BRIEF INTRODUCTION TO PROOF

▼ Negation and contrapositive of statements

- Let P, Q be compound statements. We say that $P \equiv Q$ (P and Q are equivalent) if and only if the compound statement $P \Leftrightarrow Q$ is always true, regardless of the truth value of the constituent statemends that compose P and Q.
- The following equivalences can be used to negate compound statements:

$$\boxed{\begin{array}{l|l} \overline{p \wedge q} \equiv \bar{p} \vee \bar{q} & \overline{p \veebar q} \equiv p \Leftrightarrow q \\ \overline{p \vee q} \equiv \bar{p} \wedge \bar{q} & \overline{p \Leftrightarrow q} \equiv p \veebar q \\ \overline{p \Rightarrow q} \equiv p \wedge \bar{q} & \end{array}}$$

- Quantified statements can be negated by the following rules

$$\boxed{\begin{array}{l} \overline{\forall x \in A : p(x)} \equiv \exists x \in A : \overline{p(x)} \\ \overline{\exists x \in A : p(x)} \equiv \forall x \in A : \overline{p(x)} \end{array}}$$

- Every statement of the form $P \Rightarrow Q$ is equivalent to the <u>contrapositive statement</u> $\bar{Q} \Rightarrow \bar{P}$. Consequently any proof of $P \Rightarrow Q$ also proves $\bar{Q} \Rightarrow \bar{P}$. The <u>converse statement</u> $Q \Rightarrow P$ is NOT equivalent to $P \Rightarrow Q$ and requires separate proof.

- We note that since

$$(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

the contrapositive statement of $P \Leftrightarrow Q$ is $\overline{P} \Leftrightarrow \overline{Q}$.

EXAMPLES

a) Write the negation of the definition of the limit from calculus

$$\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)$$

Solution

$$\lim_{x \to x_0} f(x) \neq \ell \Leftrightarrow$$

$$\Leftrightarrow \exists \varepsilon > 0 : \overline{\exists \delta > 0 : \forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \overline{\forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : \overline{(0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : (0 < |x - x_0| < \delta \wedge \overline{|f(x) - \ell| < \varepsilon})$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : (0 < |x - x_0| < \delta \wedge |f(x) - \ell| \geq \varepsilon)$$

b) The contrapositive to the statement
$\forall a,b \in \mathbb{R}: (ab = 0 \Rightarrow a = 0 \vee b = 0)$
is given by:

$$\forall a,b \in \mathbb{R}: (\overline{a=0 \vee b=0} \Rightarrow \overline{ab=0}) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b \in \mathbb{R}: (\overline{a=0} \wedge \overline{b=0} \Rightarrow ab \neq 0) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b \in \mathbb{R}: (a \neq 0 \wedge b \neq 0 \Rightarrow ab \neq 0).$$

c) The contrapositive to the statement
$\forall a,b \in \mathbb{R}: (a^2+b^2 = 0 \Rightarrow a = 0 \wedge b = 0)$
is given by:

$$\forall a,b \in \mathbb{R}: (\overline{a=0 \wedge b=0} \Rightarrow \overline{a^2+b^2=0}) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b \in \mathbb{R}: (\overline{a=0} \vee \overline{b=0} \Rightarrow a^2+b^2 \neq 0)$$
$$\Leftrightarrow \forall a,b \in \mathbb{R}: (a \neq 0 \vee b \neq 0 \Rightarrow a^2+b^2 \neq 0).$$

EXERCISES

(1) Write the negation of all the statements from Exercises 2 and 3 [Brief Introduction to Logic and Sets] both in terms of quantified statement notation and in English.

(2) Write the non-belonging condition $x \notin A$ for the sets given in Exercise 4 [Brief Introduction to Logic and Sets] both in terms of quantified statement notation and in English.

(3) Write the contrapositive of the following statements, both in terms of quantified statement notation and in English.

a) $\forall a \in \mathbb{R}: a \geq 3 \Rightarrow a > 5$

b) $\forall a, b \in \mathbb{R}: |a| + |b| = 0 \Rightarrow (a = 0 \wedge b = 0)$

c) $\forall a, b \in \mathbb{R}: a^2 = b^2 \Leftrightarrow (a = b \vee a = -b)$

d) $\forall a, b, c, d \in \mathbb{R}: (a < b \wedge c < d) \Rightarrow a + c < b + d$

e) $\forall a, b, c \in \mathbb{R}: (a > 0 \wedge b > c > 0) \Rightarrow ab > ac$

(Hint: $b > c > 0$ is equivalent to $b > c \wedge c > 0$)

f) $\forall a, b, c \in \mathbb{R}: a^3 + b^3 + c^3 = 3abc \Rightarrow (a + b + c = 0 \vee a = b = c)$

(Hint: $a = b = c$ is equivalent to $a = b \wedge b = c$)

Methodology for writing proofs

Proving implications

(1) To prove $\boxed{p \Rightarrow q}$

- **Direct Method**

  Assume p is true.

  [Prove q]

- **Contrapositive Method**

  We will show that $\bar{q} \Rightarrow \bar{p}$

  Assume $\bar{q}$ is true.

  [Prove $\bar{p}$]

  It follows that $p \Rightarrow q$

- **Contradiction Method**

  Assume p is true.

  To derive a contradiction, assume $\bar{q}$.

  [Prove r, using $p \wedge \bar{q}$]

  [Prove $\bar{r}$] ← Contradiction.

  It follows that q is true.

(2) To prove $\boxed{p \Leftrightarrow q}$

($\Rightarrow$): Assume p is true

[Prove q]

($\Leftarrow$): Assume q is true

[Prove p]

Proofs involving sets

Let A, B be two sets.

1. To prove $\boxed{A \subseteq B}$

[We prove $x \in A \Rightarrow x \in B$]

2. To prove $\boxed{A = B}$

[We prove $x \in A \Rightarrow x \in B$]
It follows that $A \subseteq B$ (1)
[We prove $x \in B \Rightarrow x \in A$]
It follows that $B \subseteq A$ (2)
From (1) and (2): $A = B$.

- For proofs involving sets, we recall that

$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$

$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$

$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$

$x \in \{x \in A \mid p(x)\} \Leftrightarrow x \in A \wedge p(x)$

$x \in \{\varphi(x) \mid x \in A \wedge p(x)\} \Leftrightarrow \exists y \in A : (\varphi(y) = x \wedge p(y))$

Proofs involving identities

Let a, b be two expressions.
To prove $a = b$.

- Direct Method

$$a = \cdots = \cdots = \\ = \cdots = \cdots = b$$

- Indirect Method

$$a = \cdots = \cdots = c \quad (1)$$
$$b = \cdots = \cdots = c \quad (2)$$

From (1) and (2): $a = b$.

Proofs involving quantified statements

1. To prove $\boxed{\forall x \in A : p(x)}$

Let $x \in A$ be given.
[Prove $p(x)$]
It follows that $\forall x \in A : p(x)$.

2. To prove $\boxed{\exists x \in A : p(x)}$

- 1st method

[Define an $x \in A$]
[Prove that $p(x)$ is true]
It follows that $\exists x \in A : p(x)$

(Note that x can be indirectly defined by deducing a statement of the form $\exists x \in B : r(x)$ via a theorem or by constructing it from other variables that have been indirectly defined via existential statements)

▶ <u>2nd method</u>

$p(x) \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftarrow x \in S$

Choose an $x \in S$. Show that $x \in A \wedge p(x)$.

It follows that $\exists x \in A : p(x)$.