THEORY QUESTIONS ON VECTOR SPACES

I Internal operations

1. What is the definition of an operation?
2. What is the definition of an internal operation?
3. Let $*$ be an internal operation on a set $A \neq \emptyset$. State the necessary and sufficient condition for the following statements
   a) $*$ is associative
   b) $*$ is commutative
   c) e is a unit element of $(A,*)$
   d) $*$ is NOT associative
   e) $*$ is NOT commutative
   f) e is NOT a unit element of $(A,*)$
4. Let $*$ be an internal operation on A with unit element $e \in A$. If $a, a' \in A$, write the necessary and sufficient condition for the statement:

   a, a' are symmetric
5. Show that if $*$ is an internal operation in A with e a unit element, then that unit element is unique. State and prove the corresponding mathematical statement.

(6) Show that if $*$ is an associative internal operation on A with a unit element $e \in A$, then any element $a \in A$ cannot have more than one symmetric element $a' \in A$. State and prove the corresponding mathematical statement.

(7) Let $*$ be an internal operation on A and let $A_1 \subseteq A$ be a subset of A. When do we say that "$*$" is closed on the set A_1?

V Groups

1. Let $*$ be an internal operation on U and let $G \subseteq U$ be a subset of U. Give the definitions for the following statements:
   a) $(G,*)$ is a group
   b) $(G,*)$ is an abelian group.

2. Let $*$ be an internal operation on U and let $G \subseteq U$ be a subset of U. Give the theorem stating the <u>sufficient conditions</u> for showing that $(G,*)$ is a group.

3. Let $(G,*)$ be a group and let $a' \in G$ be the symmetric element of $a \in G$. Prove that:
   a) $\forall a, b \in G: (a*b)' = b' * a'$
   b) $\forall a \in G: a'' = a$ (note: $a'' = (a')'$)

V Vector spaces

① What is the definition of an external operation?

② What is the definition of a real vector space?

③ Show that if $(V, +, \cdot)$ is a real vector space then $(V, +)$ is an abelian group.

④ Let $(V, +, \cdot)$ be a vector space and let $\mathbf{0}$ be the unit element of the group $(V, +)$. Show that:

a) $\forall \lambda \in \mathbb{R}: \lambda \mathbf{0} = \mathbf{0}$

b) $\forall x \in V: 0x = \mathbf{0}$