VECTOR SPACES

Internal Operations

Def: Let A, B, C be sets with $A\times B \neq \emptyset$ and $C \neq \emptyset$.

An operation is a mapping $f: A\times B \to C$ such that every $(a,b) \in A\times B$ is mapped to $afb \in C$.

- afb = the result of the operation.
- notation: We typically represent operations with notations such as:

  $a+b$, $a\cdot b$, $a*b$, $a\circ b$

  with $+, \cdot, *, \circ$ being the operations.
- remark: Let $*: A\times B \to C$ be an operation. An immediate consequence of the definition of the operation as a mapping is the following statements:

$$\boxed{\begin{aligned} &\forall a,b\in A: \forall c\in B: (a=b \Rightarrow a*c = b*c) \\ &\forall a,b\in B: \forall c\in A: (a=b \Rightarrow c*a = c*b) \end{aligned}}$$

Def: Let $A \neq \emptyset$ be a set. An internal operation on A is any mapping $f: A\times A \to A$ such that all $(a,b) \in A\times A$ are mapped to $afb \in A$.

Def : Let $*$ be an internal operation on A. Let $A_1 \subseteq A$ with $A_1 \neq \emptyset$. We say that

$$\boxed{* \text{ closed on } A_1 \Leftrightarrow \forall a, b \in A_1 : a * b \in A_1}$$

$\hookrightarrow$ Properties of operations

Def : Let $*$ be an internal operation on A. We say that

$*$ commutative $\Leftrightarrow \forall a, b \in A : a * b = b * a$

$*$ associative $\Leftrightarrow \forall a, b, c \in A : (a * b) * c = a * (b * c)$

e unit element of $(A, *) \Leftrightarrow \forall a \in A : a * e = e * a = a$

Def : Let $*$ be an internal operation on A with unit element $e \in A$. We say that

a, a' symmetric with respect to $*$ $\Leftrightarrow a * a' = a' * a = e$

- We now show that if $(A, *)$ has a unit element, then it is unique. Likewise, given a unique unit element $e \in A$, every $a \in A$ can have no more than than one symmetric element $a' \in A$.

Thm: Let $*$ be an internal operation on A.

a) $\boxed{e_1, e_2 \text{ unit elements of } (A, *) \Rightarrow e_1 = e_2}$

b)
$$\boxed{\left.\begin{array}{l} * \text{ associative} \\ (A, *) \text{ has unit element } e \in A \\ a, a_1 \text{ symmetric} \\ a, a_2 \text{ symmetric} \end{array}\right\} \Rightarrow a_1 = a_2}$$

Proof

a) Assume that e_1, e_2 unit elements of $(A, *)$. Then

$e_1 * e_2 = e_2$ [e_1 unit element]

$e_1 * e_2 = e_1$ [e_2 unit element]

Then $e_1 = e_2$ □

b)
$$\begin{aligned} a_1 &= a_1 * e && [e \text{ unit element}] \\ &= a_1 * (a * a_2) && [a, a_2 \text{ symmetric}] \\ &= (a_1 * a) * a_2 && [\text{associative}] \\ &= e * a_2 && [a, a_1 \text{ symmetric}] \\ &= a_2 && [e \text{ unit element}] \end{aligned}$$
□

EXAMPLES

a) Addition in $\mathbb{R}$.

1) "$+$" in $\mathbb{R}$ is associative and commutative
2) $0 \in \mathbb{R}$ is a unit element of $(\mathbb{R}, +)$
3) If $a \in \mathbb{R}$, then $-a$ is symmetric to a with respect to "$+$".

b) Multiplication in $\mathbb{R}$

1) "$\cdot$" in $\mathbb{R}$ is associative and commutative
2) $1 \in \mathbb{R}$ is a unit element of $(\mathbb{R}, \cdot)$
3) If $a \in \mathbb{R} - \{0\}$, then $1/a$ is symmetric to a with respect to "$\cdot$".

c) Multiplication in $M_n(\mathbb{R})$

1) "$\cdot$" in $M_n(\mathbb{R})$ is associative but NOT commutative
2) The identity matrix $I = [\delta_{ab}]$ with
$$\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$
is the unique unit element of $(M_n(\mathbb{R}), \cdot)$ since $\forall A \in M_n(\mathbb{R}): AI = IA$.
3) If $\det(A) \neq 0$, then A^{-1} is the symmetric element of A because $AA^{-1} = A^{-1}A = I$.

EXAMPLES

a) Let $A = \mathbb{R} - \{\lambda\}$ and define

$x * y = xy - \lambda(x+y) + \lambda(\lambda+1)$

i) Show that "$*$" is closed on A.

ii) Show that "$*$" is commutative

iii) Show that "$*$" has a unit element on A.

Solution

i) It is sufficient to show that $\forall x, y \in A : x * y \in A$.
Let $x, y \in A$ be given. To derive a contradiction, let us assume that $x * y \notin A$. Then:

$$x * y \notin A \Leftrightarrow x * y \notin \mathbb{R} - \{\lambda\} \Leftrightarrow x * y = \lambda \Leftrightarrow xy - \lambda(x+y) + \lambda(\lambda+1) = \lambda$$
$$\Leftrightarrow xy - \lambda x - \lambda y + \lambda^2 + \lambda = \lambda \Leftrightarrow xy - \lambda x - \lambda y + \lambda^2 = 0 \Leftrightarrow$$
$$\Leftrightarrow x(y-\lambda) - \lambda(y-\lambda) = 0 \Leftrightarrow (x-\lambda)(y-\lambda) = 0 \Leftrightarrow$$
$$\Leftrightarrow x - \lambda = 0 \vee y - \lambda = 0 \Leftrightarrow x = \lambda \vee y = \lambda \Leftrightarrow x \notin \mathbb{R} - \{\lambda\} \vee y \notin \mathbb{R} - \{\lambda\}$$
$$\Leftrightarrow x \notin A \vee y \notin A \longleftarrow \text{contradiction, since } x \in A \wedge y \in A.$$

It follows that $x * y \in A$. We have thus shown that

$(\forall x, y \in A : x * y \in A) \Rightarrow$ "$*$" closed under A.

ii) Sufficient to show that $\forall x, y \in A : x * y = y * x$

Let $x, y \in A$ be given. Then:

$$x * y = xy - \lambda(x+y) + \lambda(\lambda+1) = yx - \lambda(y+x) + \lambda(\lambda+1) = y * x$$

It follows that

$(\forall x, y \in A : x * y = y * x) \Rightarrow$ "$*$" commutative on A.

iii) It is sufficient to find an $e \in A$ such that

$\forall x \in A : x * e = e * x = x$

We note that

$$x * e - x = ex - \lambda(e+x) + \lambda(\lambda+1) - x =$$
$$= ex - \lambda e - \lambda x + \lambda^2 + \lambda - x =$$
$$= x(e-\lambda-1) - \lambda(e-\lambda-1) =$$
$$= (x-\lambda)(e-\lambda-1)$$

and therefore

$$x * e = x \Leftrightarrow x * e - x = 0 \Leftrightarrow (x-\lambda)(e-\lambda-1) = 0 \Leftrightarrow$$
$$\Leftrightarrow x - \lambda = 0 \vee e - \lambda - 1 = 0 \Leftrightarrow e - \lambda - 1 = 0 \quad [\text{since } x \in A \Rightarrow x \neq \lambda]$$
$$\Leftrightarrow e = \lambda + 1.$$

It follows that for $e = \lambda + 1$, $\forall x \in A : x * e = x$
"$*$" commutative on A $\Big\} \Rightarrow$

$\Rightarrow \forall x \in A : (x * e = e * x = x) \Rightarrow e = \lambda + 1$ is a unit element of "$*$" on A.

→ We note that in the argument above:

a) We use proof by contradiction in part (i) to show that $x * y \in A$.

b) We have also used the following theorem:
$\forall a, b \in \mathbb{R} : (ab = 0 \Leftrightarrow a = 0 \vee b = 0)$.

b) We define $x*y = xy + 2ax + by$, $\forall x, y \in \mathbb{R}$. Find all $a, b \in \mathbb{R}$ such that "$*$" is associative on $\mathbb{R}$.

Solution

Let $x, y, z \in \mathbb{R}$. We note that

$$
\begin{aligned}
x*(y*z) &= x*(yz+2ay+bz) = \\
&= x(yz+2ay+bz) + 2ax + b(yz+2ay+bz) = \\
&= \underline{xyz} + \underline{2axy} + bxz + 2ax + \underline{byz} + \underline{2aby} + b^2 z = \\
(x*y)*z &= (xy+2ax+by)*z = \\
&= (xy+2ax+by)z + 2a(xy+2ax+by) + bz = \\
&= xyz + 2axz + byz + 2axy + 2a^2x + 2aby + bz = \\
&= \underline{xyz} + \underline{2axy} + 2axz + 4a^2x + \underline{byz} + \underline{2aby} + bz
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
x*(y*z) - (x*y)*z &= (bxz + 2ax + b^2 z) - (2axz + 4a^2x + bz) = \\
&= (b-2a)xz + (2a-4a^2)x + (b^2-b)z = \\
&= (b-2a)xz + 2a(1-2a)x + b(b-1)z.
\end{aligned}
$$

It follows that:

"$*$" is associative on $\mathbb{R} \Leftrightarrow \forall x, y, z \in \mathbb{R}: x*(y*z) = (x*y)*z$

$\Leftrightarrow \forall x, y, z \in \mathbb{R}: (x*(y*z) - (x*y)*z = 0) \Leftrightarrow$

$\Leftrightarrow \forall x, z \in \mathbb{R}: (b-2a)xz + 2a(1-2a)x + b(b-1)z = 0 \overset{*}{\Leftrightarrow}$

$$
\Leftrightarrow \begin{cases} b-2a=0 \\ 2a(1-2a)=0 \\ b(b-1)=0 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a=0 \\ b=0 \end{cases} \vee \underbrace{\begin{cases} b=2a \\ a=1/2 \\ b=0 \end{cases} \vee \begin{cases} b=2a \\ a=0 \\ b=1 \end{cases}}_{\text{contradictions}} \vee \begin{cases} b=2a \\ a=1/2 \\ b=1 \end{cases}
$$

$$\Leftrightarrow \begin{cases} a=0 \\ b=0 \end{cases} \vee \begin{cases} a=1/2 \\ b=1 \end{cases} \Leftrightarrow (a,b) \in \{(0,0),(1/2,1)\}$$

↳ In the above solution, the main argument is:
"$*$" associative on $\mathbb{R} \Leftrightarrow \cdots \Leftrightarrow$
$\Leftrightarrow (a,b) \in \{(0,0),(1/2,1)\}$
The preceding calculations are the <u>preamble</u> of the solution. The purpose of the pream

EXERCISES

① Show that $a*b = a+b+5$ defined on $\mathbb{R}$ is both commutative and associative.

② We define on $*\mathbb{R}$ the operation $a*b = ab+a+b$.
Show that

a) $*$ is commutative and associative

b) Find the unit element of $*$

c) Find which elements of $\mathbb{R}$ have an inverse with respect to the operation $*$.

③ We define on $\mathbb{R}$ the following operations:
$x*y = x^2y^2$ and $x \circ y = y(x+y)$
Explore whether

a) the operations are commutative

b) the operations are associative

c) the operations have a unit element

d) every element of $\mathbb{R}$ has an inverse.

④ We define on the set $(0,+\infty)$ the operation

$$x*y = \frac{xy}{x+y}, \quad \forall x,y \in (0,+\infty)$$

Show that

a) $*$ is commutative and associative, using the definition

b) $\forall x,y \in (0,+\infty)$: $\dfrac{1}{x} * \dfrac{1}{y} = \dfrac{1}{x+y}$

c) Use (b) to provide an alternate proof of (a).

(5) We define on $\mathbb{R}$ the operation $a*b=(a-1)b^2-(a-1)+ab$.
Find (if it exists) the unit element of this operation.

(6) Given the set

$$A=\left\{\begin{bmatrix} a & 0 \\ 2a & 0 \end{bmatrix} \middle| \, a\in\mathbb{R}\right\}$$

show that the matrix $E=\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ is a unit element with respect to regular matrix multiplication restricted to the set A. Also show that A is closed with respect to matrix multiplication.

(7) Let A be a set with an operation $*$ and unit element e such that
$\forall a,b,c,d\in A: (a*b)*(c*d)=(a*c)*(b*d)$
Show that $*$ is associative and commutative.

(8) Let A be a set with an operation $*$ and unit element e such that
$\forall x,y,z\in A: (x*y)*z=x*(z*y)$
Show that $*$ is associative and commutative.

$\hookrightarrow$ From exercise 6 we see that the same operation may have a different unit element, if it is restricted into a smaller set.

V Groups

Def: Let G be a set with "$*$" an internal operation on U with $G \subseteq U$. We say that:

a) $(G,*)$ is a group if and only if:
   1) "$*$" is closed on G
   2) $\forall a,b,c \in G : a*(b*c) = (a*b)*c$
   3) $\exists\, e \in G : \forall a \in G : e*a = a*e = a$
   4) $\forall a \in G : \exists a' \in G : a'*a = a*a' = e$

b) $(G,*)$ is an abelian group if and only if:
   1) $(G,*)$ is a group
   2) $\forall a,b \in G : a*b = b*a$.

- Therefore, $(G,*)$ is a group if and only if "$*$" is closed on G, "$*$" is associative, has a unit element, and every element of G has a symmetric element. $(G,*)$ is an abelian group if and only if it is already a group and furthermore "$*$" is commutative.

EXAMPLES

a) $(\mathbb{R},+)$ and $(\mathbb{R}-\{0\},\cdot)$ are abelian groups.

b) $(M_n(\mathbb{R}),+)$ is an abelian group.

c) We define the general linear group

$$GL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

Then $(GL(n,\mathbb{R}), \cdot)$ is a group but NOT an abelian group. Note that

1) Matrix multiplication is NOT commutative
2) We need the restriction $\det(A) \neq 0$ to ensure that every A has a symmetric element.

Sufficient condition

- To show that $(G,*)$ is a group, we may in fact weaken conditions (c) and (d) according to the following theorem:

Thm: Let $(G,*)$ with G a set and "$*$" an internal operation on U with $G \subseteq U$. Assume that:

a) $\forall a,b \in G : a*b \in G$
b) $\forall a,b,c \in G : a*(b*c) = (a*b)*c$
c) $\exists e \in G : \forall a \in G : e*a = a$
d) $\forall a \in G : \exists a' \in G : a'*a = e$

Then $(G,*)$ is a group.

Proof

Let $a \in G$ be given. Let a' be the symmetric element of a such that $a'*a = e$. (exists by hypothesis (d)).

Let $a'' \in G$ be the symmetric element of a' such that $a'' * a' = e$ (exists by hypothesis (d)).
It is sufficient to show that $a * a' = e \wedge a * e = a$.
We note that:

$$\begin{aligned}
a * a' &= e * (a * a') = && [\text{hypothesis (c)}] \\
&= (a'' * a') * (a * a') = && [\text{definition}] \\
&= a'' * [a' * (a * a')] = && [\text{associative}] \\
&= a'' * [(a' * a) * a'] = && [\text{associative}] \\
&= a'' * (e * a') = && [\text{definition}] \\
&= a'' * a' = && [\text{hypothesis (c)}] \\
&= e && [\text{hypothesis (d)}]
\end{aligned}$$

and

$$\begin{aligned}
a * e &= a * (a' * a) = && [\text{definition}] \\
&= (a * a') * a = && [\text{associative}] \\
&= e * a = && [\text{definition}] \\
&= a && [\text{hypothesis (c)}]
\end{aligned}$$

From (a), (b), (c), (d) and the above results it follows that $(G, *)$ is a group $\square$

↳ Consequences of group definition

Thm: Let $(G, *)$ be a group. Then:

$$\boxed{\begin{array}{l} \forall a, b \in G : (a * b)' = b' * a' \\ \forall a \in G : a'' = a \end{array}}$$

Proof

a) To show $\forall a,b \in G: (a*b)' = b' * a'$

Let $a,b \in G$ be given. Then

$$
\begin{aligned}
(a*b)*(b'*a') &= a*[b*(b'*a')] && \text{[associative]}\\
&= a*[(b*b')*a'] && \text{[associative]}\\
&= a*(e*a') && \text{[}b,b'\text{ symmetric]}\\
&= a*a' && \text{[unit element]}\\
&= e && \text{[}a,a'\text{ symmetric]}
\end{aligned}
$$

and

$$
\begin{aligned}
(b'*a')*(a*b) &= b'*[a'*(a*b)] && \text{[associative]}\\
&= b'*[(a'*a)*b] && \text{[associative]}\\
&= b'*(e*b) && \text{[}a',a\text{ symmetric]}\\
&= b'*b && \text{[unit element]}\\
&= e && \text{[}b',b\text{ symmetric]}
\end{aligned}
$$

It follows, by uniqueness of the symmetric element, that

$$
\left\{ \begin{array}{l} (a*b)*(b'*a') = e \\ (b'*a')*(a*b) = e \end{array} \right. \Rightarrow (a*b)' = b'*a'.
$$

b) To show $\forall a \in G: a'' = a$

Let $a \in G$ be given. Then

$$
\begin{aligned}
a'' &= a''*e && \text{[unit element]}\\
&= a''*(a'*a) && \text{[}a',a\text{ symmetric]}\\
&= (a''*a')*a && \text{[associative]}\\
&= e*a && \text{[}a'',a'\text{ symmetric]}\\
&= a && \text{[unit element].}
\end{aligned}
$$

□

↳ For the multiplication group $(M_n(\mathbb{R}), \cdot)$ of matrices, this theorem gives:
$\forall A, B \in M_n(\mathbb{R}): (AB)^{-1} = B^{-1}A^{-1}$
$\forall A \in M_n(\mathbb{R}): (A^{-1})^{-1} = A$.

EXAMPLE

Show that $(\mathbb{R}-\{1/3\}, *)$ with $a*b = a+b-3ab$ is an abelian group.

Proof

• Closure: Let $a, b \in \mathbb{R}-\{1/3\}$ with $a \neq 1/3$ and $b \neq 1/3$ be given.

To show that $a*b \neq 1/3$, assume that $a*b = 1/3$. It follows that:

$$\begin{aligned}(a*b) - (1/3) &= a+b-3ab-(1/3) = (b-3ab)-(1/3-a) = \\ &= b(1-3a)-(1/3)(1-3a) = \\ &= (1-3a)(b-(1/3)) = 3(1/3-a)(b-1/3);\end{aligned}$$

and therefore:

$$\begin{aligned}a*b = 1/3 &\Rightarrow (a*b)-1/3 = 0 \Rightarrow 3(1/3-a)(b-1/3) = 0 \\ &\Rightarrow 1/3-a = 0 \vee b-1/3 = 0 \Rightarrow \\ &\Rightarrow a = 1/3 \vee b = 1/3 \leftarrow \text{Contradiction.}\end{aligned}$$

Therefore: $a*b \neq 1/3 \Rightarrow a*b \in \mathbb{R}-\{1/3\}$.

Thus: $\forall a, b \in \mathbb{R}-\{1/3\}: a*b \in \mathbb{R}-\{1/3\} \Rightarrow$
$\Rightarrow$ "$*$" closed on $\mathbb{R}-\{1/3\}$.

• <u>Commutative</u> : Let $a, b \in \mathbb{R}-\{1/3\}$ be given. Then:

$a * b = a+b-3ab = b+a-3ba = b*a, \ \forall a, b \in G \Rightarrow$
$\Rightarrow$ "$*$" commutative.

• <u>Associative</u> : Let $a, b, c \in \mathbb{R}-\{1/3\}$ be given. Then:

$$\begin{aligned} a*(b*c) &= a*(b+c-3bc) = \\ &= a+(b+c-3bc)-3a(b+c-3bc) = \\ &= a+b+c-3bc-3ab-3ac+9abc = \\ &= (a+b+c)-3(ab+bc+ca)+9abc \qquad (1) \end{aligned}$$

and

$$\begin{aligned} (a*b)*c &= (a+b-3ab)*c = \\ &= (a+b-3ab)+c-3(a+b-3ab)c \\ &= a+b-3ab+c-3ac-3bc+9abc \\ &= (a+b+c)-3(ab+bc+ca)+9abc \qquad (2) \end{aligned}$$

From (1) and (2):
$\forall a, b, c \in \mathbb{R}-\{1/3\}: \ a*(b*c) = (a*b)*c \Rightarrow$
$\Rightarrow$ "$*$" associative.

• <u>Unit element</u> : Let $a \in \mathbb{R}-\{1/3\}$ be given.
We solve the equation:
$e*a = a \Leftrightarrow e+a-3ea = a \Leftrightarrow e-3ea = 0 \Leftrightarrow$
$\Leftrightarrow e(1-3a) = 0 \Leftrightarrow e = 0 \lor 1-3a = 0. \qquad (3)$
Note that $a \in \mathbb{R}-\{1/3\} \Rightarrow a \neq 1/3 \Rightarrow 1-3a \neq 0$
and therefore $(3) \Leftrightarrow e = 0$.
Thus $\forall a \in \mathbb{R}-\{1/3\}: \ 0*a = a$.

• Symmetric elements:

Let $a \in \mathbb{R}-\{1/3\}$ be given. We solve the equation

$b * a = 0 \Leftrightarrow b + a - 3ba = 0 \Leftrightarrow b(1-3a) + a = 0 \Leftrightarrow$

$\Leftrightarrow b(1-3a) = -a \Leftrightarrow b(3a-1) = a.$

Since $a \in \mathbb{R}-\{1/3\} \Rightarrow a \neq 1/3 \Rightarrow 3a-1 \neq 0$, and therefore:

$$b * a = 0 \Leftrightarrow b = \frac{a}{3a-1}$$

To show that $\dfrac{a}{3a-1} \neq \dfrac{1}{3}$, assume that $\dfrac{a}{3a-1} = \dfrac{1}{3}$

Then:

$$\frac{a}{3a-1} = \frac{1}{3} \Leftrightarrow 3a = 3a - 1 \Leftrightarrow \underline{0a = -1} \leftarrow \text{inconsistent}$$

It follows that $\dfrac{a}{3a-1} \neq \dfrac{1}{3} \Rightarrow b = \dfrac{a}{3a-1} \in \mathbb{R}-\left\{\dfrac{1}{3}\right\}$

• It follows that $(\mathbb{R}-\{1/3\}, *)$ is an abelian group.

EXERCISES

(9) Given the set $A=\{x\in\mathbb{R}\mid -1<x<1\}$ we define the operation $*$ with $a*b=(a+b)/2$.
Explore whether $(A,*)$ is a group.

(10) Given the set $G=\{x\in\mathbb{R}\mid -1<x<1\}$ we define the operation $*$ with $a*b=\dfrac{a+b}{1+ab}$.

Show that $(G,*)$ is an abelian group.

(11) We define on $\mathbb{R}$ the operation $*$ with $x*y=x+y+1$.
Show that $(\mathbb{R},*)$ is a group.

(12) We define on $G=\mathbb{R}-\{2\}$ the operation $*$ with $x*y=2(x+y-1)-xy$. Show that $(G,*)$ is an abelian group.

(13) We define on $G=\mathbb{R}-\{1\}$ the operation $*$ with $x*y=xy-x-y+2$. Show that $(G,*)$ is an abelian group.

(14) We define on $G=(-\sqrt{2},\sqrt{2})$ the operation $*$ with $x*y=\dfrac{2x+2y}{xy+2}$. Show that $(G,*)$ is an abelian group.

(15) Let $(G,*)$ be a group, and let $x,y\in G$ such that $x*y=y$. Show that x is the unit element of $(G,*)$.

(16) Let $(G,*)$ be a group such that
$\forall a,b\in G:\ (a*b)*(a*b)=(a*a)*(b*b)$
Show that $(G,*)$ is an abelian group.

▼ Vector spaces

Def: An *external operation* on A with *coefficients* from G is any mapping $f: G\times A \to A$ such that every $(\lambda, a) \in G\times A$ is mapped into $\lambda a \in A$.

▶ *notation*: For external operations we prefer to use multiplicative notation. In the expression $\lambda a \in A$ we say that λ is the *coefficient* of λa.

Def: Let $(V, +, \cdot)$ be endowed with an internal operation "$+$": $V\times V \to V$ and an external operation "$\cdot$": $\mathbb{R}\times V \to V$. We say that $(V, +, \cdot)$ is a *real vector space* if and only if the following conditions are satisfied:

a) $(V, +)$ is a group

b) $\forall \lambda \in \mathbb{R} : \forall x, y \in V : \lambda(x+y) = \lambda x + \lambda y$

c) $\forall \lambda, \mu \in \mathbb{R} : \forall x \in V : (\lambda+\mu)x = \lambda x + \mu x$

d) $\forall \lambda, \mu \in \mathbb{R} : \forall x \in V : \lambda(\mu x) = (\lambda\mu)x$

e) $\forall x \in V : 1x = x$.

▶ In the above definition, $\lambda+\mu$ and $\lambda\mu$ represent regular addition and multiplication in $\mathbb{R}$.

$\hookrightarrow$ $(V,+)$ is an abelian group

We will now show that if $(V,+,\cdot)$ is a real vector space then, although not demanded by the above definition, $(V,+)$ will be an abelian group. The proof is dependent on the following general property of groups:

Lemma : Let $(G,*)$ be a group. Then:

$$\boxed{\forall a,b,c \in G : (c*a = c*b \vee a*c = b*c \Rightarrow a = b)}$$

Proof

Let $a,b,c \in G$ be given. Let $e \in G$ be the unit element of G.

Case 1 : Assume that $c*a = c*b$. Then :

$$a = e*a = (c'*c)*a = c'*(c*a) = c'*(c*b)$$
$$= (c'*c)*b = e*b = b.$$

Case 2 : Assume that $a*c = b*c$. Then

$$a = a*e = a*(c*c') = (a*c)*c' = (b*c)*c' =$$
$$= b*(c*c') = b*e = b \quad \square$$

Thm : $\boxed{(V,+,\cdot) \text{ real vector space} \Rightarrow (V,+) \text{ abelian group}}$

Proof

By definition:

$(V,+,\cdot)$ real vector space $\Rightarrow (V,+)$ group $\quad (1)$

Let $x,y \in V$ be given. Then

$(1+1)(x+y) = (1+1)x + (1+1)y = x+x+y+y \quad (2)$

$(1+1)(x+y) = 1(x+y) + 1(x+y) = x+y+x+y \quad (3)$

From (2) and (3), using the above lemma we have:

$$x+x+y+y = x+y+x+y \Rightarrow x+x+y = x+y+x \Rightarrow$$
$$\Rightarrow x+y = y+x.$$

It follows that

$$\left.\begin{array}{r} \forall x,y \in V : x+y = y+x \Rightarrow \text{"+" commutative} \\ (V,+) \text{ group} \end{array}\right\} \Rightarrow$$

$\Rightarrow (V,+)$ abelian group. $\square$

Properties of vector spaces.

- Let $\mathbb{0} \in V$ be the unit element of the abelian group $(V,+)$.
- Denote as $-x$ the symmetric element of $x \in V$.
- By definition, we know that for all $\lambda, \mu \in \mathbb{R}$ and for all $x,y,z \in V$, we have:

$(x+y)+z = x+(y+z)$	$\lambda(x+y) = \lambda x + \lambda y$
$x+y = y+x$	$(\lambda+\mu)x = \lambda x + \mu x$
$x+\mathbb{0} = x$	$\lambda(\mu x) = (\lambda\mu)x$
$x+(-x) = \mathbb{0}$	$1x = x$

- We will now show that:

① $\boxed{\forall \lambda \in \mathbb{R} : \lambda \mathbf{0} = \mathbf{0}}$

<u>Proof</u>

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given. Then:
$\lambda x + \lambda \mathbf{0} = \lambda (x + \mathbf{0}) = \lambda x = \lambda x + \mathbf{0} \Rightarrow$
$\Rightarrow \lambda \mathbf{0} = \mathbf{0}.$ □

② $\boxed{\forall x \in V : 0x = \mathbf{0}}$

<u>Proof</u>

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given. Then
$\lambda x + 0x = (\lambda + 0)x = \lambda x = \lambda x + \mathbf{0} \Rightarrow 0x = \mathbf{0}.$ □

③ $\boxed{\forall \lambda \in \mathbb{R} : \forall x \in V : (\lambda x = \mathbf{0} \Rightarrow \lambda = 0 \lor x = \mathbf{0})}$

<u>Proof</u>

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given with $\lambda x = \mathbf{0}$.
<u>Case 1</u>: If $\lambda = 0 \Rightarrow \lambda = 0 \lor x = \mathbf{0}$
<u>Case 2</u>: If $\lambda \neq 0 \Rightarrow \lambda^{-1}\lambda = 1$. It follows that

$x = 1x = (\lambda^{-1}\lambda)x = \lambda^{-1}(\lambda x) = \lambda^{-1}\mathbf{0} = \mathbf{0} \Rightarrow$
$\Rightarrow \lambda = 0 \lor x = \mathbf{0}$ □

④ $\boxed{\forall \lambda \in \mathbb{R} : \forall x \in V : (-\lambda)x = \lambda(-x) = -\lambda x}$

Proof

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given. We note that
$(-\lambda)x + \lambda x = [(-\lambda)+\lambda]x = 0x = \mathbf{0} \Rightarrow \lambda x$ symmetric of $(-\lambda)x$
$\Rightarrow (-\lambda)x = -\lambda x$.
Similarly:
$\lambda(-x) + \lambda x = \lambda[(-x)+x] = \lambda\mathbf{0} = \mathbf{0} \Rightarrow$
$\Rightarrow \lambda x$ symmetric of $\lambda(-x) \Rightarrow \lambda(-x) = -\lambda x$.
It follows that $(-\lambda)x = \lambda(-x) = -\lambda x$ □

- From the above properties we can also show that:

$$\boxed{\begin{array}{l}
\forall \lambda \in \mathbb{R}-\{0\} : \forall x,y \in V : (\lambda x = \lambda y \Rightarrow x = y) \\
\forall \lambda,\mu \in \mathbb{R} : \forall x \in V - \{\mathbf{0}\} : (\lambda x = \mu x \Rightarrow \lambda = \mu) \\
\forall \lambda,\mu \in \mathbb{R} : \forall x,y \in V : \begin{cases} \lambda(x-y) = \lambda x - \lambda y \\ (\lambda-\mu)x = \lambda x - \mu x \end{cases} \\
\forall x \in V : (-1)x = -x
\end{array}}$$

Basic Vector Spaces

(1) The space $\mathbb{R}^2$

For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ we define:

$(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2 \wedge y_1 = y_2$

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$\forall \lambda \in \mathbb{R}: \lambda (x_1, y_1) = (\lambda x_1, \lambda y_1)$

Then $(\mathbb{R}^2, +, \cdot)$ is a vector space.

(2) The space $\mathbb{R}^n$

The previous vector space can be generalized for n dimensions as follows:

Let $[n] = \{1, 2, 3, \dots, n\}$, let $\lambda \in \mathbb{R}$, and let $x, y \in \mathbb{R}^n$ with $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.

We define:

$x = y \Leftrightarrow \forall a \in [n]: x_a = y_a$

Let $z \in \mathbb{R}^n$ with $z = (z_1, z_2, \dots, z_n)$. Then define:

$z = x + y \Leftrightarrow \forall a \in [n]: z_a = x_a + y_a$

$z = \lambda x \Leftrightarrow \forall a \in [n]: z_a = \lambda x_a$

Then $(\mathbb{R}^n, +, \cdot)$ is a vector space.

(3) The space $F(A)$

We define $F(A)$ as the set of all functions $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$. Let $\lambda \in \mathbb{R}$ and let $f, g, h \in F(A)$.
We define:

$f = g \Leftrightarrow \forall x \in A: f(x) = g(x)$

$h = f + g \Leftrightarrow \forall x \in A: h(x) = f(x) + g(x)$

$h = \lambda f \Leftrightarrow \forall x \in A: h(x) = \lambda f(x)$.

Then $(F(A), +, \cdot)$ is a vector space.

(4) The space $M_{nm}(A)$

Recall that we have defined $M_{nm}(\mathbb{R})$ as the set of all $n \times m$ matrices. Combined with matrix addition "+" and scalar multiplication "$\cdot$", $(M_{nm}(\mathbb{R}), +, \cdot)$ is a vector space.

EXAMPLES

a) Show vector addition, defined on $\mathbb{R}^2$ is associative.

Solution

Sufficient to show that

$$\forall x, y, z \in \mathbb{R}^2 : \ x + (y+z) = (x+y) + z$$

Let $x, y, z \in \mathbb{R}^2$ be given with $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then

$$\begin{aligned} x+(y+z) &= (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] = \\ &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) = \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2) = \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) = \\ &= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) = \\ &= (x+y) + z \end{aligned}$$

It follows that

$$\forall x, y, z \in \mathbb{R}^2 : x + (y+z) = (x+y) + z \Rightarrow$$

$\Rightarrow$ "+" associative on $\mathbb{R}^2$.

b) Show that function addition, defined on $F(A)$ with $A \subseteq \mathbb{R}$ is associative.

Solution

Sufficient to show that

$$\forall f,g,h \in F(A) : \forall x \in A : (f+(g+h))(x) = ((f+g)+h)(x)$$

Let $f,g,h \in F(A)$ and $x \in A$ be given. Then

$$\begin{aligned}(f+(g+h))(x) &= f(x)+(g+h)(x) = \\ &= f(x)+g(x)+h(x) = \\ &= (f+g)(x)+h(x) = \\ &= ((f+g)+h)(x)\end{aligned}$$

It follows that

$$\forall f,g,h \in F(A) : \forall x \in A : (f+(g+h))(x) = ((f+g)+h)(x) \Rightarrow$$

$$\Rightarrow \forall f,g,h \in F(A) : f+(g+h) = (f+g)+h$$

$\Rightarrow$ "+" associative on $F(A)$.

EXERCISES

(17) Give the detailed proof that $\mathbb{R}^2$ is a vector space with respect to vector addition and scalar multiplication, defined as:

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$\lambda(x, y) = (\lambda x, \lambda y)$, $\forall \lambda \in \mathbb{R}$

(18) Give the detailed proof that $F(A)$ with $A \subseteq \mathbb{R}$ is a vector space with respect to function addition and scalar multiplication, defined as

$h = f + g \iff \forall x \in A : h(x) = f(x) + g(x)$

$h = \lambda f \iff \forall x \in A : h(x) = \lambda f(x).$

Vector subspaces

<u>Def</u>: Let $(V, +, \cdot)$ be a vector space. We say that

$$\boxed{V_0 \text{ subspace of } V \Leftrightarrow \begin{cases} V_0 \subseteq V \wedge V_0 \neq \emptyset \\ (V_0, +, \cdot) \text{ is a vector space} \end{cases}}$$

Subspace criteria

① Main subspace criterion

<u>Thm</u>: Let $(V, +, \cdot)$ be a vector space and let $V_0 \subseteq V$ and $V_0 \neq \emptyset$. Then:

$$\boxed{V_0 \text{ subspace of } V \Leftrightarrow \forall \lambda \in \mathbb{R} : \forall x, y \in V_0 : (x + y \in V_0 \wedge \lambda x \in V_0)}$$

② Condensed subspace criterion

<u>Thm</u>: Let $(V, +, \cdot)$ be a vector space and let $V_0 \subseteq V$ and $V_0 \neq \emptyset$. Then:

$$\boxed{V_0 \text{ subspace of } V \Leftrightarrow \forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_0 : \lambda x + \mu y \in V_0}$$

Proof

($\Rightarrow$): Assume that V_0 subspace of $V \Rightarrow$

$\Rightarrow \forall \lambda \in \mathbb{R} : \forall x, y \in V_0 : (x+y \in V_0 \wedge \lambda x \in V_0)$.

Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in V_0$ be given. Then:

$$\left.\begin{array}{l} \lambda \in \mathbb{R} \wedge x \in V_0 \Rightarrow \lambda x \in V_0 \\ \mu \in \mathbb{R} \wedge y \in V_0 \Rightarrow \mu y \in V_0 \end{array}\right\} \Rightarrow \lambda x + \mu y \in V_0$$

It follows that

$\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_0 : \lambda x + \mu y \in V_0$.

($\Leftarrow$): Assume that:

$$\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_0 : (\lambda x + \mu y \in V_0).$$

Let $\lambda \in \mathbb{R}$ and $x, y \in V_0$ be given. Then:

$1x + 1y \in V_0 \Rightarrow x + y \in V_0$

$\lambda x + 0y \in V_0 \Rightarrow \lambda x \in V_0$

It follows that

$\forall \lambda \in \mathbb{R} : \forall x, y \in V_0 : (\lambda x \in V_0 \wedge x+y \in V_0) \Rightarrow$

$\Rightarrow V_0$ subspace of V.

(3) $\rightarrow$ Unit element belongs to every subspace

Thm: Let $(V, +, \cdot)$ be a vector space with $\mathbf{0} \in V$ the unit element of the group $(V, +)$ and let $V_0 \subseteq V$ and $V_0 \neq \emptyset$. Then:

$$\boxed{V_0 \text{ subspace of } V \Rightarrow \mathbf{0} \in V_0}$$

Proof

Assume that V_0 is a subspace of V. Since $V_0 \neq \emptyset$, choose an $x \in V_0$.

Then: $x \in V_0 \Rightarrow 0x \in V_0 \Rightarrow \mathbf{0} \in V_0$ $\square$

$\to$ The contrapositive statement is:

$$\boxed{\mathbf{0} \notin V_0 \Rightarrow V_0 \text{ NOT a subspace of } V}$$

Thus showing $\mathbf{0} \notin V_0$ is sufficient to show that V_0 is not a subspace of V.

4 $\to$ Intersection of subspaces

Thm : Let $(V, +, \cdot)$ be a vector space. Then:

$$\boxed{\begin{cases} V_1 \text{ subspace of } V \\ V_2 \text{ subspace of } V \end{cases} \Rightarrow V_1 \cap V_2 \text{ subspace of } V}$$

Proof

Assume that V_1, V_2 are subspaces of V.

Then: $\begin{cases} \forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_1 : \lambda x + \mu y \in V_1 \\ \forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_2 : \lambda x + \mu y \in V_2. \end{cases}$

Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in V_1 \cap V_2$ be given. Then:

$$\begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_1 \cap V_2 \end{cases} \Rightarrow \begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_1 \end{cases} \wedge \begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_2 \end{cases} \Rightarrow$$

$$\Rightarrow \lambda x + \mu y \in V_1 \wedge \lambda x + \mu y \in V_2 \Rightarrow$$

$$\Rightarrow \lambda x + \mu y \in V_1 \cap V_2.$$

It follows that

$\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_1 \cap V_2 : \lambda x + \mu y \in V_1 \cap V_2 \Rightarrow$

$\Rightarrow V_1 \cap V_2$ subspace of V. $\square$

EXAMPLES

a) Let $V = \{(a,b) \in \mathbb{R}^2 \mid 2a+3b=0\}$. Show that V is a subspace of $\mathbb{R}^2$.

Solution

Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in V$ be given.

$x \in V \Rightarrow \exists a_1, b_1 \in \mathbb{R} : (x = (a_1, b_1) \wedge 2a_1 + 3b_1 = 0)$

$y \in V \Rightarrow \exists a_2, b_2 \in \mathbb{R} : (y = (a_2, b_2) \wedge 2a_2 + 3b_2 = 0)$

It follows that

$$\begin{aligned}\lambda x + \mu y &= \lambda(a_1, b_1) + \mu(a_2, b_2) = \\ &= (\lambda a_1, \lambda b_1) + (\mu a_2, \mu b_2) = \\ &= (\lambda a_1 + \mu a_2, \lambda b_1 + \mu b_2) \equiv (c_1, c_2) \Rightarrow\end{aligned}$$

$$\Rightarrow \begin{cases} c_1 = \lambda a_1 + \mu a_2 \\ c_2 = \lambda b_1 + \mu b_2 \end{cases} \Rightarrow$$

$$\begin{aligned}\Rightarrow 2c_1 + 3c_2 &= 2(\lambda a_1 + \mu a_2) + 3(\lambda b_1 + \mu b_2) = \\ &= \lambda(2a_1 + 3b_1) + \mu(2a_2 + 3b_2) = \\ &= \lambda 0 + \mu 0 = 0 \Rightarrow \lambda x + \mu y = (c_1, c_2) \in V\end{aligned}$$

It follows that

$$\left.\begin{array}{l} \forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V : \lambda x + \mu y \in V \\ \emptyset \neq V \subseteq \mathbb{R}^2 \end{array}\right\} \Rightarrow V \text{ subspace of } \mathbb{R}^2 \quad \square$$

↳ Note that the belonging condition for V is:

$x \in V \Leftrightarrow \exists a, b \in \mathbb{R} : (x = (a,b) \wedge 2a + 3b = 0)$.

b) Let $V = \{f \in F(\mathbb{R}) \mid f \text{ continuous in } \mathbb{R}\}$. Show that V is a subspace of $F(\mathbb{R})$.

Solution

Let $\lambda, \mu \in \mathbb{R}$ and $f, g \in V$ be given. Then

$f \in V \Rightarrow f$ continuous in $\mathbb{R} \Rightarrow \forall x_0 \in \mathbb{R} : \lim\limits_{x \to x_0} f(x) = f(x_0)$

$g \in V \Rightarrow g$ continuous in $\mathbb{R} \Rightarrow \forall x_0 \in \mathbb{R} : \lim\limits_{x \to x_0} g(x) = g(x_0)$.

It follows that:

$$\lim_{x \to x_0} [(\lambda f + \mu g)(x)] = \lim_{x \to x_0} [(\lambda f)(x) + (\mu g)(x)] =$$

$$= \lim_{x \to x_0} [\lambda f(x) + \mu g(x)] = \lambda \lim_{x \to x_0} f(x) + \mu \lim_{x \to x_0} g(x) =$$

$$= \lambda f(x_0) + \mu g(x_0) = (\lambda f)(x_0) + (\mu g)(x_0) =$$

$$= (\lambda f + \mu g)(x_0), \quad \forall x_0 \in \mathbb{R} \Rightarrow$$

$\Rightarrow \lambda f + \mu g$ continuous in $\mathbb{R} \Rightarrow \lambda f + \mu g \in V$.

It follows that:

$\forall \lambda, \mu \in \mathbb{R} : \forall f, g \in V : \lambda f + \mu g \in V \Rightarrow$

$\Rightarrow V$ subspace of $\mathbb{R}^2$ $\square$.

c) Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix and let $V = \{X \in M_n(\mathbb{R}) \mid AX = XA\}$. Show that V is a subspace of $M_n(\mathbb{R})$.

Solution

Let $\lambda, \mu \in \mathbb{R}$ and $X, Y \in V$ be given. Then:

$X \in V \Rightarrow AX = XA$

$Y \in V \Rightarrow AY = YA$.

It follows that

$$A(\lambda X + \mu Y) = A(\lambda X) + A(\mu Y) = \lambda(AX) + \mu(AY) =$$
$$= \lambda(XA) + \mu(YA) = (\lambda X)A + (\mu Y)A =$$
$$= (\lambda X + \mu Y)A \Rightarrow \lambda X + \mu Y \in V$$

and therefore:

$\forall \lambda, \mu \in \mathbb{R} : \forall X, Y \in V : \lambda X + \mu Y \in V \Rightarrow$

$\Rightarrow V$ subspace of $M_n(\mathbb{R})$.

d) Show that $V = \{f \in F(\mathbb{R}) \mid f \text{ even}\}$ is a subspace of $F(\mathbb{R})$. Recall that we define on $F(\mathbb{R})$:

f even $\Leftrightarrow \forall x \in \mathbb{R}: f(-x) = f(x)$

<u>Solution</u>

Let $\lambda, \mu \in \mathbb{R}$ and $f, g \in V$ be given.

$$f, g \in V \Rightarrow \begin{cases} f \text{ even} \\ g \text{ even} \end{cases} \Rightarrow \begin{cases} \forall x \in \mathbb{R}: f(-x) = f(x) \\ \forall x \in \mathbb{R}: g(-x) = g(x) \end{cases} \quad (1)$$

Let $x \in \mathbb{R}$ be given. Then

$$\begin{aligned}(\lambda f + \mu g)(-x) &= (\lambda f)(-x) + (\mu g)(-x) = \\ &= \lambda f(-x) + \mu g(-x) = \lambda f(x) + \mu g(x) = \\ &= (\lambda f)(x) + (\mu g)(x) = (\lambda f + \mu g)(x)\end{aligned}$$

It follows that

$\forall x \in \mathbb{R}: (\lambda f + \mu g)(-x) = (\lambda f + \mu g)(x) \Rightarrow$

$\Rightarrow \lambda f + \mu g$ even $\Rightarrow \lambda f + \mu g \in V$

Consequently:

$\forall \lambda, \mu \in \mathbb{R}: \forall f, g \in V: \lambda f + \mu g \in V \Rightarrow$

$\Rightarrow V$ subspace of $F(\mathbb{R})$.

EXERCISES

(19) Show that $V = \{(x,y) \in \mathbb{R}^2 \mid 3x + 7y = 0\}$ is a subspace of $\mathbb{R}^2$.

(20) Show that

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 2z = 0 \wedge x - y - 3z = 0\}$$

is a subspace of $\mathbb{R}^3$.

(21) Show that $V = \{(x,y) \in \mathbb{R}^2 \mid 4x + y = 2\}$ is NOT a subspace of $\mathbb{R}^2$.

(22) Show that $V = \{f \in F(\mathbb{R}) \mid f \text{ odd}\}$ is a subspace of $F(\mathbb{R})$.
Recall that: f odd $\Leftrightarrow \forall x \in \mathbb{R} : f(-x) = -f(x)$.

(23) Show that $V = \{f \in F(\mathbb{R}) \mid f \text{ periodic}\}$ is a subspace of $F(\mathbb{R})$. Recall that

$$f \text{ periodic} \Leftrightarrow \exists T > 0 : \forall x \in \mathbb{R} : f(x+T) = f(x)$$

(24) Show that

$$V = \{f \in F(\mathbb{R}) \mid \forall a, b \in \mathbb{R} : |f(a) - f(b)| \leq K|a-b|\}$$

is a subspace of $F(\mathbb{R})$, with $K \in (0, +\infty)$.
[Hint: Use the following properties of absolute values:

$$\forall a, b \in \mathbb{R} : |a+b| \leq |a| + |b|$$

$$\forall a, b \in \mathbb{R} : |ab| = |a||b| \quad]$$

(25) Show that

$$V = \{f \in F(\mathbb{R}) \mid f \text{ differentiable in } \mathbb{R} \wedge f' + 3f = 0\}.$$

is a subspace of $F(\mathbb{R})$.

(26) Let $A, B \in M_n(\mathbb{R})$ be two $n \times n$ matrices and let
$V = \{ X \in M_n(\mathbb{R}) \mid AX + XB = 0 \}$.
Show that V is a subspace of $M_n(\mathbb{R})$.

(27) Let $A \in M_n(\mathbb{R})$ be a non-singular $n \times n$ matrix and let $V = \{ X \in M_n(\mathbb{R}) \mid AXA^{-1} = I \}$
Show that V is NOT a subspace of $M_n(\mathbb{R})$.

(28) Let V be a vector space and let A, B be subspaces of V. We define
$A + B = \{ x + y \mid x \in A \wedge y \in B \}$
Show that $A + B$ is a subspace of V.

▼ Subspaces spanned by vectors

Let $(V, +, \cdot)$ be a vector space and let $x_1, x_2, \ldots, x_n \in V$ be n vectors of V.

Def : The set V_0 spanned by $\{x_1, x_2, \ldots, x_n\}$ is defined as:

$$\boxed{V_0 = \text{span}\{x_1, x_2, \ldots, x_n\} = \left\{ \sum_{a=1}^{n} \lambda_a x_a \,\middle|\, \forall a \in [n] : \lambda_a \in \mathbb{R} \right\}}$$

- We note that the belonging condition for V_0 reads:

$$\boxed{x \in \text{span}\{x_1, x_2, \ldots, x_n\} \Leftrightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : x = \sum_{a=1}^{n} \lambda_a x_a}$$

- We will now show that V_0 is a subspace of V.

Thm : $\left.\begin{array}{l} A = \{x_1, x_2, \ldots, x_n\} \subseteq V \\ (V, +, \cdot) \text{ vector space} \end{array}\right\} \Rightarrow \begin{array}{l} \text{span}(A) \text{ subspace} \\ \text{of } V. \end{array}$

Proof

Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in \text{span}(A)$ be given.
It follows that:

$$x \in \operatorname{span}(A) \Rightarrow \exists p_1, p_2, \dots, p_n \in \mathbb{R}: \; x = \sum_{a=1}^{n} p_a x_a$$

$$y \in \operatorname{span}(A) \Rightarrow \exists q_1, q_2, \dots, q_n \in \mathbb{R}: \; y = \sum_{a=1}^{n} q_a x_a$$

and therefore:

$$\lambda x + \mu y = \lambda \sum_{a=1}^{n} p_a x_a + \mu \sum_{a=1}^{n} q_a x_a =$$
$$= \sum_{a=1}^{n} (\lambda p_a + \mu q_a) x_a \Rightarrow \lambda x + \mu y \in \operatorname{span}(A).$$

It follows that
$\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in \operatorname{span}(A) : \lambda x + \mu y \in \operatorname{span}(A) \Rightarrow$
$\Rightarrow \operatorname{span}(A)$ subspace of V. □

Basic properties of spanned spaces

Let $A \subseteq V$ and $B \subseteq V$ with A, B finite sets. Then

$$\boxed{\begin{array}{l} A \subseteq \operatorname{span}(A) \\ A \subseteq B \Rightarrow \operatorname{span}(A) \subseteq \operatorname{span}(B) \end{array}}$$

Proof

a) To show $A \subseteq \operatorname{span}(A)$.
Let $A = \{x_1, x_2, \dots, x_n\}$. Let $u \in A$ be given.
Then $u \in A \Rightarrow \exists a \in [n] : u = x_a$

Define $\lambda_b = \begin{cases} 1 & , \text{if } a=b \\ 0 & , \text{if } a \neq b \end{cases}$

Then $u = x_a = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n \Rightarrow u \in span(A)$.

It follows that $\forall u \in A : u \in span(A) \Rightarrow A \subseteq span(A)$.

b) To show $A \subseteq B \Rightarrow span(A) \subseteq span(B)$

For $A = B$, the statement is trivial, so we assume with no loss of generality that $A \neq B$ and write

$A = \{x_1, x_2, \dots, x_p\}$ and $B = \{x_1, x_2, \dots, x_n\}$ with $p < n$.

Let $u \in span(A)$ be given. Since

$u \in span(A) \Rightarrow \exists \lambda_1, \lambda_2, \dots, \lambda_p \in \mathbb{R} : u = \lambda_1 x_1 + \dots + \lambda_p x_p$

Therefore,

$$u = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_p x_p =$$
$$= \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_p x_p + 0\, x_{p+1} + \dots + 0\, x_n \Rightarrow$$
$$\Rightarrow u \in span(B).$$

It follows that

$\forall u \in span(A) : u \in span(B) \Rightarrow span(A) \subseteq span(B)$

EXAMPLES

a) Define the vector space spanned by the vectors $x_1 = (1,3,0)$ and $x_2 = (0,2,-1)$.

Solution

We note that

$$ax_1 + bx_2 = a(1,3,0) + b(0,2,-1) =$$
$$= (a,3a,0) + (0,2b,-b)$$
$$= (a, 3a+2b, -b)$$

It follows that

$$V = \text{span}\{x_1, x_2\} = \{ax_1 + bx_2 \mid a,b \in \mathbb{R}\} =$$
$$= \{(a, 3a+2b, -b) \mid a,b \in \mathbb{R}\}.$$

b) Show that $V = \{(a+b, 2b, b-3a) \mid a,b \in \mathbb{R}\}$ is a vector space.

Solution

We note that

$$(a+b, 2b, b-3a) = (a,0,-3a) + (b,2b,b) =$$
$$= a(1,0,-3) + b(1,2,1)$$
$$= ax + by$$

with $x = (1,0,-3)$ and $y = (1,2,1)$. It follows that

$$V = \{(a+b, 2b, b-3a) \mid a,b \in \mathbb{R}\} = \{ax+by \mid a,b \in \mathbb{R}\}$$
$$= \text{span}\{x,y\} \Rightarrow V \text{ subspace of } \mathbb{R}^2 \Rightarrow$$
$$\Rightarrow (V,+,\cdot) \text{ is a vector space.}$$

c) Define by description the vector subspace of $F(\mathbb{R})$ spanned by the functions:

$f(x) = \sin x \quad , \forall x \in \mathbb{R}$

$g(x) = \cos x \quad , \forall x \in \mathbb{R}.$

Solution

Let $a, b \in \mathbb{R}$ and note that

$$(af+bg)(x) = (af)(x) + (bg)(x) = af(x) + bg(x) =$$
$$= a\sin x + b\cos x \; , \forall x \in \mathbb{R}$$

It follows that

$$V = \text{span}\{f, g\} = \{af + bg \mid a, b \in \mathbb{R}\} =$$

$$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : h = af + bg\}$$
$$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : h(x) = (af + bg)(x)\}$$
$$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : h(x) = a\sin x + b\cos x\}.$$

d) Show that the space defined as

$$V = \{f \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = (ax+b)\sin x + (ax^2+bx+b)\cos x\}$$

is a subspace of $F(\mathbb{R})$

Solution

$f \in V \Leftrightarrow \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} :$

$$: f(x) = (ax+b)\sin x + (ax^2+bx+b)\cos x =$$
$$= ax\sin x + b\sin x + ax^2\cos x + b(x+1)\cos x$$
$$= a(x\sin x + x^2\cos x) + b(\sin x + (x+1)\cos x)$$
$$= ag_1(x) + bg_2(x) = (ag_1)(x) + (bg_2)(x) =$$
$$= (ag_1 + bg_2)(x)$$

with $g_1, g_2 \in V$ defined as

$\forall x \in \mathbb{R}: \; g_1(x) = x\sin x + x^2\cos x$

$\forall x \in \mathbb{R}: \; g_2(x) = \sin x + (x+1)\cos x$

It follows that:

$$f \in V \Leftrightarrow \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = (ag_1 + bg_2)(x) \Leftrightarrow$$
$$\Leftrightarrow \exists a, b \in \mathbb{R} : f = ag_1 + bg_2$$
$$\Leftrightarrow f \in \operatorname{span}\{g_1, g_2\}$$

and therefore

$V = \operatorname{span}\{g_1, g_2\} \Rightarrow V$ subspace of $F(\mathbb{R})$.

e) Consider the space

$$V = \left\{ \begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

Show that V is a subspace of $M_3(\mathbb{R})$.

<u>Solution</u>

We note that

$$\begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ a & 2a & 3a \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2b & b & -2b \\ 0 & b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2c & c \\ c & 0 & 0 \\ 3c & 0 & 0 \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} =$$

$$= aA_1 + bA_2 + cA_3$$

and it follows that

$$V = \{aA_1 + bA_2 + cA_3 \mid a,b,c \in \mathbb{R}\} =$$

$$= \text{span}\{A_1, A_2, A_3\} \Rightarrow V \text{ subspace of } M_3(\mathbb{R}).$$

EXERCISES

(29) Define the vector space spanned by:

a) $x_1 = (2,1,3)$ and $x_2 = (1,3,5)$

b) $x_1 = (1,-1,3,2)$ and $x_2 = (2,-2,5,3)$

c) $x_1 = (0,3,2,5)$, $x_2 = (1,-3,-4,2)$, and $x_3 = (-2,1,3,-1)$

(30) Show that the following sets are vector spaces that are subspaces of $\mathbb{R}^n$ for some n.

a) $V = \{(a+3b, b, 2a) \mid a,b \in \mathbb{R}\}$

b) $V = \{(a-2b+c, 3b, c+2a, b-c) \mid a,b,c \in \mathbb{R}\}$

c) $V = \{(2a-b, b+a, 4b) \mid a,b \in \mathbb{R}\}$.

(31) Define by description the vector spaces of $F(\mathbb{R})$ spanned by

a) $\begin{cases} \forall x \in \mathbb{R} : f(x) = e^x \\ \forall x \in \mathbb{R} : g(x) = e^{-x} \end{cases}$

b) $\begin{cases} \forall x \in \mathbb{R} : f(x) = 2x \\ \forall x \in \mathbb{R} : g(x) = x^2 - 1 \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R} : f(x) = xe^x \\ \forall x \in \mathbb{R} : g(x) = (x+1)^2 e^x \\ \forall x \in \mathbb{R} : h(x) = (x-1)^2 e^x \end{cases}$

(32) Show that the following sets are vector spaces that are subspaces of $F(\mathbb{R})$.

a) $V = \{f \in F(\mathbb{R}) \mid \exists a,b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = x^2(ax+b)\}$

b) $V = \{f \in F(\mathbb{R}) \mid \exists a,b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = e^{-x}(a\sin x + b\cos x)\}$

c) $V = \{f \in F(\mathbb{R}) \mid \exists a,b,c \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = xe^x(ax^2+bx+c)\}$

(33) Show that the following sets are vector spaces that are subspaces of $M_n(\mathbb{R})$ for some n.

a) $V = \left\{ \begin{bmatrix} a+b & 3b \\ 2b & a-b \end{bmatrix} \middle| \, a,b \in \mathbb{R} \right\}$

b) $V = \left\{ \begin{bmatrix} a+c & 2a+b \\ 2a-b & a-c \end{bmatrix} \middle| \, a,b,c \in \mathbb{R} \right\}$

c) $V = \left\{ \begin{bmatrix} a+2c & 2a-b & 3c \\ b+c & a+c & 2a+c \\ 3c & 2b-c & a+b \end{bmatrix} \middle| \, a,b,c \in \mathbb{R} \right\}$

Linear Independence

Let $(V, +, \cdot)$ be a vector space and let $A = \{x_1, x_2, \ldots, x_n\} \subseteq V$ be a set of vectors in V.

Def:
$$\boxed{\begin{array}{l} A \text{ linearly dependent} \Leftrightarrow \exists x \in A : x \in \operatorname{span}(A - \{x\}) \\ A \text{ linearly independent} \Leftrightarrow A \text{ NOT linearly dependent} \end{array}}$$

- It follows that A is <u>linearly dependent</u> if at least one vector $x \in A$ belongs to the subspace $\operatorname{span}(A - \{x\})$ generated by all vectors in A except for x.
- By negating the definition of linear dependence, we can rewrite the definition of linear independence as follows:

$$\boxed{A \text{ linearly independent} \Leftrightarrow \forall x \in A : x \notin \operatorname{span}(A - \{x\})}$$

Characterization of linear independence/dependence

Thm:
$$\boxed{\begin{array}{l} A \text{ linearly dependent} \Leftrightarrow \\ \Leftrightarrow \exists (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : \begin{cases} (\lambda_1, \lambda_2, \ldots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \end{cases} \end{array}}$$

Proof

$(\Rightarrow)$: Assume that A is linearly dependent.
Since A linearly dependent $\Rightarrow$
$\Rightarrow \exists x \in A : x \in \text{span}(A-\{x\})$.
Without loss of generality assume a reordering of the elements of A such that:
$x_n \in \text{span}(A-\{x_n\}) \Rightarrow x_n \in \text{span}\{x_1, x_2, \ldots, x_{n-1}\}$
$\Rightarrow \exists \mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R} : x_n = \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_{n-1} x_{n-1}$
It follows that:
$x_n = \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_{n-1} x_{n-1} \Rightarrow$
$\Rightarrow \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_{n-1} x_{n-1} - x_n = 0$
For $(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\mu_1, \mu_2, \ldots, \mu_{n-1}, -1)$ we have

$$\begin{cases} (\lambda_1, \lambda_2, \ldots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \end{cases}$$

This concludes the argument.
$(\Leftarrow)$: Assume that:

$$\exists (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : \begin{cases} (\lambda_1, \lambda_2, \ldots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \end{cases}$$

Note that
$(\lambda_1, \lambda_2, \ldots, \lambda_n) \neq \mathbf{0} \Rightarrow \lambda_1 \neq 0 \vee \lambda_2 \neq 0 \vee \cdots \vee \lambda_n \neq 0$.
Assume without loss of generality that $\lambda_1 \neq 0$.
It follows that:

$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \Rightarrow$

$\Rightarrow \lambda_1 x_1 = -\lambda_2 x_2 - \cdots - \lambda_n x_n \Rightarrow$

$\Rightarrow x_1 = \frac{-\lambda_2}{\lambda_1} x_2 + \frac{-\lambda_3}{\lambda_1} x_3 + \cdots + \frac{-\lambda_n}{\lambda_1} x_n \Rightarrow$

$\Rightarrow x_1 \in \text{span}\{x_2, x_3, \ldots, x_n\} \Rightarrow$

$\Rightarrow x_1 \in \text{span}(A - \{x_1\}) \Rightarrow A$ linearly dependent. $\square$

The negation of the previous theorem gives the following equivalent statement.

$$\boxed{\begin{array}{l} A \text{ linearly independent} \Leftrightarrow \\ \Leftrightarrow \forall (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : (\lambda_1 x_1 + \cdots + \lambda_n x_n = \mathbf{0} \Rightarrow \lambda_1 = \cdots = \lambda_n = 0) \end{array}}$$

- Note that for $A = \{u\}$ with $u \in \mathbb{R}^n$ and $u \neq \mathbf{0}$, A is linearly independent because
$$\left.\begin{array}{l} \lambda u = \mathbf{0} \Rightarrow \lambda = 0 \lor u = \mathbf{0} \\ u \neq \mathbf{0} \end{array}\right\} \Rightarrow \lambda = 0.$$
- For $A = \{\mathbf{0}\}$, A is linearly dependent because $1\mathbf{0} = 0$ and $1 \neq 0$.

Properties of linear dependence/independence

(1) $\boxed{B \subset A \land B \text{ linearly dependent} \Rightarrow A \text{ linearly dependent}}$

Proof

Let $A = \{x_1, x_2, \dots, x_n\}$ and $B = \{x_1, x_2, \dots, x_p\}$ with $p < n$ (since $B \subset A$).

Assume that B is linearly dependent.

Since: B linearly dependent $\Rightarrow$

$$\Rightarrow \exists (\mu_1, \mu_2, \dots, \mu_p) \in \mathbb{R}^p : \begin{cases} (\mu_1, \mu_2, \dots, \mu_p) \neq \mathbf{0} & (1) \\ \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_p x_p = \mathbf{0} & (2) \end{cases}$$

Define $(\lambda_1, \lambda_2, \dots, \lambda_n) = (\mu_1, \mu_2, \dots, \mu_p, 0, \dots, 0)$

From (1): $(\lambda_1, \lambda_2, \dots, \lambda_n) \neq \mathbf{0}$. (3)

Furthermore:

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \mu_1 x_1 + \dots + \mu_p x_p + 0 x_{p+1} + \dots + 0 x_n$$
$$= \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_p x_p = \mathbf{0} \quad (4)$$

From (3) and (4):

$A = \{x_1, \dots, x_n\}$ is linearly dependent. $\square$

② $\rightarrow$ $\boxed{\begin{cases} B \subset A \\ A \text{ linearly independent} \end{cases} \Rightarrow B \text{ linearly independent}}$

Proof

Assume that A linearly independent and $B \subset A$.

To show that B linearly independent, assume that B is NOT linearly independent.

Since: B NOT linearly independent $\Rightarrow$

$$\left.\begin{array}{l} \Rightarrow B \text{ linearly dependent} \\ B \subset A \end{array}\right\} \Rightarrow$$

$\Rightarrow A$ linearly dependent $\leftarrow$ Contradiction with hypothesis.

It follows that B linearly independent. $\square$

(3) $\rightarrow$ $\boxed{\begin{rcases} A \text{ linearly independent} \\ A \cup \{u\} \text{ linearly dependent} \end{rcases} \Rightarrow u \in \operatorname{span}(A).}$

<u>Proof</u>

Let $A = \{x_1, x_2, \dots, x_n\}$.

Assume that A is linearly independent and $A \cup \{u\}$ linearly dependent.

Since: $A \cup \{u\}$ linearly dependent $\Rightarrow$

$$\Rightarrow \exists (\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1} : \begin{cases} (\lambda_0, \lambda_1, \dots, \lambda_n) \neq \mathbf{0} & (1) \\ \lambda_0 u + \lambda_1 x_1 + \dots + \lambda_n x_n = \mathbf{0} & (2) \end{cases}$$

We claim that $\lambda_0 \neq 0$.

To show that $\lambda_0 \neq 0$, assume that $\lambda_0 = 0$.

From (2):

$$\begin{rcases} \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \mathbf{0} \\ A = \{x_1, x_2, \dots, x_n\} \text{ linearly independent} \end{rcases} \Rightarrow$$

$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0 \Rightarrow$

$\Rightarrow (\lambda_1, \dots, \lambda_n) = \mathbf{0} \Rightarrow (\lambda_0, \lambda_1, \dots, \lambda_n) = \mathbf{0} \leftarrow$ Contradiction.

It follows that $\lambda_0 \neq 0$, and therefore from (2):

$\lambda_0 u = -\lambda_1 x_1 - \lambda_2 x_2 - \dots - \lambda_n x_n \Rightarrow$

$\Rightarrow u = -(\lambda_1/\lambda_0)x_1 - (\lambda_2/\lambda_0)x_2 - \dots - (\lambda_n/\lambda_0)x_n$

$\Rightarrow u \in \text{span}\{x_1, x_2, \dots, x_n\} \Rightarrow$

$\Rightarrow u \in \text{span}(A)$. $\square$

EXAMPLES

a) Let $x, y \in \mathbb{R}^3$ with $x = (3,1,2)$ and $y = (1,0,3)$.
Show that x, y are linearly independent.

<u>Solution</u>

Let $\underline{a, b \in \mathbb{R}}$ be given and assume that $ax + by = \mathbf{0}$.
We note that:

$$\underline{ax + by = \mathbf{0}} \Rightarrow a(3,1,2) + b(1,0,3) = (0,0,0) \Rightarrow$$

$$\Rightarrow \begin{cases} 3a + b = 0 \\ a + 2b = 0 \\ 2a + 3b = 0 \end{cases} \Rightarrow \begin{cases} 3(-2b) + b = 0 \\ a = -2b \\ 2(-2b) + 3b = 0 \end{cases} \Rightarrow \begin{cases} -6b + b = 0 \\ a = -2b \\ -4b + 3b = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} -5b = 0 \\ a = -2b \\ -b = 0 \end{cases} \Rightarrow \begin{cases} a = -2b \\ b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \end{cases} \Rightarrow \underline{a = b = 0}.$$

It follows that

$$\forall a, b \in \mathbb{R} : (ax + by = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$$

$\Rightarrow x, y$ linearly independent.

$\hookrightarrow$ Note that the steps taken to solve $ax + by = \mathbf{0}$ are valid in both directions:

$$ax + by = \mathbf{0} \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftrightarrow a = b = 0$$

however the definition of linear independence only requires the "$\Rightarrow$" direction.

b) Let $x, y, z \in \mathbb{R}^3$ with $x = (1,2,2)$, $y = (3,1,4)$, and $z = (-1,3,0)$. Show that x, y, z is linearly dependent.

<u>Solution</u>

Let $a, b, c \in \mathbb{R}$. Note that

$$ax + by + cz = \mathbf{0} \Leftrightarrow a(1,2,2) + b(3,1,4) + c(-1,3,0) = (0,0,0)$$

$$\Leftrightarrow \begin{cases} a + 3b - c = 0 \\ 2a + b + 3c = 0 \\ 2a + 4b = 0 \end{cases} \Leftrightarrow \begin{cases} (-2b) + 3b - c = 0 \\ 2(-2b) + b + 3c = 0 \\ a = -2b \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} +b - c = 0 \\ -4b + b + 3c = 0 \\ a = -2b \end{cases} \Leftrightarrow \begin{cases} b - c = 0 \\ -3b + 3c = 0 \\ a = -2b \end{cases} \Leftrightarrow \begin{cases} b - c = 0 \\ a = -2b \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c = b \\ a = -2b. \end{cases} \Leftrightarrow (a,b,c) = (-2b, b, b) = b(-2,1,1)$$

Thus: For $(a,b,c) = (-2,1,1) \Rightarrow$

$$\Rightarrow \quad -2x + y + z = \mathbf{0} \Rightarrow x, y, z \text{ linearly dependent.}$$

↳ Note that in solving $\alpha x + by + cz = \mathbf{0}$ we only need the "$\Leftarrow$" direction so we can claim that: $(a,b,c) = (-2,1,1) \Rightarrow ax + by + cz = \mathbf{0}$. Contrast this remark with the previous example.

c) Let $f, g \in F(\mathbb{R})$ with $f(x) = \sin x, \forall x \in \mathbb{R}$ and $g(x) = \cos x, \forall x \in \mathbb{R}$. Show that f, g are linearly independent.

Solution

Let $a, b \in \mathbb{R}$ be given such that $af + bg = \mathbb{0}$.
We note that

$$af + bg = \mathbb{0} \Rightarrow \forall x \in \mathbb{R} : af(x) + bg(x) = 0 \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R} : a \sin x + b \cos x = 0. \quad (1)$$

From (1), for $x = 0$:

$a \sin 0 + b \cos 0 = 0 \Rightarrow a \cdot 0 + b \cdot 1 = 0 \Rightarrow b = 0$

From (1), for $x = \pi/2$:

$a \sin(\pi/2) + b \cos(\pi/2) = 0 \Rightarrow a \cdot 1 + b \cdot 0 = 0 \Rightarrow a = 0$

It follows that

$\forall a, b \in \mathbb{R} : (af + bg = \mathbb{0} \Rightarrow a = b = 0) \Rightarrow$
$\Rightarrow f, g$ linearly independent. $\square$

d) Let $f, g \in F(\mathbb{R})$ with $f(x) = 2x, \forall x \in \mathbb{R}$ and $g(x) = x^2, \forall x \in \mathbb{R}$. Show that f, g linearly independent.

Solution

Let $a, b \in \mathbb{R}$ be given such that $af + bg = \mathbb{0}$.
We note that:

$\underline{af+bg=\mathbf{0}} \Rightarrow \forall x\in\mathbb{R}: af(x)+bg(x)=0 \Rightarrow$
$\Rightarrow \forall x\in\mathbb{R}: a(2x)+bx^2=0.$ (1).

From (1), for $x=1$: $2a+b=0$ (2)

From (1), for $x=2$: $4a+4b=0$ (3)

From (2) and (3):

$$\begin{cases} 2a+b=0 \\ 4a+4b=0 \end{cases} \Rightarrow \begin{cases} 2a+b=0 \\ a+b=0 \end{cases} \Rightarrow \begin{cases} a+(a+b)=0 \\ a+b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ a+b=0 \end{cases}$$

$$\Rightarrow \begin{cases} a=0 \\ 0+b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases} \Rightarrow \underline{a=b=0.}$$

It follows that

$\forall a,b\in\mathbb{R}: (af+bg=\mathbf{0} \Rightarrow a=b=0) \Rightarrow$
$\Rightarrow f,g$ linearly independent.

e) Let $f,g,h\in F(\mathbb{R})$ with $f(x)=\cos x$, $\forall x\in\mathbb{R}$,
$g(x)=\cos x\cos 2x$, $\forall x\in\mathbb{R}$, and $h(x)=\sin x\sin 2x$, $\forall x\in\mathbb{R}$.
Show that f,g,h are linearly dependent

<u>Solution</u>

We note that

$f(x)=\cos x=\cos(2x-x)=\cos 2x\cos x+\sin 2x\sin x=$
$=g(x)+h(x)$, $\forall x\in\mathbb{R} \Rightarrow f=g+h \Rightarrow f\in \operatorname{span}\{g,h\}$
$\Rightarrow f,g,h$ linearly dependent.

EXERCISES

(34) Let $x, y \in \mathbb{R}^3$ with $x = (1,2,1)$ and $y = (3,-1,1)$. Show that x, y are linearly independent.

(35) Let $x, y, z \in \mathbb{R}^4$ with $x = (2,1,1,3)$, $y = (-1,2,1,-1)$, and $z = (0,5,3,1)$. Show that x, y, z are linearly dependent.

(36) Let $f, g, h \in F(\mathbb{R})$ be 3 functions that belong to ~~$F(\mathbb{R})$~~ the vector space $F(\mathbb{R})$. Show that given the following definitions, f, g, h are linearly independent.

a) $\begin{cases} \forall x \in \mathbb{R} : f(x) = 3x \\ \forall x \in \mathbb{R} : g(x) = x+2 \\ \forall x \in \mathbb{R} : h(x) = (x-1)^2 \end{cases}$ b) $\begin{cases} \forall x \in \mathbb{R} : f(x) = \sin x \\ \forall x \in \mathbb{R} : g(x) = \cos x \\ \forall x \in \mathbb{R} : h(x) = x \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R} : f(x) = 1-x \\ \forall x \in \mathbb{R} : g(x) = 1+x \\ \forall x \in \mathbb{R} : h(x) = 1-x^2 \end{cases}$

(37) Let $f, g, h \in F(\mathbb{R})$ be 3 functions that belong to the vector space $F(\mathbb{R})$. Show that given the following definitions, f, g, h are linearly dependent.

a) $\begin{cases} \forall x \in \mathbb{R} : f(x) = x-1 \\ \forall x \in \mathbb{R} : g(x) = x^3-1 \\ \forall x \in \mathbb{R} : h(x) = x-x^3 \end{cases}$ b) $\begin{cases} \forall x \in \mathbb{R} : f(x) = \sin^2 x \\ \forall x \in \mathbb{R} : g(x) = \cos 2x \\ \forall x \in \mathbb{R} : h(x) = 2-\cos 2x \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R} : f(x) = \cos 2x \\ \forall x \in \mathbb{R} : g(x) = \cos^2 x \\ \forall x \in \mathbb{R} : h(x) = \sin^2 x \end{cases}$ (Hint: Use your trigonometric identities from precalculus)

(38) Let $f, g, h \in F(\mathbb{R})$ with

$$\begin{cases} \forall x \in \mathbb{R} : f(x) = 1 \\ \forall x \in \mathbb{R} : g(x) = e^x \\ \forall x \in \mathbb{R} : h(x) = e^{2x} \end{cases}$$

Show that f, g, h are linearly independent.
(Hint: Starting with $\forall x \in \mathbb{R} : a + be^x + ce^{2x} = 0$ we can obtain additional equations by differentiating twice with respect to x. Then set $x = 0$ to obtain a 3×3 system of equations for a, b, c).

(39) Let $f, g, h \in F(\mathbb{R})$ with

$$\begin{cases} \forall x \in \mathbb{R} : f(x) = 1 \\ \forall x \in \mathbb{R} : g(x) = e^x \\ \forall x \in \mathbb{R} : h(x) = xe^x \end{cases}$$

Show that f, g, h are linearly independent.

(40) Let $f_1, f_2, \ldots, f_n \in F(\mathbb{R})$ with
$\forall k \in [n] : \forall x \in \mathbb{R} : f_k(x) = \sin(kx)$
with $[n] = \{1, 2, 3, \ldots, n\}$.

a) For any $k, m \in [n]$, evaluate the integral

$$I_{km} = \int_{-\pi}^{\pi} f_k(x) f_m(x)\, dx.$$

(Hint: Distinguish between the cases $k = m$ and $k \neq m$ and use the identity

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b)$$

to do the integral).

b) Use (a) to show that $f_1, f_2, \ldots, f_n$ are linearly independent.

(41) Let $a, b, c \in (0, 2\pi)$ with $a \neq b \neq c \neq a$. Show that

a) $\sin(x+a)\sin(c-b) + \sin(x+b)\sin(a-c) + \sin(x+c)\sin(b-a) = 0$

b) Let $f, g, h \in F(\mathbb{R})$ with

$$\begin{cases} \forall x \in \mathbb{R} : f(x) = \sin(x+a) \\ \forall x \in \mathbb{R} : g(x) = \sin(x+b) \\ \forall x \in \mathbb{R} : h(x) = \sin(x+c) \end{cases}$$

Show that f, g, h are linearly dependent.

(42) Let $x, y, z \in V$ with V a vector space. Show that

a) x, y, z linearly independent $\Longrightarrow$ $x+y, y+z, z+x$ linearly independent.

b) x, y, z linearly independent

$\Longrightarrow$ $x+y, y-x, y+z-2x$ linearly independant.

Linear Independence in $\mathbb{R}^n$

In the previous examples we have used the following characterizations directly to establish linear independence and linear dependence:

- For $A = \{x_1, x_2, \dots, x_n\} \subset V$

  a) A linearly dependent $\Leftrightarrow$
  $$\exists (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \begin{cases} (\lambda_1, \lambda_2, \dots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \end{cases}$$

  b) A linearly independent $\Leftrightarrow$
  $$\forall (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \Rightarrow$$
  $$\Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = \mathbf{0})$$

For the special case of the vector space $V = \mathbb{R}^n$, linear dependence and independence can be determined via the following specialized theory:

Def: Let $A = \{x_1, x_2, \dots, x_k\} \subseteq \mathbb{R}^n$ be a set of k n-dimensional vectors. We define a corresponding matrix
$$M = \mathrm{Mat}(A) = [x_1 \; x_2 \cdots x_k] \in M_{nk}(\mathbb{R})$$
as an $n \times k$ matrix where for $a \in \mathbb{N}$ with $1 \leq a \leq k$, the a^{th} column of M consists of the components of the vector x_a.
In other words: $M_{ab} = (x_b)_a$

Def: Let $M \in M_{nk}(\mathbb{R})$ be an $n \times k$ matrix with $k \leq n$ (i.e more rows than columns). We define the set $Sub(M)$ of submatrices of M as the set of all matrices $S \in M_k(\mathbb{R})$ obtained from M by deleting any arbitrary selection of $n-k$ rows.

↳ For a square matrix $M \in M_n(\mathbb{R})$, no rows can be deleted therefore $Sub(M) = \{M\}$.

EXAMPLE

For $x_1 = (2,5,3,1)$ and $x_2 = (3,1,4,7)$ it follows that

$$M = Mat(\{x_1, x_2\}) = [x_1 \ x_2] = \begin{bmatrix} 2 & 3 \\ 5 & 1 \\ 3 & 4 \\ 1 & 7 \end{bmatrix}, \text{ and}$$

$$Sub(M) = \left\{ \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} \right\}$$ □

Thm: Let $A = \{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{R}^n$ with $k \leq n$. Then

$$\boxed{\begin{array}{l} A \text{ linearly independent} \Leftrightarrow \exists M \in Sub(Mat(A)) : \det(M) \neq 0 \\ A \text{ linearly dependent} \Leftrightarrow \forall M \in Sub(Mat(A)) : \det(M) = 0 \end{array}}$$

For the case $k=n$, the above theorem reduces to the following simpler statement:

Corollary: Let $A = \{x_1, x_2, \dots, x_n\} \subseteq \mathbb{R}^n$. Then

$$\boxed{\begin{array}{l} \{x_1, \dots, x_n\} \text{ linearly independent} \Leftrightarrow \det([x_1 \cdots x_n]) \neq 0 \\ \{x_1, \dots, x_n\} \text{ linearly dependent} \Leftrightarrow \det([x_1 \cdots x_n]) = 0 \end{array}}$$

EXAMPLES

a) Show that the vectors $x_1 = (1,0,2)$, $x_2 = (1,1,2)$, and $x_3 = (2,2,2)$ are linearly dependent.

Solution

$$\det([x_1\ x_2\ x_3]) = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{vmatrix} \begin{array}{c} (-1) \\ \leftarrow \\ \leftarrow \end{array} = \begin{vmatrix} 1 & 1 & 2 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} =$$

$$= (+1)\cdot 2 \cdot \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = (+1)\cdot 2 \cdot ((-1)\cdot 1 - 0 \cdot 1) =$$

$$= 2(-1-0) = -2 \neq 0 \Rightarrow \{x_1, x_2, x_3\} \text{ linearly dependent.}$$

b) Show that $x_1 = (1,2,1)$ and $x_2 = (2,-1,1)$ are linearly independent.

Solution

Let $$B = [x_1\ x_2] = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow Sub(B) = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \right\}$$

Since $$\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 1(-1) - 2\cdot 2 = -1-4 = -5 \neq 0 \Rightarrow$$

$$\Rightarrow \exists M \in Sub(B) : \det(M) \neq 0 \Rightarrow \{x_1, x_2\} \text{ linearly independent.}$$

c) Show that $x_1 = (1,2,0,1)$, $x_2 = (2,1,3,2)$, and $x_3 = (2,2,2,2)$ are linearly dependent

Solution

Let $B = [x_1\ x_2\ x_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \Rightarrow$

$$\Rightarrow \operatorname{Sub}(B) = \left\{ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \right\} = \{B_1, B_2, B_3, B_4\}.$$

We note that

$$\det(B_1) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} (-1) = \begin{vmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{vmatrix} =$$

$$= (+1)\cdot 2 \cdot \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = (+1)\cdot 2 \cdot [1\cdot 1 - (-1)(-1)] =$$

$$= 2(1-1) = 0$$

$$\det(B_2) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 0, \quad \det(B_3) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 0,$$

$$\det(B_4) = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{vmatrix} = - \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 3 & 2 \end{vmatrix} = + \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= \det(B_1) = 0$$

and therefore

$\forall M \in Sub(B) : \det(M) = 0 \Rightarrow$

$\Rightarrow \{x_1, x_2, x_3\}$ linearly dependent.

d) Let $x=(1,2,1)$, $y=(1,1,0)$, $z=(a,2a+3,2)$. Find all $a\in\mathbb{R}$ such that x,y,z are linearly dependent.

<u>Solution</u>

$$\det([x\ y\ z]) = \begin{vmatrix} 1 & 1 & a \\ 2 & 1 & 2a+3 \\ 1 & 0 & 2 \end{vmatrix} \begin{matrix} (-1) \\ \leftarrow \\ \end{matrix} = \begin{vmatrix} 1 & 1 & a \\ 1 & 0 & a+3 \\ 1 & 0 & 2 \end{vmatrix} =$$

$$= (-1)\cdot 1\cdot \begin{vmatrix} 1 & a+3 \\ 1 & 2 \end{vmatrix} = -\left(1\cdot 2 - 1\cdot(a+3)\right) = -2+(a+3) =$$

$$= a+1.$$

It follows that

x,y,z linearly dependent $\Leftrightarrow \det([x\ y\ z])=0$

$\Leftrightarrow a+1=0 \Leftrightarrow a=-1$

Thus:

x,y,z linearly dependent $\Leftrightarrow a=-1$.

e) Let $x=(3,9,1)$ and $y=(a,2a-1,1-3a)$. Find all $a\in\mathbb{R}$ such that x,y linearly independent.

Solution

Let $M=[x\ y]=\begin{bmatrix}3 & a\\ 9 & 2a-1\\ 1 & 1-3a\end{bmatrix} \Rightarrow$

$\Rightarrow Sub(M)=\left\{\begin{bmatrix}3 & 0\\ 9 & 2a-1\end{bmatrix}, \begin{bmatrix}3 & a\\ 1 & 1-3a\end{bmatrix}, \begin{bmatrix}9 & 2a-1\\ 1 & 1-3a\end{bmatrix}\right\}$

$=\{M_1, M_2, M_3\}$

and note that

$\det M_1 = \begin{vmatrix}3 & 0\\ 9 & 2a-1\end{vmatrix} = 3(2a-1) = 6a-3$

$\det M_2 = \begin{vmatrix}3 & a\\ 1 & 1-3a\end{vmatrix} = 3(1-3a)-1\cdot a = 3-9a-a = 3-10a$

$\det M_3 = \begin{vmatrix}9 & 2a-1\\ 1 & 1-3a\end{vmatrix} = 9(1-3a)-(2a-1) = 9-27a-2a+1 =$

$= -29a+10$

It follows that:

x,y linearly independent $\Leftrightarrow \forall A\in Sub(M): \det A\neq 0 \Leftrightarrow$

$\Leftrightarrow \det M_1\neq 0 \wedge \det M_2 \neq 0 \wedge \det M_3 \neq 0 \Leftrightarrow$

$\Leftrightarrow 6a-3\neq 0 \wedge 3-10a\neq 0 \wedge -29a+10\neq 0$

$\Leftrightarrow a\neq 1/2 \wedge a\neq 3/10 \wedge a\neq 10/29$

$\Leftrightarrow a\in\mathbb{R}-\{1/2, 3/10, 10/29\}.$

EXERCISES

(43) Show that the following vectors are linearly independent

a) $x=(1,2)$ and $y=(-1,1)$

b) $x=(3,1,1)$ and $y=(0,4,5)$

c) $x=(2,1,0,3)$, $y=(1,3,3,1)$, and $z=(3,4,3,2)$

(44) Show that the following vectors are linearly dependent

a) $x=(3,2)$, $y=(4,-1)$, and $z=(5,-2)$

b) $x=(9,-3,7)$, $y=(1,8,8)$, and $z=(5,-5,1)$

c) $x=(2,-1,5,7)$, $y=(3,1,5,-2)$, and $z=(1,1,1,-4)$

(45) Let $x=(1,3,-1)$, $y=(1,a,4)$, and $z=(3,-2,b)$. Find the set of all $a,b\in\mathbb{R}$ such that x,y,z are linearly independent on $\mathbb{R}^3$.

(46) Find all $a\in\mathbb{R}$ such that $x=(1,1,1)$, $y=(1,a,-1)$, and $z=(a,1,1)$ are linearly independent on $\mathbb{R}^3$.

(47) Find all $a\in\mathbb{R}$ such that $x=(3,1,-4,6)$, $y=(1,1,4,4)$, and $z=(1,0,-4,a)$ are linearly dependent on $\mathbb{R}^4$.

(48) Find the set of all $(a,b)\in\mathbb{R}^2$ such that $x=(3,-2,-1,3)$, $y=(1,0,2,4)$, and $z=(1,-3,a,b)$ are linearly dependent on $\mathbb{R}^4$.

(49) Show that the vectors $x=(1,3,5,p)$, $y=(a,3a,5a,p)$, and $z=(-b,-3b,-5b,r)$ are always linearly dependent on $\mathbb{R}^4$

(50) Let $x=(1,a,a^2)$, $y=(1,b,b^2)$, and $z=(1,c,c^2)$. Show that x,y,z linearly dependent $\Leftrightarrow x=y \vee y=z \vee z=x$.

Basis and dimension of vector spaces

Let $(V,+,\cdot)$ be a vector space and let $B=\{x_1,\dots,x_n\}\subseteq V$. We use the notation $|B|=n$ to denote the cardinality of B (i.e. the number of elements in the set B).

Def : $\boxed{B \text{ basis of } V \Leftrightarrow \begin{cases} B \text{ linearly independent} \\ V=\operatorname{span}(B) \end{cases}}$

Thm : Assume that B is a basis of the vector space V. Then:

$$\boxed{\forall u\in V : \exists!\,(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n : u=\lambda_1x_1+\dots+\lambda_nx_n}$$

(For all $u\in V$, there is a unique $(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n$ such that $u=\lambda_1x_1+\dots+\lambda_nx_n$).

Proof

Assume that B is a basis of V. Let $u\in V$ be given.

$$\left.\begin{matrix} B \text{ basis of } V \Rightarrow V=\operatorname{span}(B) \\ u\in V\end{matrix}\right\} \Rightarrow u\in\operatorname{span}(B)\Rightarrow$$

$\Rightarrow \exists\,(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n : u=\lambda_1x_1+\lambda_2x_2+\dots+\lambda_nx_n$.

To show that $(\lambda_1,\dots,\lambda_n)$ is unique, assume that it is not unique and therefore:

$$\exists (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n : \begin{cases} (\mu_1, \dots, \mu_n) \neq (\lambda_1, \dots, \lambda_n) \\ \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_n x_n = u \end{cases}$$

Then :

$$\sum_{a=1}^{n} (\lambda_a - \mu_a) x_a = \sum_{a=1}^{n} \lambda_a x_a - \sum_{a=1}^{n} \mu_a x_a = u - u = \mathbb{0} \quad (1)$$

and

B basis of $V \Rightarrow x_1, x_2, \dots, x_n$ linearly independent. (2)

From (1) and (2) :

$\forall a \in [n] : \lambda_a - \mu_a = 0 \Rightarrow \forall a \in [n] : \lambda_a = \mu_a \Rightarrow$

$\Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = (\mu_1, \mu_2, \dots, \mu_n) \leftarrow$ Contradiction

It follows that $(\lambda_1, \lambda_2, \dots, \lambda_n)$ is unique.

and therefore

$\forall u \in V : \exists ! (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : u = \lambda_1 x_1 + \dots + \lambda_n x_n$. $\square$

$\hookrightarrow$ This result shows that the basis B functions as a <u>coordinate system</u> for the vector space V which allows every vector $u \in V$ to be written as

$$u = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$$

in a unique way. The numbers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ are the <u>coordinates</u> of the vector u with respect to the coordinate system defined by the basis B.

Dimension of vector space V

Let $(V,+,\cdot)$ be a vector space and let $A=\{x_1,\dots,x_n\} \subseteq V$ and $B=\{y_1,y_2,\dots,y_m\} \subseteq V$. We show that:

(1) $\boxed{\left.\begin{cases} B \text{ basis of } V \\ |A|>|B| \end{cases}\right\} \Rightarrow A \text{ linearly dependent}}$

Proof

$$\left.\begin{array}{l} B \text{ basis of } V \Rightarrow V=\operatorname{span}(B) \\ \forall a\in[n]: x_a\in V \end{array}\right\} \Rightarrow \forall a\in[n]: x_a \in \operatorname{span}(B)$$

$$\Rightarrow \forall a\in[n]: \exists (M_{a1},\dots,M_{am})\in\mathbb{R}^m : x_a=\sum_{b=1}^{m} M_{ab}y_b \quad (1)$$

Let $(\lambda_1,\lambda_2,\dots,\lambda_m)\in\mathbb{R}^m$ and solve:

$$\lambda_1x_1+\lambda_2x_2+\dots+\lambda_nx_n=\mathbb{0} \Leftrightarrow \sum_{a=1}^{n}\lambda_a x_a=\mathbb{0}$$

$$\overset{(1)}{\Leftrightarrow} \sum_{a=1}^{n}\lambda_a\left[\sum_{b=1}^{m}M_{ab}y_b\right]=0 \Leftrightarrow \sum_{b=1}^{m}\left[\sum_{a=1}^{n}\lambda_a M_{ab}\right]y_b=0 \quad (2)$$

Since B basis of $V \Rightarrow y_1,y_2,\dots,y_m$ linearly independent (3)
From (2) and (3), it follows that

$$\sum_{a=1}^{n}\lambda_a M_{ab}=0,\ \forall b\in[m]. \quad (4).$$

Since (4) is a system of m equations with n unknowns and since $|A|>|B| \Rightarrow n>m$ it follows that (4) is either inconsistent or has non-zero solutions. Since $\forall a \in [n]: \lambda_a = 0$ satisfies (4), it follows that (4) is not inconsistent and therefore it has a non-zero solution $(\lambda_1, \lambda_2, \dots, \lambda_n) \neq \mathbf{0}$.

Therefore:

$$\begin{cases} (\lambda_1, \lambda_2, \dots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \mathbf{0} \end{cases} \Rightarrow$$

$\Rightarrow x_1, x_2, \dots, x_n$ linearly dependent

$\Rightarrow A$ linearly dependent.

2 $\rightarrow$ $\boxed{\begin{cases} B_1 \text{ basis of } V \\ B_2 \text{ basis of } V \end{cases} \Rightarrow |B_1| = |B_2|}$

Proof

Assume that B_1 and B_2 are basis of V.

To show that $|B_1| = |B_2|$, assume with no loss of generality that $|B_1| > |B_2|$. Then:

$$\begin{cases} |B_1| > |B_2| \\ B_2 \text{ basis of } V \end{cases} \Rightarrow B_1 \text{ linearly dependent} \Rightarrow$$

$\Rightarrow B_1$ NOT linearly independent $\Rightarrow$

$\Rightarrow B_1$ NOT basis of $V \leftarrow$ Contradiction.

Similar argument if $|B_1| < |B_2|$. It follows that $|B_1| = |B_2|$.

From this statement we conclude that any basis B of a vector space V has the same number n of elements. We call this number, the <u>dimension</u> of V and write $\dim V = n$.

Let V be a vector space with $\dim V = n \in \mathbb{N}$ and let $\{x_1, x_2, \dots, x_p\} \subseteq V$. From property (1) it immediately follows that

$$\boxed{p > \dim V \Rightarrow \{x_1, x_2, \dots, x_p\} \text{ linearly dependent}}$$

The contrapositive statement gives:

$$\boxed{\{x_1, x_2, \dots, x_p\} \text{ linearly independent} \Rightarrow p \leq \dim V}$$

Let V be a vector space and let $\mathbf{0}$ be the unit element of the group $(V, +)$. Then:

a) $\{\mathbf{0}\}$ is a subspace of V

b) $\{\mathbf{0}\} = \operatorname{span}\{\mathbf{0}\}$

c) However, $\{\mathbf{0}\}$ does not have a basis since $\{\mathbf{0}\}$ is linearly dependent.

d) Consequently, the dimension of $\{\mathbf{0}\}$ is <u>defined</u> to be $\dim\{\mathbf{0}\} = 0$

It is possible to have vector spaces with no finite set basis B. For example $F(A)$, the set of all functions $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$.

Dimension and canonical basis of $\mathbb{R}^n$

- We define the n-dimensional vectors

$e_1 = (1,0,0,\dots,0)$
$e_2 = (0,1,0,\dots,0)$
$e_3 = (0,0,1,\dots,0)$
$\vdots$
$e_n = (0,0,0,\dots,1)$

Then it follows that

a) $B=\{e_1,e_2,\dots,e_n\}$ is a basis of $\mathbb{R}^n$

b) $\dim(\mathbb{R}^n) = n$

Proof

$$a) \quad \det([e_1\, e_2 \cdots e_n]) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} =$$

$$= \det(I) = 1 \neq 0 \Rightarrow$$

$\Rightarrow B=\{e_1,e_2,\dots,e_n\}$ is linearly independent (1)

Since $B \subseteq \mathbb{R}^n \Rightarrow \text{span}(B) \subseteq \mathbb{R}^n$. (2)

It is sufficient to show that $\mathbb{R}^n \subseteq \text{span}(B)$.

Let $x=(x_1,x_2,\dots,x_n) \in \mathbb{R}^n$ be given. Then

$$x = (x_1,x_2,\dots,x_n) =$$
$$= (x_1,0,\dots,0)+(0,x_2,\dots,0)+\cdots+(0,0,\dots,x_n)$$
$$= x_1(1,0,\dots,0)+x_2(0,1,\dots,0)+\cdots+x_n(0,0,\dots,1)$$
$$= x_1e_1+x_2e_2+\cdots+x_ne_n \Rightarrow$$

$\Rightarrow x \in \mathrm{span}\{e_1, e_2, \dots, e_n\} \Rightarrow x \in \mathrm{span}(B)$.

It follows that $\forall x \in \mathbb{R}^n : x \in \mathrm{span}(B) \Rightarrow \mathbb{R}^n \subseteq \mathrm{span}(B)$. (3)

From (1), (2), (3):

$$\left\{\begin{array}{l} B \text{ linearly independent} \\ \mathrm{span}(B) \subseteq \mathbb{R}^n \\ \mathbb{R}^n \subseteq \mathrm{span}(B) \end{array}\right. \Rightarrow \left\{\begin{array}{l} B \text{ linearly independent} \\ \mathrm{span}(B) = \mathbb{R}^n \end{array}\right. \Rightarrow$$

$\Rightarrow B$ basis of $\mathbb{R}^n$.

b) B basis of $\mathbb{R}^n \Rightarrow$

$\Rightarrow \dim V = |B| = |\{e_1, e_2, \dots, e_n\}| = n$. $\square$

$\hookrightarrow$ Using similar arguments, it can be shown that

$\dim M_{mn}(\mathbb{R}) = m \cdot n$

$\dim M_n(\mathbb{R}) = n^2$.

Basis of a vector space with known dimension

- Let V be a vector space with $\dim V = n$ and let $A = \{x_1, x_2, \ldots, x_n\} \subseteq V$. The problem is to explore whether A is a basis of V.
- We note that by definition:

$$\boxed{A \text{ linearly dependent} \Rightarrow A \text{ NOT basis of } V}$$

What happens if A is linearly independent?

$$\boxed{\left.\begin{array}{l} A \subseteq V \text{ linearly independent} \\ \dim V = |A| \end{array}\right\} \Rightarrow A \text{ is basis of } V}$$

Proof

Assume that $\dim V = n$ and $A = \{x_1, x_2, \ldots, x_n\} \subseteq V$ be linearly independent.

It is sufficient to show that $\mathrm{span}(A) \subseteq V \wedge V \subseteq \mathrm{span}(A)$.

(a) To show $\mathrm{span}(A) \subseteq V$:

Since $\left\{\begin{array}{l} A \subseteq V \\ V \text{ vector space} \end{array}\right. \Rightarrow \underline{\mathrm{span}(A) \subseteq V}$.

(b) To show $V \subseteq \mathrm{span}(A)$.

Let $u \in V$ be given.

Case 1 : If $\exists a \in [n] : u = x_a$

Then since $\left.\begin{array}{l} x_a \in A \\ A \subseteq \operatorname{span}(A) \end{array}\right\} \Rightarrow u = x_a \in \operatorname{span}(A) \Rightarrow$

$\Rightarrow \underline{u \in \operatorname{span}(A)}$.

Case 2 : If $\forall a \in [n] : u \neq x_a$

Then, it follows that

$|\{x_1, x_2, \dots, x_n, u\}| = n+1 > n = \dim V \Rightarrow$

$\Rightarrow \left.\begin{array}{l} \{x_1, x_2, \dots, x_n, u\} \text{ linearly dependent} \\ \{x_1, x_2, \dots, x_n\} \text{ linearly independent} \end{array}\right\} \Rightarrow$

$\Rightarrow u \in \operatorname{span}\{x_1, x_2, \dots, x_n\} = \operatorname{span}(A)$

It follows that

$\forall u \in V : u \in \operatorname{span}(A) \Rightarrow \underline{V \subseteq \operatorname{span}(A)}$

In both cases above we find that $V \subseteq \operatorname{span}(A)$.

It follows that

$\left\{\begin{array}{l} V \subseteq \operatorname{span}(A) \\ \operatorname{span}(A) \subseteq V \\ A \text{ linearly independent} \end{array}\right. \Rightarrow \left\{\begin{array}{l} V = \operatorname{span}(A) \\ A \text{ linearly independent} \end{array}\right. \Rightarrow$

$\Rightarrow A$ basis of V $\square$

$\hookrightarrow$ Recall that we have shown previously that

$\left.\begin{array}{l} B \text{ basis of } V \\ |A| > |B| \end{array}\right\} \Rightarrow A$ linearly dependent

It follows that

$$\boxed{\begin{array}{l} p > \dim V \Rightarrow \{x_1, x_2, \dots, x_p\} \subseteq V \text{ linearly dependent} \\ \{x_1, x_2, \dots, x_p\} \text{ linearly independent} \Rightarrow p \leq \dim V. \end{array}}$$

EXAMPLES

a) Let $x = (2,1,3)$, $y = (1,3,0)$, and $z = (1,2,3)$.
Show that $B = \{x,y,z\}$ is a basis of $\mathbb{R}^3$.

Solution

$$\det([x\ y\ z]) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ 3 & 0 & 0 \end{vmatrix} =$$

(-1)

$$= (+1)\cdot 3 \cdot \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 3\left(1\cdot 1 - (-1)\cdot 3\right) = 3(1+3) = 12 \neq 0$$

$$\left.\begin{array}{l} \Rightarrow \{x,y,z\} \text{ linearly independent} \\ \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow \{x,y,z\} \text{ basis of } \mathbb{R}^3.$$

b) Let $x = (1,1,0)$, $y = (2,0,1)$, and $z = (6,2,2)$. Show that $B = \{x,y,z\}$ is NOT basis of $\mathbb{R}^3$.

Solution

$$\det([x\ y\ z]) = \begin{vmatrix} 1 & 2 & 6 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} \overset{(-1)}{\leftarrow} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{vmatrix} =$$

$$= (+1)\cdot 1 \cdot \begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} = (-2)\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = (-2)\cdot 0 = 0$$

$\Rightarrow \{x,y,z\}$ linearly dependent $\Rightarrow \{x,y,z\}$ NOT basis of $\mathbb{R}^3$.

c) Show that $B = \{(a, a+1), (a+1, a+2)\}$ is a basis of $\mathbb{R}^2$ for all $a \in \mathbb{R}$.

Solution

Define $x = (a, a+1)$ and $y = (a+1, a+2)$. It follows that

$$\det([x\ y]) = \begin{vmatrix} a & a+1 \\ a+1 & a+2 \end{vmatrix} = a(a+2) - (a+1)^2 =$$

$$= (a^2+2a) - (a^2+2a+1) =$$

$$= a^2+2a-a^2-2a-1 = -1 \neq 0 \Rightarrow$$

$\Rightarrow x, y$ linearly independent (1).

Also: $|B| = |\{x, y\}| = 2 = \dim \mathbb{R}^2$ (2)

From (1) and (2): B basis of $\mathbb{R}^2$.

d) Let $B = \{(3a-1, a), (3a, a+1)\}$. Find all $a \in \mathbb{R}$ such that B is a basis of $\mathbb{R}^2$.

Solution

Define $x = (3a-1, a)$ and $y = (3a, a+1)$. Then

$$\det([x\ y]) = \begin{vmatrix} 3a-1 & 3a \\ a & a+1 \end{vmatrix} = (3a-1)(a+1) - 3a^2 =$$

$$= 3a^2+3a-a-1-3a^2 = 2a-1.$$

Since $|B| = |\{x, y\}| = 2 = \dim \mathbb{R}^2$, it follows that

B basis of $\mathbb{R}^2 \Leftrightarrow x, y$ linearly independent $\Leftrightarrow$

$\Leftrightarrow \det([x\ y]) \neq 0 \Leftrightarrow$

$\Leftrightarrow 2a-1 \neq 0 \Leftrightarrow 2a \neq 1 \Leftrightarrow a \neq 1/2$

$\Leftrightarrow a \in \mathbb{R} - \{1/2\}$.

e) Let $B=\{x,y\}$ be a basis of $\mathbb{R}^2$. Let $u=x+3y$ and $v=2x-y$. Show that $B=\{u,v\}$ is also a basis of $\mathbb{R}^2$.

<u>Solution</u>

$B=\{x,y\}$ basis of $\mathbb{R}^2 \Rightarrow x,y$ linearly independent $\Rightarrow$

$\Rightarrow \forall a,b\in\mathbb{R}: (ax+by=\mathbf{0} \Rightarrow (a,b)=(0,0))$ (1)

Let $a,b\in\mathbb{R}$ be given and assume that $au+bv=\mathbf{0}$.

We note that

$$au+bv = a(x+3y)+b(2x-y) = ax+3ay+2bx-by = $$
$$= (a+2b)x+(3a-b)y$$

and therefore

$$au+bv=\mathbf{0} \Rightarrow (a+2b)x+(3a-b)y=\mathbf{0} \overset{(1)}{\Rightarrow} \begin{cases} a+2b=0 \\ 3a-b=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1\cdot(-1)-2\cdot 3 = -1-6=-7\neq 0$

it follows that $a=b=0 \Rightarrow (a,b)=(0,0)$.

We have thus shown that

$\forall a,b\in\mathbb{R}: (au+bv=\mathbf{0} \Rightarrow (a,b)=(0,0)) \Rightarrow$

$\Rightarrow u,v$ linearly independent $\Rightarrow$

$\Rightarrow B=\{u,v\}$ basis of $\mathbb{R}^2$ (since $|B|=2=\dim\mathbb{R}^2$). $\square$

EXERCISES

(51) Show that the following sets are a basis for $\mathbb{R}^2$.

a) $B = \{(1,1),(0,1)\}$

b) $B = \{(a,0),(a,b)\}$ with $ab \neq 0$

c) $B = \{(\cos\vartheta, \sin\vartheta), (-\sin\vartheta, \cos\vartheta)\}$ with $\vartheta \in \mathbb{R}$.

d) $B = \{(\cos\vartheta - \sin\vartheta, -\cos\vartheta - \sin\vartheta), (\cos\vartheta + \sin\vartheta, \cos\vartheta - \sin\vartheta)\}$

(52) Find all $a \in \mathbb{R}$ such that the following sets are a basis of $\mathbb{R}^2$

a) $x = (a-1, 3)$ and $y = (-a+1, a+1)$

b) $x = (a-1, a^2-2a+1)$ and $y = (0, a+1)$

(53) Let $B = \{x,y\}$ be a basis of $\mathbb{R}^2$. Show that $B' = \{u,v\}$ with $u = 3x-y$ and $v = x+2y$ is also a basis of $\mathbb{R}^2$.

(54) Let $x = (2,1,0)$, $y = (2,1,1)$, $z = (2,2,1)$. Show that $B = \{x,y,z\}$ is a basis of $\mathbb{R}^3$.

(55) Let $x = (-1,1,1)$, $y = (1,a^2,2)$, and $z = (-2,2a,1)$. Find all $a \in \mathbb{R}$ such that $B = \{x,y,z\}$ is a basis of $\mathbb{R}^3$.

(56) Let $B = \{x,y,z\}$ be a basis of $\mathbb{R}^3$, and let $u = 2x+y$, $v = z$, $w = u+2v$. Show that $B' = \{u,v,w\}$ is also a basis of $\mathbb{R}^3$.

(57) Show that $B = \{x,y,z,w\}$ is a basis of $\mathbb{R}^4$ with

a) $x = (0,1,1,1)$, $y = (1,0,1,1)$, $z = (1,1,0,1)$, and $w = (1,1,1,0)$

b) $x = (2,-1,0,1)$, $y = (1,3,2,0)$, $z = (0,-1,-1,0)$, and $w = (-2,1,2,1)$

c) $x = (1,-1,2,0)$, $y = (1,1,2,0)$, $z = (3,0,0,1)$, and $w = (2,1,-1,0)$

(58) Let $B=\{x,y,z,w\}$ be a basis of $\mathbb{R}^4$. Let $u=x+y$, $v=z+w$, $p=-x+z+w$, and $q=w-y$. Show that $B'=\{u,v,p,q\}$ is also a basis of $\mathbb{R}^4$.

(59) Let V be a vector space with $\dim V=4$. Let $x_1,x_2,x_3,x_4 \in V$ with x_1,x_2,x_3 linearly independent. Define $u=\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3+\lambda_4 x_4$ with $\lambda_1+\lambda_2+\lambda_3+\lambda_4 \neq 1$. Show that $B=\{u-x_1, u-x_2, u-x_3, u-x_4\}$ is a basis of V.

Dimension of span(B)

- Let V be a vector space, let $B = \{x_1, \dots, x_n\} \subseteq V$ be a set of n vectors and consider the subspace of V given by $V_0 = \text{span}(B)$. The problem is to determine the dimension $\dim V_0$.
- By the basis definition, it follows immediately that

$$\boxed{B \text{ linearly independent} \Rightarrow \dim(\text{span}(B)) = |B| = n}$$

So the question becomes, what if B is linearly dependent?

Thm: Let V be a vector space, and $x_1, x_2, \dots, x_n \in V$, and let $p < n$. Then

$$\boxed{\left.\begin{array}{l} \{x_1, \dots, x_p\} \text{ linearly independent} \\ \forall u \in \{x_{p+1}, \dots, x_n\} : \{x_1, \dots, x_p, u\} \text{ linearly dependent} \end{array}\right\} \Rightarrow \\ \Rightarrow \{x_1, \dots, x_p\} \text{ basis of } \text{span}\{x_1, \dots, x_n\}}$$

Proof

Since $\{x_1, \dots, x_p\} \subseteq \{x_1, \dots, x_n\} \Rightarrow$

$$\Rightarrow \text{span}(\{x_1, \dots, x_p\}) \subseteq \text{span}(\{x_1, \dots, x_n\}) \quad (1)$$

It is sufficient to show that

$$\text{span}(\{x_1, \dots, x_n\}) \subseteq \text{span}(\{x_1, \dots, x_p\})$$

Preliminary argument:

Let $a \in \mathbb{N}$ with $1 \leq a \leq n-p$. Then

$x_{p+a} \in \{x_{p+1}, \ldots, x_n\} \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow \{x_1, x_2, \ldots, x_p, x_{p+a}\} \text{ linearly dependent} \\ \{x_1, x_2, \ldots, x_p\} \text{ linearly independent} \end{array}\right\} \Rightarrow$$

$\Rightarrow x_{p+a} \in \operatorname{span}\{x_1, \ldots, x_p\} \Rightarrow$

$\Rightarrow \exists \mu_{a1}, \mu_{a2}, \ldots, \mu_{ap} \in \mathbb{R} : x_{p+a} = \mu_{a1} x_1 + \cdots + \mu_{ap} x_p.$

Main argument:

Let $u \in \operatorname{span}(\{x_1, \ldots, x_n\})$ be given. Then

$\exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : u = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n.$

It follows that

$$u = \sum_{a \in [n]} \lambda_a x_a = \sum_{b \in [p]} \lambda_b x_b + \sum_{a \in [n-p]} \lambda_{p+a} x_{p+a} =$$

$$= \sum_{b \in [p]} \lambda_b x_b + \sum_{a \in [n-p]} \lambda_{p+a} \left[\sum_{b \in [p]} \mu_{ab} x_b \right]$$

$$= \sum_{b \in [p]} \lambda_b x_b + \sum_{b \in [p]} \left[\sum_{a \in [n-p]} \lambda_{p+a} \mu_{ab} \right] x_b =$$

$$= \sum_{b \in [p]} \left[\lambda_b + \sum_{a \in [n-p]} (\lambda_{p+a} \mu_{ab}) \right] x_b \Rightarrow$$

$\Rightarrow u \in \operatorname{span}\{x_1, x_2, \ldots, x_p\}.$

It follows that span$\{x_1,$

$\forall u \in \operatorname{span}(\{x_1, \ldots, x_n\}) : u \in \operatorname{span}(\{x_1, \ldots, x_p\}) \Rightarrow$

$\Rightarrow \operatorname{span}(\{x_1, \ldots, x_n\}) \subseteq \operatorname{span}(\{x_1, \ldots, x_p\}). \quad (2)$

From (1) and (2):

$$\left.\begin{array}{l} \text{span}(\{x_1,\dots,x_p\}) = \text{span}(\{x_1,\dots,x_n\}) \\ \{x_1,\dots,x_p\} \text{ linearly independent} \end{array}\right\} \Rightarrow$$

$\Rightarrow \{x_1,\dots,x_p\}$ basis of $\text{span}(\{x_1,\dots,x_n\})$. □

Belonging condition to span(B)

Let $V = \text{span}(B)$. If B is shown to be a basis of B, then the following proposition gives a belonging condition to V. We stress that that if B is linearly dependent, then the theorem below will not work.

Prop: If $V = \text{span}(B)$ and $B = \{x_1, x_2, \dots, x_n\}$ be a basis of V. Then

$$\boxed{x \in V \Leftrightarrow x, x_1, x_2, \dots, x_n \text{ linearly dependent}}$$

Proof

($\Rightarrow$): Assume $x \in V$. Then

$x \in V \Rightarrow x \in \text{span}\{x_1, x_2, \dots, x_n\} \Rightarrow$

$\Rightarrow x, x_1, x_2, \dots, x_n$ linearly dependent.

($\Leftarrow$): Assume that $x, x_1, x_2, \dots, x_n$ linearly dependent. Then

$$\left\{\begin{array}{l} x_1, x_2, \dots, x_n \text{ basis of } V \\ x, x_1, x_2, \dots, x_n \text{ linearly dependent} \end{array}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} x_1, x_2, \dots, x_n \text{ linearly independent} \\ x, x_1, x_2, \dots, x_n \text{ linearly dependent} \end{array}\right. \Rightarrow x \in \text{span}\{x_1, x_2, \dots, x_n\}$$

$\Rightarrow x \in V$.

EXAMPLES

a) Let $V = \text{span}\{x_1, x_2, x_3, x_4\}$ with $x_1 = (1,2,0,3)$, $x_2 = (2,0,3,1)$, $x_3 = (-1,2,-3,2)$, $x_4 = (3,-2,6,-1)$. Find $\dim V$ and a belonging condition for $(a,b,c,d) \in V$.

Solution

Sufficient to find a basis B of V.

- Check x_1, x_2, x_3, x_4:

$$\det([x_1\ x_2\ x_3\ x_4]) = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & 0 & 2 & -2 \\ 0 & 3 & -3 & 6 \\ 3 & 1 & 2 & -1 \end{vmatrix} \begin{matrix} (-2) & (-3) \\ \\ \\ \\ \end{matrix} =$$

$$= \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & -4 & 4 & -8 \\ 0 & 3 & -3 & 6 \\ 0 & -5 & 5 & -10 \end{vmatrix} = 0 \Rightarrow x_1, x_2, x_3, x_4 \text{ linearly dependent.}$$

- Check x_1, x_2, x_3.

$$\text{Let } A_{123} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \text{Sub}(A_{123}) = \left\{ \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{bmatrix} \right\}$$

$$= \{ A_{123}^{(1)}, A_{123}^{(2)}, A_{123}^{(3)}, A_{123}^{(4)} \}.$$

Since:

$$\det A_{123}^{(1)} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} = 0$$

$$\det A_{123}^{(2)} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -4 & 4 \\ 3 & -5 & 5 \end{vmatrix} = 0$$

(-2) (+1)

$$\det A_{123}^{(3)} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

(+1)

$$\det A_{123}^{(4)} = \begin{vmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

(+1)

it follows that

$\forall B \in Sub(A_{123}) : (\det B = 0) \Rightarrow x_1, x_2, x_3$ linearly dependent.

- Check x_1, x_2, x_4

Let $A_{124} = [x_1\ x_2\ x_4] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{bmatrix} \Rightarrow$

$$\mathrm{Sub}(A_{124}) = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 0 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{bmatrix} \right\} =$$

$$= \{ A_{124}^{(1)}, A_{124}^{(2)}, A_{124}^{(3)}, A_{124}^{(4)} \}$$

Since

$$\det A_{124}^{(1)} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 0 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 0 \\ 0 & 3 & 6 \end{vmatrix} \rightarrow = (-1) \cdot 2 \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} =$$

$(+1)$

$$= (-1) \cdot 2 \cdot 2 \cdot 3 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = (-1) \cdot 2 \cdot 2 \cdot 3 \cdot 0 = 0$$

$$\det A_{124}^{(2)} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix} \rightarrow = (-1) \cdot 2 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} =$$

$(+1)$

$$= (-1) \cdot 2 \cdot 2 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$\det A_{124}^{(3)} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{vmatrix} (-3) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & -5 & -10 \end{vmatrix} = (+1) \cdot 1 \cdot \begin{vmatrix} 3 & 6 \\ -5 & -10 \end{vmatrix} =$$

$$= (+1) \cdot 1 \cdot 3 \cdot (-5) \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$\det A_{124}^{(4)} = \begin{vmatrix} 2 & 0 & -2 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 3 & 1 & 2 \end{vmatrix} = (+1)\cdot 2 \cdot \begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} =$$

$(+1)$

$$= (1)\cdot 2 \cdot 3 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

and therefore:

$\forall B \in Sub(A_{124}) : (\det B = 0) \Rightarrow x_1, x_2, x_4$ linearly dependent;

- Check x_1, x_2.

Let $A_{12} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow$

$$\Rightarrow Sub(A_{12}) = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix} \right\} = \{ A_{12}^{(1)}, \dots, A_{12}^{(6)} \}$$

$$\det A_{12}^{(1)} = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 1\cdot 0 - 2\cdot 2 = -4 \neq 0 \Rightarrow$$

$\Rightarrow \exists B \in Sub(A_{12}) : (\det B \neq 0) \Rightarrow x_1, x_2$ linearly independent.

- Basis of V.

It follows from the above results that:

$$\left\{\begin{array}{l} x_1, x_2 \text{ linearly independent} \\ x_1, x_2, x_3 \text{ linearly dependent} \\ x_1, x_2, x_4 \text{ linearly dependent} \end{array}\right. \Rightarrow$$

$\Rightarrow V = \operatorname{span}\{x_1, x_2, x_3, x_4\} = \operatorname{span}\{x_1, x_2\}$

and therefore:

$$\left\{\begin{array}{l} V = \operatorname{span}\{x_1, x_2\} \\ x_1, x_2 \text{ linearly independent} \end{array}\right. \Rightarrow \{x_1, x_2\} \text{ basis of } V \Rightarrow$$

$\Rightarrow \dim V = |\{x_1, x_2\}| = 2.$

- Belonging condition for V.

Let $x = (a, b, c, d) \in \mathbb{R}^4$ and define

$$A = [x_1\ x_2\ x] = \begin{bmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 0 & 3 & c \\ 3 & 1 & d \end{bmatrix} \Rightarrow$$

$$\Rightarrow \operatorname{Sub}(A) = \left\{ \begin{bmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 0 & 3 & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 3 & 1 & d \end{bmatrix}, \begin{bmatrix} 1 & 2 & a \\ 0 & 3 & c \\ 3 & 1 & d \end{bmatrix}, \begin{bmatrix} 2 & 0 & b \\ 0 & 3 & c \\ 3 & 1 & d \end{bmatrix} \right\}$$

$$= \{A_1, A_2, A_3, A_4\}.$$

We calculate the determinants of A_1, A_2, A_3, A_4:

$$\det A_1 = \begin{vmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 0 & 3 & c \end{vmatrix} \begin{matrix} (-2) \\ \hookleftarrow \\ \end{matrix} = \begin{vmatrix} 1 & 2 & a \\ 0 & -4 & -2a+b \\ 0 & 3 & c \end{vmatrix} =$$

$$= \begin{vmatrix} -4 & -2a+b \\ 3 & c \end{vmatrix} = (-4c) - 3(-2a+b) = 6a - 3b - 4c$$

$$\det A_2 = \begin{vmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 3 & 1 & d \end{vmatrix} \begin{array}{l} \\ \\ (-2) \end{array} = \begin{vmatrix} -5 & 0 & a-2d \\ 2 & 0 & b \\ 3 & 1 & d \end{vmatrix} =$$

$$= (-1)\cdot 1 \cdot \begin{vmatrix} -5 & a-2d \\ 2 & b \end{vmatrix} = -(-5b - 2(a-2d)) =$$

$$= 2a + 5b - 4d$$

$$\det A_3 = \begin{vmatrix} 1 & 2 & a \\ 0 & 3 & c \\ 3 & 1 & d \end{vmatrix} \begin{array}{l} (-3) \\ \\ \\ \end{array} = \begin{vmatrix} 1 & 2 & a \\ 0 & 3 & c \\ 0 & -5 & -3a+d \end{vmatrix} = \begin{vmatrix} 3 & c \\ -5 & -3a+d \end{vmatrix} =$$

$$= 3(-3a+d) - (-5)c = -9a + 5c + 3d$$

$$\det A_4 = \begin{vmatrix} 2 & 0 & b \\ 0 & 3 & c \\ 3 & 1 & d \end{vmatrix} \begin{array}{l} \\ \\ (-3) \end{array} = \begin{vmatrix} 2 & 0 & b \\ -9 & 0 & c-3d \\ 3 & 1 & d \end{vmatrix} = (-1)\cdot 1 \cdot \begin{vmatrix} 2 & b \\ -9 & c-3d \end{vmatrix}$$

$$= -[2(c-3d) - (-9)b] = -9b - 2c + 6d$$

Main argument:

$x \in V \Leftrightarrow x \in \operatorname{span}\{x_1, x_2\} \Leftrightarrow$

$\Leftrightarrow x_1, x_2, x$ linearly dependent $\Leftrightarrow$

$\Leftrightarrow \forall A \in \operatorname{Sub}([x_1\ x_2\ x]) : \det A = 0 \Leftrightarrow$

$\Leftrightarrow \det A_1 = 0 \wedge \det A_2 = 0 \wedge \det A_3 = 0 \wedge \det A_4 = 0$

$\Leftrightarrow 6a - 3b - 4c = 0 \wedge 2a + 5b - 4d = 0 \wedge -9a + 5c + 3d = 0$

$\wedge -9b - 2c + 6d = 0.$

b) Let $f, g, h \in F(\mathbb{R})$ with:

$\forall x \in \mathbb{R} : f(x) = 1$

$\forall x \in \mathbb{R} : g(x) = \sin^2 x$

$\forall x \in \mathbb{R} : h(x) = \cos^2 x$

Find the dimension of $V = \text{span}\{f, g, h\}$.

<u>Solution</u>

• Check f, g, h.

We note that

$$\forall x \in \mathbb{R} : (g+h)(x) = g(x) + h(x) = \sin^2 x + \cos^2 = 1 = f(x) \Rightarrow$$

$\Rightarrow f = g + h \Rightarrow f \in \text{span}\{g, h\} \Rightarrow f, g, h$ linearly dependent. (1)

• Check g, h

We will show that $\forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 g + \lambda_2 h = 0 \Rightarrow \lambda_1 = \lambda_2 = 0)$.

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be given. Assume that $\lambda_1 g + \lambda_2 h = 0$ (2)

Then

$$\forall x \in \mathbb{R} : (\lambda_1 g + \lambda_2 h)(x) = (\lambda_1 g)(x) + (\lambda_2 h)(x) = \lambda_1 g(x) + \lambda_2 h(x) = \lambda_1 \sin^2 x + \lambda_2 \cos^2 x \quad (3)$$

From (2) and (3): $\forall x \in \mathbb{R} : \lambda_1 \sin^2 x + \lambda_2 \cos^2 x = 0$

For $x = 0$: $\lambda_1 \sin^2 0 + \lambda_2 \cos^2 0 = 0 \Rightarrow 0\lambda_1 + 1\lambda_2 = 0 \Rightarrow \lambda_2 = 0$

For $x = \pi/2$: $\lambda_1 \sin^2(\pi/2) + \lambda_2 \cos^2(\pi/2) = 0 \Rightarrow 1\lambda_1 + 0\lambda_2 = 0 \Rightarrow$
$\Rightarrow \lambda_1 = 0$.

It follows that

$\forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 g + \lambda_2 h = 0 \Rightarrow \lambda_1 = \lambda_2 = 0) \Rightarrow$

$\Rightarrow g, h$ linearly independent (4).

From (1) and (4):

$$\begin{cases} f, g, h \text{ linearly dependent} \\ g, h \text{ linearly independent} \end{cases} \Rightarrow$$

$\Rightarrow V = \operatorname{span}\{f, g, h\} = \operatorname{span}\{g, h\}$. (5).

From (4) and (5):

$$\begin{cases} V = \operatorname{span}\{g, h\} \\ g, h \text{ linearly independent} \end{cases} \Rightarrow \{g, h\} \text{ basis of } V \Rightarrow$$

$\Rightarrow \dim V = |\{g, h\}| = 2.$

c) Let us define

$$M(a,b,c) = \begin{bmatrix} a & b & c \\ 3c & a-3c & b \\ 3b & -3b+3c & a-3c \end{bmatrix}$$

and consider the set

$$V = \{ M(a,b,c) \mid (a,b,c) \in \mathbb{R}^3 \}$$

Show that V is a subspace of $M_3(\mathbb{R})$ and determine the dimension $\dim V$.

Solution

We note that

$$M(a,b,c) = \begin{bmatrix} a & b & c \\ 3c & a-3c & b \\ 3b & -3b+3c & a-3c \end{bmatrix} =$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 3b & -3b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ 3c & -3c & 0 \\ 0 & 3c & -3c \end{bmatrix} =$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -3 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 3 & -3 & 0 \\ 0 & 3 & -3 \end{bmatrix}$$

$$= a A_1 + b A_2 + c A_3$$

with

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -3 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 3 & -3 & 0 \\ 0 & 3 & -3 \end{bmatrix}$$

It follows that

$V = \{M(a,b,c) \mid a,b,c \in \mathbb{R}\} =$

$= \{aA_1 + bA_2 + cA_3 \mid a,b,c \in \mathbb{R}\} =$

$= \text{span}\{A_1, A_2, A_3\}$ (1)

From (1): V subspace of $M_2(\mathbb{R})$

• Check A_1, A_2, A_3 dependence

Let $a,b,c \in \mathbb{R}$ be given. Assume that $aA_1 + bA_2 + cA_3 = \mathbb{0}$.

It follows that

$$\begin{bmatrix} a & b & c \\ 3c & a-3c & b \\ 3b & -3b+3c & a-3c \end{bmatrix} = M(a,b,c) = aA_1 + bA_2 + cA_3 = \mathbb{0} \Rightarrow$$

$\Rightarrow a = 0 \wedge b = 0 \wedge c = 0 \Rightarrow (a,b,c) = (0,0,0).$

We have thus shown

$\forall a,b,c \in \mathbb{R}: (aA_1 + bA_2 + cA_3 = \mathbb{0} \Rightarrow (a,b,c) = (0,0,0)) \Rightarrow$

$\Rightarrow A_1, A_2, A_3$ linearly independent. (2)

From (1) and (2):

$$\begin{cases} V = \text{span}\{A_1, A_2, A_3\} \\ A_1, A_2, A_3 \text{ linearly independent} \end{cases} \Rightarrow$$

$\Rightarrow \{A_1, A_2, A_3\}$ basis of $V \Rightarrow$

$\Rightarrow \dim V = |\{A_1, A_2, A_3\}| = 3.$

d) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and consider the space V given by

$V = \{X \in M_2(\mathbb{R}) : AX = XA\}$.

Show that V is a subspace of $M_2(\mathbb{R})$ and evaluate $\dim V$.

<u>Solution</u>

- Determine V.

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We note that

$$X \in V \Leftrightarrow X \in \{X \in M_2(\mathbb{R}) : AX = XA\} \Leftrightarrow AX = XA \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a+2c = a \\ c = c \\ b+2d = 2a+b \\ d = 2c+d \end{cases} \Leftrightarrow \begin{cases} 2c = 0 \\ 2d = 2a \\ 2c = 0 \end{cases} \Leftrightarrow \begin{cases} a = d \\ c = 0 \end{cases}$$

$$\Leftrightarrow X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} =$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = aA_1 + bA_2$$

with $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

It follows that

$X \in V \Leftrightarrow \exists a, b \in \mathbb{R}: X = aA_1 + bA_2 \Leftrightarrow X \in \text{span}\{A_1, A_2\}$

and therefore $V = \text{span}\{A_1, A_2\}$ (1)

From (1): V subspace of $M_2(\mathbb{R})$.

- Check dependence of A_1, A_2.

Let $a, b \in \mathbb{R}$ be given. Assume that $aA_1 + bA_2 = \mathbf{0}$.

It follows that

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = aA_1 + bA_2 = \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = 0.$$

Thus:

$\forall a, b \in \mathbb{R}: (aA_1 + bA_2 = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$

$\Rightarrow A_1, A_2$ linearly independent. (2)

From (1) and (2):

$$\left\{ \begin{array}{l} V = \text{span}\{A_1, A_2\} \\ A_1, A_2 \text{ linearly independent} \end{array} \right. \Rightarrow \{A_1, A_2\} \text{ basis of } V$$

$\Rightarrow \dim V = |\{A_1, A_2\}| = 2.$

e) Given $x=(1,2,3)$, $y=(-1,4,5)$, $z=(-5,2,1)$, and $w=(9,12,19)$. Show that $\operatorname{span}\{x,y\}=\operatorname{span}\{z,w\}$.

Solution

Let $A_1=\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}\in \operatorname{Sub}([x\ y]) \Rightarrow$

$\Rightarrow \det A_1 = 1\cdot 4-(-1)\cdot 2 = 4+2=6\neq 0 \Rightarrow$

$\Rightarrow$ x,y linearly independent. (1)

Let $A_2=\begin{bmatrix}-1 & -5\\ 4 & 2\end{bmatrix}\in \operatorname{Sub}([z\ w]) \Rightarrow$

$\Rightarrow \det A_2 = (-1)\cdot 2-(-5)\cdot 4 = -2+20 = 18\neq 0 \Rightarrow$

$\Rightarrow$ z,w linearly independent (2)

We also note that

$$\det([x\ y\ z]) = \begin{vmatrix}1 & -1 & -5\\ 2 & 4 & 2\\ 3 & 5 & 1\end{vmatrix}\begin{matrix}(-2)\ (-3)\\ \\ \\ \end{matrix} = \begin{vmatrix}1 & -1 & -5\\ 0 & 6 & 12\\ 0 & 8 & 16\end{vmatrix} =$$

$$= 6\cdot 8\cdot \begin{vmatrix}1 & -1 & -5\\ 0 & 1 & 2\\ 0 & 1 & 2\end{vmatrix} = 0 \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow x,y,z \text{ linearly dependent}\\ x,y \text{ linearly independent}\end{array}\right\} \Rightarrow \underline{z\in \operatorname{span}\{x,y\}} \quad (3)$$

$$\det([x\ y\ w]) = \begin{vmatrix}1 & -1 & 9\\ 2 & 4 & 12\\ 3 & 5 & 19\end{vmatrix}\begin{matrix}(-2)\ (-3)\\ \\ \\ \end{matrix} = \begin{vmatrix}1 & -1 & 9\\ 0 & 6 & -6\\ 0 & 8 & -8\end{vmatrix}$$

$$= 6 \cdot 8 \cdot \begin{vmatrix} 1 & -1 & 9 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 0 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow x, y, w \text{ linearly dependent} \\ \quad x, y \text{ linearly independent} \end{array}\right\} \Rightarrow \underline{w \in \operatorname{span}\{x, y\}}. \qquad (4)$$

From (3), (4):

$w, z \in \operatorname{span}\{x, y\} \Rightarrow \forall a, b \in \mathbb{R}: (az + bw) \in \operatorname{span}\{x, y\} \Rightarrow$

$\Rightarrow \operatorname{span}\{z, w\} = \{az + bw \mid a, b \in \mathbb{R}\} \subseteq \operatorname{span}\{x, y\} \Rightarrow$

$\Rightarrow \underline{\operatorname{span}\{z, w\} \subseteq \operatorname{span}\{x, y\}} \qquad (5)$

Furthermore:

$$\det([x\ z\ w]) = \begin{vmatrix} 1 & -5 & 9 \\ 2 & 2 & 12 \\ 3 & 1 & 19 \end{vmatrix} \begin{matrix} (-2) & (-3) \\ & \\ & \end{matrix} = \begin{vmatrix} 1 & -5 & 9 \\ 0 & 12 & -6 \\ 0 & 16 & -8 \end{vmatrix} =$$

$$= 6 \cdot 8 \cdot \begin{vmatrix} 1 & -5 & 9 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{vmatrix} = 0 \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow x, z, w \text{ linearly dependent} \\ \quad z, w \text{ linearly independent} \end{array}\right\} \Rightarrow \underline{x \in \operatorname{span}\{z, w\}} \qquad (6)$$

$$\det([y\ z\ w]) = \begin{vmatrix} -1 & -5 & 9 \\ 4 & 2 & 12 \\ 5 & 1 & 19 \end{vmatrix} \begin{matrix} 4 & 5 \\ & \\ & \end{matrix} = \begin{vmatrix} -1 & -5 & 9 \\ 0 & -18 & 48 \\ 0 & -24 & 64 \end{vmatrix} =$$

$$= 6 \cdot 8 \cdot \begin{vmatrix} -1 & -5 & 9 \\ 0 & -3 & 8 \\ 0 & -3 & 8 \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} y, z, w \text{ linearly dependent} \\ z, w \text{ linearly independent} \end{array}\right\} \Rightarrow \underline{y \in \operatorname{span}\{z, w\}.} \quad (7)$$

From (6) to (7):

$$x, y \in \operatorname{span}\{z, w\} \Rightarrow \forall a, b \in \mathbb{R}: (ax + by) \in \operatorname{span}\{z, w\} \Rightarrow$$

$$\Rightarrow \operatorname{span}\{x, y\} = \{ax + by \mid \alpha, b \in \mathbb{R}\} \subseteq \operatorname{span}\{z, w\} \Rightarrow$$

$$\Rightarrow \underline{\operatorname{span}\{x, y\} \subseteq \operatorname{span}\{z, w\}.} \quad (8)$$

From (5) and (8):

$$\left\{\begin{array}{l} \operatorname{span}\{z, w\} \subseteq \operatorname{span}\{x, y\} \\ \operatorname{span}\{x, y\} \subseteq \operatorname{span}\{z, w\} \end{array}\right. \Rightarrow \underline{\underline{\operatorname{span}\{x, y\} = \operatorname{span}\{z, w\}.}}$$

EXERCISES

(60) Let $x=(1,-1,2,1)$, $y=(1,2,1,0)$, and $z=(-1,1,-2,-1)$. Find a basis and the dimension of $V=\text{span}\{x,y,z\}$.

(61) Let $x=(1,4,-5,2)$ and $y=(1,2,3,1)$, and define $V=\text{span}\{x,y\}$. Explore whether $u=(2,14,-34,7)$ belongs to V.

(62) Let $x=(2,1,0)$, $y=(1,-1,2)$, and $z=(0,3,4)$, and define $V=\text{span}\{x,y,z\}$. Show that

$$(a,b,c)\in V \Leftrightarrow 2a-4b-3c=0$$

(63) Let $x=(1,1,1)$, $y=(1,-1,0)$, $z=(0,2,1)$, and $w=(3,1,2)$. Show that $\text{span}\{x,y\}=\text{span}\{z,w\}$.
(Hint: First we use linear dependance and independence to show that $z,w\in\text{span}\{x,y\}$ and $x,y\in\text{span}\{z,w\}$.)

(64) Let $x=(1,-1,2)$, $y=(2,1,3)$, and $z=(3,3,4)$. Show that $z\in\text{span}\{x,y\}$.

(65) Find the dimension and a basis for the subspace $F(\mathbb{R})$ spanned by:

a) $\begin{cases} \forall x\in\mathbb{R}: f(x)=\sin x\cos x \\ \forall x\in\mathbb{R}: g(x)=\sin 2x \\ \forall x\in\mathbb{R}: h(x)=\cos 2x \end{cases}$

b) $\begin{cases} \forall x\in\mathbb{R}: f(x)=\sin^2 x \\ \forall x\in\mathbb{R}: g(x)=\cos 2x \\ \forall x\in\mathbb{R}: h(x)=1+\cos 2x \end{cases}$

c) $\begin{cases} \forall x\in\mathbb{R}: f(x)=\sin^2 x \\ \forall x\in\mathbb{R}: g(x)=\cos^2 x \\ \forall x\in\mathbb{R}: h(x)=\cos 2x \end{cases}$

d) $\begin{cases} \forall x\in\mathbb{R}: f(x)=xe^x \\ \forall x\in\mathbb{R}: g(x)=x^2e^x \\ \forall x\in\mathbb{R}: h(x)=x^3e^x \end{cases}$

(66) Let $M(a,b) = \begin{bmatrix} 3a+b & 2a \\ 2b & a+b \end{bmatrix}$ and define

$V = \{M(a,b) \mid a,b \in \mathbb{R}\}$.

Show that V is a subspace of $M_2(\mathbb{R})$ and find a basis and the dimension of V.

(67) Let

$$M(a,b,c) = \begin{bmatrix} a+b+c & b+c & a+b \\ a-b+c & c+a & b+c \\ a+b-c & a+b & c+a \end{bmatrix}$$

and define $V = \{M(a,b,c) \mid a,b,c \in \mathbb{R}\}$. Show that V is a subspace of $M_2(\mathbb{R})$ and find a basis and the dimension of V.