GRAPH THEORY II

Adjacency matrix

Def: Let G be a graph with vertices $V(G)=\{u_1,u_2,\dots,u_n\}$.

a) We define the adjacency matrix $A(G)\in M_n(\mathbb{R})$ via

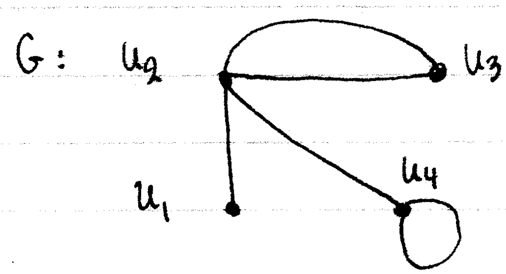
$$\forall a,b\in[n]: [A(G)]_{ab} = \begin{cases} |\{e\in E(G) \mid \psi_G(e)=\{u_a,u_b\}\}|, & \text{if } a\neq b \\ 2|\{e\in E(G) \mid \psi_G(e)=\{u_a\}\}|, & \text{if } a=b \end{cases}$$

b) We define the connectivity matrix $B(G)\in M_n(\mathbb{R})$ via

$$B(G) = A(G)+A^2(G)+\dots+A^{n-1}(G)$$

EXAMPLE

Consider the graph G given by



The corresponding adjacency matrix is:

$$A(G)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

↳ Note that loops are counted twice.

Properties of the adjacency matrix

① $A(G)$, $B(G)$ are both symmetric

$$\boxed{\begin{array}{l} \forall a,b \in [n]: [A(G)]_{ab} = [A(G)]_{ba} \\ \forall a,b \in [n]: [B(G)]_{ab} = [B(G)]_{ba} \end{array}}$$

② Column / Row sums of $A(G)$

$$\boxed{\begin{array}{l} \forall b \in [n]: \displaystyle\sum_{a=1}^{n} [A(G)]_{ab} = d(u_b) \\ \forall a \in [n]: \displaystyle\sum_{b=1}^{n} [A(G)]_{ab} = d(u_a) \end{array}}$$

Enumeration of walks

<u>notation</u>: Recall that $W(G)$ is the set of all walks on the graph G. Let $a \in \mathbb{N}^*$. The set of all walks with length a will be denoted as:

$$W_a(G) = \{ w \in W(G) \mid \ell(w) = a \}$$

The following result counts the number of walks of fixed length between two vertices.

<u>Thm</u>: Let G be a graph and let $u_a, u_b \in V(G)$ be two vertices, and let $k \in \mathbb{N}^*$. Then,

$$|\{ w \in W_k(G) \mid s(w) = u_a \wedge t(w) = u_b \}| = [A^k(G)]_{ab}$$

Remark

a) A <u>closed walk</u> $w \in W(G)$ is a walk whose starting point $s(w)$ and endpoint $t(w)$ coincide.

It follows from the previous result that the number of closed walks with length $k \in \mathbb{N}^*$ is given by:

$$\boxed{|\{w \in W_k(G) \mid s(w) = t(w)\}| = \operatorname{tr}(A^k(G))}$$

Note that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$, then $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are the eigenvalues of $A^k(G)$. We may then conclude that

$$\boxed{\operatorname{tr}(A^k(G)) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k}$$

Consequently, it is easy to enumerate the number of closed walks, as a function of the length k, via the eigenvalues of the adjacency matrix for any graph G.

<u>Graph connectivity</u>

<u>Thm</u>: Let G be a graph with connectivity matrix $B(G)$. It follows that

$$G \text{ connected} \Leftrightarrow \forall a, b \in [n] : (a \neq b \Rightarrow [B(G)]_{ab} > 0)$$

This theorem can be used algorithmically to calculate the number of components $\omega(G)$ for any graph as follows.

(a) Pick a vertex $u_a \in V(G)$

(b) Find all vertices $u_b \in V(G)$ such that $[B(G)]_{ab} > 0$. This defines a set of vertices V_0 such that $G[V_0]$ is a component of the graph G.

(c) Apply the same algorithm recursively to the graph $G - V_0$ to extract the next component.

(d) Upon removing all vertices, we will have found all graph components, and counting them will give us $\omega(G)$.

Given an algorithm that calculates $\omega(G)$, it is easy to write algorithms to calculate $\kappa(G)$ and $\lambda(G)$.

EXAMPLE

Let $A(K_4)$ be the adjacency matrix of K_4.

a) Find the characteristic polynomial of $A(K_4)$ and its eigenvalues

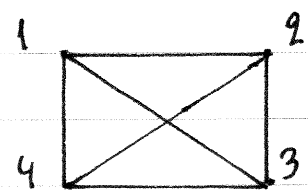
b) How many closed walks of length n does K_4 have?

c) Show that $A^4(K_4) = 6A^2(K_4) + 8A(K_4) + 3I$

d) How many open walks of length 4 does K_4 have?

Solution

a) For K_4:



the adjacency matrix is given by:

$$A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \det(A(K_4) - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 & 1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & -\lambda & 1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} \begin{matrix} \\ \\ \\ (-1)\ \lambda \end{matrix} =$$

$$= \begin{vmatrix} 0 & \lambda+1 & \lambda+1 & 1-\lambda^2 \\ 0 & -\lambda-1 & 0 & \lambda+1 \\ 0 & 0 & -\lambda-1 & \lambda+1 \\ 1 & 1 & 1 & -\lambda \end{vmatrix} = -1 \begin{vmatrix} \lambda+1 & \lambda+1 & (\lambda+1)(1-\lambda) \\ -(\lambda+1) & 0 & \lambda+1 \\ 0 & -(\lambda+1) & \lambda+1 \end{vmatrix}$$

$$= (\lambda+1)^3 \begin{vmatrix} 1 & 1 & 1-\lambda \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = (\lambda+1)^3 \begin{vmatrix} 1 & 1 & 1-\lambda \\ 0 & 1 & 2-\lambda \\ 0 & -1 & 1 \end{vmatrix} =$$

$$= (\lambda+1)^3 \begin{vmatrix} 1 & 2-\lambda \\ -1 & 1 \end{vmatrix} = (\lambda+1)^3 [1 \cdot 1 - (-1)(2-\lambda)] =$$

$$= (\lambda+1)^3 (1+2-\lambda) = (\lambda+1)^3 (3-\lambda)$$

and therefore

λ eigenvalue of $K_4 \Leftrightarrow \det(A(K_4) - \lambda I) = 0 \Leftrightarrow$

$\Leftrightarrow (\lambda+1)^3 \; (3-\lambda) = 0 \Leftrightarrow \lambda + 1 = 0 \vee 3 - \lambda = 0 \Leftrightarrow$

$\Leftrightarrow \lambda = -1 \vee \lambda = 3.$

b) The number N of closed walks with length n is given by:

$$N = |\{w \in W_n(K_4) \mid s(w) = t(w)\}| =$$
$$= tr(A^n(K_4)) = \lambda_1^n + \lambda_2^n + \lambda_3^n + \lambda_4^n =$$
$$= (-1)^n + (-1)^n + (-1)^n + 3^n =$$
$$= 3(-1)^n + 3^n$$

c) Since

$$\det(A(K_4) - \lambda I) = (\lambda+1)^3(3-\lambda) = (\lambda^3 + 3\lambda^2 + 3\lambda + 1)(3-\lambda) =$$
$$= 3\lambda^3 + 9\lambda^2 + 9\lambda + 3 - \lambda^4 - 3\lambda^3 - 3\lambda^2 - \lambda =$$
$$= -\lambda^4 + (9-3)\lambda^2 + (9-1)\lambda + 3$$
$$= -\lambda^4 + 6\lambda^2 + 8\lambda + 3 \Rightarrow$$

$$\Rightarrow -A^4(K_4) + 6A^2(K_4) + 8A(K_4) + 3I = 0$$

$$\rightarrow A^4(K_4) = 6A^2(K_4) + 8A(K_4) + 3I$$

d)

$$A^2(K_4) = A(K_4)A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} \Rightarrow$$

$$\Rightarrow A^4(K_4) = 6A^2(K_4) + 8A(K_4) + 3I =$$

$$= 6\begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} + 8\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{bmatrix}$$

and therefore the number of walks of length 4 are given by:

$$N = |\{w \in W_4(K_4) \mid s(w) \neq t(w)\}| =$$

$$= \sum_{\substack{a,b \in [4] \\ a \neq b}} [A^4(K_4)]_{ab} = 20 \cdot 12 = 240$$

EXERCISES

(42) Write the incidence matrices for the following graphs:

a) K_3
b) K_4
c) $K_{2,3}$
d) $K_{3,3}$
e) C_4
f) P_4

(43) Let $A(K_3)$ be the adjacency matrix of K_3

a) Find the characteristic polynomial of $A(K_3)$
b) Show that $A^3 = 3A + 2I$
c) How many open walks does K_3 have of length 3?
d) How many closed walks does K_3 have of length 3?
e) Calculate A^5 and answer the same questions (c), (d) for walks of length 5.

(44) Let $A(K_{2,2})$ be the adjacency matrix of $K_{2,2}$

a) Find the characteristic polynomial of $A(K_{2,2})$
b) How many cycles of length 5 are there in $K_{2,2}$?

(45) Let G be a graph with adjacency matrix

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Show that G is not Eulerian.

(46) Show the following statements:

a) $\displaystyle\sum_{a=1}^{n} \sum_{b=1}^{n} [A(G)]_{ab} = 2|E(G)|$

b) $\left.\begin{array}{l} \displaystyle\sum_{a=1}^{n} [A(G)]_{ab} \geq \frac{n}{2} \\ \text{for all } b \in [n] \end{array}\right\} \Rightarrow G \text{ Hamiltonian}$

c) $\displaystyle\sum_{a=1}^{n} [A(G)]_{ab} \geq \lambda(G)\ , \ \forall b \in [n]$

▼ The shortest path problem

Def: A weighted graph G is a graph endowed with a mapping
$$f: E(G) \longrightarrow \mathbb{R}$$
which maps every edge $e \in E(G)$ to a unique real number $f(e)$.

Def: Let G be a weighted graph with $f: E(G) \to \mathbb{R}$ and let $w \in P(G)$ be a path on G. We define the weight $f(w)$ of the path w as:

$$\forall w \in P(G): f(w) = \sum_{e \in E(w)} f(e)$$

↳ Recall that a path $w \in P(G)$ is a walk on G where vertices and edges are visited no more than one time. $E(w)$ is the set of edges visited by w.

Def: Let G be a weighted graph with $f: E(G) \to \mathbb{R}$ and let $a, b \in V(G)$ with $a \neq b$ be two vertices of G. We define the distance $f(a,b)$ between a and b as:
$$f(a,b) = \min\{f(w) \mid w \in P(G, a \to b)\}$$

• The <u>shortest path problem</u> is, given a weighted graph G with $f: E(G) \to \mathbb{R}$ and vertices $a, b \in V(G)$ to find a path $w \in P(G, a \to b)$ such that $f(w) = f(a, b)$. Equivalently, we are looking for the path $w \in P(G, a \to b)$ that minimizes $f(w)$.

The Dijkstra algorithm

The Dijkstra algorithm can be used to solve the shortest path problem for the special case:

$$\forall e \in E(G) : f(e) > 0$$

where all edges have a positive weight. The algorithm consists of two parts:

(1) <u>Preprocessing</u>: Given the initial vertex $a \in V(G)$, we define a mapping $L_k : V(G) \to \mathbb{R} \cup \{\infty\}$ with $k \in [n]$ and $n = |V(G)| - 1$ and ∞ representing a "fictional" infinite number. We also define a vertex sequence $u_0, u_1, \ldots, u_{n-1}$.

(2) <u>Postprocessing</u>: Given the terminal vertex $b \in V(G)$ and the output of the preprocessing step, the second step of the algorithm will find one (or all) shortest path from the initial vertex a to the terminal vertex b.

• Note that the postprocessing step can be repeated for different choices of a terminal vertex $b \in V(G)$ without repeating the preprocessing step. The preprocessing step needs to be repeated only if we change the initial vertex.

① → Preprocessing step

Let $a \in V(G)$ be the chosen initial vertex.

•1 We define

$$\begin{cases} u_0 = a \wedge S_0 = \{u_0\} = \{a\} \\ \forall u \in V(G) : L_0(u) = \begin{cases} 0 & , \text{ if } u = u_0 \\ \infty & , \text{ if } u \neq u_0 \end{cases} \end{cases}$$

•2 Assume that

$$\begin{cases} S_K = \{u_0, u_1, \ldots, u_K\} \\ L_K : V(G) \to \mathbb{R} \cup \{\infty\} \end{cases}$$

have already been defined.

- ▶ If $K = |V(G)| - 1$, then the algorithm terminates.
- ▶ Otherwise, we define L_{K+1} as follows:
  1) $\forall u \in S_K : L_{K+1}(u) = L_K(u)$
  2) $\forall u \in V(G) - S_K : (u, u_K \text{ not adjacent} \Rightarrow L_{K+1}(u) = L_K(u))$
  3) $\forall u \in V(G) - S_K : (u, u_K \text{ adjacent} \Rightarrow$

  $$\Rightarrow L_{K+1}(u) = \min\{L_K(u), L_K(u_K) + f(uu_K)\}$$

  with uu_K the edge between u and u_K.
- ▶ We define $u_{K+1} \in V(G)$ such that it satisfies:

  $$\begin{cases} u_{K+1} \in V(G) - S_K \\ \forall u \in V(G) - S_K : L_{K+1}(u) \geq L_{K+1}(u_{K+1}) \end{cases}$$

  Note that if multiple choices exist for u_{K+1}, we can fork the execution of the algorithm and pursue each choice. That may lead to multiple shortest paths of equal weight, if they exist.
- ▶ We define $S_{K+1} = S_K \cup \{u_{K+1}\}$.

A practical implementation of the algorithm is based on a grid of the form:

	$k=0$	$k=1$	$k=2$	$\cdots$	$k=n$
a	0	$L_1(a)$	$L_2(a)$	$\cdots$	$L_n(a)$
a_1	∞	$L_1(a_1)$	$L_2(a_1)$	$\cdots$	$L_n(a_1)$
a_2	∞	$L_1(a_2)$	$L_2(a_2)$	$\cdots$	$L_n(a_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_n	∞	$L_1(a_n)$	$L_2(a_n)$	$\cdots$	$L_n(a_n)$
		$u_0=a$	u_1	$\cdots$	u_{n-1}

We note that

1) The first column is initialized with 0 for the initial vertex a and with ∞ for all other vertices. We insert $u_0=a$ on column $k=1$.
2) Given the construction of column k, we define column $k+1$ as follows:
   - We copy from column k to column $k+1$ the numbers corresponding to the vertices $S_k=\{u_0,u_1,\dots,u_k\}$, and we <u>circle them.</u>
   - We also copy any additional vertices that are not adjacent to u_k from among the set $V(G)-S_k$.
   - We use the formula
     $$L_{k+1}(u)=\min\{L_k(u),\ L_k(u_k)+f(uu_k)\}$$
     for any remaining vertices.

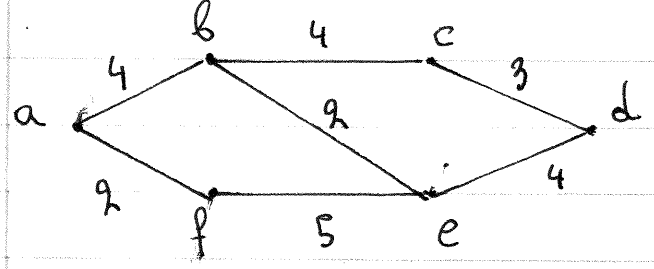
3) Given the column $k+1$, we choose from among the non-circled vertices (i.e. all $u \in V(G) - S_k$) the one that minimizes $L_{k+1}(u)$. This defines u_{k+1}, which we write under the column $k+2$.

4) We define $S_{k+1} = S_k \cup \{u_k\}$.

EXAMPLE

Consider the following graph with initial vertex $a \in V(G)$:

G:



	K=0	K=1	K=2	K=3	K=4	K=5
a	0	0	0	0	0	0
b	∞	4	$\boxed{4}$	4	4	4
c	∞	∞	∞	8	$\boxed{8}$	8
d	∞	∞	∞	∞	10	10
e	∞	∞	7	$\boxed{6}$	6	6
f	∞	$\boxed{2}$	2	2	2	2
		a	f	b	e	c

Algorithm log:

$u_0 = a$

$S_0 = \{a\}$

For $K = 1$:

copy a with circle

copy c, d, e

$L_1(b) = \min\{L_0(b), L_0(a) + f(ab)\} = \min\{\infty, 0+4\} = 4$

$L_1(f) = \min\{L_0(f), L_0(a) + f(af)\} = \min\{\infty, 0+2\} = 2$

minimum at f $\longrightarrow$ $u_1 = f$ and $S_1 = \{a, f\}$

For $k=2$:
copy a, f with circle
copy b, c, d
$L_2(e) = \min\{L_1(e), L_1(f) + f(ef)\} = \min\{\infty, 2+5\} = 7$
minimum at b $\rightarrow u_2 = b$ and $S_2 = \{a, f, b\}$
For $k=3$:
copy a, f, b with circle
copy d
$L_3(c) = \min\{L_2(c), L_2(b) + f(bc)\} = \min\{\infty, 4+4\} = 8$
$L_3(e) = \min\{L_2(e), L_2(b) + f(be)\} = \min\{7, 4+2\} = 6$
minimum at e $\rightarrow u_3 = e$ and $S_3 = \{a, f, b, e\}$
For $k=4$:
copy a, f, b, e with circle
copy c
$L_4(d) = \min\{L_3(d), L_3(e) + f(ed)\} = \min\{\infty, 6+4\} = 10$
Minimum at c $\rightarrow u_4 = c$ and $S_4 = \{a, f, b, e, c\}$
For $k=5$:
copy a, f, b, e, c with circle.
$L_5(d) = \min\{L_4(d), L_4(c) + f(cd)\} = \min\{10, 8+3\} = 10$
done!

$\hookrightarrow$ Note that L_5 gives the distance between the initial vertex a and all other vertices.
e.g. $f(a,c) = L_5(c) = 8$, etc...

(2) → Postprocessing

The postprocessing algorithm is illustrated by the following example where we seek the shortest path from vertex "a" to vertex "d" in the previous graph G :

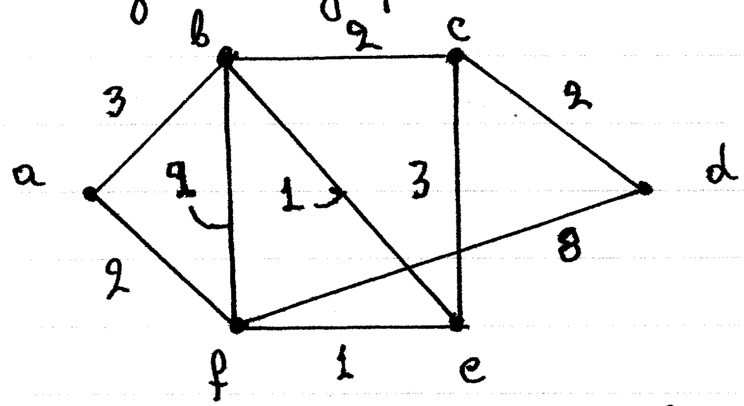
	K=1	K=2	K=3	K=4	K=5	K=6	
a	0	0	0	0	0	0	
b	∞	4	4	4	4	4	
c	∞	∞	∞	8	8	8	
d	∞	∞	∞	∞	10	10	$\boxed{d}$
e	∞	∞	7	6	6	6	
f	∞	2	2	2	2	2	
		$\boxed{a}$	f	$\boxed{b}$	$\boxed{e}$	c	

Shortest path:
abed

- 1 On the last column we circle the number corresponding to the terminal vertex, which we show on the side
- 2 We move left, if the number to the left is equal. Otherwise, we square the vertex under the current column and move to the number corresponding to that vertex. Either way, we circle the next number.
- 3 Repeat until we get to column K=1.
- 4 The squared vertices form the shortest path

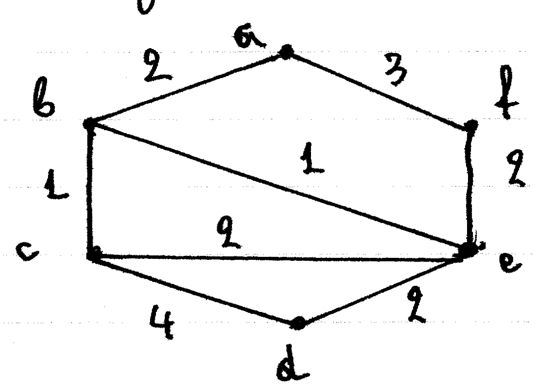
EXERCISES

(47) Apply Dijksta's algorithm to the following weighted graph.



Find the shortest path from a to d.

(48) Similarly, for the following graph



Find the shortest path from a to d.

Tree graphs

To give a rigorous definition of <u>trees</u>, we need the following preliminary definitions first.

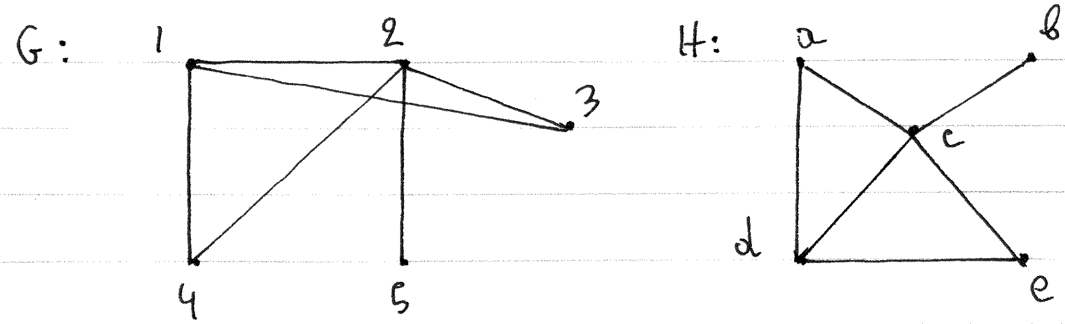
Preliminaries

<u>Def</u>: Let $f: A \to B$ be a mapping and let $S \subseteq A$ be given. We define
$$f(S) = \{ f(x) \mid x \in S \}$$
or equivalently:
$$y \in f(S) \Leftrightarrow \exists x \in S : f(x) = y$$
We also say that
a) f <u>one-to-one</u> $\Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$
b) f <u>onto</u> $\Leftrightarrow f(A) = B$
c) f <u>bijection</u> $\Leftrightarrow \begin{cases} f \text{ one-to-one} \\ f \text{ onto.} \end{cases}$

<u>Def</u>: (<u>Isomorphic graphs</u>)
Let G, H be two graphs. We say that $G \cong H$ (G isomorphic to H) if and only if there exist mappings
$$f : V(G) \to V(H) \quad \text{and} \quad g : E(G) \to E(H)$$
such that
$$\begin{cases} f, g \text{ are bijections} \\ \forall e \in E(G) : \psi_H(g(e)) = f(\psi_G(e)) \end{cases}$$

interpretation : The bijections f, g represent a relabelling of the vertices and edges of G. We say that $G \cong H$ when after such a relabelling, the vertices are connected by the edges the same way.

EXAMPLE



$G \cong H$ with bijections

$f(1) = d$ (both have degree 3)

$f(2) = c$ (both have degree 4)

$f(3) = a$ (3 connected with 1,2)

$f(4) = e$ (4 also connected with 1,2)

$f(5) = b$ (both have degree 1)

and

$g(12) = dc$ $\qquad$ $g(14) = de$

$g(23) = ca$ $\qquad$ $g(24) = ce$

$g(31) = ad$ $\qquad$ $g(25) = bc$

Def: (Subgraphs)
Let G, H be two graphs. We say that

$$H \subseteq G \Leftrightarrow \begin{cases} V(H) \subseteq V(G) \\ E(H) \subseteq E(G) \\ \forall e \in E(H): \psi_H(e) = \psi_G(e) \end{cases}$$

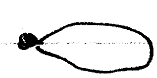
notation: We denote the set of all subgraphs of a graph G as $\mathcal{P}(G)$ such that
$$H \in \mathcal{P}(G) \Leftrightarrow H \subseteq G$$

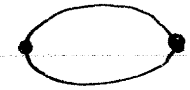
Definition of trees

A tree is defined as a connected acyclic graph. In detail, this is done via the following definitions:

Def: Let G be a graph. We say that
a) G cyclic $\Leftrightarrow \exists H \in \mathcal{P}(G): \exists n \in \mathbb{N}^*: H \cong C_n$
b) G acyclic $\Leftrightarrow$ G not cyclic
$\Leftrightarrow \forall H \in \mathcal{P}(G): \forall n \in \mathbb{N}^*: H \not\cong C_n$

Intuitively, G is cyclic if and only if there is a subgraph H of G such that it is isomorphic to the cycle graph C_n for some n. Recall that:

C_1: 

C_2: 

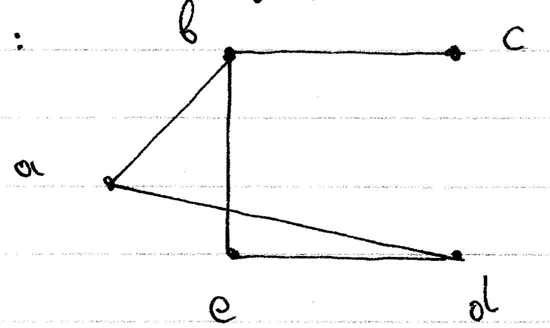
therefore loops and multiple edges automatically form cycles. It follows that:

$$\boxed{G \text{ acyclic} \Rightarrow G \text{ simple}}$$

EXAMPLE

Consider the graph

G :



Since

$G[\{a,b,d,e\}] \cong C_4 \Rightarrow G$ cyclic.

<u>Def</u> : Let G be a graph. We say that

$$G \text{ \underline{tree}} \Leftrightarrow \begin{cases} G \text{ connected} \\ G \text{ acyclic} \end{cases}$$

Properties of trees – Summary

① $\boxed{G \text{ tree} \Rightarrow G \text{ simple}}$

② A graph is a tree if and only if for any two distinct vertices there is a unique path that connects them

Thm: Let G be a graph. Then,
$$G \text{ tree} \Leftrightarrow \forall a, b \in V(G) : (a \neq b \Rightarrow |P(G, a \to b)| = 1)$$

③ A tree always disconnects to two components upon removing any one edge.

Thm: Let G be a graph. Then,
$$G \text{ tree} \Rightarrow \forall e \in E(G) : \omega(G - \{e\}) = 2$$

④ Relation between number of edges and vertices

Thm: Let G be a graph. Then,
$$G \text{ tree} \Leftrightarrow \begin{cases} |E(G)| = |V(G)| - 1 \\ G \text{ connected} \end{cases}$$

⑤ Trees and connectivity

Thm: Let G be a graph. Then,
$$G \text{ tree} \Rightarrow \kappa(G) = \lambda(G) = \delta(G) = 1$$

● → Properties of trees

① → $\boxed{G \text{ tree} \Rightarrow G \text{ simple}}$

Proof

$$G \text{ tree} \Rightarrow \begin{cases} G \text{ acyclic} \\ G \text{ connected} \end{cases} \Rightarrow$$

$\Rightarrow G$ acyclic

$\Rightarrow G$ simple. $\square$

② → A graph G is a tree if and only if for any two distinct vertices there is a unique path that connects them.

$$\boxed{\begin{array}{l} \text{Thm: Let } G \text{ be a graph. Then} \\ G \text{ tree} \Leftrightarrow \forall a,b \in V(G) : (a \neq b \Rightarrow |P(G, a \to b)| = 1) \end{array}}$$

Proof

($\Rightarrow$): Assume that G is a tree. Let $a, b \in V(G)$ be given and assume that $a \neq b$. To show that $|P(G, a \to b)| = 1$ we assume that $|P(G, a \to b)| \neq 1$ and derive a contradiction. We distinguish between the following cases:

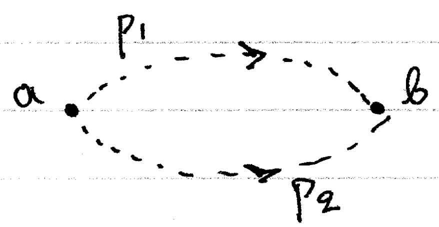
Case 1: Assume that $|P(G, a \to b)| = 0$. Then,

$|P(G, a \to b)| = 0 \Rightarrow G$ not connected
$\Rightarrow G$ not a tree.

which is a contradiction.

Case 2 : Assume that $|P(G, a \to b)| \geq 2$.

Choose paths $p_1, p_2 \in P(G, a \to b)$ such that $p_1 \neq p_2$.



Then p_1, p_2 together form a cycle and therefore G cyclic $\Rightarrow G$ not a tree which is again a contradiction.

It follows that $\underline{|P(G, a \to b)| = 1}$

We have thus shown that

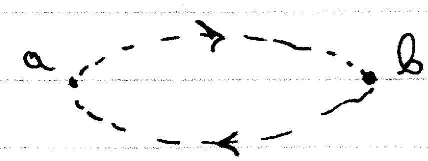
$\forall a, b \in V(G) : (a \neq b \Rightarrow |P(G, a \to b)| = 1)$

($\Leftarrow$): Assume that

$\forall a, b \in V(G) : (a \neq b \Rightarrow |P(G, a \to b)| = 1)$

It immediately follows that <u>G connected.</u> (1)

To show that G is acyclic, assume that G is cyclic. Let p be a cycle of G and let $a, b \in V(G)$ be two vertices on the cycle p. Then, there are at least two paths that connect a to b, therefore



$$|P(G, a \to b)| \geq 2$$

This contradicts the hypothesis. It follows that G acyclic (2)

From Eq.(1) and Eq.(2): G is a tree.

③ $\rightarrow$ A tree always disconnects to two components upon removing any one edge
$\downarrow$

Thm : Let G be a graph. Then
$$G \text{ tree} \Rightarrow \forall e \in E(G) : \omega(G - \{e\}) = 2$$

Proof

Assume that G is a tree. Let $e \in E(G)$ be given. Then
G tree $\Rightarrow$ G connected $\Rightarrow \omega(G) = 1$.
We know that for any graph:
$$\omega(G) \leq \omega(G - \{e\}) \leq \omega(G) + 1 \Rightarrow$$
$$\Rightarrow 1 \leq \omega(G - \{e\}) \leq 2 \Rightarrow$$
$$\Rightarrow \omega(G - \{e\}) = 1 \lor \omega(G - \{e\}) = 2 \qquad (1)$$
Let $a, b \in V(G)$ such that $\psi_G(e) = \{a, b\}$.
If $\omega(G - \{e\}) = 1 \Rightarrow G - \{e\}$ connected $\Rightarrow$
$$\Rightarrow \exists p \in P(G - \{e\}) : s(p) = a \land t(p) = b.$$
The path p does not use the edge e. However, the edge e defines a second path $p' = (a, e, b)$.
It follows that $|P(G, a \to b)| \geq 2 \Rightarrow G$ not a tree which is a contradiction.
We conclude that $\omega(G - \{e\}) \neq 1 \qquad (2)$
From Eq.(1) and Eq.(2): $\omega(G - \{e\}) = 2$.
We have thus shown that
$\forall e \in E(G) : \omega(G - \{e\}) = 2$. $\square$

④ → Relation between number of edges and vertices

Thm: Let G be a graph. Then
$$G \text{ tree} \Leftrightarrow \begin{cases} |E(G)| = |V(G)| - 1 \\ G \text{ connected} \end{cases}$$

Proof: $(\Rightarrow)$

Assume that G is a tree. Then, by definition

G tree $\Rightarrow$ G acyclic $\wedge$ G connected

$\Rightarrow$ G connected (1)

To show $|E(G)| = |V(G)| - 1$ we use induction on $|V(G)|$

- For $|V(G)| = 1$, the only tree that can be defined is:

  G: $\cdot a$

  or equivalently:
$$\begin{cases} V(G) = \{a\} \\ E(G) = \varnothing \\ \psi_G = \varnothing \end{cases}$$

Then $|E(G)| = |\varnothing| = 0 = 1 - 1 = |V(G)| - 1$

- For $|V(G)| \leq n$, we assume that the statement has been shown for all possible trees.
- Consider any tree G with $|V(G)| = n+1$.

Choose any edge $e \in E(G)$. From property ③ we have $\omega(G - \{e\}) = 2$. Let G_1, G_2 be the connected components of $G - \{e\}$ such that $G - \{e\} = G_1 \cup G_2$. Note that

G_1, G_2 cannot be cyclic because otherwise G would not be a tree. It follows that

G_1, G_2 trees $\Rightarrow \begin{cases} |E(G_1)| = |V(G_1)| - 1 \\ |E(G_2)| = |V(G_2)| - 1 \end{cases}$, via the induction hypothesis.

We also note that G has the edges of G_1, G_2, and the edge e., therefore

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| + 1 \\ &= [|V(G_1)| - 1] + [V(G_2) - 1] + 1 \\ &= |V(G_1)| + |V(G_2)| - 1 \\ &= |V(G_1 \cup G_2)| - 1 = |V(G - \{e\})| - 1 \\ &= |V(G)| - 1. \end{aligned}$$

$(\Leftarrow)$: Assume that

$$\begin{cases} |E(G)| = |V(G)| - 1 \\ G \text{ connected} \end{cases}$$

To show that G is a tree, we assume that G is not a tree, to arrive a contradiction. Remove as many edges from G as necessary to eliminate all cycles in G. Let $E_0 \subseteq E(G)$ be the set of all the edges removed. Then, using the $(\Rightarrow)$ result:

$$G - E_0 \text{ tree} \Rightarrow |E(G - E_0)| = |V(G - E_0)| - 1 \Rightarrow$$

$$\begin{aligned} \Rightarrow |E(G)| &= |E(G - E_0)| + |E_0| = \\ &= |V(G - E_0)| - 1 + |E_0| = \\ &= [|V(G)| - 1] + |E_0| = |E(G)| + |E_0| \Rightarrow \end{aligned}$$

$$\Rightarrow |E_0| = 0.$$

This contradicts the assumption that G is not a tree

It follows that G is a tree

5 → Trees and connectivity

Thm: Let G be a ~~tree~~ graph. Then

$$G \text{ tree} \Rightarrow \kappa(G) = \lambda(G) = \delta(G) = 1$$

Proof

Assume that G is a tree. Then,

$$|E(G)| = |V(G)| - 1 \Rightarrow$$

$$\Rightarrow \delta(G) \leq \frac{2|E(G)|}{|V(G)|} = \frac{2(|V(G)|-1)}{|V(G)|} < 2$$

$$\Rightarrow \delta(G) = 0 \vee \delta(G) = 1.$$

We also note that

$$\begin{aligned} G \text{ tree} &\Rightarrow G \text{ acyclic} \wedge G \text{ connected} \\ &\Rightarrow G \text{ connected} \\ &\Rightarrow \kappa(G) \geq 1 \wedge \lambda(G) \geq 1 \wedge \delta(G) \geq 1 \end{aligned}$$

It follows that

$$\delta(G) = 1$$

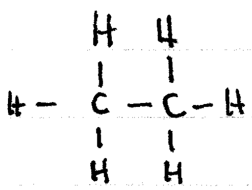
$$1 \leq \kappa(G) \leq \delta(G) = 1 \Rightarrow \kappa(G) = 1$$

$$1 \leq \lambda(G) \leq \delta(G) = 1 \Rightarrow \lambda(G) = 1. \qquad \square$$

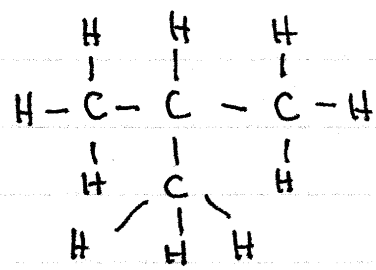
EXAMPLE / APPLICATION

A saturated hydrocarbon is a molecule C_aH_b in which
a) Every C atom has 4 simple bonds
b) Every H atom has 1 simple bond
c) No sequence of bonds forms a cycle.
Show that (a), (b), (c) imply that $b = 2a + 2$

examples



C_2H_6



C_4H_{10}

Solution

Let G be the graph representing the molecule C_aH_b where the C, H atoms are vertices and the bonds are edges. Let V_1 be the vertices corresponding to C atoms and let V_2 be the vertices corresponding to H atoms. Since C atoms have 4 bonds and H atoms have 1 bond:

$$\begin{cases} \forall u \in V_1 : d(u) = 4 \\ \forall u \in V_2 : d(u) = 1 \end{cases}$$

We also note that $|V_1| = a$ and $|V_2| = b$. It follows that $|V(G)| = |V_1| + |V_2| = a+b$

and

$$|E(G)| = \frac{1}{2}\sum_{u\in V(G)} d(u) = \frac{1}{2}\sum_{u\in V_1} d(u) + \frac{1}{2}\sum_{u\in V_2} d(u) =$$

$$= \frac{4|V_1|}{2} + \frac{|V_2|}{2} = \frac{4a+b}{2}$$

Since no sequence of bonds forms a cycle,

G is a tree $\Rightarrow |E(G)| = |V(G)| - 1 \Rightarrow \frac{4a+b}{2} = a+b-1$

$$\Rightarrow 4a+b = 2(a+b) - 2$$

$$\Rightarrow 4a+b = 2a+2b-2$$

$$\Rightarrow 4a+b-2a-2b+2 = 0$$

$$\Rightarrow 2a-b+2 = 0 \Rightarrow \underline{b = 2a+2.}$$

EXAMPLE / APPLICATION

Consider an unsaturated hydrocarbon molecule C_aH_b such that

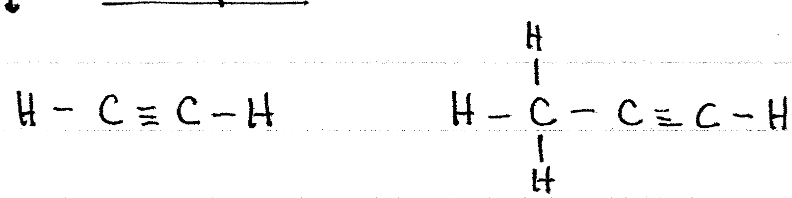
a) Every C atom has 4 simple bonds, except there also exists one triple bond between C atoms.

b) Every H atom has 1 simple bond

c) No sequence of bonds forms a cycle.

Then, show that $b = 2a - 2$.

$\mapsto$ examples



C_2H_2 $\qquad$ C_3H_4

Solution

Let G be a graph representing the molecule C_aH_b. Let V_1 be the vertices corresponding to C_1 atoms with 4 simple bonds. Let V_2 be the vertices corresponding to C atoms with one triple bond and one simple bond. Let V_3 be the vertices corresponding to H atoms with one simple bond. It follows that

$$\begin{cases} |V_1| + |V_2| = a \\ |V_3| = b \end{cases}$$

and

$$\begin{cases} \forall u \in V_1 : d(u) = 4 \\ \forall u \in V_2 : d(u) = 2 \\ \forall u \in V_3 : d(u) = 1 \end{cases}$$

Also, since there is only one triple bond, involving two C atoms, it follows that $|V_2| = 2$.

Then:

$$|V(G)| = |V_1| + |V_2| + |V_3| = (|V_1| + |V_2|) + |V_3| = a + b$$

and

$$|V_1| + |V_2| = a \Rightarrow |V_1| = a - |V_2| = a - 2$$

and

$$|E(G)| = \frac{1}{2} \sum_{u \in V(G)} d(u) =$$

$$= \frac{1}{2} \sum_{u \in V_1} d(u) + \frac{1}{2} \sum_{u \in V_2} d(u) + \frac{1}{2} \sum_{u \in V_3} d(u)$$

$$= \frac{4|V_1|}{2} + \frac{2|V_2|}{2} + \frac{1|V_3|}{2} =$$

$$= \frac{4(a-2) + 2 \cdot 2 + b}{2} = \frac{4a - 8 + 4 + b}{2} =$$

$$= \frac{4a + b - 4}{2}$$

Since no sequence of bonds forms a cycle, G is a tree $\Rightarrow |E(G)| = |V(G)| - 1 \Rightarrow$

$$\Rightarrow \frac{4a + b - 4}{2} = a + b - 1 \Rightarrow$$

$\Rightarrow 4a+b-4 = 2a+2b-2 \Rightarrow$

$\Rightarrow 4a+b-4-2a-2b+2=0$

$\Rightarrow 2a-b-2=0 \Rightarrow \underline{b=2a-2}$

EXAMPLE

a) Show that

K_a is a tree $\Leftrightarrow$ $a=1 \vee a=2$

Method: For problems of this time, it is possible to construct a direct argument using $\Leftrightarrow$ (if and only if) at every step, via the following proposition:

G tree $\Leftrightarrow$ G connected $\wedge$ $|E(G)| = |V(G)|-1$

Usually, the condition that G is connected can be removed if it is unconditionally true.

Solution

K_a has "a" vertices, and every vertex connects with one edge with the other $a-1$ vertices. It follows that:

$|V(K_a)| = a$ and

$\forall u \in V(K_a): d(u) = a-1$

and therefore:

$$|E(K_a)| = \frac{1}{2}\sum_{u\in V(K_a)} d(u) = \frac{1}{2}(a-1)|V(K_a)| = \frac{a(a-1)}{2}$$

It follows that

$$|E(K_a)| - |V(K_a)| + 1 = \frac{a(a-1)}{2} - a + 1 = \frac{1}{2}\left[a^2 - a - 2a + 2\right] =$$

$$= (1/2)(a^2 - 3a + 2) = (1/2)(a-1)(a-2)$$

and therefore:

$$\underline{K_a \text{ free}} \Leftrightarrow K_a \text{ connected} \wedge |E(K_a)| = |V(K_a)| - 1$$

$$\Leftrightarrow |E(K_a)| - |V(K_a)| + 1 = 0$$

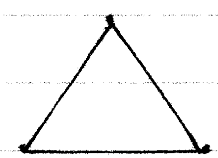
$$\Leftrightarrow (1/2)(a-1)(a-2) = 0$$

$$\Leftrightarrow a - 1 = 0 \vee a - 2 = 0$$

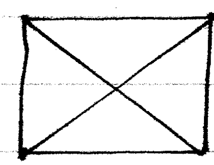
$$\Leftrightarrow \underline{a = 1 \vee a = 2}$$

$\mapsto$ Note that:

K_1: 

K_3: 

K_2:

K_4: 

EXERCISES

(49) Consider a generalised hydrocarbon molecule C_aH_b that has t triple bonds and d double bonds. Show that: $b = 2a - 2(t+d-1)$.

↳ Hint / Outline : Possible configurations for the bonds of C atoms are:

$$-\overset{|}{\underset{|}{C}}- \quad \text{or} \quad \overset{|}{\underset{|}{C}}= \quad \text{or} \quad =C= \quad \text{or} \quad \equiv C-$$

and for H atoms: $H-$

Consequently, we define the following vertex sets:

V_1 = all C atoms with 4 simple bonds

V_2 = all C atoms with 1 simple and 1 triple bond.

V_3 = all C atoms with 2 double bonds

V_4 = all C atoms with 1 double and 2 simple bonds.

V_5 = all H atoms with 1 simple bond.

Then we note that $2|V_3| + |V_4| = 2d$ and $|V_2| = 2t$ (explain why) and show that $|V(G)| = a+b$ and $|E(G)| = \frac{1}{2}\left[2a + b + 2|V_1| + |V_4|\right]$

Then we show that $2|V_1| + |V_4| = 2a - 2t - 2d$ and the rest is up to you.

(50) Let G be a tree. Use the fact that $\delta(G) = 1$ and $\kappa(G) = 1$ to show that

a) G is not Eulerian

b) G is not Hamiltonian

c) Can we establish (a) and (b) via an alternate argument directly from the tree definition?

(51) A forest is a graph whose components $G_1, G_2, \ldots, G_n$ are trees. Show that if G is a forest, then

$$|E(G)| = |V(G)| - \omega(G)$$

(Hint: for all a, we note that $|E(G_a)| = |V(G_a)| - 1$).

(52) Let $K_{a,b}$ be the complete bipartite graph. Show that

$K_{a,b}$ is a tree $\iff$ $a = 1$ or $b = 1$

(53) Show that the path graph P_a is a tree for all $a \geq 2$.

(54) Let C_a be the cycle graph. Show that for $a > 2$, C_a is not a tree.

$\hookrightarrow$ Use the statement

G tree $\iff (|E(G)| = |V(G)| - 1 \wedge G$ connected$)$.

(55) Let G be a regular graph with at least three vertices. Show that G cannot be a tree.

↳ <u>Hint/Outline</u>: We argue by contradiction. To show that G is not a tree, assume that G is a tree. Then show that if ρ is the common degree of all vertices, then ρ satisfies:

$$\rho = 2\,\frac{|V(G)|-1}{|V(G)|}$$

Next show that $|V(G)| \geqslant 3$ implies $1 < \rho < 2$ which is a contradiction since ρ has to be an integer. It will help to write $\rho = 2f(|V(G)|)$ with $f(x) = \dfrac{x-1}{x}$ and use calculus to study the graph of f in the interval $[3, +\infty)$.

▼ The minimum spanning tree problem

- Let G be a graph. We say that a tree T is a spanning tree of G if and only if

(a) T is a tree
(b) T is a subgraph of G
(c) All the vertices of G are also vertices of T, and vice versa.

$$\boxed{T \text{ spanning tree of } G \Leftrightarrow \begin{cases} T \subseteq G \\ T \text{ is a tree} \\ V(T) = V(G) \end{cases}}$$

- The set of all spanning trees of G is denoted as

$$\boxed{\tau(G) = \{ T \subseteq G \mid T \text{ spanning tree of } G \}}$$

- Thm: (Cayley) The complete graph K_n has n^{n-2} spanning trees.

$$\boxed{|\tau(K_n)| = n^{n-2}}$$

- <u>The problem</u> : Let $f : E(G) \to \mathbb{R}$ be a weight function that maps every edge $e \in E(G)$ to a number $f(e) \in \mathbb{R}$.

If $T \in \tau(G)$ is a spanning tree of G then the weight associated with T is given by

$$\boxed{w(T) = \sum_{e \in E(T)} f(e)}$$

A tree T_0 is a minimum spanning tree if and only if it minimizes $f(T)$.

$$\boxed{T_0 \text{ minimum spanning tree of } G \iff \forall T \in \tau(G) : f(T_0) \leq f(T).}$$

- A graph always has at least one minimum spanning tree and it is not necessarily unique.

- <u>Kruskal's Algorithm</u>

1) Choose $e_1 \in E(G)$ that minimizes $f(e_1)$
2) Assume we have chosen $e_1, e_2, \dots, e_k$.
   Choose $e_{k+1} \in E(G) - \{e_1, e_2, \dots, e_k\}$ such that

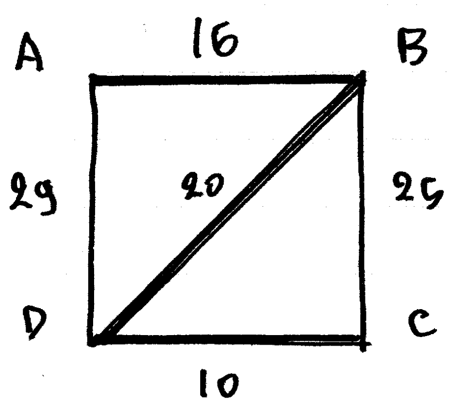
(a) $f(e_{k+1})$ is minimum
(b) The induced graph $G[\{e_1, e_2, \dots, e_{k+1}\}]$ is acyclic.

3) Repeat 2 until e_{k+1} cannot be found.

Upon completion, we have the edges
$e_1, e_2, \dots, e_n$
and the minimum spanning tree is:

$$T_0 = G[\{e_1, e_2, \dots, e_n\}]$$

<u>example</u>

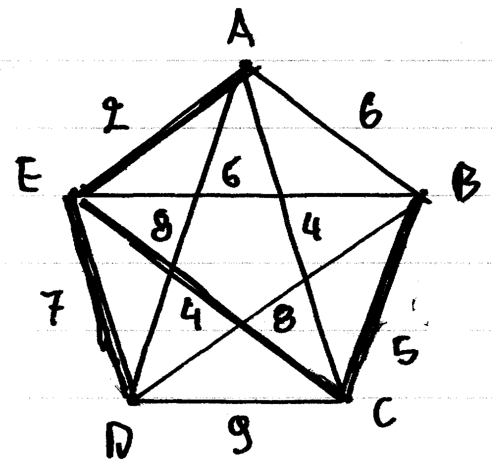


Add DC
Add AB
Add BD

Minimum spanning tree.

$$T_0 = G[\{AB, BD, DC\}]$$

EXAMPLE



Add AE

From AC, CE choose CE

Reject AC (cycle AECA)

Add BC

Reject AB (cycle AECBA)

Reject BE (cycle ECBE)

Add ED

We now have a spanning tree

Thus : $T_0 = G[\{AE, ED, EC, BC\}]$

EXERCISES

(56) Show that if G connected then

$$|E(G)| \geq |V(G)| - 1$$

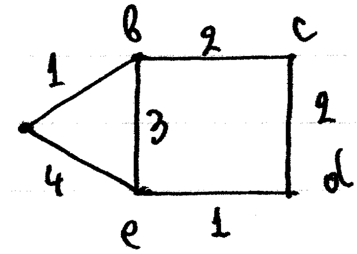
(Hint: Consider the spanning tree of G)

(57) How many spanning trees do the following graphs have?

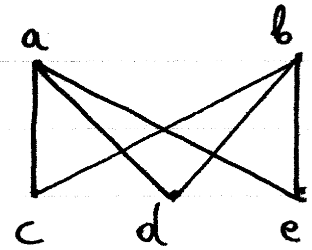
a) K_3 b) K_4 c) K_5

(58) Find the minimum spanning tree for the following graphs:

a)



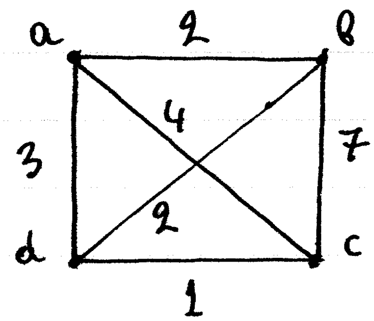
c)



with

$ac = 1$, $ad = 3$, $ae = 2$
$bc = 4$, $bd = 5$, $be = 3$

b)



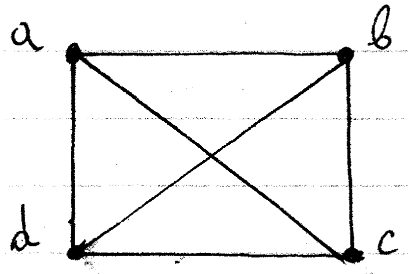
(59) Show that $|\tau(K_{2,n})| = n2^{n-1}$.
(Hint: Try $K_{2,4}$ first as an example then generalize)

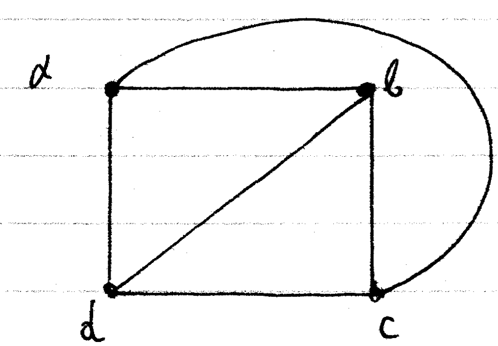
▼ Planar graphs

Let G be a graph. We say that G is <u>planar</u> if and only if it can be embedded on a two-dimensional plane so that no two edges intersect, except at the vertices. A graph whose definition does not seem to be planar could still be planar, using a different embedding. The definition of "planar" requires the existence of at least one such embedding.

EXAMPLE

K_4 is planar. Consider an embedding which is not planar and a different embedding which is planar.





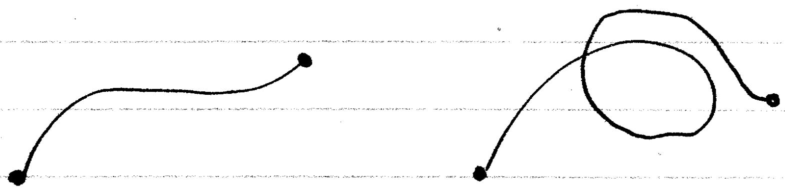
non-planar embedding of K_4

planar embedding of K_4

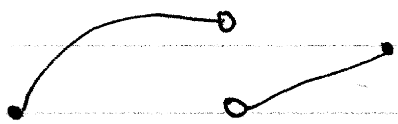
● → Rigorous definition

A rigorous definition of planar graphs is based on the concept of a <u>simple curve</u>. Intuitively, a simple curve is defined as a curve that does not cross itself. We also need the curve to be continuous, with no interruptions.

<u>EXAMPLE</u>



"simple curve" not simple curve



not simple due to lack of continuity.

<u>Def</u> (<u>Simple curves</u>)

(a) A curve (c) defined as

$$(c) : (x,y) = f(t) \quad , \quad t \in [a,b]$$

is the set of points given by:

$$(c) = \{ f(t) \mid t \in [a,b] \} =$$
$$= \{ (x,y) \in \mathbb{R}^2 \mid \exists t \in [a,b] : (x,y) = f(t) \}$$

(b) We say that

$$(c) \text{ simple curve} \Leftrightarrow \begin{cases} f \text{ continuous on } [a,b] \\ f \text{ one-to-one} \end{cases}$$

Remarks

- We denote $C_2[a,b]$ as the set of all mappings $f:[a,b]\to\mathbb{R}^2$ that define a simple curve.
- Note that given two such mappings $f,g\in C_2[0,1]$, the claim that the corresponding curves do not intersect, except possibly at the endpoints can be written concisely as:

$$f((0,1))\cap g((0,1))=\emptyset$$

Using open intervals is needed in order to ignore the endpoints of the two curves.

Based on the above, we define planar graphs as follows:

Def: Let G be a graph. We say that G is planar if and only if there exist

(a) a mapping $f:V(G)\to\mathbb{R}^2$ of vertices to points on the plane $\mathbb{R}^2$

(b) a mapping $g:E(G)\to C_2[0,1]$ of edges onto mappings that represent simple curves.

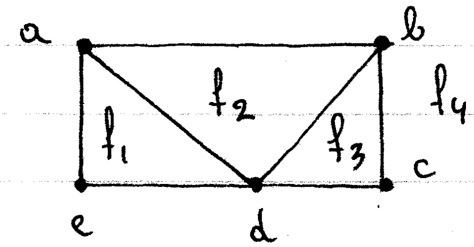
such that the following condition is satisfied:

$$\begin{cases}\forall e\in E(G): [g(e)](\{0,1\})=f(\psi_G(e))\\ \forall e_1,e_2\in E(G): [g(e_1)]((0,1))\cap[g(e_2)]((0,1))=\emptyset\end{cases}$$

Faces of planar graphs

- A planar graph partitions the plane into regions. We call these region faces and the set of all faces of the graph G is denoted $F(G)$. This includes the infinite face which represents the area outside the overall perimeter of the graph G.

EXAMPLE



$F(G) = \{f_1, f_2, f_3, f_4\}$ with f_4 the infinite face.

- For every edge $e \in E(G)$, on either side of the edge e there are one or two faces. We say that those faces are incident upon the edge e, and we define an incidence mapping

$$f_G : E(G) \to \mathcal{P}_1(F(G)) \cup \mathcal{P}_2(F(G))$$

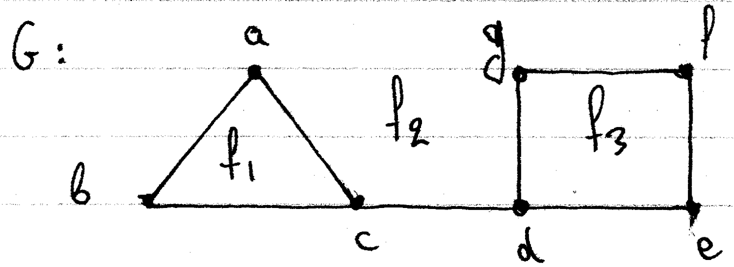
such that

$$\forall e \in E(G) : f_G(e) = \{f_0 \in F(G) \mid f_0 \text{ incident to } e\}$$

$$\boxed{\begin{array}{l}\textbf{Def}: \text{Let } G \text{ be a planar graph and let } f_1, f_2 \in F(G) \\ \quad \text{be two faces of } G. \text{ We say that} \\ f_1, f_2 \text{ adjacent} \iff \exists e \in E(G) : f_G(e) = \{f_1, f_2\}\end{array}}$$

EXAMPLE

Consider the graph



Then:

$$\begin{cases} f_G(ab) = f_G(bc) = f_G(ac) = \{f_1, f_2\} \\ f_G(cd) = \{f_2\} \\ f_G(de) = f_G(ef) = f_G(fg) = f_G(gd) = \{f_2, f_3\}. \end{cases}$$

- Recall that given a connected graph G and an edge $e \in E(G)$, we say that
  $$e \text{ bridge} \iff \omega(G - \{e\}) > \omega(G)$$
  It can be shown that:

Prop: Let G be a planar graph and let $e \in E(G)$ be an edge. Then:
$$e \text{ bridge} \iff |f_G(e)| = 1$$

Dual graph and face degree

Def: Let G be a planar graph. We define the dual graph G^* such that:

$$\begin{cases} V(G^*) = F(G) \\ E(G^*) = E(G) \\ \forall e \in E(G^*): \ \psi_{G^*}(e) = f_G(e) \end{cases}$$

Informally, the vertices of G^* are the faces of G. G^* and G have the same edges. In G^* and edge connects faces of G if in G the two faces are incident to that edge. The definition of G^* allows us to define the degree of faces as follows:

Def: Let G be a planar graph with dual graph G^*. Let $f \in F(G)$ be a face of G. We define the degree $d_G(f)$ such that:

$$d_G(f) = d_{G^*}(f)$$

Note that G^* sees f as a vertex and thus $d_{G^*}(f)$ is the vertex degree of f ~~not~~ with respect to G^*, which was previously defined. An immediate consequence is a handshaking lemma for faces.

Lemma: Let G be a planar graph. Then,

$$\sum_{f \in F(G)} d_G(f) = 2|E(G)|$$

Proof

Let G^* be the dual graph of G. Then:

$$\sum_{f \in F(G)} d_G(f) = \sum_{f \in V(G^*)} d_{G^*}(f) \quad \text{[def. of face degree]}$$
$$= 2|E(G^*)| \quad \text{[handshaking lemma on } G^*]$$
$$= 2|E(G)| \quad \text{[definition of } G^*]$$

which proves the lemma.

Def: Let G be a planar graph. We say that

a) G face-regular $\Leftrightarrow \exists a \in \mathbb{N} : \forall f \in F(G) : d_G(f) = a$

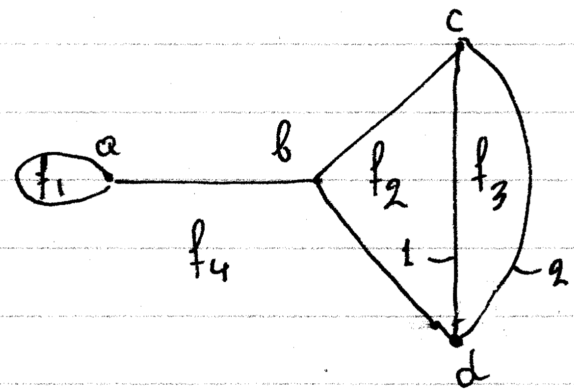
b) G face-regular with regularity $a \in \mathbb{N} \Leftrightarrow$
$\Leftrightarrow \forall f \in F(G) : d_G(f) = a$

An immediate consequence of the handshaking lemma for faces is that

Prop: Let G be a planar graph. Then
G face-regular with face-regularity $\alpha \in \mathbb{N} \Rightarrow$
$\Rightarrow a|F(G)| = 2|E(G)|$

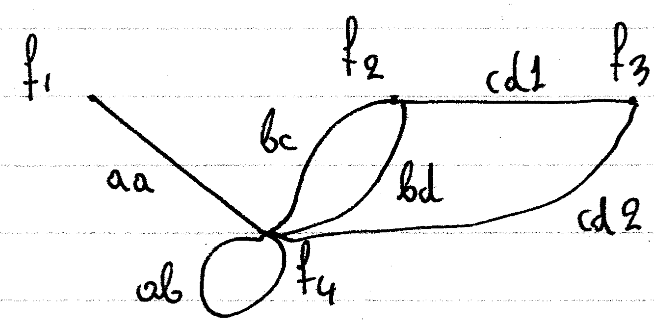
EXAMPLE

For G:



the dual graph is:

G^*:



- Note that a loop in G becomes a bridge in G^*.
- A bridge in G becomes a loop in G^*
- $d_G(f_1) = |\{aa\}| = 1$

  $d_G(f_2) = |\{bc, bd, cd1\}| = 3$

  $d_G(f_3) = |\{cd1, cd2\}| = 2$

  $d_G(f_4) = |\{aa, ab, bc, bd, cd2\}| + |\{ab\}| =$

  $= 5 + 1 = 6.$

Face Boundary and face degree

- Let G be a planar graph. Intuitively, a walk $w \in W(G)$ is the boundary of a face $f \in F(G)$ if and only if it is a closed walk that uses every edge that is not a bridge once, and uses every edge that is indeed a bridge twice, and includes all edges that are incident to the face f.
- The set of all such walks shall be denoted as $\mathcal{B}(f)$.

Writing this definition rigorously is difficult, but it can be effected by introducing some edge-counting notation:

<u>notation</u>: Let G be a graph, let $w \in W(G)$ be a walk on G, and let $e \in E(G)$ be an arbitrary edge. The number of times the edge e is used on the walk w is denoted as $|e|_w$ with

$$\forall e \in E(G): |e|_w = |\{k \in [\ell(w)] \mid e_k(w) = e\}|$$

<u>Def</u>: Let G be a planar graph, let $f \in F(G)$ be a face, and let $w \in W(G)$ be a walk. We say that

$$w \in \mathcal{B}(f) \Leftrightarrow \begin{cases} w \text{ closed walk} \\ \forall e \in E(G): (f \in f_G(e) \Leftrightarrow e \in E(w)) \\ \forall e \in E(w): |e|_w = \begin{cases} 1 & \text{, if } e \text{ is not bridge} \\ 2 & \text{, if } e \text{ is bridge} \end{cases} \end{cases}$$

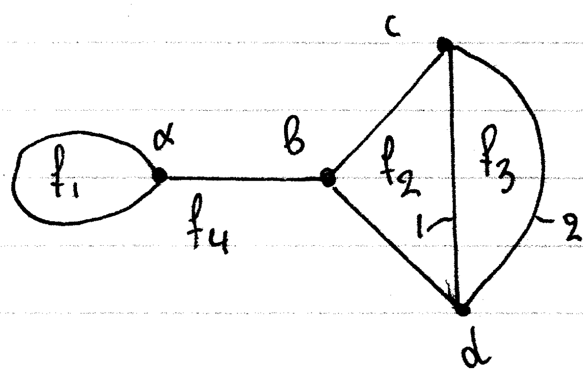
The main result is that the length of all walks $w \in \mathcal{B}(f)$ is equal to $d_G(f)$

> **Prop:** Let G be a planar graph, and let $f \in F(G)$ be a face of G. Then
> $\forall w \in \mathcal{B}(f): \ \ell(w) = d_G(f)$

EXAMPLE

Consider the previous example

G:



$(a, \underline{aa}, a) \in \mathcal{B}(f_1) \Rightarrow d_G(f_1) = 1$

$(b, \underline{bc}, c, \underline{cd1}, d, \underline{db}, b) \in \mathcal{B}(f_2) \Rightarrow d_G(f_2) = 3$

$(c, \underline{cd2}, d, \underline{cd1}, c) \in \mathcal{B}(f_3) \Rightarrow d_G(f_3) = 2$

$(a, \underline{aa}, a, \underline{ab}, b, \underline{bc}, c, \underline{cd2}, d, \underline{bd}, b, \underline{ab}, a) \in \mathcal{B}(f_4)$

$\Rightarrow d_G(f_4) = 6$

Properties of graphs

1 Trees are planar

Thm: Let G be a graph. Then
$$G \text{ tree} \Rightarrow \begin{cases} G \text{ planar} \\ |F(G)| = 1 \end{cases}$$

To show that a tree is planar, we need to define a general embedding for all trees onto $\mathbb{R}^2$. This can be done by choosing a vertex and exploiting the length of the unique path to all other vertices. To show that $|F(G)| = 1$, we note that G has the infinite face, so $|F(G)| \geq 1$. To show that $|F(G)| < 2$ we note that if G has additional faces, those faces are enclosed by cycles, and if G has cycles then it is not a tree. This proof sketch can be made more precise, but doing so is a lot of work.

2 Euler's formula

Thm: Let G be a graph. Then
$$\begin{cases} G \text{ planar} \\ G \text{ connected} \end{cases} \Rightarrow |V(G)| - |E(G)| + |F(G)| = 2$$

Proof

We use the Kruskal algorithm to remove edges that are part of cycles in G until all cycles are eliminated and we obtain a spanning tree T. Let E_0 be the set of all edges removed. Since every edge $e \in E_0$ was part of some cycle, removing each $e \in E_0$ corresponded to merging two faces, thus reducing the number of faces by 1. It follows that $|F(G)| - |E_0| = |F(G - E_0)| = |F(T)| = 1$, since T is a tree. We also note that since

$$T \text{ spanning tree of } G \Rightarrow \begin{cases} |V(T)| = |V(G)| \\ |E(T)| = |V(T)| - 1 \end{cases}$$

and since

$$T = G - E_0 \Rightarrow |E(G)| = |E(T)| + |E_0|$$

From the above, we have:

$$\begin{aligned} |V(G)| - |E(G)| + |F(G)| &= |V(T)| - [|E(T)| + |E_0|] + [1 + |E_0|] \\ &= |V(T)| - |E(T)| - |E_0| + 1 + |E_0| = \\ &= |V(T)| - |E(T)| + 1 = \\ &= |V(T)| - [|V(T)| - 1] + 1 = \\ &= |V(T)| - |V(T)| + 1 + 1 = \\ &= 1 + 1 = 2. \end{aligned}$$

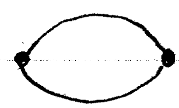
③ Planarity and girth

Def: Let G be a graph. We define the girth $g(G)$ of the graph G as:

$$g(G) = \min\{n \in \mathbb{N}^* \mid \exists H \in \mathcal{P}(G) : H \cong C_n\}$$

Remarks

a) The girth $g(G)$ is equal to the length of the shortest cycle C_n contained in the graph G. Recall that:

C_1 :  and C_2: 

b) If G has a loop then $g(G) = 1$
If G has a multiple edge and no loops, then $g(G) = 2$.
If G is simple (no loops and no multiple edges) then $g(G) \geq 3$.

c) For any face $f \in F(G)$ the length of a boundary $W \in \mathcal{B}(f)$ cannot be shorter than $g(G)$. It follows that:

$$\boxed{G \text{ planar} \Rightarrow \forall f \in F(G) : d_G(f) \geq g(G)}$$

Our main result is the following theorem:

Thm: Let G be a graph. Then:

$$\left.\begin{cases} G \text{ planar} \\ G \text{ connected} \\ G \text{ simple} \end{cases}\right\} \Rightarrow |E(G)| \leqslant \frac{g(G)[|V(G)|-2]}{g(G)-2}$$

Remarks

a) In practice, we use the contrapositive statement to show that a graph is NOT planar:

$$\left.\begin{array}{l} G \text{ connected} \wedge G \text{ simple} \\ |E(G)| > \dfrac{g(G)[|V(G)|-2]}{g(G)-2} \end{array}\right\} \Rightarrow G \text{ not planar}$$

b) It is worth noting that

$$\left.\begin{cases} G \text{ connected} \\ G \text{ simple} \end{cases}\right\} \Rightarrow |V(G)| \geqslant 3$$

which ensures the numerator of the RHS of the inequality is positive. Likewise,

$$G \text{ simple} \Rightarrow g(G) \geqslant 3$$

so the denominator will be positive too.

Proof of theorem

Assume that G is planar, connected, and simple. Since $\forall f \in F(G) : d_G(f) \geqslant g(G)$

$$\Rightarrow 2|E(G)| = \sum_{f\in F(G)} d_G(f) \geqslant \sum_{f \in F(G)} g(G) = g(G)|F(G)| \Rightarrow$$

$$\Rightarrow |F(G)| \leqslant \frac{2}{g(G)} |E(G)|$$

and since

$$\begin{cases} G \text{ planar} \\ G \text{ connected} \end{cases} \Rightarrow |V(G)| - |E(G)| + |F(G)| = 2 \Rightarrow$$

$$\Rightarrow |E(G)| = -2 + |V(G)| + |F(G)| \leqslant$$

$$\leqslant -2 + |V(G)| + \frac{2}{g(G)} |E(G) \Rightarrow$$

$$\Rightarrow \left[1 - \frac{2}{g(G)}\right] |E(G)| \leqslant |V(G)| - 2$$

$$\Rightarrow \frac{g(G) - 2}{g(G)} |E(G)| \leqslant |V(G)| - 2 \qquad (1)$$

and since

G simple $\Rightarrow g(G) \geqslant 3 \Rightarrow g(G) - 2 \geqslant 3 - 2 = 1 > 0 \quad (2)$

combining Eq. (1) and Eq. (2); we get:

$$|E(G)| \leqslant \frac{g(G)[|V(G)| - 2]}{g(G) - 2} \qquad \text{D}$$

Note that with $x = g(G)$, the function

$$f(x) = \frac{x}{x-2}, \quad \forall x \in [3, +\infty)$$

has derivative

$$f'(x) = \frac{d}{dx}\left[\frac{x}{x-2}\right] = \frac{(x)'(x-2) - x(x-2)'}{(x-2)^2}$$

$$= \frac{(x-2)-x}{(x-2)^2} = \frac{-2}{(x-2)^2} < 0 \quad , \quad \forall x \in [3, +\infty) \Rightarrow$$

$\Rightarrow f$ decreasing on $[3, +\infty)$

This means that the inequality in the above theorem becomes tighter as we increase $g(G)$. It also follows that:

$$\boxed{\left.\begin{array}{l} G \text{ planar} \\ G \text{ connected} \\ G \text{ simple} \end{array}\right\} \Rightarrow |E(G)| \leq 3(|V(G)|-2)}$$

Proof

Since $\left\{\begin{array}{l} G \text{ simple} \\ G \text{ connected} \end{array}\right. \Rightarrow \left\{\begin{array}{l} g(G) \geq 3 \\ |V(G)| \geq 3 \end{array}\right.$

it follows that

$$\left.\begin{array}{l} G \text{ planar} \\ G \text{ connected} \\ G \text{ simple} \end{array}\right\} \Rightarrow |E(G)| \leq \frac{g(G)[|V(G)|-2]}{g(G)-2} =$$

$$= f(g(G))\,[|V(G)|-2]$$

$$\leq f(3)\,[|V(G)-2]$$

$$= \frac{3}{3-2}\,[|V(G)|-2]$$

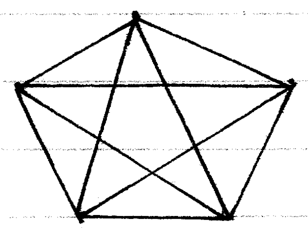
$$= 3(|V(G)|-2) \quad \square$$

APPLICATIONS

a) Show that K_5 is not planar.

<u>Solution</u>

K_5:



We note that $|V(K_5)| = 5$ and $\forall u \in V(K_5): d(u) = 4$, so

$$2|E(K_5)| = \sum_{u \in V(K_5)} d(u) = \sum_{u \in V(K_5)} 4 = 4|V(K_5)| = 4 \cdot 5 = 20 \Rightarrow$$

$$\Rightarrow |E(K_5)| = 10$$

Also

$$3(|V(K_5)| - 2) = 3(5-2) = 3 \cdot 3 = 9 < 10 = |E(K_5)| \Rightarrow$$

$$\Rightarrow |E(K_5)| > 3(|V(K_5)| - 2) \qquad (1)$$

and

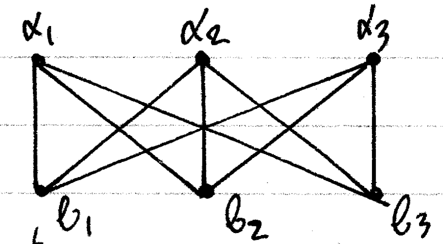
$$\left\{ \begin{array}{l} K_5 \text{ connected} \\ K_5 \text{ simple} \end{array} \right. \qquad (2)$$

From Eq.(1) and Eq.(2): K_5 not planar.

8) Show that $K_{3,3}$ is not planar

<u>Solution</u>

$K_{3,3}$:



We note that

$|V(K_{3,3})| = 3+3 = 6$

and

$\forall u \in V(K_{3,3}) : d(u) = 3$

$\Rightarrow 2|E(K_{3,3})| = \sum_{u \in V(K_{3,3})} d(u) = \sum_{u \in V(K_{3,3})} 3 = 3|V(K_{3,3})| =$

$= 3 \cdot 6 = 18 \Rightarrow$

$\Rightarrow |E(K_{3,3})| = 9.$

Since $K_{3,3}$ simple $\Rightarrow g(G) \geq 3$.

▸ We claim that $K_{3,3}$ cannot have a 3-cycle.

To show this, assume that

$w = (u_1, e_1, u_2, e_2, u_3, e_3, u_1)$

is a 3-cycle. We distinguish between the following cases.

<u>Case 1</u>: Assume that $u_1 \in V_1$. Then,

$u_1 \in V_1 \Rightarrow u_2 \in V_2 \rightarrow u_3 \in V_1 \Rightarrow u_1 \in V_2$

which is a contradiction.

<u>Case 2</u>: Assume that $u_1 \in V_2$. Then,

$u_1 \in V_2 \Rightarrow u_2 \in V_1 \Rightarrow u_3 \in V_2 \Rightarrow u_1 \in V_1$

which is also a contradiction.

In both cases we have a contradiction, so it follows that $K_{3,3}$ has no 3-cycles, therefore $g(G) \geq 4$.
We exhibit a 4-cycle:
$W = (a_1, a_1b_1, b_1, a_2b_1, a_2, a_2b_2, b_2, a_1b_2, a_1)$
$\Rightarrow g(G) \leq 4$.
It follows that $g(K_{3,3}) = 4$, and therefore:

$$\left.\begin{array}{l} \dfrac{g(K_{3,3})[|V(K_{3,3})|-2]}{g(K_{3,3})-2} = \dfrac{4(6-2)}{4-2} = \dfrac{4 \cdot 4}{2} = 8 \\ |E(K_{3,3})| = 9 \end{array}\right\} \Rightarrow$$

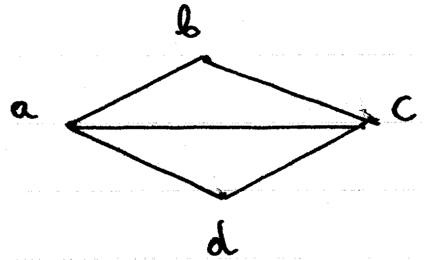
$$\Rightarrow |E(K_{3,3})| > \frac{g(K_{3,3})[|V(K_{3,3})|-2]}{g(K_{3,3})-2} \Rightarrow$$

$\Rightarrow K_{3,3}$ not planar.

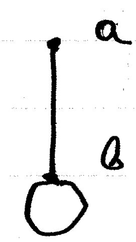
EXERCISES

(60) For the following planar graphs, identify the faces, the degree of each face and then draw the dual graph.

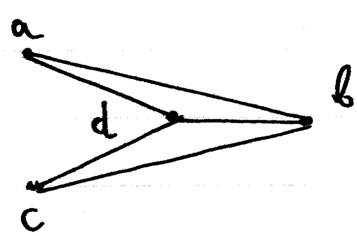
a)



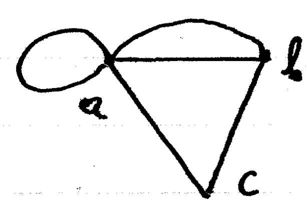
b)



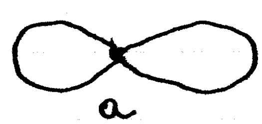
c)



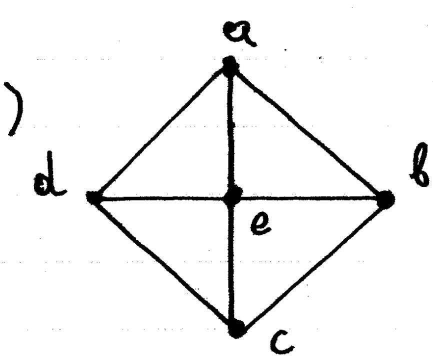
d)



e)



f)



(61) Show that if a connected planar graph is face-regular with face-regularity 4 and has 10 vertices then it must have 8 faces.

(62) Show that a face-regular planar connected graph G with face regularity r must satisfy

$$(a-2)|F(G)| = 2|V(G)| - 4$$

Then show that if $|V(G)| \geq 3$ then $a \geq 3$.

(63) Consider a planar graph G which is simple, connected and regular with regularity a and face-regular with regularity b. Show that $2a+2b-ab$ divides $2ab$.

(Hint: First show that

$$(2a+2b-ab)|E(G)| = 2ab)$$

(64) Show that the following graphs are planar:

a) K_2 b) K_4

b) K_3 c) $K_{2,a}$, for $a \geq 1$

(65) Having shown that $K_{2,a}$ is planar, how many faces does it have, as a function of a?

(66) Show that

$a \geq 5 \Rightarrow K_a$ not planar

$a \geq 3 \Rightarrow K_{3,a}$ not planar

$a \geq 3$ and $b \geq 3 \Rightarrow K_{a,b}$ not planar.

(67) Show that a regular graph with regularity $r > 6$ which is also simple and connected can never be a planar graph.

(68) Show that if G is a planar connected graph which is face regular with face regularity a, then

a) $a-2$ divides $2|V(G)|-4$

b) $a=2 \Rightarrow |V(G)|=2$

c) $|V(G)| \geq 2 \Rightarrow a \geq 2$

[Hint: First show that

$$(a-2)|F(G)| = 2|V(G)|-4).$$