = \frac{\mu}{x} + 1$

$$\begin{cases} f(x,\mu) = 0 \\ f_x(x,\mu) = 0 \end{cases} \Leftrightarrow \begin{cases} \mu \ln x + x - 1 = 0 \\ \mu/x + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} -x \ln x + x - 1 = 0 \\ \mu = -x \end{cases}$$

Let $g(x) = -x \ln x + x - 1$. Note the obvious solution $x = 1$ since $g(1) = -1 \ln 1 + 1 - 1 = -0 + 0 = 0$. We now show the solution is unique.

$$g'(x) = -(x)'\ln x - x(\ln x)' + 1 = -\ln x - x\frac{1}{x} + 1 =$$
$$= -\ln x - 1 + 1 = -\ln x$$

It follows that $g \nearrow (0,1)$ and $g \searrow (1, +\infty)$ thus $\forall x \in (0,1) \cup (1, +\infty)$: $g(x) < 0$.

We conclude that the solution $x = 1$ is unique and therefore a bifurcation may occur when $(x_0, \mu_0) = (1, -1)$. Note that

$$f_\mu(x,\mu) = \ln x \Rightarrow f_\mu(1,-1) = \ln 1 = 0$$

$$f_{x\mu}(x,\mu) = \frac{1}{x} \Rightarrow f_{x\mu}(1,-1) = \frac{1}{1} = 1 \neq 0$$

$$f_{xx}(x,\mu) = -\frac{\mu}{x^2} \Rightarrow f_{xx}(1,-1) = -\frac{-1}{1^2} = 1 \neq 0$$

It follows that there is a transcritical bifurcation at $(x_0, \mu_0) = (1, -1)$.

c) <u>Pitchfork bifurcation</u> : $\dot{x} = -x + \mu \tanh x$

Let $f(x,\mu) = -x + \mu \tanh x \Rightarrow$

$\Rightarrow f_x(x,\mu) = -1 + \mu(1 - \tanh^2 x)$

$$\begin{cases} f(x,\mu) = 0 \\ f_x(x,\mu) = 0 \end{cases} \Leftrightarrow \begin{cases} -x + \mu \tanh x = 0 \\ -1 + \mu(1 - \tanh^2 x) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu \tanh x = x \\ -1 + \mu - (\mu \tanh x)\tanh x = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu \tanh x = x \\ -1 + \mu - x \tanh x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu \tanh x = x \\ \mu = 1 + x \tanh x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1+x\tanh x)\tanh x = x & (1) \\ \mu = 1 + x\tanh x \end{cases}$$

Since $\tanh 0 = 0$, $x=0$ is an obvious solution of (1).
We now show that this solution is unique.
Let $g(x) = (1+x\tanh x)\tanh x - x =$

$$= \tanh x + x\tanh^2 x - x \Rightarrow$$

$$\Rightarrow g'(x) = (1-\tanh^2 x) + \tanh^2 x + x(2\tanh x)(1-\tanh^2 x) - 1$$

$$= \underline{1} - \underline{\underline{\tanh^2 x}} + \underline{\underline{\tanh^2 x}} + 2x\tanh x - 2x\tanh^3 x - \underline{1}$$

$$= 2x\tanh x - 2x\tanh^3 x =$$

$$= 2x\tanh x\,(1-\tanh^2 x)$$

Note that $-1 < \tanh x < 1 \Rightarrow 1 - \tanh^2 x > 0$ thus

x		0	
$2x$	$-$	0	$+$
$\tanh x$	$-$	0	$+$
$1-\tanh^2 x$	$+$		$+$
$g'(x)$	$+$	0	$+$
$g(x)$	$\nearrow$		$\nearrow$

Since $g \nearrow (-\infty, 0)$ and $g \nearrow (0, +\infty)$ and $g(0)=0$, it follows that $x=0$ is a unique solution of $g(x)=0$.

For $x = 0 \Rightarrow \mu = 1 + 0\tanh 0 = 1$ thus there is a bifurcation at $(x_0, \mu_0) = (0, 1)$.

Now, we note that:

$f_\mu(x,\mu) = \tanh x \Rightarrow f_\mu(0,1) = \tanh 0 = 0$

$f_{x\mu}(x,\mu) = 1 - \tanh^2 x \Rightarrow f_{x\mu}(0,1) = 1 - \tanh^2 0 = 1 - 0 = 1 \neq 0$

$$f_{xx}(x,\mu) = -\mu \frac{\partial^2}{\partial x^2} \tanh^2 x =$$

$$= -2\mu \tanh x \cdot (1 - \tanh^2 x) \Rightarrow$$

$\Rightarrow f_{xx}(0,1) = -2 \cdot 1 \cdot 0 \cdot (1-0) = 0$, thus we rule out transcritical.

$$f_{xxx}(x,\mu) = \frac{\partial}{\partial x}\left[-2\mu \tanh x + 2\mu \tanh^3 x\right] =$$

$$= -2\mu(1 - \tanh^2 x) + 6\mu \tanh^2 x (1 - \tanh^2 x)$$

$$= 2\mu(1 - \tanh^2 x)[-1 + 3\tanh^2 x] \Rightarrow$$

$$\Rightarrow f_{xxx}(0,1) = 2 \cdot 1 \cdot (1-0)[-1 + 3 \cdot 0] =$$

$$= 2 \cdot 1 \cdot 1 \cdot (-1) = -2 \neq 0.$$

It follows that $(x_0, \mu_0) = (0,1)$ are pitchfork bifurcation. Since

$f_{xxx}(0,1) f_{x\mu}(0,1) = (-2) \cdot 1 = -2 < 0 \Rightarrow$

$\Rightarrow$ there are 3 fixed points for $\mu > 1$.

EXERCISES

(5) Show that the following systems undergo a saddle-node bifurcation. Find the value $\mu=\mu_0$ where such a bifurcation occurs

a) $\dot{x} = 1+\mu x+x^2$

b) $\dot{x} = \mu+x-\ln(1+x)$

c) $\dot{x} = \mu+\dfrac{x}{2}-\dfrac{x}{1+x}$

d) $\dot{x} = 1-\dfrac{\mu x^2}{2}+\dfrac{x^4}{4}$

(6) Likewise, show that the following systems undergo a transcritical bifurcation. Find the value $\mu=\mu_0$ where the bifurcation occurs.

a) $\dot{x} = \mu x+x^2$

b) $\dot{x} = \mu x-\ln(1+x)$

c) $\dot{x} = \mu x-x(1-x)$

d) $\dot{x} = x(\mu-e^x)$

e) $\dot{x} = \mu x+x^4$

(7) Likewise, show that the following systems undergo a pitchfork bifurcation. Find the value $\mu=\mu_0$ where the bifurcation occurs.

a) $\dot{x} = \mu x+x^3$

b) $\dot{x} = \mu x-x^3$

c) $\dot{x} = \mu x-\sinh(x)$

d) $\dot{x} = x+\dfrac{\mu x}{1+x^2}$

e) $\dot{x} = \mu x+\sin x$

More on sufficient conditions for bifurcation events

We will now derive the sufficient conditions for classifying bifurcation events. The proofs are based on the implicit function theorem.

Implicit function theorem

First we define the ball $B((x_0,y_0),\varepsilon)$ as:

$$B((x_0,y_0),\varepsilon) = \{(x,y) \in \mathbb{R}^2 \mid (x-x_0)^2+(y-y_0)^2<\varepsilon^2\}$$

The implicit function theorem states:

Thm: Assume that the function $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}^2$ satisfies

a) $f(x_0,y_0)=0$

b) $\forall (x,y) \in B((x_0,y_0),\varepsilon): f_y(x,y) \neq 0$

c) f_x, f_y continuous at $B((x_0,y_0),\varepsilon)$

Then, there is a unique function g such that $\forall (x,y) \in B((x_0,y_0),\varepsilon) : f(x,g(x)) = 0$

Note that condition (b) can be weakened to $f_y(x_0,y_0) \neq 0$. Then, combined with (c) it follows that there is an ε for which both (b) and (c) are satisfied.

(1) → Saddle-node bifurcation conditions

Let us assume that

$$f(x_0, \mu_0) = 0 \quad (1)$$

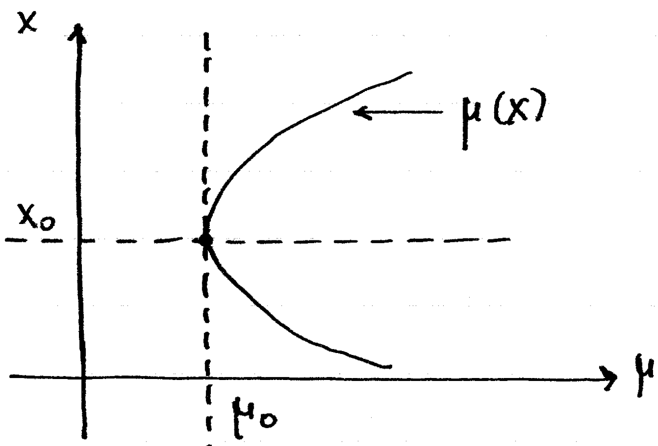
$$f_x(x_0, \mu_0) = 0 \quad (2)$$

$$f_\mu(x_0, \mu_0) \neq 0 \quad (3)$$

$$f_{xx}(x_0, \mu_0) \neq 0 \quad (4)$$

• <u>Analysis</u>

The typical bifurcation diagram for a saddle-node bifurcation is shown below:



We see that we have to show that there is a unique function $\mu(x)$ such that $f(x, \mu(x)) = 0, \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ with

$$\mu'(x_0) = \frac{d}{dx}\mu(x_0) = 0$$

$$\mu''(x_0) = \frac{d^2}{dx^2}\mu(x_0) \neq 0$$

The condition $\mu'(x_0) = 0$ ensures that the bifurcation curve is tangent to $\mu = \mu_0$. The condition $\mu''(x_0) \neq 0$ ensures that x_0 is a minimum or maximum so that the bifurcation curve $\mu(x)$ remains on the same half-plane defined by $\mu = \mu_0$.

• <u>Construction</u>: Since $f(x_0, y_0) = 0$ and $f_\mu(x_0, \mu_0) \neq 0$, it follows that the implicit function theorem applies and therefore there is a unique function $\mu(x)$ such that

$$\forall x \in (x_0-\varepsilon, x_0+\varepsilon) : f(x, \mu(x)) = 0 \qquad (5)$$

Thus $\mu(x)$ is hereby constructed.

- <u>Proof</u>: We will now show that $\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0$.

Differentiating (5) with respect to x gives:

$$f_x(x, \mu(x)) + f_\mu(x, \mu(x))\,\mu'(x) = 0 \qquad (6)$$

For $x = x_0$:

$f_x(x_0, \mu(x_0)) = f_x(x_0, \mu_0) = 0$ and

$f_\mu(x_0, \mu(x_0)) = f_\mu(x_0, \mu_0) \neq 0$

thus:

$$\mu'(x_0) = \frac{-f_x(x_0, \mu_0)}{f_\mu(x_0, \mu_0)} = 0 \qquad (7)$$

Differentiating (6) one more time with respect to x gives:

$$f_{xx} + f_{x\mu} \cdot \mu' + (f_{\mu x} + f_{\mu\mu} \cdot \mu')\mu' + f_\mu \cdot \mu'' = 0 \Rightarrow$$

$$\Rightarrow f_{xx} + (2f_{x\mu} + f_{\mu\mu} \cdot \mu')\mu' + f_\mu \cdot \mu'' = 0$$

evaluated at $(x, \mu(x))$. For $x = x_0$, $\mu'(x_0) = 0$, and therefore:

$$f_{xx}(x_0, \mu_0) + f_\mu(x_0, \mu_0) \cdot \mu''(x_0) = 0$$

Since $f_{xx}(x_0, \mu_0) \neq 0$ and $f_\mu(x_0, \mu_0) \neq 0$, it follows that

$$\mu''(x_0) = \frac{-f_{xx}(x_0, \mu_0)}{f_\mu(x_0, \mu_0)} \neq 0.$$

• <u>Stability</u>: We will now show that the two fixed-points that emerge one one of the two half-planes defined by $\mu = \mu_0$ on the bifurcation diagram have opposite stability.

From (6):

$f_x(x, \mu(x)) = -f_\mu(x, \mu(x))\mu'(x)$, $\forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

Since $f_\mu(x_0, \mu_0) \neq 0$, we can choose $\varepsilon > 0$ small enough so that

$\forall x \in (x_0 - \varepsilon, x_0 + \varepsilon): f_\mu(x, \mu(x)) \neq 0$

Thus $f_\mu(x, \mu(x))$ maintains its sign in $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

Since $\mu''(x_0) \neq 0$ and $\mu'(x_0) = 0$, we expect that $\mu'(x_0)$ changes sign from $x \in (x_0 - \varepsilon, x_0)$ to $x \in (x_0, x_0 + \varepsilon)$. Thus, so does $f_x(x, \mu(x))$ and it follows that the two fixedpoints, when they exist, have opposite stability.

② → Transcritical Bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0$ (1)

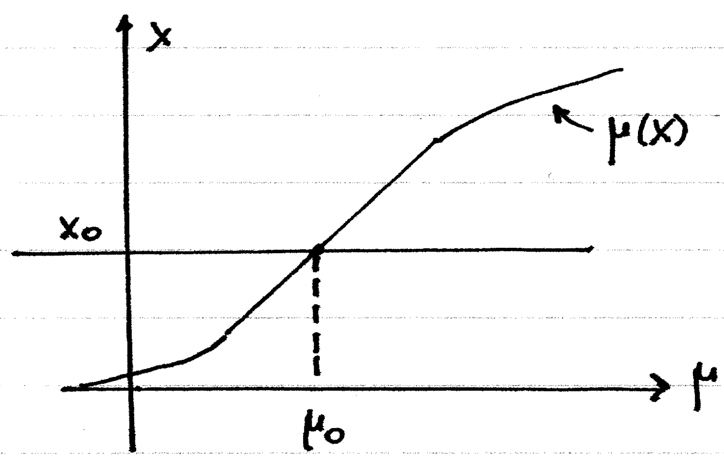
$f_x(x_0, \mu_0) = 0$ (2)

$f_\mu(x_0, \mu_0) = 0$ (3)

$f_{xx}(x_0, \mu_0) \neq 0$ (4)

$f_{x\mu}(x_0, \mu_0) \neq 0$ (5)

The typical bifurcation diagram for a transcritical bifurcation is shown below:



• <u>Analysis</u> : We see that there are two bifurcation curves passing through (x_0, μ_0) :

a) The line (ℓ): $x = x_0$ (independent of the parameter μ)

b) The line (ℓ_2): $\mu = \mu(x)$ passing from one half-plane to the other, separated by $\mu = \mu_0$, with $\mu_0 = \mu(x_0)$.

It follows that :

$f(x_0, \mu) = 0 \quad , \; \forall \mu \in (\mu_0 - \varepsilon_1, \mu_0 + \varepsilon_1)$
$f(x, \mu(x)) = 0 \, , \; \forall x \in (x_0 - \varepsilon_2, x_0 + \varepsilon_2)$
Note that to get two distinct curves pass through (x_0, μ_0) it is necessary to violate the implicit function theorem. Since $f(x_0, \mu_0) = 0$, to violate the theorem we require that $f_\mu(x_0, \mu_0) = 0$.
Let us now define

$$F(x, \mu) = \begin{cases} f(x, \mu)/(x - x_0) & , \; x \neq x_0 \\ f_x(x, \mu) & , \; x = x_0 \end{cases}$$

It follows that $f(x, \mu) = (x - x_0) F(x, \mu)$, thus we assume that $x = x_0$ is a bifurcation curve. We also note that $F(x, \mu)$ retains continuity because

$$\lim_{x \to x_0} F(x, \mu) = \lim_{x \to x_0} \frac{f(x, \mu)}{x - x_0} \overset{\text{L'Hospital}}{=} \lim_{x \to x_0} \frac{f_x(x, \mu)}{1 - 0} =$$

$$= \lim_{x \to x_0} f_x(x, \mu) = f_x(x_0, \mu) = F(x_0, \mu).$$

Note that L'Hospital applies since $\lim_{x \to x_0} f(x, \mu) = f(x_0, \mu) = 0$.

We will now show that $F(x, \mu)$ has a unique curve passing through (x_0, μ_0) and accross $\mu = \mu_0$.

• <u>Construction</u>: We note that

$F(x_0, \mu_0) = f_x(x_0, \mu_0) = 0$ and
$F_\mu(x_0, \mu_0) = f_{x\mu}(x_0, \mu_0) \neq 0$

therefore the implicit function theorem applies.
It follows that there is a unique function $\mu(x)$ such that $F(x,\mu(x))=0$ for all x near x_0.
$x=\mu(x)$ is a bifurcation curve since
$f(x,\mu(x)) = (x-x_0)F(x,\mu(x)) = (x-x_0)\cdot 0 = 0$

- <u>Proof</u> : We will now show that the curve $x=\mu(x)$ passes accross $\mu=\mu_0$. To do that, it is sufficient to show that $\mu'(x_0)\neq 0$.

Since $F(x,\mu(x))=0 \Rightarrow$

$\Rightarrow F_x(x,\mu(x)) + F_\mu(x,\mu(x))\mu'(x) = 0 \Rightarrow$

$$\Rightarrow \mu'(x_0) = \frac{-F_x(x_0,\mu(x_0))}{F_\mu(x_0,\mu(x_0))} = \frac{-f_{xx}(x_0,\mu(x_0))}{f_{x\mu}(x_0,\mu(x_0))}$$

Since $f_{xx}(x_0,\mu_0)\neq 0$ and $f_{x\mu}(x_0,\mu_0)\neq 0$, $\mu'(x_0)$ is well-defined and $\mu'(x_0)\neq 0$.
It follows that $x=\mu(x)$ does not have a min or max at $x=x_0$, thus it will go accross the line $\mu=\mu_0$.

- <u>Stability</u>: We will now show that both bifurcation lines $(l_1): x=x_0$ and $(l_2): x=\mu(x)$ change stability upon crossing the point (x_0,μ_0).

a) For the line $(l_1): x=x_0$:

$$f_x(x_0,\mu) = f_x(x_0,\mu_0) + \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm =$$

$$= \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$$

Since $f_{x\mu}(x_0, \mu_0) \neq 0 \Rightarrow$

$\Rightarrow \exists \varepsilon > 0 : \forall \mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon) : f_{x\mu}(x_0, \mu) \neq 0$

Thus $f_{x\mu}(x_0, \mu)$ maintains its sign in $(\mu_0 - \varepsilon, \mu_0 + \varepsilon)$ therefore $f_x(x_0, \mu)$ changes sign from $\mu > \mu_0$ to $\mu < \mu_0$.

b) For the line $(\ell_2) : x = \mu(x)$

$$f_x(x, \mu(x)) = \frac{\partial}{\partial x}\left[(x - x_0) F(x, \mu(x))\right] =$$

$$= F(x, \mu(x)) + (x - x_0) F_x(x, \mu(x))$$

$$= (x - x_0) F_x(x, \mu(x))$$

Here we have used $F(x, \mu(x)) = 0$.

At $x = x_0$: $F_x(x_0, \mu(x_0)) = f_{xx}(x_0, \mu(x_0)) =$

$$= f_{xx}(x_0, \mu_0) \neq 0 \Rightarrow$$

$\Rightarrow \exists \varepsilon > 0 : \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon) : F_x(x, \mu(x)) \neq 0$

Thus $F_x(x, \mu(x))$ does not change sign in $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ but $x - x_0$ does change from negative to positive. It follows that $f_x(x, \mu(x))$ changes sign accross $x = x_0$

From (a) and (b) above we conclude that since for both curves f_x changes sign accross the point (x_0, μ_0), the stability for both curves also changes.

③ → Pitchfork bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0$ (1)

$f_x(x_0, \mu_0) = 0$ (2)

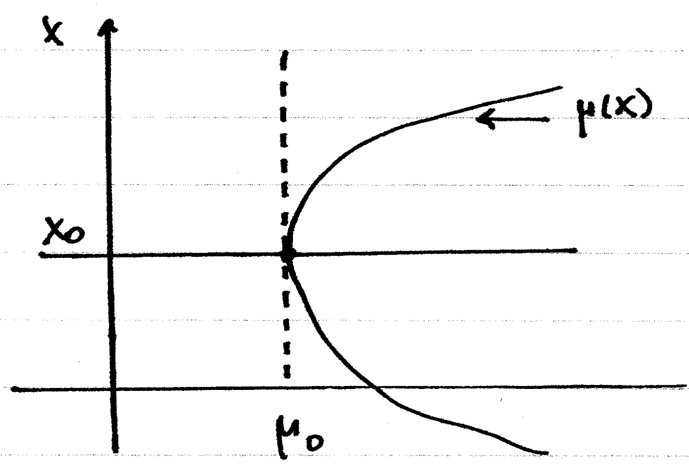
$f_\mu(x_0, \mu_0) = 0$ (3)

$f_{xx}(x_0, \mu_0) = 0$ (4)

$f_{x\mu}(x_0, \mu_0) \neq 0$ (5)

$f_{xxx}(x_0, \mu_0) \neq 0$ (6)

The typical bifurcation diagram for a pitchfork bifurcation is shown below:



• <u>Analysis</u>: The bifurcation diagram has two lines:

(a) The line (l_1): $x = x_0$ which is independent of μ.

(b) The curve (l_2): $\mu = \mu(x)$ which is tangent to the line (l): $\mu = \mu_0$. It follows that μ must satisfy $\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0$.

Both lines intersect at (x_0, μ_0).

Again, in order to have two curves passing through

(x_0, μ_0) it is necessary to violate the implicit function theorem. Since $f(x_0,\mu_0)=0$, it is thus necessary to have $f_\mu(x_0,\mu_0)=0$.
Again, let us define

$$F(x,\mu) = \begin{cases} f(x,\mu)/(x-x_0) & , \text{ if } x \neq x_0 \\ f_x(x_0,\mu) & , \text{ if } x = x_0 \end{cases}$$

Similarly with our transcritical bifurcation argument, it follows that

$$f(x,\mu) = (x-x_0)F(x,\mu)$$
$$\lim_{x\to x_0} F(x,\mu) = F(x_0,\mu).$$

Thus $(\ell_1): x = x_0$ is by definition a bifurcation line.

- <u>Construction</u>: We note that

$$F(x_0,\mu_0) = f_x(x_0,\mu_0) = 0$$
$$F_\mu(x_0,\mu_0) = f_{x\mu}(x_0,\mu_0) \neq 0$$

It follows that the implicit function theorem applies and thus there is a unique function $\mu(x)$ such that $F(x,\mu(x))=0$. It follows that
$f(x,\mu(x)) = (x-x_0)F(x,\mu(x)) = (x-x_0)\cdot 0 = 0$
Thus $\mu(x)$ has been constructed.

- <u>Proof</u>: We will now show that $\mu'(x_0)=0$ and $\mu''(x_0)\neq 0$. Using a calculation similar to the one we did for the saddle-node proof, it follows that

$$\mu'(x_0) = \frac{-F_x(x_0,\mu_0)}{F_\mu(x_0,\mu_0)} = \frac{-f_{xx}(x_0,\mu_0)}{f_{x\mu}(x_0,\mu_0)} = 0$$

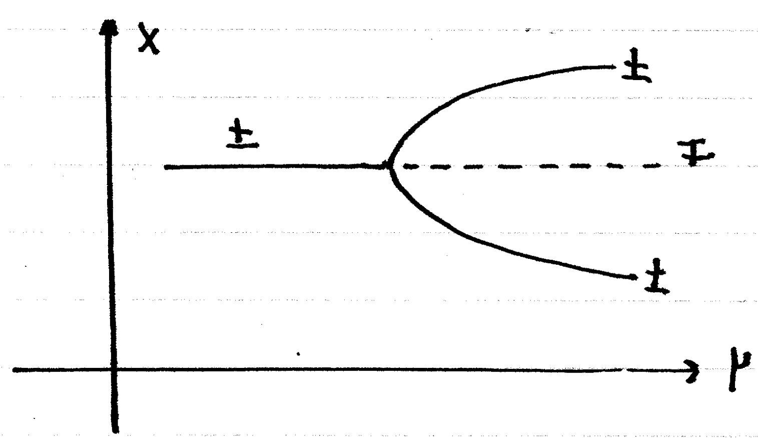
because $f_{xx}(x_0,\mu_0) = 0$

and therefore

$$\mu''(x_0) = \frac{-F_{xx}(x_0,\mu_0)}{F_\mu(x_0,\mu_0)} = \frac{-f_{xxx}(x_0,\mu_0)}{f_{x\mu}(x_0,\mu_0)} \neq 0$$

because $f_{xxx}(x_0,\mu_0) \neq 0$.

- <u>Stability</u>: We will now show that the inner fixed point changes stability accross the point (x_0,μ_0). We will also show that the outer fixed points after, the pitchfork occurs, have the same stability with each other as well as with the inner fixe dpoint BEFORE the fixed point. This is all shown in the diagram below:



a) For the line $x = x_0$:

$$f_x(x_0,\mu) = f_x(x_0,\mu_0) + \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm =$$
$$= \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$$

Since $f_{x\mu}(x_0,\mu_0) \neq 0 \Rightarrow$
$\Rightarrow f_{x\mu}(x_0,m)$ does not change sign accross $m=\mu_0$
$\Rightarrow f_x(x,\mu)$ changes sign accross $\mu = \mu_0$
$\Rightarrow$ The fixed point on the line $x = x_0$ changes stability.

b) For the line $\mu = \mu(x)$:

$$f_x(x,\mu(x)) = \frac{\partial}{\partial x}\Big[(x-x_0)\,F(x,\mu(x))\Big] =$$
$$= F(x,\mu(x)) + (x-x_0)\,F_x(x,\mu(x))$$
$$= (x-x_0)\,F_x(x,\mu(x)) =$$
$$= (x-x_0)\left[-F_\mu(x,\mu(x))\,\mu'(x)\right]$$

Here we used the identity
$$F_x(x,\mu(x)) + F_\mu(x,\mu(x))\,\mu'(x) = 0$$
We note that accross $x = x_0$:

$x - x_0$ changes sign, and
$\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0 \Rightarrow \mu'(x_0)$ changes sign
and $F_\mu(x_0,\mu(x_0)) = F_\mu(x_0,\mu_0) = f_{x\mu}(x_0,\mu_0) \neq 0 \Rightarrow$

$\Rightarrow F_\mu(x,\mu(x))$ does not change sign.
Thus $f_x(x,\mu(x))$ does not change sign. It follows that the two outer fixed points have the same stability.
c) We now compare the stability of the outer fixed points with the inner fixed point. Recall that f_x for these fixed points is:

inner point: $f_x(x_0,\mu) = \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$

outer points: $f_x(x,\mu(x)) = -\mu'(x)(x-x_0)F_\mu(x,\mu(x))$

We assume, with no loss of generality, that $\mu''(x_0)>0$. This implies that $\mu(x)$ has a minimum at $x=x_0$, so the 3 fixed points occur when $\mu > \mu_0$. We may thus assume that $\underline{\mu>\mu_0}$. It also follows that when x is near x_0, $\mu'(x)$ is increasing, and therefore:

$x-x_0<0 \Rightarrow \mu'(x)<0$

$x-x_0>0 \Rightarrow \mu'(x)>0$

Thus: $\mu'(x)(x-x_0)>0$ when x is near x_0.
It follows that:

$f_x(x,\mu(x))$ opposite sign as $F_\mu(x,\mu(x))$
same sign as $F_\mu(x_0,\mu_0)$ (x near x_0)
same sign as $f_{x\mu}(x_0,\mu_0)$
same sign as $f_{x\mu}(x_0,m)$ (m near μ_0)
same sign as $\int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm = f_x(x_0,\mu)$ (use $\mu>\mu_0$).

Thus $f_x(x,\mu(x))$ has opposite sign from $f_x(x_0,\mu)$, thus outer and inner points have opposite stability.

EXERCISES

(8) Identify the bifurcations that the following systems undergo, find the parameter values $\mu = \mu_0$ where the bifurcations occur, and classify them as saddle-node, transcritical, or pitchfork.

a) $\dot{x} = \mu \sin x - \sin 2x$

b) $\dot{x} = \mu + \cos x + \cos 2x$

c) $\dot{x} = \mu + \sin x + \cos 2x$

d) $\dot{x} = \dfrac{\sin x}{\mu + \cos x}$

e) $\dot{x} = \dfrac{\sin x}{\mu + \sin x}$

(9) Consider a system with a saddle-node bifurcation satisfying the relevant sufficient conditions at (x_0, μ_0). Show that

a) If $f_{xx}(x_0, \mu_0) f_\mu(x_0, \mu_0) < 0$, then we have 2 fixed-points for $\mu > \mu_0$.

b) If $f_{xx}(x_0, \mu_0) f_\mu(x_0, \mu_0) > 0$, then we have 2 fixed-points for $\mu < \mu_0$.

c) Discuss the stability of the two fixed points for the above cases. Distinguish the case $f_\mu(x_0, \mu_0) > 0$ vs. $f_\mu(x_0, \mu_0) < 0$.

(10) Consider a system with a transcritical bifurcation at (x_0, μ_0) that satisfies the relevant sufficient conditions. Assume $x = x_0$ is a fixed point for all μ. Show that:

a) If $f_{x\mu}(x_0,\mu_0)>0$, then $x=x_0$ transitions from unstable to stable with increasing μ.

b) If $f_{x\mu}(x_0,\mu_0)<0$, then $x=x_0$ transitions from stable to unstable with increasing μ.

c) If $f_{xx}(x_0,\mu_0)>0$, then the other fixed point transitions from unstable to stable with increasing x.

d) If $f_{xx}(x_0,\mu_0)<0$, then the other fixed point transitions from stable to unstable with increasing x.

(11) Consider a system with a pitchfork bifurcation at (x_0,μ_0) that satisfies the relevant sufficient conditions. Show that

a) If $f_{xxx}(x_0,\mu_0)f_{x\mu}(x_0,\mu_0)>0$, then there are 3 fixed points at $\mu<\mu_0$

b) If $f_{xxx}(x_0,\mu_0)f_{x\mu}(x_0,\mu_0)<0$, then there are 3 fixed points at $\mu>\mu_0$.

c) Discuss the stability of the fixed points for the above two cases. Distinguish the case $f_{x\mu}(x_0,\mu_0)>0$ vs. $f_{x\mu}(x_0,\mu_0)<0$.