DETERMINANTS AND LINEAR SYSTEMS

Determinants

Determinants are used to find the inverse of $n \times n$ matrices, and solve $n \times n$ linear systems.

Leibnitz definition of determinants

1) Permutations

Let $[n] = \{1, 2, 3, \dots, n\}$. A permutation σ is a reshuffling of the order of the elements of n. Formally, σ is a bijection $\sigma : [n] \to [n]$ whereby each element of $[n]$ is mapped into a distinct element of $[n]$.

S_n = set of all permutations on $[n]$

EXAMPLE

For $n = 3$:

$S_3 = \{(1,2,3), (2,3,1), (3,1,2), (3,2,1), (1,3,2), (2,1,3)\}$ are the six permutations of $[3]$.

For $\sigma = (2,3,1)$: $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$.

2) Permutation parity

Let $\sigma \in S_n$ be a permutation of $[n]$. We define the parity $s(\sigma)$ of σ as:

$$\boxed{s(\sigma) = \text{sign}\left[\prod_{b=1}^{n-1} \prod_{a=b+1}^{n} (\sigma(a) - \sigma(b))\right]}$$

with sign(x) defined as

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x>0 \\ 0, & \text{if } x=0 \\ -1, & \text{if } x<0 \end{cases}$$

For $\sigma \in S_n$, $s(\sigma)=1$ or $s(\sigma)=-1$. We say that

σ even permutation $\Leftrightarrow$ $s(\sigma)=1$

σ odd permutation $\Leftrightarrow$ $s(\sigma)=-1$

EXAMPLE

For $\sigma=(3,1,4,2)$, the parity of σ is:

$$s(\sigma)=\operatorname{sign}\left[\prod_{b=1}^{3}\prod_{a=b+1}^{4}(\sigma(a)-\sigma(b))\right]=$$

$$=\operatorname{sign}[(\sigma(2)-\sigma(1))(\sigma(3)-\sigma(1))(\sigma(4)-\sigma(1))(\sigma(3)-\sigma(2)) \times(\sigma(4)-\sigma(2))(\sigma(4)-\sigma(3))]$$

$$=\operatorname{sign}[(1-3)(4-3)(2-3)(4-1)(2-1)(2-4)]$$

$$=\operatorname{sign}[(-2)(1)(-1)(3)(1)(-2)]=-1.$$

- A transposition is a permutation that switches only two elements of $[n]$. Every permutation can be constructed as a sequence of transpositions. An even permutation can be constructed by an even number of transpositions. An odd permutation requires an odd number of transpositions.

EXAMPLE

a) For $\sigma = (3,1,4,2)$, we can construct σ with 3 transpositions:

$$(1,2,3,4) \longrightarrow (3,2,1,4) \longrightarrow (3,2,4,1)$$

$$\longrightarrow (3,1,4,2)$$

and therefore σ is odd.

b) For $n=3$, S_3 has 3 even permutations and 3 odd permutations:

$A = \{\sigma \in S_3 \mid \sigma \text{ even}\}$
$= \{(1,2,3), (2,3,1), (3,1,2)\} \leftarrow$ even permutations

$B = \{\sigma \in S_3 \mid \sigma \text{ odd}\}$
$= \{(3,2,1), (1,3,2), (2,1,3)\} \leftarrow$ odd permutations

3) Determinants

We now use permutations to define determinants as follows:

$$\boxed{\forall A \in M_n(\mathbb{R}) : \det(A) = \sum_{\sigma \in S_n} \left[s(\sigma) \prod_{a=1}^{n} A_{a,\sigma(a)} \right]}$$

1×1, 2×2, 3×3 determinants

For $n=1$: $\qquad |A_{11}| = A_{11}$

For $n=2$: $\qquad \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$

For $n=3$: we use the Sarrus scheme:

$$\left|\begin{array}{ccc|cc} A_{11} & A_{12} & A_{13} & A_{11} & A_{12} \\ A_{21} & A_{22} & A_{23} & A_{21} & A_{22} \\ A_{31} & A_{32} & A_{33} & A_{31} & A_{32} \end{array}\right. =$$

$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33}$$

▸ Note that there are 3 positive terms corresponding to the 3 even permutations of S_3 and 3 negative terms corresponding to the 3 odd permutations.

Fundamental properties of determinants

1) $I \in M_n(\mathbb{R})$ identity matrix $\Rightarrow \det(I) = 1$
2) $\forall A \in M_n(\mathbb{R})$: $\det(A^T) = \det(A)$

3) $\forall A, B \in M_n(\mathbb{R})$: $\det(AB) = \det(A)\det(B)$

4) $\forall A \in M_n(\mathbb{R})$: $(A \text{ non-singular} \Leftrightarrow \det(A) \neq 0)$

$\forall A \in M_n(\mathbb{R})$: $(A \text{ singular} \Leftrightarrow \det(A) = 0)$

$\hookrightarrow$ It follows that the set $GL(n, \mathbb{R})$ of non-singular matrices satisfies

$$GL(n,\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det A \neq 0 \}$$

5) $\forall A \in GL(n,\mathbb{R})$: $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$\hookrightarrow$ <u>Determinant of lower/upper triangular matrices</u>

- Let $A \in M_n(\mathbb{R})$ be a matrix. We say that

A lower-triangular $\Leftrightarrow \forall a, b \in [n]: (a < b \Rightarrow A_{ab} = 0)$

A upper-triangular $\Leftrightarrow \forall a, b \in [n]: (a > b \Rightarrow A_{ab} = 0)$

- It can be shown that if A is upper-triangular or lower-triangular, its determinant $\det(A)$ is given by the product of all diagonal components:

$$\boxed{\begin{array}{l} \forall A \in M_n(\mathbb{R}): \left(A \text{ lower-triangular} \Rightarrow \det A = \prod_{a=1}^{n} A_{aa}\right) \\ \forall A \in M_n(\mathbb{R}): \left(A \text{ upper-triangular} \Rightarrow \det A = \prod_{a=1}^{n} A_{aa}\right) \end{array}}$$

EXAMPLES

a) Evaluate the determinant of

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 3\cdot 1 - 5\cdot 2 = 3 - 10 = -7.$$

b) Evaluate the determinant of

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \text{ using the Sarrus rule.}$$

Solution

$$\det(A) = \left|\begin{array}{ccc|cc} 1 & 2 & -2 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 0 & 3 \end{array}\right| =$$

$$= 1\cdot 1\cdot 1 + 2\cdot 0\cdot 0 + (-2)\cdot 1\cdot 3 - 0\cdot 1\cdot(-2) - 3\cdot 0\cdot 1 - 1\cdot 1\cdot 2$$

$$= 1 + 0 - 6 - 0 - 0 - 2 = 1 - 6 - 2 = 1 - 8 = -7.$$

c) Evaluate the determinant of

$$A = \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Solution:

A upper triangular $\Rightarrow$

$\Rightarrow \det A = A_{11} A_{22} A_{33} A_{44}$

$= 1(-1)\cdot 2\cdot 7 = -14.$

d) Let $A, B \in M_n(\mathbb{R})$. Show that $AB = I \Rightarrow BA = I$.

<u>Solution</u>

Assume that $AB = I$. It follows that

$\det(A)\det(B) = \det(AB) = \det(I) = 1 \Rightarrow$

$\Rightarrow \det(A)\det(B) \neq 0 \overset{*}{\Rightarrow}$

$\Rightarrow \det(A) \neq 0 \wedge \det(B) \neq 0 \Rightarrow$

$\Rightarrow A, B$ are non-singular

Let A^{-1}, B^{-1} be the corresponding inverse matrices. Then

$$
\begin{aligned}
BA &= I(BA) && \text{[identity matrix]} \\
&= (A^{-1}A)(BA) && [A^{-1} \text{ inverse of } A] \\
&= A^{-1}[A(BA)] && \text{[associative property]} \\
&= A^{-1}[(AB)A] && \text{[associative property]} \\
&= A^{-1}IA && [\text{hypothesis } AB = I] \\
&= A^{-1}A && \text{[identity matrix]} \\
&= I && [A^{-1} \text{ inverse of } A]
\end{aligned}
$$

$*\hookrightarrow$ Note that we use the contrapositive of the statement

$\forall a, b \in \mathbb{R}: (ab = 0 \Leftrightarrow (a = 0 \vee b = 0))$

which is given by

$\forall a, b \in \mathbb{R}: (ab \neq 0 \Leftrightarrow (a \neq 0 \wedge b \neq 0))$.

EXERCISES

① Which of the following permutations are odd and which are even? Show using both the product definition and enumeration of transpositions.

a) $\sigma = (1,3,2,4)$ c) $\sigma = (2,3,4,1)$

b) $\sigma = (3,1,4,2)$ d) $\sigma = (1,4,3,2)$

② Calculate the following determinants:

a) $\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}$ b) $\begin{vmatrix} 2a & a+1 \\ a-1 & a \end{vmatrix}$

c) $\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix}$ d) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

③ Given the matrix

$$A = \begin{bmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{bmatrix}$$

show that

$$A \text{ singular} \Leftrightarrow x = 1$$

④ Solve with respect to x the following equations:

a) $\begin{vmatrix} 1 & 3x-4 \\ -1 & 4x+1 \end{vmatrix} = 0$ b) $\begin{vmatrix} x-1 & x^2-1 \\ 1-x^2 & x^3-1 \end{vmatrix} = 0$

(5) Let $A, B \in M_n(\mathbb{R})$. Show that
AB singular $\Rightarrow$ (A singular $\vee$ B singular)

(6) <u>Rotation matrix.</u>
Consider the rotation matrix
$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$
Show that $\det R(\vartheta) = 1$.

(7) <u>Complex number matrix</u>
Let $z = a + bi \in \mathbb{C}$ be a complex number with $a, b \in \mathbb{R}$, and define
$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
Show that $\det M(z) = |z|$, $\forall z \in \mathbb{C}$.

Co-factor expansion of determinants

Let $A \in M_n(\mathbb{R})$ be a square matrix. Let $a, b \in [n]$.
The minor matrix $M_{ab}(A)$ is defined as the $(n-1)\times(n-1)$ square matrix obtained from A by deleting: (a) The a^{th} row of A
(b) The b^{th} column of A.

The formal definition of $M_{ab}(A)$ is given by:

$$\boxed{\forall c,d \in [n-1] : (M_{ab}(A))_{cd} = \begin{cases} A_{cd} & \text{, if } c<a \wedge d<b \\ A_{c,d+1} & \text{, if } c<a \wedge d \geq b \\ A_{c+1,d} & \text{, if } c \geq a \wedge d<b \\ A_{c+1,d+1} & \text{, if } c \geq a \wedge d \geq b \end{cases}}$$

EXAMPLE

$$\text{Given } A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 5 & 7 & 2 \\ 3 & 1 & 5 & 2 \\ 1 & 4 & 7 & 3 \end{bmatrix} \Rightarrow$$

Note that $A_{23} = 7$

$$\Rightarrow M_{23}(A) = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

- We may use minor matrices to calculate determinants <u>recursively</u> as follows:

1) Row Expansion

$$\boxed{\forall a \in [n]: \det A = \sum_{b=1}^{n} (-1)^{a+b} A_{ab} \det(M_{ab}(A))}$$

2) Column expansion

$$\boxed{\forall b \in [n]: \det A = \sum_{a=1}^{n} (-1)^{a+b} A_{ab} \det(M_{ab}(A))}$$

EXAMPLE

$$\begin{vmatrix} 3 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} =$$

sign of $(-1)^{a+b} \leftrightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$= (-1)\cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + (+1)\cdot 5 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + (-1)\cdot 1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} =$$

$$= -(1\cdot 1 - 2\cdot 3) + 5(3\cdot 1 - 2\cdot 2) - (3\cdot 3 - 1\cdot 2) =$$

$$= -(1-6) + 5(3-4) - (9-2) =$$

$$= -(-5) + 5(-1) - 7 = 5 - 5 - 7 = -7.$$

▸ <u>Method</u>: Zero is your FRIEND.

<u>example</u>

$$\begin{vmatrix} 4 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & 3 & 1 \end{vmatrix} \rightarrow =$$

$$= 4 \cdot (-2) \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 4 \cdot (-2) \cdot (2 \cdot 1 - 3 \cdot 3)$$

$$= (-8)(2-9) = (-8)(-7) = 56$$

EXERCISE

(34) Evaluate the determinants

(a) $\begin{vmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & 9 & 2 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 1 & 5 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 3 & 7 \\ 2 & 0 & 2 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 3 & 2 & 7 & 5 \\ 5 & 0 & 7 & 0 & 0 \\ 2 & 0 & 2 & 3 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 8 & 0 & 1 & 4 & 2 \end{vmatrix}$

Simplification of determinants

The calculation of determinants can be simplified considerably by using the following properties:

1) If we transpose 2 rows or 2 columns, the determinant changes sign.

e.g. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

2) If 2 rows or 2 columns are identical, then the determinant is equal to 0.

e.g. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

3) If we multiply a row or column by $\lambda \in \mathbb{R}$, then the determinant itself is multiplied by λ.

e.g. $$\lambda \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & \lambda a_2 & a_3 \\ b_1 & \lambda b_2 & b_3 \\ c_1 & \lambda c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It follows that

- 1 We can pull out a common factor from any row or column.

e.g. $\begin{vmatrix} 3 & 2 & 7 \\ 5 & 4 & -1 \\ 8 & -8 & 12 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 & 7 \\ 5 & 2 & -1 \\ 8 & -4 & 12 \end{vmatrix} = 2 \cdot 4 \cdot \begin{vmatrix} 3 & 1 & 7 \\ 5 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

- 2 If all the elements of a row or column are 0, then the determinant is 0.

e.g. $\begin{vmatrix} 2 & 0 & 3 \\ 7 & 0 & 2 \\ 1 & 0 & 4 \end{vmatrix} = 0$

- 3 If $A \in M_{nn}(\mathbb{R}) \rightarrow \det(\lambda A) = \lambda^n \det(A), \forall \lambda \in \mathbb{R}$.

4) When every element of a row or column is written as a sum of two numbers, then the determinant can be rewritten as the sum of two determinants.

e.g. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$

5) If we add to the elements of a row (or column) the elements of another row (or column) multiplied by a common factor λ, then the value of the determinant does not change.

e.g. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+\lambda a_1 & b_2+\lambda a_2 & b_3+\lambda a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6) When the elements above OR bellow the diagonal are all 0, then the determinant is equal to the product of the diagonal elements.

e.g. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

So, to simplify a determinant

- 1 Check if common factors can be pulled out via (3)
- 2 Use (1), (5) to diagonalize the determinant (i.e. create ZEROES!) so you can then use (6)
- 3 If you run into identical rows or columns then use (2).

EXAMPLES

a) Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & -4 & -3 & 3 \\ 0 & 4 & 0 & 1 \\ 1 & 6 & 0 & 1 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & -4 & -3 & 3 \\ 0 & 4 & 0 & 1 \\ 1 & 6 & 0 & 1 \end{vmatrix} \begin{matrix} \cdot(-2) \quad \cdot(-1) \\ \\ \\ \\ \end{matrix} =$$

$$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 2+(-2)\cdot 1 & -4+(-2)\cdot 2 & -3+(-2)(-1) & 3+(-2)\cdot 2 \\ 0 & 4 & 0 & 1 \\ 1+(-1)\cdot 1 & 6+(-1)\cdot 2 & 0+(-1)(-1) & 1+(-1)2 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -8 & -1 & -1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 1 & -1 \end{vmatrix} = (+1)\cdot 1 \cdot \begin{vmatrix} -8 & -1 & -1 \\ 4 & 0 & 1 \\ 4 & 1 & -1 \end{vmatrix} =$$

$$= 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} \cdot 1 = 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1-2 & 1-1 & -1-1 \end{vmatrix} =$$

$$= 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{vmatrix} = 4 \cdot (-1)(-1)\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} =$$

$$= 4\left(1(-2) - 1(-1)\right) = 4(-2+1) = -4$$

↳ We use determinant properties to zero out a column or a row. Then we perform a co-factor expansion. This reduces the size of the determinant. We repeat, all the way down to 2x2 size.

8) Show that

$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} = 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

Solution

$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} (a+b)+(b+c)+(c+a) & b+c & c+a \\ (c+a)+(a+b)+(b+c) & a+b & b+c \\ (b+c)+(c+a)+(a+b) & c+a & a+b \end{vmatrix} =$$

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & c+a & a+b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 1 & a+b & b+c \\ 1 & c+a & a+b \end{vmatrix} (-1)$$

$$= 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 0 & (a+b)-(b+c) & (b+c)-(c+a) \\ 0 & (c+a)-(b+c) & (a+b)-(c+a) \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 0 & a-c & b-a \\ 0 & a-b & b-c \end{vmatrix} = 2(a+b+c)\cdot(+1)(1)\begin{vmatrix} a-c & b-a \\ a-b & b-c \end{vmatrix}$$

$$= 2(a+b+c)\left[(a-c)(b-c)-(b-a)(a-b)\right]$$

$$= 2(a+b+c)\left[(a-c)(b-c)+(a-b)^2\right] =$$
$$= 2(a+b+c)(\underline{ab}-ac-bc+c^2+a^2\underline{-2ab}+b^2)$$
$$= 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

EXERCISES

(9) Evaluate the following determinants:

a) $\begin{vmatrix} 3 & 5 & 8 \\ 3 & 6 & 9 \\ 3 & 7 & 4 \end{vmatrix}$ b) $\begin{vmatrix} x & x+1 & x+3 \\ y & y+1 & y+3 \\ 1 & 1 & 1 \end{vmatrix}$

c) $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$ d) $\begin{vmatrix} 1 & -2 & 0 & 3 \\ 1 & 1 & 2 & 1 \\ 3 & 1 & -1 & 4 \\ 5 & 1 & 2 & -1 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$ f) $\begin{vmatrix} -1 & 0 & 2 & 1 & -3 \\ 1 & 2 & 3 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 4 & 5 & 1 & -1 \\ 0 & 1 & 2 & 0 & -2 \end{vmatrix}$

(10) Solve the following equations

a) $\begin{vmatrix} x-3 & 4 & x \\ 3x-2 & -6 & 2x-1 \\ 4x-3 & 2 & x^2-3 \end{vmatrix} = 0$ b) $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & x \\ 1 & 2 & 1 & x^2 \\ 1 & 3 & 3 & x^3 \end{vmatrix} = 0$

(11) Show that

a) $$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

b) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$$

c) $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

d) $$\begin{vmatrix} x+y & z & z \\ y & z+x & y \\ x & x & z+y \end{vmatrix} = 4xyz$$

e) $$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$$

f) $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} = ab$$

g) $$\begin{vmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{vmatrix} = a^2+b^2+c^2+1$$

h) $$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = a(b-a)(c-b)(d-c)$$

i) $$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = 0$$

j) $$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0$$

k) $$\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ a^4 & 1 & a & a^2 & a^3 \\ a^3 & a^4 & 1 & a & a^2 \\ a^2 & a^3 & a^4 & 1 & a \\ a & a^2 & a^3 & a^4 & 1 \end{vmatrix} = (1-a^5)^4$$

l) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (ab+bc+ca) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

m) $\begin{vmatrix} a_1 & b_1 & a_1x^2+b_1x+c_1 \\ a_2 & b_2 & a_2x^2+b_2x+c_2 \\ a_3 & b_3 & a_3x^2+b_3x+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

n) $\begin{vmatrix} 1 & a & b & 1 \\ 1 & a & a & a \\ a & 1 & ab & b \\ a & a & ab & 1 \end{vmatrix} = (a-b)(a-1)(1-ab)$

o) $\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = 4(a^2+b^2)(c^2+d^2)$

p) $\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ b & b & a & b \\ a & a & a & b \end{vmatrix} = -(a-b)^4$

Matrix inverse, in general

Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have

$$ad-bc \neq 0 \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For larger matrices, we use the following theory:

Def: Let $A \in M_n(\mathbb{R})$ be a square matrix. We define the <u>adjugate matrix</u> $\operatorname{adj}(A)$ such that

$$\forall a,b \in [n]: [\operatorname{adj}(A)]_{ab} = (-1)^{a+b} \det(M_{ba}(A))$$

Thm: Let $A \in GL(n,\mathbb{R})$ be a non-singular square matrix. Then

$$A^{-1} = \left(\frac{1}{\det(A)}\right) \operatorname{adj}(A)$$

EXAMPLE

Find the matrix inverse of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

Solution

Since,

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} \begin{matrix} (-2) & (-1) \\ & \\ & \end{matrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1+(-2)\cdot 0 & -1+(-2)(-1) \\ 0 & 2+(-1)\cdot 0 & 5+(-1)(-1) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 6 \end{vmatrix} = (+1)\cdot 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} = 1\cdot 6 - 1\cdot 2 = 4$$

and

$$[\operatorname{adj}(A)]_{11} = (-1)^{1+1}\det(M_{11}(A)) = \begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = 1\cdot 5 - (-1)\cdot 2$$

$$= 5+2 = 7$$

$$[\operatorname{adj}(A)]_{12} = (-1)^{1+2}\det(M_{21}(A)) = -\begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix} =$$

$$= -(0\cdot 5 - (-1)\cdot 2) = -(0+2) = -2$$

$$[\operatorname{adj}(A)]_{13} = (-1)^{1+3}\det(M_{31}(A)) = \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} =$$

$$= 0(-1) - (-1)\cdot 1 = 1$$

$$[\mathrm{adj}(A)]_{21} = (-1)^{2+1} \det(M_{12}(A)) = -\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} =$$

$$= -(2\cdot 5 - (-1)1) = -(10+1) = -11$$

$$[\mathrm{adj}(A)]_{22} = (-1)^{2+2} \det(M_{22}(A)) = \begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix} =$$

$$= 1\cdot 5 - (-1)\cdot 1 = 5+1 = 6$$

$$[\mathrm{adj}(A)]_{23} = (-1)^{2+3} \det(M_{32}(A)) = -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} =$$

$$= -[1(-1) - (-1)2] = -(-1+2) = -1$$

$$[\mathrm{adj}(A)]_{31} = (-1)^{3+1} \det(M_{13}(A)) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2\cdot 2 - 1\cdot 1 = 3$$

$$[\mathrm{adj}(A)]_{32} = (-1)^{3+2} \det(M_{23}(A)) = -\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} =$$

$$= -(1\cdot 2 - 0\cdot 1) = -2$$

$$[\mathrm{adj}(A)]_{33} = (-1)^{3+3} \det(M_{33}(A)) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1\cdot 1 - 2\cdot 0 = 1$$

it follows that

$$\mathrm{adj}(A) = \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow A^{-1} = \frac{1}{\det A} \mathrm{adj}(A) = \frac{1}{4} \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

EXERCISES

(12) Find the inverse matrix A^{-1} for the following matrices

a) $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

(13) If $(2I-A)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, evaluate the matrix A.

(14) If $A^{-1}B^{-1} = \begin{bmatrix} 5 & 0 \\ 2 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ then evaluate the matrix B.

(15) If $A^{-1}B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$, then evaluate the matrix BA.

(16) Solve for the matrix $X \in M_3(\mathbb{R})$:

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 5 \\ 5 & 0 & 8 \end{bmatrix} X = 7 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

▼ $n \times n$ linear system of equations

- An $n \times n$ linear system of equations is a system of the form:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n = b_n \end{cases} \qquad (1)$$

- This system can be rewritten as a matrix equation:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or equivalently as

$$Ax = b \qquad (2)$$

with $A \in M_n(\mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$.

- We say that

Eq.(2) is homogeneous $\Leftrightarrow$ $b = \mathbf{0}$

Eq.(2) is inhomogeneous $\Leftrightarrow$ $b \neq \mathbf{0}$

- Solution techniques for solving linear systems include:

a) Matrix inverse method
b) Cramer's rule
c) Gaussian elimination

Matrix inverse method

We have already explained that if $\det A \neq 0$, then any linear system can be solved via the property

$$\forall A \in GL(n, \mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b)$$

However, due to the difficulty of calculating A^{-1}, this method is recommended only for 2×2 systems, as was explained earlier.

Cramer's rule

We write

$$A = [A_1 \ A_2 \ A_3 \cdots A_n]$$

with $A_1, A_2, A_3, \ldots, A_n \in M_{n1}(\mathbb{R})$ the columns of the matrix A, and define the determinants

$$D = \det A = \det([A_1 \ A_2 \ A_3 \cdots A_n])$$
$$D_1 = \det([b \ A_2 \ A_3 \cdots A_n])$$
$$D_2 = \det([A_1 \ b \ A_3 \cdots A_n])$$

$$D_3 = \det([A_1 \ A_2 \ b \cdots A_n])$$

$$\vdots$$

$$D_n = \det([A_1 \ A_2 \ A_3 \cdots \ b])$$

Note that for any $k \in [n]$, in the determinant D_k, we replace the k^{th} column of A with the column matrix b.

- Cramer's method for solving linear systems is based on the following theorem:

<u>Thm</u>: Given the linear system $Ax=b$ with $A \in M_n(\mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$.

a) If $D \neq 0$, then $Ax=b$ has a unique solution given by $\forall k \in [n]: x_k = D_k / D$.

b) $\begin{cases} D = 0 \\ \exists k \in [n]: D_k \neq 0 \end{cases} \Rightarrow Ax=b$ has no solutions

<u>Remark</u>: Note that the theorem is inconclusive when
$$\begin{cases} D = 0 \\ \forall k \in [n]: D_k = 0 \end{cases}$$
Then, the system needs to be investigated via Gaussian Elimination method.

a) $\begin{cases} \lambda x + (\lambda-2)y = \lambda+1 & (1) \\ (\lambda+1)x - (\lambda-2)y = \lambda \end{cases}$

Solution

$$D = \begin{vmatrix} \lambda & \lambda-2 \\ \lambda+1 & -(\lambda-2) \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda & 1 \\ \lambda+1 & -1 \end{vmatrix} = (\lambda-2)[\lambda(-1) - (\lambda+1)\cdot 1]$$

$$= (\lambda-2)(-\lambda-\lambda-1) = -(\lambda-2)(2\lambda+1)$$

$$D_x = \begin{vmatrix} \lambda+1 & \lambda-2 \\ \lambda & -(\lambda-2) \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda+1 & 1 \\ \lambda & -1 \end{vmatrix} = (\lambda-2)[(\lambda+1)(-1) - 1\cdot\lambda].$$

$$= (\lambda-2)(-\lambda-1-\lambda) = -(\lambda-2)(2\lambda+1)$$

$$D_y = \begin{vmatrix} \lambda & \lambda+1 \\ \lambda+1 & \lambda \end{vmatrix} = \lambda^2 - (\lambda+1)^2 = (\lambda-(\lambda+1))(\lambda+(\lambda+1)) =$$

$$= (\lambda-\lambda-1)(\lambda+\lambda+1) = -(2\lambda+1)$$

Note that $D = 0 \Leftrightarrow -(\lambda-2)(2\lambda+1) = 0 \Leftrightarrow$

$\Leftrightarrow \lambda - 2 = 0 \vee 2\lambda+1 = 0 \Leftrightarrow \lambda = 2 \vee \lambda = -1/2 \Leftrightarrow$

$\Leftrightarrow \lambda \in \{2, -1/2\}$.

Case 1: If $\lambda \in \mathbb{R} - \{-1/2, 2\}$, then the system has a unique solution given by:

$$x = \frac{D_x}{D} = \frac{-(\lambda-2)(2\lambda+1)}{-(\lambda-2)(2\lambda+1)} = 1$$

$$y = \frac{D_y}{D} = \frac{-(2\lambda+1)}{-(\lambda-2)(2\lambda+1)} = \frac{1}{\lambda-2}$$

Case 2: If $\lambda = 2$, then

$D = 0 \wedge D_x = 0 \wedge D_y = -5 \neq 0 \Rightarrow$

$\Rightarrow$ the system is inconsistent (i.e. no solutions).

Case 3 : If $\lambda = -1/2$, then

$$D = 0 \wedge D_x = 0 \wedge D_y = 0$$

so we have to solve the system explicitly:

$$(1) \Leftrightarrow \begin{cases} (-1/2)x + (-1/2-2)y = -1/2+1 \\ (-1/2+1)x - (-1/2-2)y = -1/2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -x + (-1-4)y = -1+2 \\ (-1+2)x - (-1-4)y = -1 \end{cases} \Leftrightarrow \begin{cases} -x - 5y = 1 \\ x + 5y = -1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x + 5y = 1 \Leftrightarrow x = 1 - 5y \Leftrightarrow (x,y) = (1-5y, y)$$

$$\Leftrightarrow (x,y) \in \{(1-5y, y) \mid y \in \mathbb{R}\}.$$

To summarize, the solution set is:

$$S = \begin{cases} \{(1, 1/(\lambda-2))\} & , \text{ if } \lambda \in \mathbb{R} - \{-1/2, 2\} \\ \emptyset & , \text{ if } \lambda = 2 \\ \{(1-5y, y) \mid y \in \mathbb{R}\} & , \text{ if } \lambda = -1/2 \end{cases}$$

EXERCISES

(17) Use Cramer's rule to solve the following linear systems:

a) $\begin{cases} x-y=0 \\ 3x+2y=5 \end{cases}$
$(1,1)$

b) $\begin{cases} x+2y-z=0 \\ 2x-y+3z=0 \\ x+y+z=2 \end{cases}$
$(-2,2,2)$

c) $\begin{cases} x-y+z=3 \\ 2x+y-3z=10 \\ x+5y-9z=8 \end{cases}$
(no solutions)

d) $\begin{cases} 2x-y-z-w=-1 \\ x-2y+z+w=-2 \\ x+y-2z+w=4 \\ x+y+z-2w=-8 \end{cases}$
$(-2,-1,-3,1)$

e) $\begin{cases} x+y+z+w=2 \\ 2x-w+3z=9 \\ -x+2y-z+2w=-5 \\ 3x+y-w=4 \end{cases}$
$(x,y,z,w)=(1,0,2,-1)$

(18) Solve the following linear systems in terms of the parameter $a \in \mathbb{R}$.

a) $\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases}$

b) $\begin{cases} x-ay+z=a \\ x+y+z=-1 \\ ax+y-a^2z=1. \end{cases}$

c) $\begin{cases} x+y+z=1 \\ ax+by+cz=d \\ a^2x+b^2y+c^2z=d^2 \end{cases}$

d) $\begin{cases} x+y+z=1+c \\ x+(1+a)y+z=1 \\ x+y+(1+b)z=1 \end{cases}$

Method of Gaussian elimination

- We represent the linear system $Ax=b$ in terms of an <u>augmented matrix</u> M:

$$\left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1n} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} & b_m \end{array}\right] \leftarrow \text{each row represents an equation}$$

- We say that two augmented matrices M_1 and M_2 are equivalent (notation: $M_1 \sim M_2$) if and only if the corresponding linear systems have the same solution set.

Properties

- The following transformations map an augmented matrix M_1 to an equivalent augmented matrix M_2.

1) <u>Transposition</u>: We can swap any two rows (but not columns).

e.g. $$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \end{array}\right]$$

2) <u>Scalar Multiplication</u>: We can multiply any row (but not a column) with a non-zero scalar $\lambda \in \mathbb{R} - \{0\}$.

e.g. $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \cdot\lambda \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ \lambda a_2 & \lambda b_2 & \lambda c_2 & \lambda d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

3) <u>Linear combination</u>: We can add to any row (but not a column) any other row multiplied by a non-zero scalar $\lambda \in \mathbb{R}-\{0\}$

e.g. $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \begin{array}{l} \cdot\lambda \\ \leftarrow \\ \end{array} \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2+\lambda a_1 & b_2+\lambda b_1 & c_2+\lambda c_1 & d_2+\lambda d_1 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

- Note that these properties are somewhat different from the corresponding properties of determinants.

▶ <u>Method</u>

- Using these properties, we try to diagonalize the augmented matrix defering fractional arithmetic as much as possible to the very last step. We work on the augmented matrix one column at a time.
- If during this process we get a row of the form

$$0 \quad 0 \quad 0 \; \ldots \; 0 \,|\, a$$

then: a) If $a \neq 0$, the system is inconsistent and we stop work.

b) If $a=0$, then the row corresponds to an identity and may then be deleted from the augmented matrix.

EXAMPLES

a) $\begin{cases} 2x-y=1 \\ x+y=3 \\ 3x+y=0 \end{cases}$ ⟵● An <u>overdetermined</u> system: more equations than unknowns

<u>Solution</u>

$$M = \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 3 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{array}\right] \begin{array}{c} (-2) \quad (-3) \\ \\ \\ \end{array} \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1+(-2)\cdot 1 & 1+(-2)3 \\ 0 & 1+(-3)\cdot 1 & 0+(-3)3 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -5 \\ 0 & -2 & -9 \end{array}\right] \begin{array}{l} \\ \cdot 2 \\ \cdot (-3) \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & 6 & 27 \end{array}\right] \begin{array}{l} \\ \cdot 1 \\ \\ \end{array} \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & 0 & 27-10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & 0 & 17 \end{array}\right]$$

therefore the system has no solutions.

b) $\begin{cases} x+z+4w+2v=3 \\ y+2w-v=-1 \\ -x+3y+2z=-2 \end{cases}$ ⟵● An <u>underdetermined</u> system: less equations than unknowns.

Solution

Since,

$$\begin{cases} x+z+4w+2v=3 \\ y+2w-v=-1 \\ -x+3y+2z=-2 \end{cases} \iff \begin{cases} 1x+0y+1z+4w+2v=3 \\ 0x+1y+0z+2w-v=-1 \\ -1x+3y+2z+0w+0v=-2 \end{cases} \qquad (1)$$

the corresponding augmented matrix is:

$$M=\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ -1 & 3 & 2 & 0 & 0 & -2 \end{array}\right] \cdot 1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 3+0 & 2+1 & 0+4 & 0+2 & -2+3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 3 & 3 & 4 & 2 & 1 \end{array}\right] (-3)$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3+(-3)\cdot 0 & 4+(-3)2 & 2+(-3)(-1) & 1+(-1)(-3) \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right] \cdot 3 \quad \sim$$

$$\sim \left[\begin{array}{ccccc|c} 3 & 0 & 3 & 12 & 6 & 9 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right] (-1) \quad \sim$$

$$\sim \left[\begin{array}{ccccc|c} 3 & 0 & 0 & 12+(-1)(-2) & 6+(-1)5 & 9+(-1)4 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 3 & 0 & 0 & 14 & 1 & 5 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right]$$

and it follows that

$$\text{Eq.}(1) \Leftrightarrow \begin{cases} 3x + 14w + v = 5 \\ y + 2w - v = -1 \\ 3z - 2w + 5v = 4 \end{cases} \Leftrightarrow \begin{cases} 3x = 5 - 14w - v \\ y = -1 - 2w + v \\ 3z = 4 + 2w - 5v \end{cases}$$

$$\Leftrightarrow \begin{cases} x = (5/3) - (14/3)w - (1/3)v \\ y = -1 - 2w + v \\ z = (4/3) + (2/3)w - (5/3)v \end{cases}$$

$$\Leftrightarrow (x,y,z) = ((5/3) - (14/3)w - (1/3)v,$$
$$-1 - 2w + v, (4/3) + (2/3)w - (5/3)v)$$
$$= (5/3, -1, 4/3) + w(-14/3, -2, 2/3) +$$
$$+ v(-1/3, 1, -5/3)$$

$$\Leftrightarrow (x,y,z,w,v) = (5/3, -1, 4/3, 0, 0) + w(-14/3, -2, 2/3, 1, 0)$$
$$+ v(-1/3, 1, -5/3, 0, 1)$$

We conclude that the solution set is given by

$$S = \{a + wb + vc \mid w, v \in \mathbb{R}\}$$

with $a, b, c \in \mathbb{R}^5$ given by

$$a = (5/3, -1, 4/3, 0, 0)$$
$$b = (-14/3, -2, 2/3, 1, 0)$$
$$c = (-1/3, 1, -5/3, 0, 1)$$

EXERCISES

(19) Solve the following systems using Gaussian Elimination

a) $\begin{cases} x-2y=-4 \\ 3x+y=9 \\ x+5y=17 \end{cases}$ $(2,3)$

b) $\begin{cases} x+z=4 \\ 2x-y+3z=9 \\ 2y-z=1 \\ 3x+y-2z=-1 \end{cases}$ $(1,2,3)$

c) $\begin{cases} x-y-2z=6 \\ 3x-3y-6z=1 \end{cases}$ (inconsistent)

d) $\begin{cases} x+y+w=4 \\ y+w+z=-2 \\ x+w+z=1 \end{cases}$ $(z+6,\ z+3,\ -2z-5,\ z)$

e) $\begin{cases} x+2y+4z=0 \\ y-2z=0 \\ x+8z=0 \end{cases}$ $(-8z,\ 2z,\ z)$