BRIEF INTRODUCTION TO LOGIC AND SETS

▼ Basic concepts

The basic concepts we wish to introduce informally are
a) Propositions
b) Sets
c) Predicates - Quantified statements.

Propositions

- A proposition p is any statement which is true or false.
- Given two propositions p, q we define the following composite propositions.

1) Conjunction $p \wedge q$: "p is true and q is true"
   - ▶ True if both p and q are true, otherwise false.
2) Disjunction: $p \vee q$: "p is true or q is true (or both)"
   - ▶ True if at least one of the two statements p or q is true, otherwise false.
3) Negation $\overline{p}$: "p is not true"
   - ▶ True if p is false. False if p is true.
4) Exclusive Disjunction $p \veebar q$: "either p or q is true (not both)"
   - ▶ True if either p or q but not both is true. Otherwise false.

5) Implication $p \Rightarrow q$: "If p is true then q is true"
True if the truth of p implies the truth of q. Note that if p is false, then we presume that $p \Rightarrow q$ is true regardless of whether q is true or false. If p is true and q is false then $p \Rightarrow q$ is false.

6) Equivalence $p \Leftrightarrow q$: "p is true if and only if q is true"
True if p and q always have the same truth value. False if p and q have opposite truth values.

Sets

- A <u>set</u> A is an <u>unordered</u> collection of <u>elements</u>. An element can be a number, a derived object (i.e. vectors, matrices, etc.) or another set.
- A set with a finite number of elements can be defined by listing the elements.
  e.g.: $A = \{2, 3, 6, 9, 12\}$.
- <u>Notation</u>: Let A, B be sets and let x be an element.
  1) $x \in A$: x belongs to A
     x is an element of A
  2) $x \notin A$: x does not belong to A
     x is not an element of A
  3) $A = B$: A and B have the same elements.
  4) $A \subseteq B$: All the elements of A belong to B

- We note that: $A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$
- <u>Special sets</u>

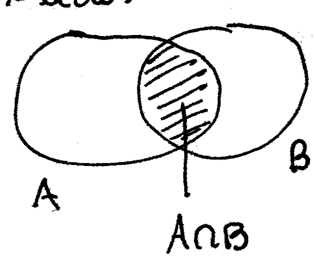
1) $\emptyset = \{\}$. The empty set.
   The empty set is the set that has no elements.
2) $\mathbb{C}$ = the set of all complex numbers
3) $\mathbb{R}$ = the set of all real numbers.
4) $\mathbb{Q}$ = the set of all rational numbers.
5) $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ = the set of all integers.
6) $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ = the set of all natural numbers.
7) For $n \in \mathbb{N}$: $[n] = \{1, 2, 3, \ldots, n\}$.

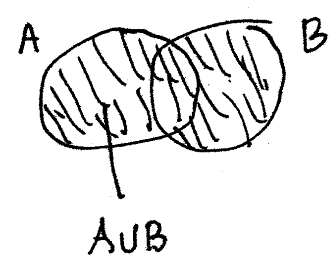
- We note that: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
- <u>Set operations</u>

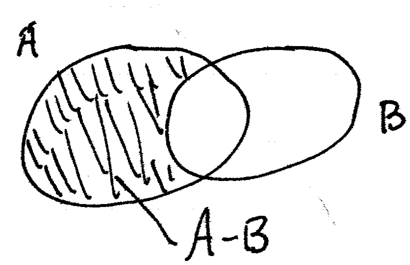
Let A, B be two sets. We define the following set operations:

1) Intersection: $A \cap B$
   $x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$
2) Union: $A \cup B$
   $x \in A \cup B \Leftrightarrow x \in A \vee x \in B$
3) Difference: $A - B$
   $x \in A - B \Leftrightarrow x \in A \wedge x \notin B$

We represent these operations with Venn Diagrams as follows:







- <u>Predicates and quantified statements</u>
- A predicate $p(x)$ is a statement about x which is true or false depending on the value of x.
- Note that x can also be an ordered collection of elements $x = (x_1, x_2, \dots, x_n)$. Then we write $p(x)$ as $p(x_1, x_2, \dots, x_n)$.
- Given a predicate $p(x)$ and a set A, we define the following quantified statements:

1) $\forall x \in A : p(x)$

   <u>For all</u> $x \in A$, $p(x)$ is satisfied.

2) $\exists x \in A : p(x)$

   <u>There is at least one</u> $x \in A$ such that $p(x)$ is satisfied.

3) $\exists! x \in A : p(x)$

   <u>There is a unique</u> $x \in A$ such that $p(x)$ is satisfied.

- If A is a finite set, then the above quantified statements are abbreviations for conjunction, disjunction, and exclusive disjunction: For example:

$$(\forall x \in \{a, b, c\} : p(x)) \Leftrightarrow (p(a) \wedge p(b) \wedge p(c))$$
$$(\exists x \in \{a, b, c\} : p(x)) \Leftrightarrow (p(a) \vee p(b) \vee p(c))$$

- Quantifiers can be nested to give compound quantified statements. For example:

1) $\forall x \in A : \exists y \in B : p(x, y)$

   For all $x \in A$, there is a $y \in B$, such that $p(x, y)$ is satisfied.

2) $\exists x \in A : \forall y \in B : p(x,y)$

There is an $x \in A$ such that for all $y \in B$, $p(x,y)$ is satisfied.

- Important quantified statements from algebra

$\forall a,b \in \mathbb{R} : (ab = 0 \Leftrightarrow a = 0 \lor b = 0)$

$\forall a,b \in \mathbb{R} : (a^2 + b^2 = 0 \Leftrightarrow a = 0 \land b = 0)$

$\forall a,b \in \mathbb{R} : (|a| + |b| = 0 \Leftrightarrow a = 0 \land b = 0)$

- Definitions of sets

There are 3 methods for defining sets:

1) By listing: For finite sets we can simply list the elements.

e.g.: $A = \{3, 7, 10, 12\}$

2) By predicate: $A = \{x \in U \mid p(x)\}$

with U a predefined set and $p(x)$ a predicate.

Belonging condition: $x \in A \Leftrightarrow (x \in U \land p(x))$

e.g.: We can use definition by predicate to define intervals:

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[n] = \{x \in \mathbb{N} \mid 1 \leq x \leq n\} = \{1, 2, \ldots, n\}$

3) By mapping: $A = \{\varphi(x) \mid x \in U \land p(x)\}$

with $\varphi(x)$ some expression of x, U a predefined set, and $p(x)$ a predicate.

Belonging condition: $y \in A \Leftrightarrow \exists x \in U : (\varphi(x) = y \land p(x))$

EXAMPLES

a) The set of complex numbers:

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$.

$z \in \mathbb{C} \Leftrightarrow \exists a, b \in \mathbb{R} : z = a + bi$

b) The set of rational numbers:

$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \wedge b \in \mathbb{N} - \{0\}\}$

$x \in \mathbb{Q} \Leftrightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N} - \{0\} : x = a/b$.

c) The set of even integers

$A = \{2k \mid k \in \mathbb{Z}\}$

$x \in A \Leftrightarrow \exists k \in \mathbb{Z} : x = 2k$

d) The set of odd integers

$A = \{2k+1 \mid k \in \mathbb{Z}\}$

$x \in A \Leftrightarrow \exists k \in \mathbb{Z} : x = 2k + 1$.

e) $A = \{a^2 + b^2 \mid a, b \in \mathbb{R} \wedge a + 3b < 1\}$

$x \in A \Leftrightarrow \exists a, b \in \mathbb{R} : (x = a^2 + b^2 \wedge a + 3b < 1)$

- <u>Cartesion product</u>

We use definition by mapping to define the cartesian product between sets.

- An <u>ordered pair</u> (a, b) is an <u>ordered</u> collection of two elements a and b. We call a and b the <u>components</u> of (a, b).
- We note that: $(a, b) = (c, d) \Leftrightarrow (a = c \wedge b = d)$.

- Let A, B be two sets. We define the Cartesian product
$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$.
We also define:
$A^2 = A \times A = \{(a,b) \mid a \in A \wedge b \in A\}$

EXAMPLE

For $A = \{1,2,3\}$ and $B = \{5,6\}$. Calculate $A \times B$, A^2, B^2.

Solution

$A \times B = \{1,2,3\} \times \{5,6\} =$
$= \{(1,5),(1,6),(2,5),(2,6),(3,5),(3,6)\}$

$A^2 = A \times A = \{1,2,3\} \times \{1,2,3\} =$
$= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

$B^2 = B \times B = \{5,6\} \times \{5,6\} =$
$= \{(5,5),(5,6),(6,5),(6,6)\}$

$\longmapsto$ The above can be generalized as follows

- An ordered n-tuplet $(x_1, x_2, \dots, x_n)$ is an ordered collection of n elements $x_1, x_2, \dots, x_n$.
- Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.
We note that:
$x = y \iff \forall a \in [n] : x_a = y_a$
- Let $A_1, A_2, \dots, A_n$ be n sets. We define:
$A_1 \times A_2 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n) \mid \forall a \in [n] : x_a \in A_a\}$
- Special case:
$A_1 \times A_2 \times A_3 = \{(x_1, x_2, x_3) \mid x_1 \in A_1 \wedge x_2 \in A_2 \wedge x_3 \in A_3\}$.

EXERCISES

① Let $A = [7]$, $B = \{x \in A \mid x > 4\}$, and $C = \{x-1 \mid x \in B\}$.
List the elements of

a) B b) C c) $B \cap C$ d) $B \cup C$

e) $A - B$ f) $B - C$ g) $C - B$

② Write out the following statements in English

a) $\forall a \in A : \exists b \in B : (a,b) \in f$

b) $\exists a \in A : \forall b \in B : a+b > 3$

c) $\forall a \in A : \exists b \in B : (ab > 2 \wedge a+b > 1)$

d) $\forall a, b \in A : \exists c \in B : \forall d \in A : ab + bd < 3$

e) $\exists a \in A : \forall b \in B : (ab > 3 \Rightarrow b > 2)$

f) $\forall a \in A : \exists b \in B : (3a > b \veebar a+b < 0)$

③ Write the following statements symbolically using quantifiers.

a) Every real number is equal to itself.

b) There is a real number x such that $3x - 1 = 2(x+3)$

c) For every real number x, there is a natural number n such that $n > x$.

d) For every real number x, there is a complex number y such that $y^2 = x$.

e) There is a real number x such that for all real numbers y we have $x + y = 0$.

f) For all $\varepsilon > 0$, there is a $\delta > 0$ such that for all real numbers x, if $x_0 - \delta < x < x_0 + \delta$ then $|f(x) - a| < \varepsilon$.

g) There is a real number b such that for all natural numbers n we have $a_n < b$.

h) For all $\varepsilon > 0$, there is a natural number n_0 such that for any two natural numbers n_1 and n_2, if $n_1 > n_0$ and $n_2 > n_0$, then $|a_{n_1} - a_{n_2}| < \varepsilon$.

i) For any $M > 0$, there is a natural number n_0, such that for any other natural number n, if $n > n_0$ then $a_n > M$.

④ Write the belonging condition $x \in A$ for the following sets, using quantifiers.

a) $A = \{x^2 + 1 \mid x \in \mathbb{Q} \wedge 2x < 1\}$

b) $A = \{3x + 1 \mid x \in \mathbb{Z} \wedge x$ is a prime number$\}$

c) $A = \{x \in \mathbb{R} \mid x^2 + 3x > 0\}$

d) $A = \{a^3 + b^3 + c^3 \mid a, b \in \mathbb{R} \wedge c \in \mathbb{Q} \wedge a + b + c = 0\}$

e) $A = \{x \in \mathbb{R} \mid x^2 + 2x < 0 \vee 3x + 1 > -4 + x\}$

f) $A = \{a^2 - b^2 \mid a \in \mathbb{N} \wedge b \in \mathbb{R} \wedge a + b > 5\}$

g) $A = \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z} : x = 3k\}$

h) $A = \{ab \mid a, b \in \mathbb{R} \wedge (a + b > 2 \vee a - b < -3)\}$

i) $A = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : y^2 + y = x\}$

j) $A = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R} : x < y^2 + 1\}$

k) $A = \{a + b \mid a, b \in \mathbb{R} \wedge (ab > 1 \Rightarrow a^2 + b^2 > 2)\}$

l) $A = \{abc \mid a, b, c \in \mathbb{R} \wedge (a + b > 2 \,\underline{\vee}\, a - c < 3)\}$

m) $A = \{2a + 3b \mid a, b \in \mathbb{R} \wedge ab > 1 \wedge a - b < 0\}$

5 List the elements for the following cartesian products

a) $A \times B$ with $A = \{2,3,4\}$ and $B = \{7,8\}$

b) $A \times B$ with $A = \{1\}$ and $B = \{3,9\}$

c) $A \times B$ with $A = \{3\}$ and $B = \{5\}$

d) $[2] \times [3]$

e) $A \times B$ with $A = [5] - [2]$ and $B = [2] \cap [4]$

f) $A \times B \times C$ with $A = [3] - \{1\}$, $B = [3] \cap [6]$, and $C = [2]$.

g) $A \times B \times C$ with $A = \{2\}$, $B = [2]$, $C = [4] - [2]$.