# GRAPH THEORY

## Preliminaries

Let $A$ be a finite set. We define the following notation:

a) $|A|$ is the number of elements of the set $A$.

b) $\mathcal{P}(A)$ is the set of all subsets of $A$, i.e.

$$X \in \mathcal{P}(A) \Leftrightarrow X \subseteq A$$

c) $\mathcal{P}_a(A)$ is the set of all subsets of $A$ with $a$ elements, i.e.

$$X \in \mathcal{P}_a(A) \Leftrightarrow (X \subseteq A \wedge |X| = a)$$

### EXAMPLE

For $A = \{a, b, c\}$, we have:

$|A| = |\{a, b, c\}| = 3$

$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$

$\mathcal{P}_0(A) = \{\emptyset\}$

$\mathcal{P}_1(A) = \{\{a\}, \{b\}, \{c\}\}$

$\mathcal{P}_2(A) = \{\{a, b\}, \{b, c\}, \{c, a\}\}$

$\mathcal{P}_3(A) = \{\{a, b, c\}\}$

**Def**: Let $f: A \to B$ be a mapping from a set $A$ to a set $B$. We say that

$f$ one-to-one $\Leftrightarrow \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$

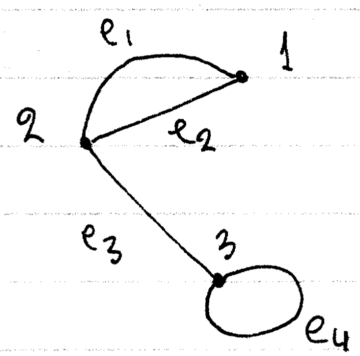
interpretation: A mapping $f: A \to B$ is one-to-one if and only if each element of $A$ is mapped to a distinct element of $B$.

## Graphs – Basic definitions

**Def**: A *graph $G$* is a triplet $G = (V(G), E(G), \psi_G)$ that consists of
a) a set of *vertices* $V(G)$
b) a set of *edges* $E(G)$
c) an *incidence mapping* $\psi_G : E(G) \to \mathcal{P}_1(V(G)) \cup \mathcal{P}_2(V(G))$ that maps every edge to one or two vertices.

### EXAMPLE

$G$:




$V(G) = \{1,2,3\}$
$E(G) = \{e_1, e_2, e_3, e_4\}$
$\psi_G(e_1) = \{1,2\}$
$\psi_G(e_2) = \{1,2\}$
$\psi_G(e_3) = \{2,3\}$
$\psi_G(e_4) = \{3\} \leftarrow$ a *loop*

Equivalently, $\psi_G$ can be rewritten as:
$\psi_G = \{(e_1, \{1,2\}), (e_2, \{1,2\}), (e_3, \{2,3\}), (e_4, \{3\})\}$

## Elementary definitions about graphs

**Def**: Let $G$ be a graph. We say that
a) The vertex $u \in V(G)$ is incident to the edge $e \in E(G)$ if and only if the edge $e$ connects $u$ with itself or with another vertex:
$\forall u \in V(G) : \forall e \in E(G) : u$ incident to $e \Leftrightarrow u \in \psi_G(e)$
b) The edge $e \in E(G)$ is a loop if and only if it connects a vertex with itself.
$\forall e \in E(G) : (e \text{ loop} \Leftrightarrow |\psi_G(e)| = 1)$

**Def**: Let $G$ be a graph and let $u \in V(G)$ be a vertex of $G$. The degree $d(u)$ of $u$ is given by
$d(u) = |\{e \in E(G) \mid u \in \psi_G(e)\}| + |\{e \in E(G) \mid u \in \psi_G(e) \land e \text{ loop}\}|$

The degree $d(u)$ is the number of edges $e \in E(G)$ to which $u$ is incident, with the loops being counted twice. In the previous example:
$d(1) = 2 \land d(2) = 3 \land d(3) = 3$

**Def**: Let $G$ be a graph. We define:
a) The minimum degree $\delta(G) = \min\limits_{u \in V(G)} d(u)$
b) The maximum degree $\Delta(G) = \max\limits_{u \in V(G)} d(u)$

**Remark**: It follows from this definition that
$$\forall u \in V(G): \delta(G) \leqslant d(u) \leqslant \Delta(G)$$

**Lemma**: **(The Handshaking lemma)**

Let $G$ be a graph. Then:

$$\sum_{u \in V(G)} d(u) = 2|E(G)|$$

↳ Although this is not a formal proof, a simple explanation of the handshaking lemma is that every edge is usually shared by two vertices. As a result, adding up the degrees of all vertices counts the edges twice. Although loops attach to only one vertex, the loop, by definition, adds 2 to the degree of that vertex, so loops are also counted twice

# EXAMPLES

a). Show that it is not possible to define a graph with 3 vertices of degree 2 and 5 vertices of degree 3.

Solution

Assume a graph $G$ exists with $V(G) = \{u_1, u_2, u_3, v_1, v_2, v_3, v_4, v_5\}$ such that

$$\begin{cases} \forall a \in [3]: \ d(u_a) = 2 \\ \forall a \in [5]: \ d(v_a) = 3 \end{cases}$$

It follows that

$$2|E(G)| = \sum_{u \in V(G)} d(u) = \sum_{a \in [3]} d(u_a) + \sum_{b \in [5]} d(v_a) =$$

$$= 3 \cdot 2 + 5 \cdot 3 = 6 + 15 = 21 \Rightarrow$$

$\Rightarrow |E(G)| = 21/2 \leftarrow$ contradiction since $|E(G)|$ is a natural number. It follows that $G$ cannot be constructed.

b) A graph has twice as many vertices as edges. Show that there is at least one vertex with degree less than 2.

Solution

Let $G$ be the graph with $2|E(G)| = |V(G)|$. Then:

$$2|E(G)| = \sum_{u \in V(G)} d(u) \geqslant \sum_{u \in V(G)} \delta(G) = \delta(G)\,|V(G)| =$$

$$= \delta(G)\, 2|E(G)| \Rightarrow 2|E(G)| \geqslant \delta(G)\, 2|E(G)| \Rightarrow$$

$$\Rightarrow \delta(G) \leqslant 1 \Rightarrow \delta(G) = 0 \lor \delta(G) = 1 \Rightarrow$$

$\Rightarrow \exists u \in V(G): (d(u) = 0 \vee d(u) = 1)$

$\Rightarrow \exists u \in V(G): d(u) < 2.$

c) Let $G$ be a graph with $\Delta(G) = 2$. Show that the graph cannot have more edges than vertices.

Solution

Since $\Delta(G) = 2 \Rightarrow \forall u \in V(G): d(u) \leq 2$

$$\Rightarrow 2|E(G)| = \sum_{u \in V(G)} d(u) \leq \sum_{u \in V(G)} \Delta(G) = \Delta(G)|V(G)|$$

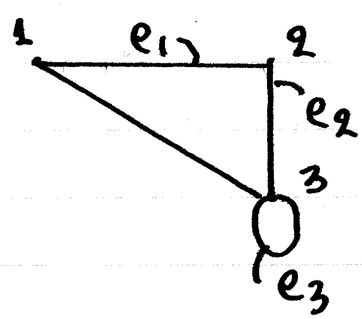
$$= 2|V(G)| \Rightarrow 2|E(G)| \leq 2|V(G)| \Rightarrow$$

$$\Rightarrow |E(G)| \leq |V(G)|.$$

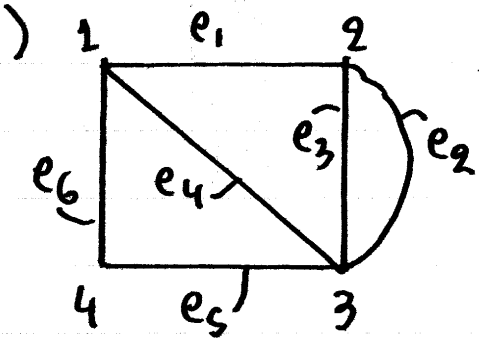
# EXERCISES

① For the following graphs, list $V(G)$, $E(G)$, and the values of the incidence mapping $\psi_G$:

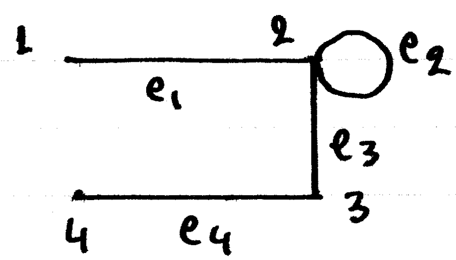
a)



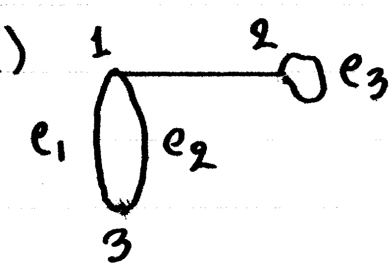

b)




c)




d)




② For the graphs of the previous exercise, list the degrees of each vertex and write $\delta(G)$ and $\Delta(G)$.

③ Show that it is not possible to create a graph with 9 vertices such that the degree of every vertex is 3.

(4) Show that it is not possible to create a graph with 7 vertices of degree 3 and 2 vertices of degree 2.

(5) Let $G$ be a graph with 10 vertices such that

$$\delta(G) = \Delta(G) = 2$$

How many edges does $G$ have?

(6) Let $G$ be a graph with $|V(G)| = 8$ such that $\Delta(G) = 4$. Show that $|E(G)| < 36$.

(7) Let $G$ be a graph such that $|V(G)| = |E(G)|$. Show that $\delta(G) < 3$

(8) Let $G$ be a graph with $\Delta(G) = 4$. Show that

$$|E(G)| \leq 2|V(G)|$$

(9) A graph with 4 edges has a vertex with degree 4, a vertex with degree 1 and one more vertex. What is the degree of the third vertex?

## Types of graphs

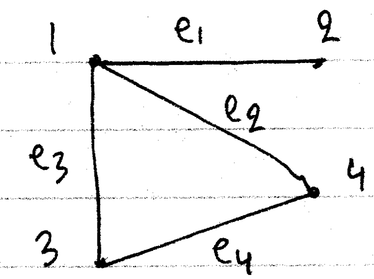
① → Simple graphs

**Def** : A graph $G$ is simple if and only if it has no loops and no multiple edges, i.e.

$$G \text{ simple} \Leftrightarrow \begin{cases} \forall e \in E(G) : |\psi_G(e)| = 2 \\ \psi_G \text{ is one-to-one} \end{cases}$$

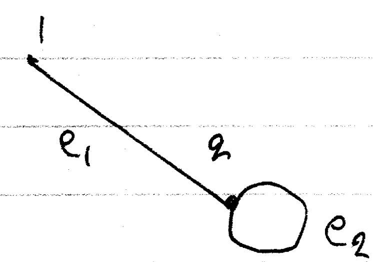
EXAMPLE

$G_1$:



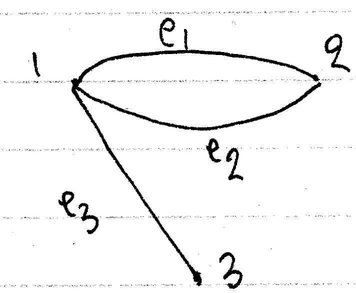


$G_2$:



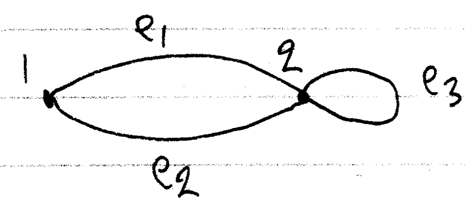


$G_3$:





$G_4$:





$G_1$ is simple.

$G_2, G_3, G_4$ are NOT simple.

## (2) → Regular graphs

**Def**: Let $G$ be a graph. We say that

a) $G$ is <u>regular</u> if and only if all vertices have the same degree, i.e.

$G$ regular $\Leftrightarrow \forall u_1, u_2 \in V(G): d(u_1) = d(u_2)$

$\Leftrightarrow \exists a \in \mathbb{N}: \forall u \in V(G): d(u) = a$

b) $G$ is <u>a-regular</u> if and only if all vertices have degree equal to $a$, i.e

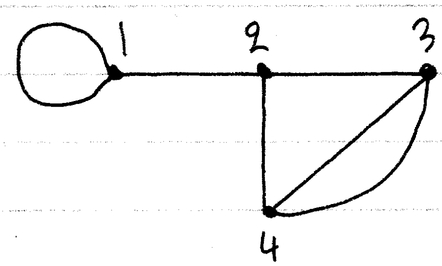
$G$ a-regular $\Leftrightarrow \forall u \in V(G): d(u) = a$

**Remark**: Note the negation of this definition:

$G$ not regular $\Leftrightarrow \exists u_1, u_2 \in V(G): d(u_1) \neq d(u_2)$

### EXAMPLE

$G$:




$d(1) = d(2) = d(3) = d(4) = 3 \Rightarrow$

$\Rightarrow \forall u \in V(G): d(u) = 3$

$\Rightarrow G$ 3-regular

Also: $G$ is regular.

## (3) Complete graphs

A complete graph is a simple graph such that any two distinct vertices are connected by an edge. The formal definition reads:

**Def**: Let $G$ be a graph. We say that

$$G \text{ complete} \iff \begin{cases} G \text{ simple} \\ \forall u_1, u_2 \in V(G) : (u_1 \neq u_2 \implies \exists e \in E(G) : \psi_G(e) = \{u_1, u_2\}) \end{cases}$$

Given the number of vertices $n$, there is a unique graph such that

$$\begin{cases} G \text{ complete} \\ |V(G)| = n \end{cases}$$

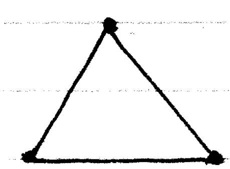
which we denote as $K_n$.

EXAMPLES

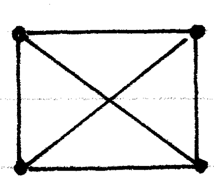
$K_1$:

$K_2$:

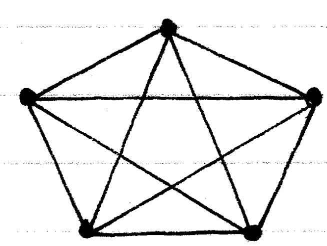
$K_3$:



$K_4$:



$K_5$:



## ④ → Null graphs

A null graph is a graph with no edges. The formal definition is:

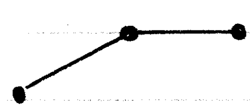
> **Def**: Let $G$ be a graph. We say that $G$ is a null graph $\iff E(G) = \emptyset$

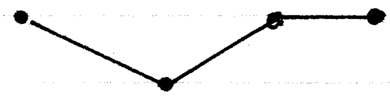
The null graph with $n$ vertices is unique and denoted as $N_n$.

## ⑤ → The path graph $P_n$

> **Def**: We distinguish between the following cases.
>
> Case 1: For $n = 1$, we define
>
> $$V(P_1) = \{u_1\} \wedge E(P_1) = \emptyset \wedge \psi_{P_1} = \emptyset$$
>
> Case 2: For $n \geq 2$, we define
>
> $$\begin{cases} V(P_n) = \{u_1, u_2, \dots, u_n\} \\ E(P_n) = \{e_1, e_2, \dots, e_{n-1}\} \\ \forall k \in [n-1] : \psi_{P_n}(e_k) = \{u_k, u_{k+1}\} \end{cases}$$

### EXAMPLES

$P_1$:  $P_2$:  $P_3$: 

$P_4$: 

## 6 → The cycle graph $C_n$

**Def**: We distinguish between the following cases:

**Case 1**: For $n=1$, we have

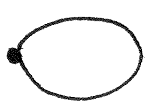
$$\begin{cases} V(C_1) = \{u_1\} \\ E(C_1) = \{e_1\} \\ \psi_{C_1}(e_1) = \{u_1\} \end{cases}$$

**Case 2**: For $n \geq 2$, we have

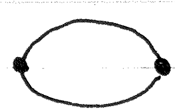
$$\begin{cases} V(C_n) = \{u_1, u_2, \dots, u_n\} \\ E(C_n) = \{e_1, e_2, \dots, e_n\} \\ \forall k \in [n-1] : \psi_{C_n}(e_k) = \{u_k, u_{k+1}\} \\ \psi_{C_n}(e_n) = \{u_n, u_1\} \end{cases}$$

### EXAMPLES

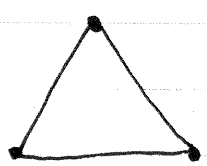
$C_1$ :



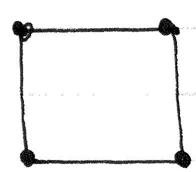
$C_2$ :



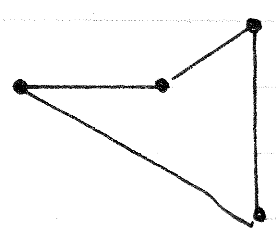
$C_3$ :



$C_4$ :



$C_5$ :



## (7) → Bipartite graphs

- A graph $G$ is called <u>bipartite</u> if and only if its vertex $V(G)$ can be partitioned to two sets $V_1$ and $V_2$ such that every edge of $G$ connects a vertex in $V_1$ with a vertex in $V_2$. The formal definition is:

> <u>Def</u>: Let $G$ be a graph. We say that
>
> a) $G$ <u>bipartite with vertex partition</u> $V_1, V_2 \iff$
>
> $$\begin{cases} V_1 \cup V_2 = V(G) \\ V_1 \cap V_2 = \varnothing \\ \forall e \in E(G): \begin{cases} |\psi_G(e) \cap V_1| = 1 \\ |\psi_G(e) \cap V_2| = 1 \end{cases} \end{cases}$$
>
> b) $G$ bipartite $\iff \exists V_1, V_2 \in \mathcal{P}(V(G))$ : $G$ bipartite with vertex partition $V_1, V_2$.

- A complete bipartite graph is bipartite with some vertex partition $V_1, V_2$, simple, and every vertex of $V_1$ is connected with every vertex of $V_2$ with exactly one edge. The formal definition is:

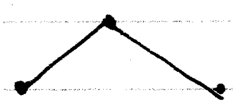
> <u>Def</u>: Let $G$ be a graph. We say that:
>
> $G$ <u>complete bipartite with vertex partition</u> $V_1, V_2 \iff$
>
> $$\iff \begin{cases} G \text{ bipartite with vertex partition } V_1, V_2 \\ G \text{ simple} \\ \forall u_1 \in V_1 : \forall u_2 \in V_2 : \exists e \in E(G) : \psi_G(e) = \{u_1, u_2\} \end{cases}$$

- There is a unique graph $G$ such that
$$\begin{cases} G \text{ complete bipartite with vertex partition } V_1, V_2 \\ |V_1| = n \wedge |V_2| = m \end{cases}$$
with $n, m \in \mathbb{N}^*$, and it is denoted as $K_{n,m}$.

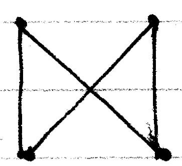
EXAMPLES

$K_{1,1}$:

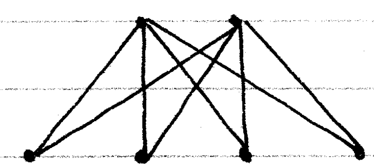
$K_{1,2}$:



$K_{2,2}$:



$K_{2,4}$:



# EXAMPLES

a) Evaluate $\delta(K_{a,b})$, $\Delta(K_{a,b})$, and $|E(K_{a,b})|$ for the complete bipartite graph $K_{a,b}$.

Solution

Let $V(K_{a,b}) = V_1 \cup V_2$ with $|V_1| = a$ and $|V_2| = b$.

Each vertex of $V_1$ connects to all vertices of $V_2$, therefore

$\forall u \in V_1 : d(u) = |V_2| = b.$ (1)

Similarly, each vertex of $V_2$ connects to all vertices of $V_1$, and therefore:

$\forall u \in V_2 : d(u) = |V_1| = a$ (2)

It follows that

$$\delta(K_{a,b}) = \min_{u \in V(K_{a,b})} d(u) = \min\{a, b\}$$

$$\Delta(K_{a,b}) = \max_{u \in V(K_{a,b})} d(u) = \max\{a, b\}$$

$$2|E(K_{a,b})| = \sum_{u \in V(K_{a,b})} d(u) = \sum_{u \in V_1} d(u) + \sum_{u \in V_2} d(u) =$$

$$= |V_1| b + |V_2| a = ab + ba = 2ab \Rightarrow$$

$$\Rightarrow |E(K_{a,b})| = ab.$$

b) Show that $P_{10}$ is not regular.

Solution

Let $V(P_{10}) = \{u_1, u_2\} \cup V_0$ with $u_1, u_2$ the endpoint vertices and $V_0 = \{v_1, v_2, \ldots, v_8\}$ the interior vertices. We note that $d(u_i) = 1$ and $d(v_i) = 2$. It follows that

$\exists u, v \in V(P_{10}) : d(u) \neq d(v)$

$\Rightarrow \overline{\forall u, v \in V(P_{10}) : d(u) = d(v)}$

$\Rightarrow$ $P_{10}$ is not regular.

c) Show that if $G$ is regular, then

$$|E(G)| = \frac{\delta(G)\,|V(G)|}{2}$$

Solution

Assume $G$ is regular. Then

$G \text{ regular} \Rightarrow \forall u, v \in V(G) : d(u) = d(v)$

$\Rightarrow \exists a \in \mathbb{N} : \forall u \in V(G) : d(u) = a$

It follows that

$$\delta(G) = \min_{u \in V(G)} d(u) = \min_{u \in V(G)} a = a$$

and therefore:

$$|E(G)| = \frac{1}{2} \sum_{u \in V(G)} d(u) = \frac{1}{2} \sum_{u \in V(G)} a =$$

$$= \frac{a\,|V(G)|}{2} = \frac{\delta(G)\,|V(G)|}{2}$$

# EXERCISES

(10) Draw the following graphs:

a) $K_4$ d) $K_{1,3}$ g) $P_4$
b) $K_5$ e) $K_{2,2}$ h) $C_3$
c) $K_6$ f) $K_{3,3}$ i) $C_4$

(11) Which of the graphs in the previous exercise are regular?

(12) For $a, b$ integers $a > 0$ and $b > 0$ evaluate the following:

a) $\delta(K_a)$ e) $\Delta(K_a)$ i) $|E(K_a)|$
b) $\delta(K_{a,b})$ f) $\Delta(K_{a,b})$ j) $|E(K_{a,b})|$
c) $\delta(P_a)$ g) $\Delta(P_a)$ k) $|E(P_a)|$
d) $\delta(C_a)$ h) $\Delta(C_a)$ l) $|E(C_a)|$

[You can check your general answers by testing them when $a=2$, $b=3$ or $a=4$, $b=3$]

(13) Show that

$K_{a,b}$ regular $\Leftrightarrow a = b$

(14) Show that $K_{5,7}$ is not regular.

(15) Show that
G regular $\Leftrightarrow \delta(G) = \Delta(G)$.

(16) Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$.
If $|V_1| = a$ and $|V_2| = a+2$
show that
$$|E(G)| \leq a^2 + 2a$$

(17) Show that we cannot build a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = 4$ and $|V_2| = 3$ and $|E(G)| > 14$.

## Graph operations

We define the following graph operations.

### ① → Induced subgraph

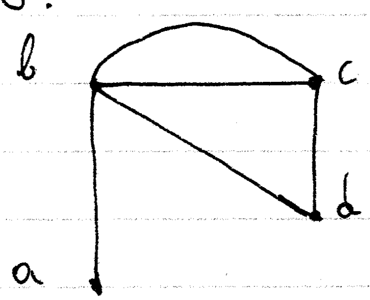
<u>Def</u>: Let $G$ be a graph and let $V_0 \subseteq V(G)$. We define the vertex induced subgraph $G[V_0]$ such that

$$V(G[V_0]) = V_0$$
$$E(G[V_0]) = \{e \in E(G) \mid \psi_G(e) \subseteq V_0\}$$
$$\forall e \in E(G[V_0]) : \psi_{G[V_0]}(e) = \psi_G(e)$$

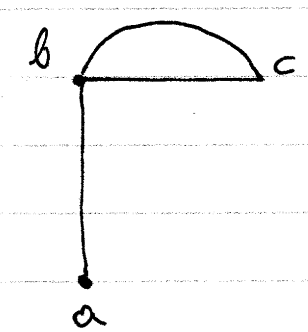
- Intuitively, the vertex induced subgraph $G[V_0]$ consists of the vertices $V_0$, the edges that are incident only to vertices in $V_0$, connected similarly as in $G$.

EXAMPLE

$G$:




$G[\{a,b,c\}]$:




Note that the removal of the vertex $d$ removes the two edges that are incident to it.

## (2) Vertex subtraction

**Def**: Let $G$ be a graph and let $V_0 \subseteq V(G)$. We define the graph $G - V_0$ such that
$$G - V_0 = G[V(G) - V_0]$$

Intuitively, $G - V_0$ is the graph obtained by deleting from $G$, the vertices in $V_0$ and all edges to which these vertices are incident.

## (3) Edge-induced subgraph

**Def**: Let $G$ be a graph and let $E_0 \subseteq E(G)$. We define the graph $G[E_0]$ such that
$$\begin{cases} V(G[E_0]) = V(G) \\ E(G[E_0]) = E_0 \\ \forall e \in E_0 : \psi_{G[E_0]}(e) = \psi_G(e) \end{cases}$$

Intuitively, $G[E_0]$ consists of all the vertices of the original graph $G$ but only the edges that belong to $E_0$.

## (4) Edge subtraction

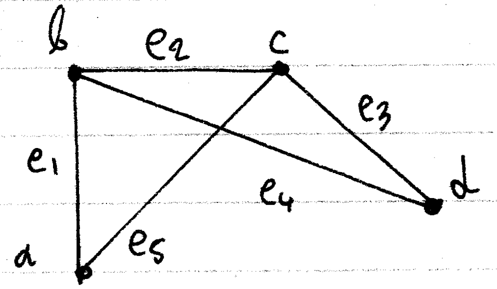
**Def**: Let $G$ be a graph and let $E_0 \subseteq E(G)$. We define the graph $G - E_0$ as:

$$G - E_0 = G[E(G) - E_0]$$

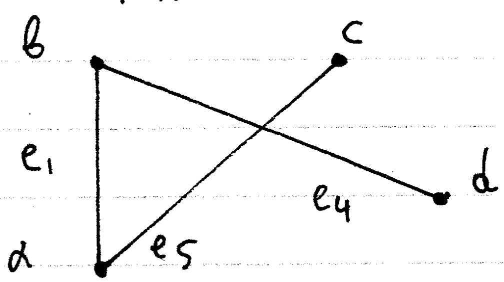
- Note that, unlike vertex subtraction, subtracting edges does not remove vertices under any circumstances.

**EXAMPLE**

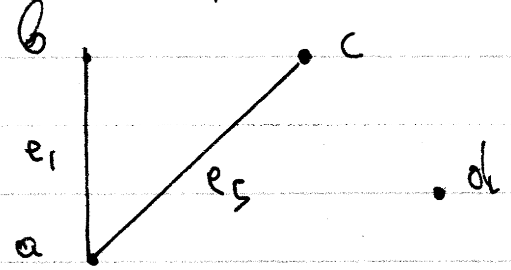
$G$:




$G - \{e_2, e_3\}$:




$G - \{e_2, e_3, e_4\}$:



## (5) Graph union

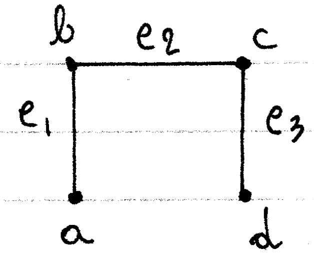
A necessary condition for defining the graph union $G_1 \cup G_2$ of two graphs $G_1, G_2$ is that $G_1, G_2$ should not share any edges, though they may share vertices.
The formal definition is:

**Def**: Let $G_1, G_2$ be two graphs such that $E(G_1) \cap E(G_2) = \emptyset$.
We define the graph union $G = G_1 \cup G_2$ such that

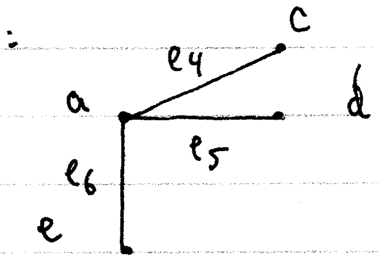
$$\begin{cases} V(G) = V(G_1) \cup V(G_2) \\ E(G) = E(G_1) \cup E(G_2) \\ \forall e \in E(G) : \psi_G(e) = \begin{cases} \psi_{G_1}(e) & \text{, if } e \in E(G_1) \\ \psi_{G_2}(e) & \text{, if } e \in E(G_2) \end{cases} \end{cases}$$

### EXAMPLE

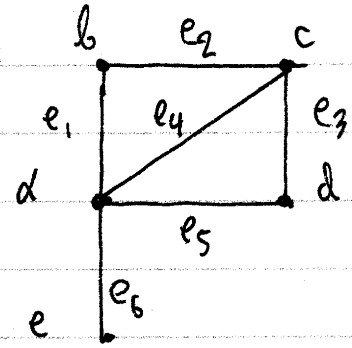
$G_1$:




$G_2$:




$G_1 \cup G_2$:

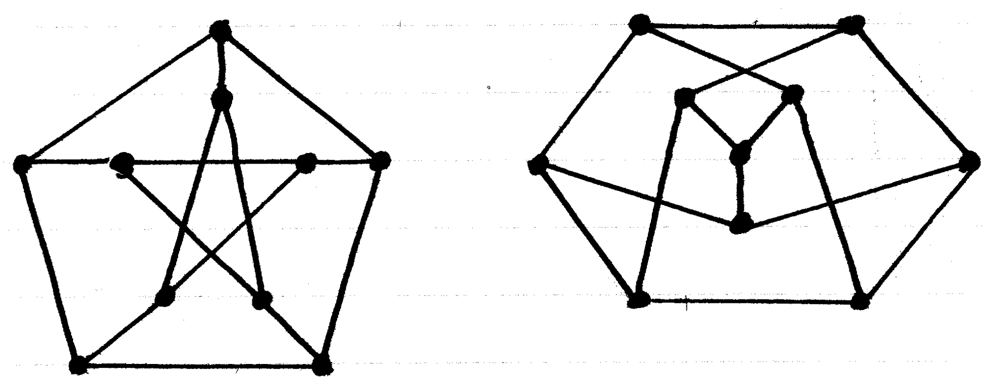


This definition generalizes to the union of $n$ graphs as follows:

**Def**: Let $G_1, G_2, \ldots, G_n$ be graphs such that
$$\forall k, m \in [n] : (k \neq m \Rightarrow E(G_k) \cap E(G_m) = \emptyset)$$
We define the graph $G = G_1 \cup G_2 \cup \cdots \cup G_n$ such that:
$$\begin{cases} V(G) = \bigcup\limits_{a \in [n]} V(G_a) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n) \\ E(G) = \bigcup\limits_{a \in [n]} E(G_a) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n) \\ \forall e \in E(G) : \forall k \in [n] : (e \in E(G_k) \Rightarrow \psi_G(e) = \psi_{G_k}(e)) \end{cases}$$

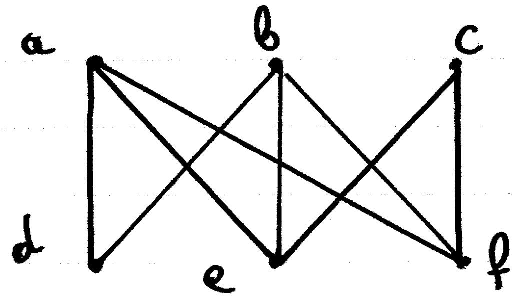
# EXERCISES

(18) Show that the following graphs are isomorphic



(Hint: Look at the "cycles")

(19) Consider the graph $K_{3,3} = G$




Draw the following:

a) $G[\{a,b,d\}]$
b) $G[\{a,d,e,f\}]$
c) $G[\{a,b,d,e\}]$
d) $G-\{a\}$
e) $G-\{a,b\}$
f) $G-\{a,d\}$
g) $G-\{c,d,e\}$
h) $G-\{d,e,f\}$
i) $G-\{a,c,e,f\}$

(20) In the previous exercise, let

$G_1 = G - \{a, c, e, f\}$

$G_2 = G[\{a, b, e\}]$

Draw $G_1 \cup G_2$.

[Hint : List $V(G_1), E(G_1), V(G_2), E(G_2)$ first].

(21) In the previous exercise show that

$G[\{a, d\}] \cup G[\{b, e\}] \neq G[\{a, d, b, e\}]$

## Connected graphs

### Walks, trails, and paths

**Def**: Let $G$ be a graph. A walk $w$ is an $n$-tuple of alternating vertices and edges of the form

$$w = (u_0, e_1, u_1, e_2, u_2, \dots, e_n, u_n)$$

such that

$$\begin{cases} \forall a \in [n] \cup \{0\} : u_a \in V(G) \\ \forall a \in [n] : e_a \in E(G) \\ \forall a \in [n] : \psi_G(e_a) = \{u_{a-1}, u_a\} \end{cases}$$

- Features of a walk

a) Starting vertex: $s(w) = u_0$

b) Terminal vertex: $t(w) = u_n$

c) Length of walk: $\ell(w) = n$

d) Vertex function $u[w] : [n] \cup \{0\} \to V(G)$ is given by
$\forall a \in [n] \cup \{0\} : u[w](a) = u_a$

e) Edge function $e[w] : [n] \to E(G)$ is given by
$\forall a \in [n] : e[w](a) = e_a$

f) Vertex set $V(w)$ is given by
$V(w) = u[w]([n] \cup \{0\}) = \{u_a \mid a \in [n] \cup \{0\}\}$

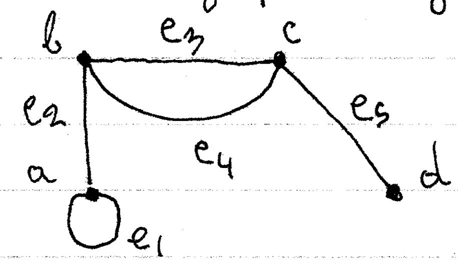
g) Edge set $E(w)$ is given by
$E(w) = e[w]([n]) = \{e_a \mid a \in [n]\}$

- The set of all walks on $G$ is denoted as $\underline{\underline{W(G)}}$.

## EXAMPLE

Consider the graph $G$ given by:

$G$:




Then, $w = (a, e_1, a, e_2, b, e_4, c)$ is a walk with
$s(w) = a \wedge t(w) = c \wedge l(w) = 3$
$V(w) = \{a, b, c\}$
$E(w) = \{e_1, e_2, e_4\}$
We also note that
$u[w](0) = a \wedge u[w](1) = a \wedge u[w](2) = b \wedge u[w](3) = c$
$e[w](1) = e_1 \wedge e[w](2) = e_2 \wedge e[w](3) = e_4$

- A trail is a walk in which all edges visited are distinct. A path is a walk in which all edges and vertices visited are distinct. The formal definition is as follows:

**Def**: Let $G$ be a graph and let $w \in W(G)$ be a walk. We say that

$w$ trail $\Leftrightarrow$ $e[w]$ is one-to-one

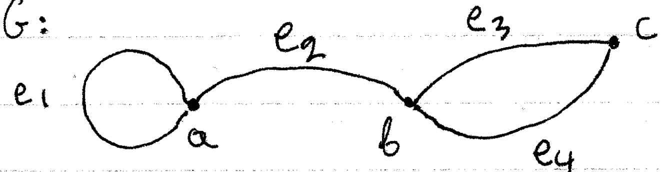
$w$ path $\Leftrightarrow \begin{cases} w \text{ trail} \\ u[w] \text{ is one-to-one} \end{cases}$

Recall that for a general mapping $f: A \to B$, we say that $f$ one-to-one $\iff \forall x_1, x_2 \in A: (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.

## EXAMPLE

Consider the graph $G$ given by

$G$:




and the following walks:

$W_1 = (a, e_1, a, e_2, b)$

$W_3 = (c, e_4, b, e_2, a)$

Then, $W_1$ is a trail but not a path, and $W_2$ is a path.

<u>notation</u>: Let $G$ be a graph and let $u, v \in V(G)$ be vertices. Then, we define

a) The set $T(G)$ of all trails

$T(G) = \{w \in W(G) \mid w \text{ trail}\}$

b) The set $P(G)$ of all paths

$P(G) = \{w \in W(G) \mid w \text{ paths}\}$

c) The set $T(G, u \to v)$ of all trails that connect $u$ to $v$

$T(G, u \to v) = \{w \in T(G) \mid s(w) = u \wedge t(w) = v\}$

d) The set $P(G, u \to v)$ of all paths that connect $u$ to $v$

$P(G, u \to v) = \{w \in P(G) \mid s(w) = u \wedge t(w) = v\}$

- Note that $W(G)$ is an infinite set (e.g. you can go back and forth between two vertices indefinitely) but $T(G)$ and $P(G)$ are both finite sets, because as you construct a trail or a path you ~~rece~~ inevitably run out of edges, since every edge can be used only once.

▸ Connected graphs

**Def:** Let $G$ be a graph. We say that
$$G \text{ connected} \Leftrightarrow \forall u,v \in V(G): (u \neq v \Rightarrow |P(G, u \to v)| \geq 1)$$

interpretation: A graph $G$ is connected if and only if for any two distinct vertices $u, v$ there is at least one path from $u$ to $v$.

- The following graphs are obviously connected.

a) Complete graph $K_n$
b) Path graph $P_n$
c) Cycle graph $C_n$
d) The complete bipartite graph $K_{m,n}$

- It is useful to note that:

$$\boxed{G \text{ connected} \Rightarrow \forall u \in V(G): d(u) > 0}$$

## Graph components

**Thm:** Let $G$ be a graph which is not connected. Then, the vertex set $V(G)$ can be partitioned to $n$ subsets $V_1, V_2, \dots, V_n$ such that:

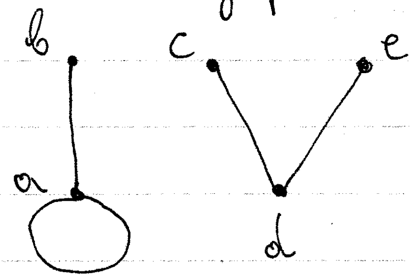
$$\begin{cases} \forall a, b \in [n] : (a \neq b \Rightarrow V_a \cap V_b = \emptyset) \\ V_1 \cup V_2 \cup \dots \cup V_n = V(G) \\ \forall a \in [n] : G[V_a] \text{ connected} \\ G[V_1] \cup G[V_2] \cup \dots \cup G[V_n] = G \end{cases}$$

- The connected subgraphs $G[V_1], G[V_2], \dots, G[V_n]$ are the <u>components</u> of $G$.
- We denote $w(G) = n$ to be the number of components of $G$.
- An obvious consequence of the above is that
  $G$ connected $\Leftrightarrow w(G) = 1$
  $G$ not connected $\Leftrightarrow w(G) > 1$.

### EXAMPLE

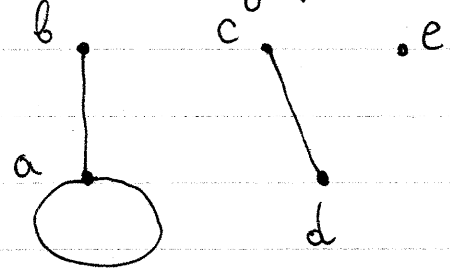
Consider the graph $G$ given by:

$G$:



This graph has two components
$G = G[\{a,b\}] \cup G[\{c,d,e\}]$
and thus $\omega(G) = 2$. Note that if we attempt a finer vertex partition, such as
$G' = G[\{a,b\}] \cup G[\{c,d\}] \cup G[\{e\}]$
we obtain the graph $G' = G - \{de\} \neq G$

$G'$:




➢ <u>Bridges</u>

> **Thm:** Let $G$ be a graph. Then, we have:
> $\forall e \in E(G): \omega(G) \leq \omega(G - \{e\}) \leq \omega(G) + 1$

<u>interpretation:</u> For any graph $G$, removing one edge may or may not increase the number of components by 1.

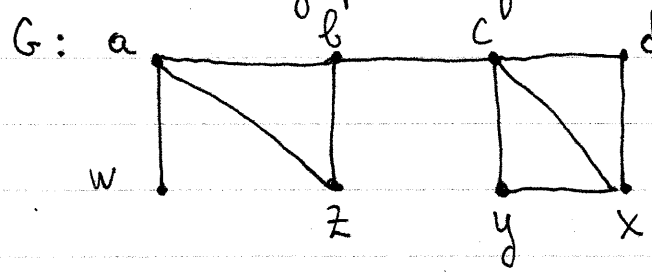
- This theorem cannot be generalized to the deletion of vertices.

> **Def:** Let $G$ be a graph and let $e \in E(G)$. We say that
> $e$ bridge of $G \iff \omega(G - \{e\}) > \omega(G)$.

<u>Interpretation</u> : An edge $e \in E(G)$ is a <u>bridge</u> of $G$ if and only if removing the edge $e$ from the graph $G$ increases the number of components in the resulting graph.

## EXAMPLE

Consider the graph $G$ given by:




The edges $aw$ and $bc$ are bridges of $G$, because

$$\begin{cases} G - \{aw\} = G[\{w\}] \cup G[\{a,b,c,d,x,y,z\}] \\ G - \{bc\} = G[\{a,b,z,w\}] \cup G[\{c,d,x,y\}] \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \omega(G - \{aw\}) = 2 > \omega(G) \\ \omega(G - \{bc\}) = 2 > \omega(G) \end{cases} \Rightarrow$$

$\Rightarrow aw, bc$ bridges of $G$.

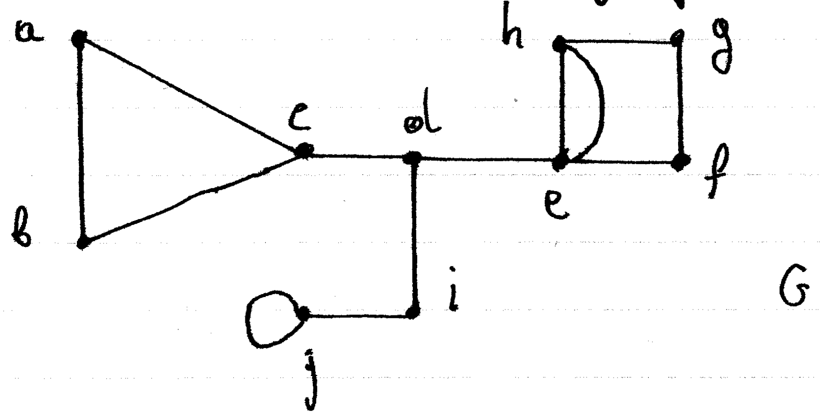
<u>Def</u>: Let $G$ be a graph. We say that

$G$ <u>strongly linked</u> $\Leftrightarrow \forall e \in E(G) : G - \{e\}$ connected

$G$ <u>weakly linked</u> $\Leftrightarrow \begin{cases} G \text{ connected} \\ \exists e \in E(G) : G - \{e\} \text{ not connected} \end{cases}$

# EXERCISES

(22) Consider the following graph




a) List the components of the following graphs:

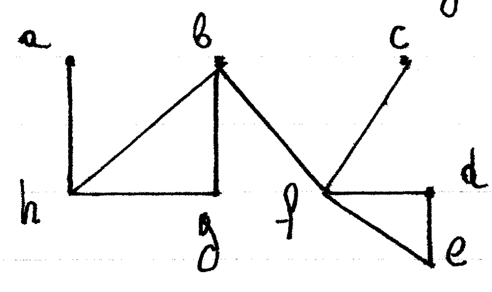
$G_1 = G - \{c\}$  $G_4 = G - \{e\}$

$G_2 = G - \{d\}$  $G_5 = G - \{he, gf\}$

$G_3 = G - \{i\}$  $G_6 = G - \{di\}$

b) What are the bridges of the graph $G$?

(23) Consider the following graph



a) List the components of the following graphs:

$G_1 = G - \{bf\}$ $\qquad$ $G_4 = G - \{ah, bg\}$

$G_2 = G - \{g\}$ $\qquad$ $G_5 = G - \{h\}$

$G_3 = G - \{b, g\}$ $\qquad$ $G_6 = G - \{f\}$

b) What are the bridges of the graph $G$?

24 Let $G$ be a connected graph and let $e \in E(G)$. Show that

$$\omega(G - \{e\}) \leq 2$$

## The Laplacian matrix

- Let $G$ be a graph with $n = |V(G)|$ vertices:

$$V(G) = \{v_1, v_2, \dots, v_n\}$$

The Laplacian matrix $L_G$ is defined as

$$\boxed{(L_G)_{ab} = \begin{cases} d(v_a) & , \text{ if } a = b \\ -1 & , \text{ if } a \neq b \text{ and } v_a \leftrightarrow v_b \\ 0 & , \text{ otherwise.} \end{cases}}$$

- If $w(G)$ is the number of components of $G$ then the characteristic polynomial of $L_G$ has a common factor $\lambda^{w(G)}$
(i.e. $0$ is a root with multiplicity $w(G)$)
Thus

$$\boxed{\begin{array}{l} \det(L_G - \lambda I) = \lambda^{w(G)} f(\lambda) \\ \text{with } f(0) \neq 0 \end{array}}$$

## Graph connectivity

### Edge connectivity $\lambda(G)$

> **Def**: Let $G$ be a graph and let $E_0 \subseteq E(G)$. We say that
> $E_0$ is an edge cutset of $G \iff$
> $$\iff \begin{cases} G - E_0 \text{ not connected} \\ \forall E_1 \in \mathcal{P}(E_0) : (E_1 \neq E_0 \Rightarrow G - E_1 \text{ connected}) \end{cases}$$

- The smallest number of edges needed to construct a cutset $E_0$ of $G$ is the edge-connectivity $\lambda(G)$ of $G$. More formally,

$$\boxed{\lambda(G) = \min\{|E_0| \mid E_0 \in \mathcal{P}(E(G)) \wedge E_0 \text{ edge cut-set of } G\}}$$

### Vertex connectivity $\kappa(G)$

> **Def**: Let $G$ be a graph and let $V_0 \subseteq V(G)$. We say that $V_0$ is a vertex cutset of $G \iff$
> $$\iff \begin{cases} G - V_0 \text{ not connected} \\ \forall V_1 \in \mathcal{P}(V_0) : (V_1 \neq V_0 \Rightarrow G - V_1 \text{ connected}) \end{cases}$$

- The smallest number of vertices needed to construct a vertex cutset $V_0$ of $G$ is the vertex connectivity $\kappa(G)$ of $G$

$$\boxed{\kappa(G) = \min\{|V_0| \mid V_0 \in \mathcal{P}(V(G)) \wedge V_0 \text{ vertex cutset of } G\}}$$

Note that

G not connected $\Longleftrightarrow \lambda(G) = \kappa(G) = 0$

G weakly linked $\Longleftrightarrow \lambda(G) = 1$

G strongly linked $\Longleftrightarrow \lambda(G) > 1$

A property of connectivity

Recall that $\delta(G)$ is the minimum degree of G:

$$\delta(G) = \min\{d(u) \mid u \in V(G)\}$$

It can be shown that:

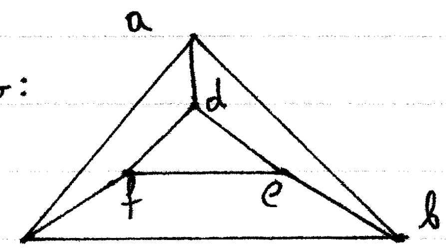
**Thm:** Let G be a graph. Then:

$$G \text{ connected} \Rightarrow \kappa(G) \leq \lambda(G) \leq \delta(G) \leq \frac{2|E(G)|}{|V(G)|}$$

# EXAMPLE

Calculate the vertex connectivity $\kappa(G)$ and edge connectivity $\lambda(G)$ for the following graph $G$:

G:




## Solution

First we note that

$d(a) = d(b) = d(c) = d(d) = d(e) = d(f) = 3 \Rightarrow$

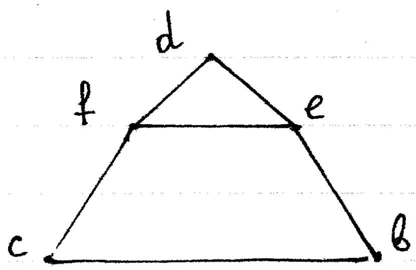
$\Rightarrow \delta(G) = \min\limits_{u \in V(G)} d(u) = 3 \Rightarrow \kappa(G) \leq \delta(G) = 3 \Rightarrow \kappa(G) \leq 3$

$\Rightarrow \kappa(G) = 0 \vee \kappa(G) = 1 \vee \kappa(G) = 2 \vee \kappa(G) = 3.$

Since $G$ connected $\Rightarrow \kappa(G) > 0$.

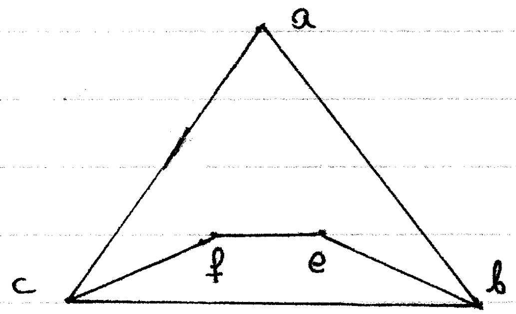
- Try deleting one vertex

a) For $G - \{a\}$ we have:




which is connected. $G - \{b\}$, $G - \{c\}$ are similarly connected.

b) For $G-\{d\}$ we have:

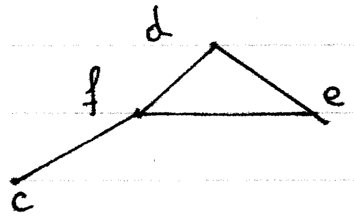



which is connected, and by symmetry, $G-\{e\}$ and $G-\{f\}$ are also connected.

It follows from (a) and (b) that $\kappa(G) > 1$.

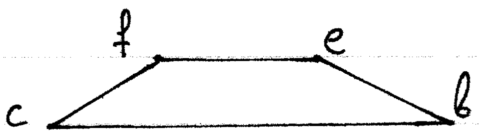
- Try deleting two vertices

a) For $G-\{a,b\}$ we have:



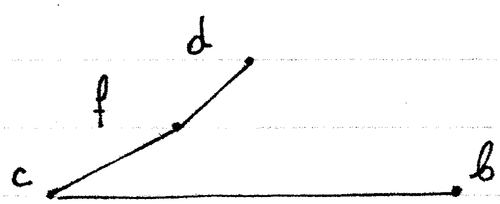

which is still connected. By symmetry, $G-\{b,c\}$ and $G-\{c,a\}$ are also connected.

b) For $G-\{a,d\}$ we have:



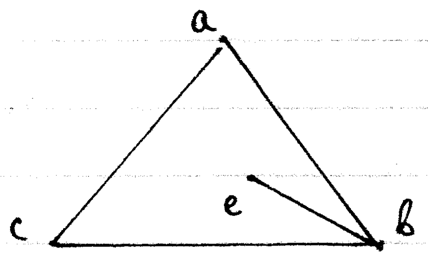

which is still connected. By symmetry, $G-\{b,e\}$ and $G-\{c,f\}$ are also connected

c) For $G-\{a,e\}$ we have:




which is still connected. By symmetry, $G-\{a,f\}$, $G-\{b,d\}$, $G-\{b,f\}$, $G-\{c,d\}$, $G-\{c,e\}$ are also connected.

d) For $G-\{d,f\}$ we have:




which is connected, and by symmetry $G-\{d,e\}$ and $G-\{e,f\}$ are also connected.

From (a), (b), (c), (d) it follows that $\kappa(G)>2$.

Since $2<\kappa(G)\leqslant\delta(G)=3 \Rightarrow \underline{\kappa(G)=3}$

and since $\kappa(G)\leqslant\lambda(G)<\delta(G) \Rightarrow 3\leqslant\lambda(G)\leqslant 3$

$\Rightarrow \underline{\lambda(G)=3}$

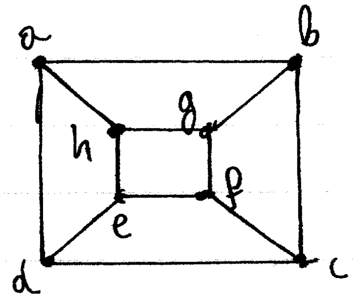
# EXERCISES

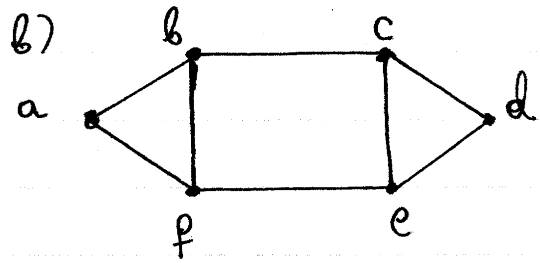
(25) Consider the complete graph $K_a$
Let $u \in V(K_a)$. Show
a) Show that $K_a - \{u\} = K_{a-1}$
b) Show that
$$\kappa(K_a) = \lambda(K_a) = \delta(K_a) = a-1$$

(26) Similarly, for the complete bipartite graph $K_{a,b}$ show that
$$\kappa(K_{a,b}) = \lambda(K_{a,b}) = \delta(K_{a,b}) = \min\{a,b\}$$

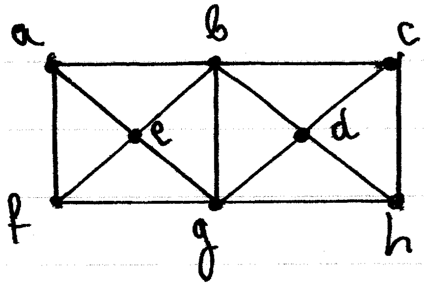
(27) Show that
a) $\kappa(P_a) = \lambda(P_a) = \delta(P_a) = 1$
b) $\kappa(C_a) = \lambda(C_a) = \delta(C_a) = 2$

(28) Calculate $\kappa(G)$ and $\lambda(G)$ for the following graphs:

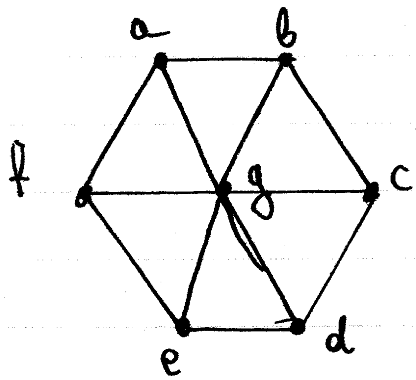
a)



b)


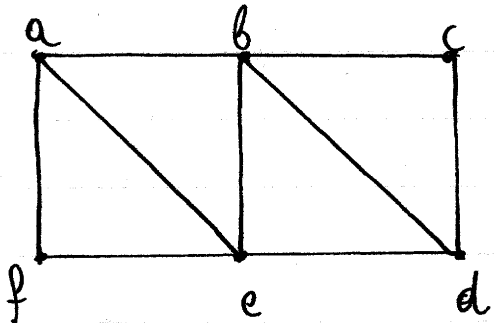
c)




d)




e)



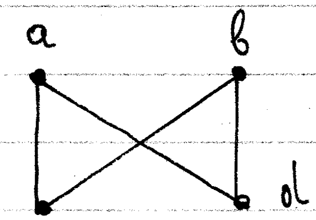
## V Eulerian graphs

The Eulerian problem: Given a connected graph $G$, is there a walk that can visit every edge of the graph once and only once and return to the starting vertex at the end? If the answer is yes, we say that $G$ is an Eulerian graph and the corresponding walk is an Eulerian trail.

**Def**: Let $G$ be a connected graph. We say that
$$G \text{ Eulerian} \Leftrightarrow \exists W \in T(G) : E(W) = E(G) \wedge s(W) = t(W)$$

### EXAMPLE

The graph $K_{2,2}$ :




is Eulerian with Eulerian trail:
$W = (a, ac, c, cb, b, bd, d, da, a)$

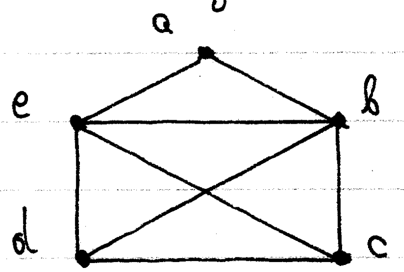
Euler solved the Eulerian problem by introducing the definitions for graph, vertex degree, and proving the following theorem:

**Thm**: Let $G$ be a connected graph. Then:
$$G \text{ Eulerian} \Leftrightarrow \forall u \in V(G) : \exists k \in \mathbb{N}^* : d(u) = 2k$$

## EXAMPLE

Consider the graph

G :




$d(c) = |\{bc, cd, ce\}| = 3 \Rightarrow G$ not Eulerian.

## EXAMPLE

A connected graph with 5 vertices and 4 edges has two vertices with degree 2. Show that the graph G is not Eulerian.

### Solution

We assume that $|V(G)| = 5$ and $|E(G)| = 4$ with

$$V(G) = \{u_1, u_2, u_3, u_4, u_5\}$$

and $d(u_1) = d(u_2) = 2$. Define $a = d(u_3) \wedge b = d(u_4) \wedge c = d(u_5)$. From the handshaking lemma:

$$\sum_{u \in V(G)} d(u) = 2|E(G)| \Rightarrow d(u_1) + d(u_2) + d(u_3) + d(u_4) + d(u_5) = 2 \cdot 4$$

$$\Rightarrow 2 + 2 + \alpha + b + c = 8$$

$$\Rightarrow a + b + c = 4.$$

$$G \text{ connected} \Rightarrow \forall u \in V(G) : d(u) > 0$$
$$\Rightarrow a > 0 \wedge b > 0 \wedge c > 0.$$
$$\Rightarrow a \geq 1 \wedge b \geq 1 \wedge c \geq 1$$

It follows that

$a+b+c = 4 \Leftrightarrow$

$\Leftrightarrow (a,b,c) \in \{(1,1,2),(1,2,1),(2,1,1)\}$

and therefore:

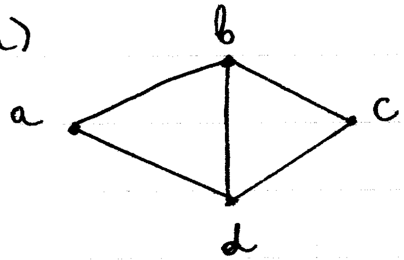
$a \text{ odd} \vee b \text{ odd} \vee c \text{ odd} \Rightarrow$

$\Rightarrow G$ not Eulerian. $\square$

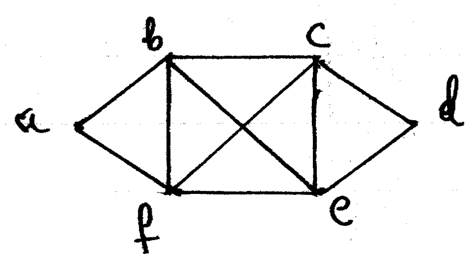
# EXERCISES

(29) Which of the following graphs is Eulerian?

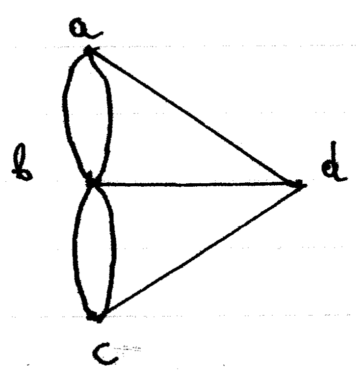
a)



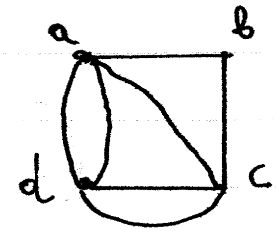

b)



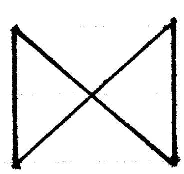

c)




d)

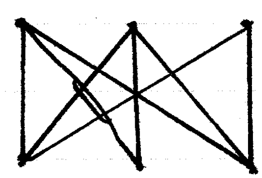



e)



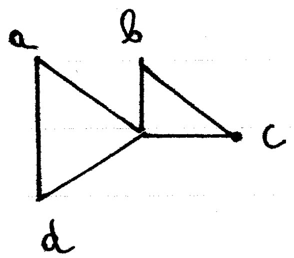
$K_{2,2}$

f)

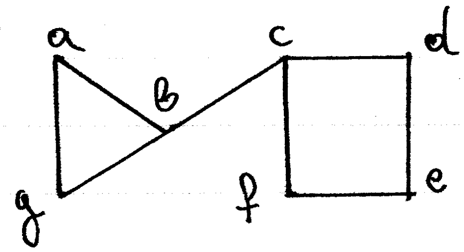


$K_{3,3}$

g)




h)



(30) Show that

a) $K_a$ Eulerian $\Leftrightarrow$ $a$ is odd

b) $K_{a,b}$ Eulerian $\Leftrightarrow$ $a$ even $\wedge$ $b$ even

c) $\forall a \in \mathbb{N}: (a \geq 2 \Rightarrow P_a$ not Eulerian$)$

d) $\forall a \in \mathbb{N}: (a \geq 3 \Rightarrow C_a$ Eulerian$)$

(31) A connected Eulerian graph has 3 vertices and 5 edges. Show that if one vertex has degree 4, then another vertex must have degree 2

(32) A connected graph with 4 edges and 4 vertices has 2 vertices of degree 2 Show that

a) $G$ not Eulerian $\Rightarrow \exists u \in V(G): d(u) = 3.$

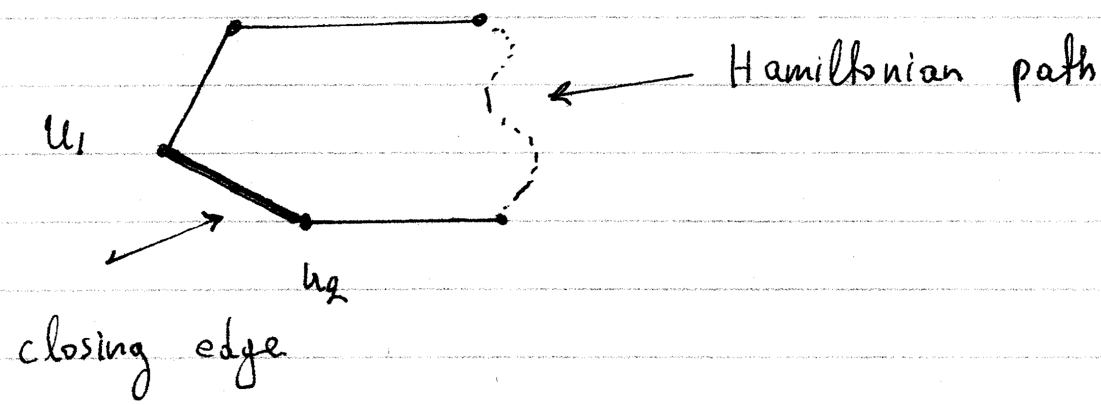
b) $G$ Eulerian $\Rightarrow$ $G$ regular.

(33) Show that a connected regular graph with an odd number of vertices is always Eulerian

(34) Show that a connected regular graph with odd number of edges and whose number of vertices is a multiple of 4 is never Eulerian.

## Hamiltonian graphs

Hamilton's Problem : Let $G$ be a connected graph. Can we construct a walk that visits every vertex of the graph once and only once, without using any edge more than once, and then close the walk with a direct edge from its terminal point back to its initial point? If yes, then we say that the graph is a Hamiltonian graph, the walk is a Hamiltonian path, and the walk together with the closing edge is a Hamiltonian circuit.




Recall that any walk where no edges or vertices are repeated is a path. The Hamiltonian circuit as a whole is not a path since the initial vertex is repeated once, as a terminal vertex. Thus, the reason for the distinction between the Hamiltonian path and the Hamiltonian circuit. Based on the above, we give the following definition:

**Def** : Let $G$ be a connected graph. We say that $G$ Hamiltonian $\iff$

$$\iff \exists u_1, u_2 \in V(G) : \exists w \in P(G, u_1 \to u_2) : \begin{cases} u_1 \neq u_2 \wedge V(w) = V(G) \\ \exists e \in E(G) : \psi_G(e) = \{u_1, u_2\} \wedge e \notin E(w) \end{cases}$$

Here, $w$ is the Hamiltonian path, $u_1$ the initial vertex, $u_2$ the terminal vertex and $e$ the closing edge.

- Note that it is not necessary for the Hamiltonian circuit to visit all the edges.

## Criteria for the Hamiltonian property

Noone has successfully solved the Hamiltonian problem by proving a practical necessary and sufficient condition. We have however the following partial results:

### 1 → A necessary condition

**Thm** : Let $G$ be a connected graph. Then:

$G$ Hamiltonian $\Rightarrow \forall V_0 \in P(V(G)) : (V_0 \neq V(G) \Rightarrow \omega(G - V_0) \leq |V_0|)$

Intuitively, if the graph $G$ is Hamiltonian, then if we subtract the vertices in $V_0 \subset V(G)$, then the resulting graph $G-V_0$ cannot have more components than the number of vertices in $V_0$.

- The <u>contrapositive</u> statement of this theorem can be used to show that a graph is not Hamiltonian

**Corollary:** Let $G$ be a connected graph. Then
$(\exists V_0 \in \mathcal{P}(V(G)) : (V_0 \neq V(G) \wedge \omega(G-V_0) > |V_0|)) \Rightarrow G$ not Hamiltonian

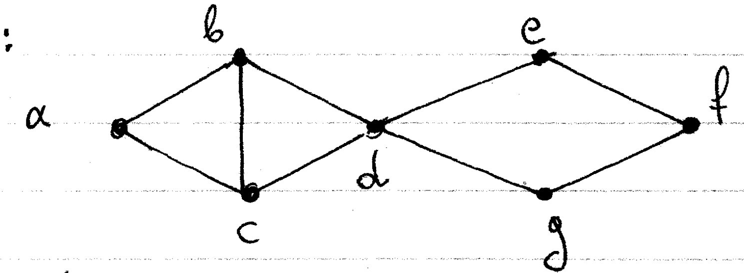
↳ In general, proving a statement of the form $p \Rightarrow q$ also proves the contrapositive statement $\bar{q} \Rightarrow \bar{p}$. Negations can be calculated according to the following rules of Boolean logic:

| Statement | It's negation |
|---|---|
| $\forall x \in A : p(x)$ | $\exists x \in A : \overline{p(x)}$ |
| $\exists x \in A : p(x)$ | $\forall x \in A : \overline{p(x)}$ |
| $p \wedge q$ | $\bar{p} \vee \bar{q}$ |
| $p \vee q$ | $\bar{p} \wedge \bar{q}$ |
| $p \Rightarrow q$ | $p \wedge \bar{q}$ |
| $p \Leftrightarrow q$ | $p \veebar q$ |
| $p \veebar q$ | $p \Leftrightarrow q$ |

# EXAMPLE

Show that the graph

G:

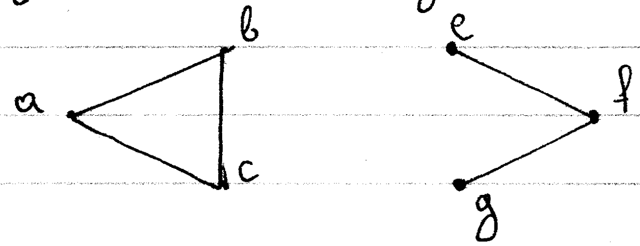



is not Hamiltonian.

## Solution

Subtracting the vertex d gives

$G-\{d\}$:




It follows that

$G-\{d\} = G[\{a,b,c\}] \cup G[\{e,f,g\}] \Rightarrow$

$\Rightarrow \omega(G-\{d\}) = 2 > 1 = |\{d\}| \Rightarrow$

$\Rightarrow \omega(G-\{d\}) > |\{d\}| \Rightarrow$

$\Rightarrow$ G not Hamiltonian.

## ② → Ore's theorem

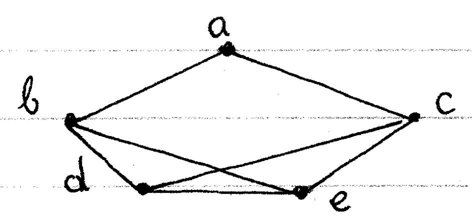
**Thm:** Let $G$ be a graph. Then:

$$\left\{\begin{array}{l} G \text{ simple and connected} \\ |V(G)| \geq 3 \\ \forall u, v \in V(G) : (u, v \text{ not adjacent} \Rightarrow \\ \qquad \Rightarrow d(u) + d(v) \geq |V(G)|) \end{array}\right. \Rightarrow G \text{ is Hamiltonian}$$

### EXAMPLE

Use Ore's theorem to show that

$G$:




is Hamiltonian

**Solution**

We note that

$$\left.\begin{array}{l} G \text{ is simple and connected.} \\ |V(G)| = |\{a, b, c, d, e\}| = 5 \geq 3 \\ d(a) + d(d) = 2+3 = 5 \geq |V(G)| \\ d(a) + d(e) = 2+3 = 5 \geq |V(G)| \\ d(b) + d(c) = 3+3 = 6 \geq |V(G)| \end{array}\right\} \Rightarrow G \text{ is Hamiltonian}$$

## ③ → Dirac's theorem

**Thm**: Let $G$ be a graph. Then

$$\left\{\begin{array}{l} G \text{ simple and connected} \\ |V(G)| \geq 3 \\ \delta(G) \geq (1/2)|V(G)| \end{array}\right. \Rightarrow G \text{ is Hamiltonian}$$

**Proof**

Assume that

$$\left\{\begin{array}{ll} G \text{ simple and connected} & (1) \\ |V(G)| \geq 3 & (2) \\ \delta(G) \geq (1/2)|V(G)| & (3) \end{array}\right.$$

Let $u, v \in V(G)$ be given and assume that $u, v$ not adjacent. Then:

$$d(u) + d(v) \geq \delta(G) + \delta(G) = 2\delta(G) \geq 2[(1/2)|V(G)|] = |V(G)| \Rightarrow$$

$$\Rightarrow d(u) + d(v) \geq |V(G)|$$

It follows that

$$\forall u, v \in V(G) : (u, v \text{ not adjacent} \Rightarrow d(u) + d(v) \geq |V(G)|) \quad (4)$$

From Eq. (1), Eq. (2), Eq. (4), via Ore's theorem, it follows that $G$ is Hamiltonian

## ④ → Bipartite graphs

**Thm**: Let $G$ be a graph. Then

$$\left\{\begin{array}{l} G \text{ connected and bipartite} \\ \exists k \in \mathbb{N} : |V(G)| = 2k+1 \end{array}\right. \Rightarrow G \text{ not Hamiltonian}$$

Proof

Assume that

$$\begin{cases} G \text{ connected and bipartite} \\ \exists k \in \mathbb{N} : |V(G)| = 2k+1 \end{cases}$$

Since, $G$ is bipartite, we choose $V_1 \subseteq V(G)$ and $V_2 \subseteq V(G)$ such that

$$\begin{cases} V_1 \cap V_2 = \emptyset \wedge V_1 \cup V_2 = V(G) \\ \forall e \in E(G) : \begin{cases} |\psi_G(e) \cap V_1| = 1 \\ |\psi_G(e) \cap V_2| = 1 \end{cases} \end{cases}$$

To show a contradiction, assume that $G$ is Hamiltonian. Then, a Hamiltonian circuit must alternate between vertices in $V_1$ and vertices in $V_2$. Because each vertex can only be visited once, it follows that

$$|V_1| = |V_2| \Rightarrow$$
$$\Rightarrow |V(G)| = |V_1| + |V_2| = |V_1| + |V_1| = 2|V_1| \Rightarrow$$
$$\Rightarrow |V(G)| \text{ is even}$$
$$\Rightarrow |V(G)| \text{ not odd}$$

which contradicts the assumption

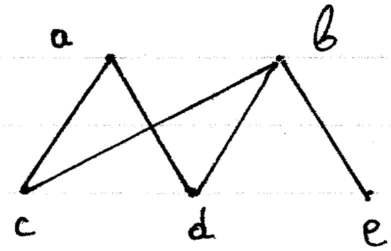
$$\exists k \in \mathbb{N} : |V(G)| = 2k+1$$

It follows that $G$ is not Hamiltonian.

# EXAMPLES

Show that the following graph is not Hamiltonian:

G:




**Solution**

Note that for $V_1 = \{a, b\}$ and $V_2 = \{c, d, e\}$:

$$\forall e \in E(G): \begin{cases} |\psi_G(e) \cap V_1| = 1 \\ |\psi_G(e) \cap V_2| = 1 \end{cases} \Rightarrow G \text{ bipartite} \qquad (1)$$

Furthermore:

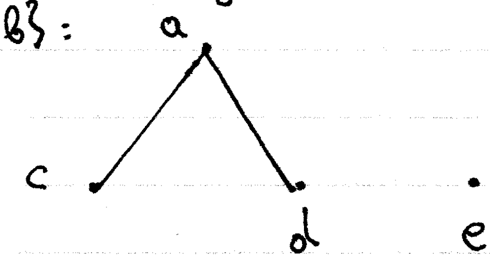
$$|V(G)| = |\{a, b, c, d, e\}| = 5 \Rightarrow |V(G)| \text{ odd} \qquad (2)$$

From (1) and (2): G is not Hamiltonian.

**2nd method**

Consider the graph

$G - \{b\}$:




Since $G - \{b\} = G[\{a, c, d\}] \cup G[\{e\}] \Rightarrow$

$\Rightarrow \omega(G - \{b\}) = 2 > 1 = |\{b\}| \Rightarrow$

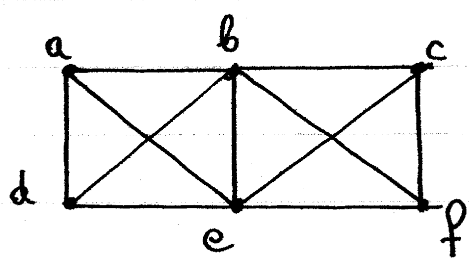
$\Rightarrow \omega(G - \{b\}) > |\{b\}|$

$\Rightarrow$ G not Hamiltonian.

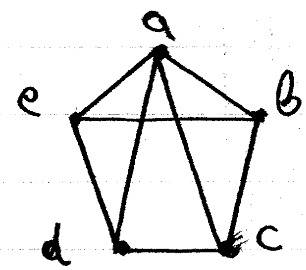
# EXERCISES

(35) Show that the following graphs are Hamiltonian

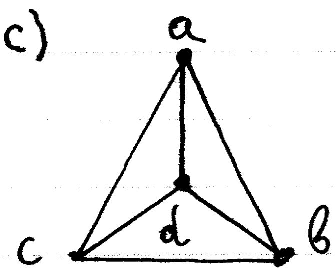
a)



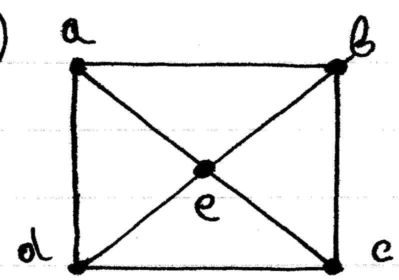

b)




c)

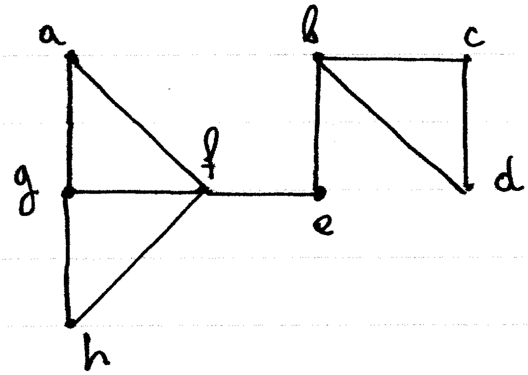



d)

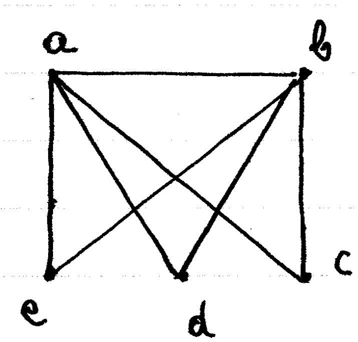



(36) Show that the following graphs are not Hamiltonian

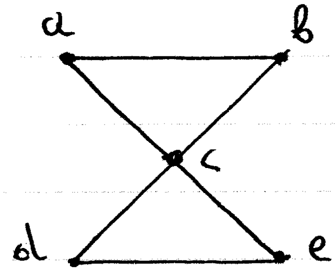
a)



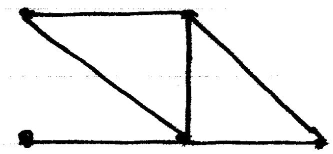

b)



c)




d)



(37) Show that $K_a$ is Hamiltonian for all $a \geqslant 3$.

(38) Show that

a) $a = b \Rightarrow K_{a,b}$ Hamiltonian

b) $a \neq b \Rightarrow K_{a,b}$ not Hamiltonian

$\hookrightarrow$ It follows from this exercise that $K_{a,b}$ Hamiltonian $\Leftrightarrow a = b$.

(39) Let $G$ be a graph with less than 7 vertices and vertex connectivity $\kappa(G) = 4$. Show that $G$ is Hamiltonian.

(40) Show that a graph $G$ with vertex connectivity $\kappa(G) = 1$ is not Hamiltonian

(41) Show that a strongly-linked graph with 4 vertices is always Hamiltonian.