BRIEF INTRODUCTION TO LOGIC AND SETS

Basic concepts

The basic concepts we wish to introduce informally are
a) Propositions
b) Sets
c) Predicates - Quantified statements.

Propositions

- A proposition \( p \) is any statement which is true or false.
- Given two propositions \( p, q \) we define the following composite propositions.
  1) Conjunction \( p \land q : \text{"p is true and q is true"} \)
     - True if both \( p \) and \( q \) are true, otherwise false.
  2) Disjunction: \( p \lor q : \text{"p is true or q is true (or both)"} \)
     - True if at least one of the two statements \( p \) or \( q \) is true, otherwise false.
  3) Negation \( \overline{p} : \text{"p is not true"} \)
     - True if \( p \) is false. False if \( p \) is true.
  4) Exclusive Disjunction \( p \lor q : \text{"either p or q is true (not both)"} \)
     - True if either \( p \) or \( q \) but not both is true.
     - Otherwise false.
5) Implication $p \Rightarrow q$: "If $p$ is true then $q$ is true"
   True if the truth of $p$ implies the truth of $q$. Note that if $p$ is false, then we presume that $p \Rightarrow q$ is true regardless of whether $q$ is true or false. If $p$ is true and $q$ is false then $p \Rightarrow q$ is false.

6) Equivalence $p \iff q$: "$p$ is true if and only if $q$ is true"
   True if $p$ and $q$ always have the same truth value.
   False if $p$ and $q$ have opposite truth values.

$\Rightarrow$ Sets

• A set $A$ is an unordered collection of elements. An element can be a number, a derived object (i.e. vectors, matrices, etc.) or another set.

• A set with a finite number of elements can be defined by listing the elements.
  e.g.: $A = \{2, 3, 6, 9, 123\}$.

• Notation: Let $A, B$ be sets and let $x$ be an element.
  1) $x \in A$: $x$ belongs to $A$
  2) $x \notin A$: $x$ does not belong to $A$
  3) $A = B$: $A$ and $B$ have the same elements.
  4) $A \subseteq B$: All the elements of $A$ belong to $B$
- We note that: \( A = B \iff (A \subseteq B \land B \subseteq A) \)

- **Special sets**
  1) \( \emptyset = \{ \} \). The empty set.

  The empty set is the set that has no elements.

  2) \( \mathbb{C} \) = the set of all complex numbers.

  3) \( \mathbb{R} \) = the set of all real numbers.

  4) \( \mathbb{Q} \) = the set of all rational numbers.

  5) \( \mathbb{Z} = \{ 0, 1, -1, 2, -2, \ldots \} \) = the set of all integers.

  6) \( \mathbb{N} = \{ 0, 1, 2, 3, \ldots \} \) = the set of all natural numbers.

  7) For \( n \in \mathbb{N} : [n] = \{ 1, 2, 3, \ldots, n \} \).

- We note that: \( \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \)

- **Set operations**

  Let \( A, B \) be two sets. We define the following set operations:

  1) Intersection: \( A \cap B \)

     \[ x \in A \cap B \iff x \in A \land x \in B \]

  2) Union: \( A \cup B \)

     \[ x \in A \cup B \iff x \in A \lor x \in B \]

  3) Difference: \( A - B \)

     \[ x \in A - B \iff x \in A \land x \notin B \]

- We represent these operations with Venn Diagrams as follows:
• Predicates and quantified statements
  
• A predicate $p(x)$ is a statement about $x$ which is true or false depending on the value of $x$.

• Note that $x$ can also be an ordered collection of elements $x = (x_1, x_2, \ldots, x_n)$. Then we write $p(x)$ as $p(x_1, x_2, \ldots, x_n)$.

• Given a predicate $p(x)$ and a set $A$, we define the following quantified statements:
  1) $\forall x \in A : p(x)$
     \[\text{For all } x \in A, \ p(x) \text{ is satisfied.}\]
  2) $\exists x \in A : p(x)$
     \[\text{There is at least one } x \in A \text{ such that } p(x) \text{ is satisfied.}\]
  3) $\exists! x \in A : p(x)$
     \[\text{There is a unique } x \in A \text{ such that } p(x) \text{ is satisfied.}\]

• If $A$ is a finite set, then the above quantified statements are abbreviations for conjunction, disjunction, and exclusive disjunction: For example:
  \[(\forall x \in \{a, b, c\} : p(x)) \iff (p(a) \land p(b) \land p(c))\]
  \[(\exists x \in \{a, b, c\} : p(x)) \iff (p(a) \lor p(b) \lor p(c))\]
  \[(\exists! x \in \{a, b, c\} : p(x)) \iff (p(a) \lor p(b) \lor p(c))\]

• Quantifiers can be nested to give compound quantified statements. For example:
  1) $\forall x \in A : \exists y \in B : p(x, y)$
     \[\text{For all } x \in A, \text{ there is a } y \in B, \text{ such that } p(x, y) \text{ is satisfied.}\]
2) \( \exists x \in A : \forall y \in B : p(x, y) \)

There is an \( x \in A \) such that for all \( y \in B \), \( p(x, y) \) is satisfied.

- Important quantified statements from algebra
  \( \forall a, b \in \mathbb{R} : (ab = 0 \iff a = 0 \lor b = 0) \)
  \( \forall a, b \in \mathbb{R} : (a^2 + b^2 = 0 \iff a = 0 \land b = 0) \)
  \( \forall a, b \in \mathbb{R} : (|a| + |b| = 0 \iff a = 0 \land b = 0) \)

- Definitions of sets

  There are 3 methods for defining sets:

  1) **By listing**: For finite sets we can simply list the elements.

  e.g.: \( A = \{3, 7, 10, 12\} \)

  2) **By predicate**: \( A = \{x \in U : p(x)\} \)

     with \( U \) a predefined set and \( p(x) \) a predicate.

     Belonging condition: \( x \in A \iff (x \in U \land p(x)) \)

     e.g.: We can use definition by predicate to define intervals:

     \( [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \)

     \( (a, b) = \{x \in \mathbb{R} : a < x < b\} \)

     \( [n] = \{x \in \mathbb{N} : 1 \leq x \leq n\} = \{1, 2, \ldots, n\} \)

  3) **By mapping**: \( A = \{\varphi(x) \mid x \in U \land p(x)\} \)

     with \( p(x) \) some expression of \( x \), \( U \) a predefined set, and \( p(x) \) a predicate.

     Belonging condition: \( y \in A \iff \exists x \in U : (\varphi(x) = y \land p(x)) \)
EXAMPLES

a) The set of complex numbers:
\[ C = \{ a + bi \mid a, b \in \mathbb{R} \} \]
\[ x \in C \iff \exists a, b \in \mathbb{R} : x = a + bi \]

b) The set of rational numbers:
\[ \mathbb{Q} = \{ \frac{a}{b} \mid a \in \mathbb{Z} \land b \in \mathbb{N} \backslash \{0\} \} \]
\[ x \in \mathbb{Q} \iff \exists a \in \mathbb{Z} : \exists b \in \mathbb{N} \backslash \{0\} : x = \frac{a}{b} \]

c) The set of even integers
\[ A = \{ 2k \mid k \in \mathbb{Z} \} \]
\[ x \in A \iff \exists k \in \mathbb{Z} : x = 2k \]

d) The set of odd integers
\[ A = \{ 2k + 1 \mid k \in \mathbb{Z} \} \]
\[ x \in A \iff \exists k \in \mathbb{Z} : x = 2k + 1 \]

e) \[ A = \{ a^2 + b^2 \mid a, b \in \mathbb{R} \land a + 3b < 1 \} \]
\[ x \in A \iff \exists a, b \in \mathbb{R} : (x = a^2 + b^2 \land a + 3b < 1) \]

- Cartesian product
We use definition by mapping to define the cartesian product between sets.

- An ordered pair \((a, b)\) is an ordered collection of two elements \(a\) and \(b\). We call \(a\) and \(b\) the components of \((a, b)\).
- We note that:\((a, b) = (c, d) \iff (a = c \land b = d)\).
• Let \(A, B\) be two sets. We define the Cartesian product
\[A \times B = \{(a, b) \mid a \in A \land b \in B\}.
\]
We also define:
\[A^2 = A \times A = \{(a, b) \mid a \in A \land b \in A\}.
\]

**Example**

For \(A = \{1, 2, 3\}\) and \(B = \{5, 6, 3\}\). Calculate \(A \times B, A^2, B^2\).

**Solution**

\[A \times B = \{1, 2, 3\} \times \{5, 6, 3\} = \{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6)\}\]

\[A^2 = A \times A = \{1, 2, 3\} \times \{1, 2, 3\} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}\]

\[B^2 = B \times B = \{5, 6, 3\} \times \{5, 6, 3\} = \{(5, 5), (5, 6), (5, 3), (6, 5), (6, 6), (6, 3)\}\]

\[\uparrow\text{ The above can be generalized as follows}\]

• An ordered \(n\)-tuple \((x_1, x_2, \ldots, x_n)\) is an ordered collection of \(n\) elements \(x_1, x_2, \ldots, x_n\).

• Let \(x = (x_1, x_2, \ldots, x_n)\) and \(y = (y_1, y_2, \ldots, y_n)\). We note that:
\[x = y \iff \forall a \in [n]: x_a = y_a\]

• Let \(A_1, A_2, \ldots, A_n\) be \(n\) sets. We define:
\[A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid \forall a \in [n]: x_a \in A_a\}\]

• Special case:
\[A_1 \times A_2 \times A_3 = \{(x_1, x_2, x_3) \mid x_1 \in A_1 \land x_2 \in A_2 \land x_3 \in A_3\}\]
EXERCISES

1. Let \( A = \{7\} \), \( B = \{x \in A | x > 43\} \), and \( C = \{x - 1 | x \in B\} \).
   List the elements of:
   a) \( B \)  
   b) \( C \)  
   c) \( B \cap C \)  
   d) \( B \cup C \)  
   e) \( A - B \)  
   f) \( B - C \)  
   g) \( C - B \)

2. Write out the following statements in English:
   a) \( \forall a \in A : \exists b \in B : (a, b) \in \emptyset \)
   b) \( \exists a \in A : \forall b \in B : a + b > 3 \)
   c) \( \forall a \in A : \exists b \in B : (ab > 1 \land a + b > 1) \)
   d) \( \forall a, b \in A : \exists c \in B : \forall d \in A : ab + bd < 3 \)
   e) \( \exists a \in A : \forall b \in B : (ab > 3 \Rightarrow b > 2) \)
   f) \( \forall a \in A : \exists b \in B : (3a > b \lor a + b < 0) \)

3. Write the following statements symbolically using quantifiers:
   a) Every real number is equal to itself.
   b) There is a real number \( x \) such that \( 3x - 1 = 2(x + 3) \)
   c) For every real number \( x \), there is a natural number \( n \) such that \( n > x \).
   d) For every real number \( x \), there is a complex number \( y \) such that \( y^2 = x \).
   e) There is a real number \( x \) such that for all real numbers \( y \) we have \( x + y = 0 \).
f) For all $\varepsilon > 0$, there is a $\delta > 0$ such that for all real numbers $x$, if $x_0 - \delta < x < x_0 + \delta$ then $|f(x) - a| < \varepsilon$.

g) There is a real number $b$ such that for all natural numbers $n$ we have $a_n < b$.

h) For all $\varepsilon > 0$, there is a natural number $n_0$ such that for any two natural numbers $n$ and $n_2$, if $n > n_0$ and $n_2 > n_0$, then $|a_n - a_{n_2}| < \varepsilon$.

i) For any $M > 0$, there is a natural number $n_0$, such that for any other natural number $n$, if $n > n_0$ then $a_n > M$.

4) Write the belonging condition $x \in A$ for the following sets, using quantifiers.

a) $A = \{x \in \mathbb{R} : x^2 + 1 > 3\}$

b) $A = \{x \in \mathbb{Z} : x$ is a prime number$\}$

c) $A = \{x \in \mathbb{R} : x^2 + 3x > 0\}$

d) $A = \{a^2 + b^2 + c^2 : a, b, c \in \mathbb{R}, a + b + c = 0\}$

e) $A = \{x \in \mathbb{R} : x^2 + 2x < 0 \lor 3x + 1 > -4 + x\}$

f) $A = \{x \in \mathbb{R} : x^2 - 6x > 5\}$

g) $A = \{x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} : x = 3y\}$

h) $A = \{ab : a, b \in \mathbb{R} \land (a + b > 2 \lor a - b < -3)\}$

i) $A = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : y^2 + y = x\}$

j) $A = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R} : x < y^2 + 1\}$

k) $A = \{ab : a, b \in \mathbb{R} \land (ab > 1 \Rightarrow a^2 + b^2 > 2)\}$

l) $A = \{a^2 + 3b : a, b \in \mathbb{R} \land (a^2 + b^2 > 2 \lor a - b < 3)\}$

m) $A = \{2a + 3b : a, b \in \mathbb{R} \land ab > 1 \land a - b < 0\}$
(5) List the elements for the following Cartesian products

a) $A \times B$ with $A = \{2, 7, 43\}$ and $B = \{7, 83\}$

b) $A \times B$ with $A = \{13\}$ and $B = \{3, 93\}$

c) $A \times B$ with $A = \{33\}$ and $B = \{93\}$

d) $[2] \times [3]$ 


