GRAPH THEORY

\section*{Preliminaries}

Let $A$ be a finite set. We define the following notation:

a) $|A|$ is the number of elements of the set $A$.

b) $\mathcal{P}(A)$ is the set of all subsets of $A$, i.e.
$$X \in \mathcal{P}(A) \iff X \subseteq A$$

c) $\mathcal{P}_a(A)$ is the set of all subsets of $A$ with $a$ elements, i.e.
$$X \in \mathcal{P}_a(A) \iff (X \subseteq A \land |X| = a)$$

\section*{Example}

For $A = \{a, b, c\}$, we have:

$|A| = |\{a, b, c\}| = 3$

$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$\mathcal{P}_0(A) = \{\emptyset\}$

$\mathcal{P}_1(A) = \{\{a\}, \{b\}, \{c\}\}$

$\mathcal{P}_2(A) = \{\{a, b\}, \{a, c\}, \{b, c\}\}$

$\mathcal{P}_3(A) = \{\{a, b, c\}\}$

\textbf{Def:} Let $f: A \to B$ be a mapping from a set $A$ to a set $B$. We say that $f$ one-to-one $\iff \forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$
interpretation: A mapping $f: A \rightarrow B$ is one-to-one if and only if each element of $A$ is mapped to a distinct element of $B$.

\textbf{Graphs - Basic definitions}

\textbf{Def}: A graph $G$ is a triplet $G = (V(G), E(G), \psi_G)$ that consists of:

a) a set of vertices $V(G)$

b) a set of edges $E(G)$

c) an incidence mapping $\psi_G: E(G) \rightarrow \mathcal{P}_1(V(G)) \cup \mathcal{P}_2(V(G))$ that maps every edge to one or two vertices.

\textbf{EXAMPLE}

$G:$

\begin{align*}
&\begin{array}{c}
\text{V}(G) = \{1, 2, 3, 4\} \\
\text{E}(G) = \{e_1, e_2, e_3, e_4\} \\
\psi_G(e_1) = \{1, 2\} \\
\psi_G(e_2) = \{1, 2\} \\
\psi_G(e_3) = \{2, 3\} \\
\psi_G(e_4) = \{3, 3\} \quad \text{--- a loop}
\end{array}
\end{align*}

Equivalently, $\psi_G$ can be rewritten as:

$\psi_G = \{\{e_1, \{1, 2\}\}, \{e_2, \{1, 2\}\}, \{e_3, \{2, 3\}\}, \{e_4, \{3, 3\}\}\}$
Elementary definitions about graphs

**Def:** Let $G$ be a graph. We say that

1. The vertex $u \in V(G)$ is **incident** to the edge $e \in E(G)$ if and only if the edge $e$ connects $u$ with itself or with another vertex:
   \[
   \forall u \in V(G), \forall e \in E(G): u \text{ incident to } e \iff u \in \psi_G(e)
   \]

2. The edge $e \in E(G)$ is a **loop** if and only if it connects a vertex with itself:
   \[
   \forall e \in E(G): (e \text{ loop} \iff |\psi_G(e)| = 1)
   \]

**Def:** Let $G$ be a graph and let $u \in V(G)$ be a vertex of $G$. The **degree** $d(u)$ of $u$ is given by

\[
   d(u) = |\{e \in E(G) | u \in \psi_G(e)\}| + |\{e \in E(G) | u \in \psi_G(e) \land e \text{ loop}\}|
\]

The degree $d(u)$ is the number of edges $e \in E(G)$ to which $u$ is incident, with the loops being counted twice. In the previous example:

\[
   d(1) = 2 \land d(2) = 3 \land d(3) = 3
\]

**Def:** Let $G$ be a graph. We define:

1. The **minimum degree** $\delta(G) = \min_{u \in V(G)} d(u)$

2. The **maximum degree** $\Delta(G) = \max_{u \in V(G)} d(u)$
Remark: It follows from this definition that \[ \forall v \in V(G): 0 \leq d(v) \leq \Delta(G) \]

**Lemma:** (The Handshaking Lemma)

Let \( G \) be a graph. Then:

\[ \sum_{v \in V(G)} d(v) = 2 |E(G)| \]

Although this is not a formal proof, a simple explanation of the handshaking lemma is that every edge is usually shared by two vertices. As a result, adding up the degrees of all vertices counts the edges twice. Although loops attach to only one vertex, the loop, by definition, adds 2 to the degree of that vertex, so loops are also counted twice.
a) Show that it is not possible to define a graph with
3 vertices of degree 2 and 5 vertices of degree 3.

Solution

Assume a graph \( G \) exists with \( V(G) = \{ u_1, u_2, u_3, V_4, V_5, V_6, V_7 \} \)
such that

\[ \forall a \in [3]: \ d(u_a) = 2 \]

\[ \forall a \in [5]: \ d(v_a) = 3 \]

It follows that

\[ 2|E(G)| = \sum_{u \in V(G)} d(u) = \sum_{a \in [3]} d(u_a) + \sum_{b \in [5]} d(v_b) = \]

\[ = 3 \cdot 2 + 5 \cdot 3 = 6 + 15 = 21 \Rightarrow \]

\[ \Rightarrow |E(G)| = 21/2 \] — contradiction since \( |E(G)| \) is a natural number. It follows that \( G \) cannot be constructed.

b) A graph has twice as many edges as vertices. Show that there is at least one vertex with degree less than 2.

Solution

Let \( G \) be the graph with \( 2|E(G)| = |V(G)| \). Then:

\[ 2|E(G)| = \sum_{u \in V(G)} d(u) \Rightarrow \sum_{u \in V(G)} \delta(G) = \delta(G) |V(G)| = \]

\[ = \delta(G) 2|E(G)| \Rightarrow 2|E(G)| \geq \delta(G) 2|E(G)| \Rightarrow \]

\[ \Rightarrow \delta(G) \leq 1 \Rightarrow \delta(G) = 0 \vee \delta(G) = 1 \Rightarrow \]
\[ \Rightarrow \exists u \in V(G) : (d(u) = 0 \lor d(u) = 1) \]
\[ \Rightarrow \exists u \in V(G) : d(u) \leq 2. \]

c) Let \( G \) be a graph with \( \Delta(G) = 2 \). Show that the graph cannot have more edges than vertices.

Solution

Since \( \Delta(G) = 2 \Rightarrow \forall u \in V(G) : d(u) \leq 2 \)

\[ \Rightarrow 2 |E(G)| = \sum_{u \in V(G)} d(u) \leq \sum_{u \in V(G)} \Delta(G) = \Delta(G) |V(G)| \]

\[ = 2 |V(G)| \Rightarrow 2 |E(G)| \leq 2 |V(G)| \Rightarrow \]

\[ \Rightarrow |E(G)| \leq |V(G)|. \]
EXERCISES

(1) For the following graphs, list \( V(G) \), \( E(G) \), and the values of the incidence mapping \( p_e \):

a) ![Graph a](image)

b) ![Graph b](image)

c) ![Graph c](image)

d) ![Graph d](image)

(2) For the graphs of the previous exercise, list the degrees of each vertex and write \( \delta(G) \) and \( \Delta(G) \).

(3) Show that it is not possible to create a graph with 9 vertices such that the degree of every vertex is 3.
4) Show that it is not possible to create a graph with 7 vertices of degree 3 and 2 vertices of degree 2.

5) Let $G$ be a graph with 10 vertices such that
\[ \delta(G) = \Delta(G) = 2 \]
How many edges does $G$ have?

6) Let $G$ be a graph with $|V(G)| = 8$ such that $\Delta(G) = 4$. Show that $|E(G)| < 36$.

7) Let $G$ be a graph such that $|V(G)| = |E(G)|$. Show that $\delta(G) < 3$

8) Let $G$ be a graph with $\Delta(G) = 4$. Show that $|E(G)| \leq 2|V(G)|$

9) A graph with 4 edges has a vertex with degree 4, a vertex with degree 1, and one more vertex. What is the degree of the third vertex?
Types of graphs

1. Simple graphs

**Def.:** A graph $G$ is simple if and only if it has no loops and no multiple edges, i.e.

- $G$ simple $\iff \forall e \in E(G): |\psi_G(e)| = 2$
- $\psi_G$ is one-to-one

**Example**

$G_1$:
- Nodes: 1, 2, 3, 4
- Edges: $e_1, e_2, e_3, e_4$

$G_2$:
- Nodes: 1, 2
- Edges: $e_1, e_2$

$G_3$:
- Nodes: 1, 2, 3
- Edges: $e_1, e_2, e_3$

$G_4$:
- Nodes: 1, 2
- Edges: $e_1, e_2, e_3$

$G_1$ is simple.
$G_2, G_3, G_4$ are NOT simple.
(2) **Regular graphs**

**Def.** Let $G$ be a graph. We say that

a) $G$ is **regular** if and only if all vertices have the same degree, i.e.

$$G \text{ regular} \iff \forall u_1, u_2 \in V(G): d(u_1) = d(u_2)$$

$$\iff \exists a \in \mathbb{N}: \forall u \in V(G): d(u) = a$$

b) $G$ is **$a$-regular** if and only if all vertices have degree equal to $a$, i.e.

$$G \text{ } a\text{-regular} \iff \forall u \in V(G): d(u) = a$$

**Remark:** Note the negation of this definition:

$G$ not regular $\iff \exists u_1, u_2 \in V(G): d(u_1) \neq d(u_2)$

**EXAMPLE**

$$G:$$

```
      1
     /|
    / | 3
    /  |
   4
    |
    2
```

$$d(1) = d(2) = d(3) = d(4) = 3 \Rightarrow$$

$$\Rightarrow \forall u \in V(G): d(u) = 3$$

$$\Rightarrow G \text{ 3-regular}$$

Also: $G$ is regular.
Complete graphs

A complete graph is a simple graph such that any two distinct vertices are connected by an edge. The formal definition reads:

\[
\text{Def: Let } G \text{ be a graph. We say that } G \text{ complete } \iff \forall u_1, u_2 \in V(G): (u_1 \neq u_2 \Rightarrow \exists e \in E(G): u_1 e u_2)
\]

Given the number of vertices \( n \), there is a unique graph such that

\[
\begin{align*}
&\{ G \text{ complete} \\
&|V(G)| = n
\end{align*}
\]

which we denote as \( K_n \).

**EXAMPLES**

\( K_1: \) \hspace{1cm} \( K_2: \) \hspace{1cm} \( K_3: \)

\( K_4: \) \hspace{1cm} \( K_5: \)
4. Null graphs

A null graph is a graph with no edges. The formal definition is:

**Def:** Let $G$ be a graph. We say that $G$ is a null graph $\iff E(G) = \emptyset$

The null graph with $n$ vertices is unique and denoted as $N_n$.

5. The path graph $P_n$

**Def:** We distinguish between the following cases.

- **Case 1:** For $n = 1$, we define
  \[ V(P_1) = \{ u_1 \} \land E(P_1) = \emptyset \land \psi_{P_1} = \emptyset \]

- **Case 2:** For $n \geq 2$, we define
  \[
  \begin{align*}
  V(P_n) &= \{ u_1, u_2, \ldots, u_n \} \\
  E(P_n) &= \{ e_1, e_2, \ldots, e_{n-1} \} \\
  \forall k \in [n-1] : \psi_{P_n}(e_k) &= \{ u_k, u_{k+1} \}
  \end{align*}
  \]

**Examples**

- $P_1$: \[
  \begin{array}{c}
  u_1
  \end{array}
  \]
- $P_2$: \[
  \begin{array}{c}
  u_1 \quad u_2
  \end{array}
  \]
- $P_3$: \[
  \begin{array}{c}
  u_1 \quad u_2 \quad u_3
  \end{array}
  \]
- $P_4$: \[
  \begin{array}{c}
  u_1 \quad u_2 \quad u_3 \quad u_4
  \end{array}
  \]
- $P_5$: \[
  \begin{array}{c}
  u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5
  \end{array}
  \]
The cycle graph $C_n$

**Def:** We distinguish between the following cases:

**Case 1:** For $n=1$, we have
- $V(C_1) = \{u_1\}$
- $E(C_1) = \{e_1\}$
- $\psi_{C_1}(e_1) = \{u_1\}$

**Case 2:** For $n \geq 2$, we have
- $V(C_n) = \{u_1, u_2, \ldots, u_n\}$
- $E(C_n) = \{e_1, e_2, \ldots, e_n\}$
- $\forall k \in [n]: \psi_{C_n}(e_k) = \{u_k, u_{k+1}\}$
- $\psi_{C_n}(e_n) = \{u_n, u_1\}$

**EXAMPLES**

- $C_1$:

- $C_2$:

- $C_3$:

- $C_4$:

- $C_5$: 
Bipartite graphs

A graph $G$ is called bipartite if and only if its vertex $V(G)$ can be partitioned to two sets $V_1$ and $V_2$ such that every edge of $G$ connects a vertex in $V_1$ with a vertex in $V_2$. The formal definition is:

**Def**: Let $G$ be a graph. We say that

1) $G$ bipartite with vertex partition $V_1, V_2$ $\iff$

\[
\begin{align*}
V_1 \cup V_2 &= V(G) \\
V_1 \cap V_2 &= \emptyset \\
\forall e \in E(G): \begin{cases}
\left|V_1 \cap V_2\right| = 0 \\
\left|\psi_G(e) \cap V_1\right| = 1
\end{cases}
\end{align*}
\]

2) $G$ bipartite $\iff \exists V_1, V_2 \in \mathcal{P}(V(G)): G$ bipartite with vertex partition $V_1, V_2$.

A complete bipartite graph is bipartite with some vertex partition $V_1, V_2$, simple, and every vertex of $V_1$ is connected with every vertex of $V_2$ with exactly one edge. The formal definition is:

**Def**: Let $G$ be a graph. We say that:

$G$ complete bipartite with vertex partition $V_1, V_2$ $\iff$

\[
\begin{cases}
G$ bipartite with vertex partition $V_1, V_2$ \\
G$ simple \\
\forall u_1 \in V_1: \forall u_2 \in V_2: \exists e \in E(G): \psi_G(e) = \{u_1, u_2\}
\end{cases}
\]
There is a unique graph $G$ such that
\[ G \text{ complete bipartite with vertex partition } V_1, V_2 \]
\[ |V_1| = n \land |V_2| = m \]
with $n, m \in \mathbb{N}$, and it is denoted as $K_{n,m}$.

**EXAMPLES**

$K_{1,1}: \quad \begin{array}{c}
\end{array}$

$K_{1,2}: \quad \begin{array}{c}
\end{array}$

$K_{2,2}: \quad \begin{array}{c}
\end{array}$

$K_{2,4}: \quad \begin{array}{c}
\end{array}$
EXAMPLES

a) Evaluate $S(k, a, b)$, $\Delta(k, a, b)$, and $|E(k, a, b)|$ for the complete bipartite graph $K_{k,a,b}$.

**Solution**

Let $V(k, a, b) = V_1 \cup V_2$ with $|V_1| = a$ and $|V_2| = b$.

Each vertex of $V_1$ connects to all vertices of $V_2$, therefore

$\forall u \in V_1 : d(u) = |V_2| = b$. \hspace{1cm} (1)

Similarly, each vertex of $V_2$ connects to all vertices of $V_1$, and therefore:

$\forall u \in V_2 : d(u) = |V_1| = a$. \hspace{1cm} (2)

It follows that

$S(k, a, b) = \min_{u \in V(k, a, b)} d(u) = \min \{ a, b \}$

$\Delta(k, a, b) = \max_{u \in V(k, a, b)} d(u) = \max \{ a, b \}$

$2|E(k, a, b)| = \sum_{u \in V(k, a, b)} d(u) = \sum_{u \in V_1} d(u) + \sum_{u \in V_2} d(u) = |V_1|b + |V_2|a = ab + ba = 2ab \Rightarrow$

$|E(k, a, b)| = ab$. 

b) Show that $P_{10}$ is not regular.

Solution

Let $V(P_{10}) = \{v_1, v_2, \ldots, v_{10}\}$ with $v_1, v_{10}$ the endpoint vertices and $v_2, v_3, \ldots, v_9$ the interior vertices. We note that $d(v_1) = 1$ and $d(v_{10}) = 9$. It follows that

$\exists u, v \in V(P_{10}) : d(u) \neq d(v)$

$\Rightarrow \forall u, v \in V(P_{10}) : d(u) = d(v)$

$\Rightarrow P_{10}$ is not regular.

c) Show that if $G$ is regular, then

$$|E(G)| = \frac{\delta(G)|V(G)|}{2}$$

Solution

Assume $G$ is regular. Then

$G$ regular $\Rightarrow \forall u, v \in V(G) : d(u) = d(v)$

$\Rightarrow \exists \alpha \in \mathbb{N} : \forall u \in V(G) : d(u) = \alpha$

It follows that

$\delta(G) = \min_{u \in V(G)} d(u) = \min_{u \in V(G)} \alpha = \alpha$

and therefore:

$$|E(G)| = \frac{1}{2} \sum_{u \in V(G)} d(u) = \frac{1}{2} \sum_{u \in V(G)} \alpha = \frac{\alpha |V(G)|}{2} = \frac{\delta(G)|V(G)|}{2}$$
EXERCISES

10. Draw the following graphs:

a) $K_4$  
 b) $K_5$  
 c) $K_6$  
 d) $K_{4,3}$  
 e) $K_{2,2}$  
 f) $K_{3,3}$  
 g) $P_4$  
 h) $C_3$  
 i) $C_4$

11. Which of the graphs in the previous exercise are regular?

12. For $a, b$ integers $a > 0$ and $b > 0$ evaluate the following:

a) $\delta(K_a)$  
 b) $\delta(K_a,b)$  
 c) $\delta(P_a)$  
 d) $\delta(C_a)$  
 e) $\Delta(K_a)$  
 f) $\Delta(K_a,b)$  
 g) $\Delta(P_a)$  
 h) $\Delta(C_a)$  
 i) $|E(K_a)|$  
 j) $|E(K_a,b)|$  
 k) $|E(P_a)|$  
 l) $|E(C_a)|$

[You can check your general answers by testing them when $a=2, b=3$ or $a=4, b=3$]

13. Show that $K_{a,b}$ regular $\iff a=b$
14. Show that $K_{5,7}$ is not regular.

15. Show that 
$\Delta(G) = k(G)$.

16. Let $G$ be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$.
If $|V_1| = a$ and $|V_2| = a+2$ 
show that 
$|E(G)| \leq a^2 + 2a$

17. Show that we cannot build a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = 4$ and $|V_2| = 3$ and $|E(G)| > 14$. 
Graph operations

We define the following graph operations.

1. Induced subgraph

**Def**: Let $G$ be a graph and let $V_0 \subseteq V(G)$. We define the vertex induced subgraph $G[V_0]$ such that

- $V(G[V_0]) = V_0$
- $E(G[V_0]) = \{ e \in E(G) \mid \psi_G(e) \subseteq V_0 \}$
- $\forall e \in E(G[V_0]) : \psi_{G[V_0]}(e) = \psi_G(e)$

Intuitively, the vertex induced subgraph $G[V_0]$ consists of the vertices $V_0$, the edges that are incident only to vertices in $V_0$, connected similarly as in $G$.

**Example**

$G$:

```
  b
 / \   \  
  a   c  d
```

$G[\{a,b,c\}]$:

```
  a
```

Note that the removal of the vertex $d$ removes the two edges that are incident to it.
2. **Vertex subtraction**

**Def:** Let $G$ be a graph and let $V_0 \subseteq V(G)$. We define the graph $G-V_0$ such that $G-V_0 = G[V(G)-V_0]$.

Intuitively, $G-V_0$ is the graph obtained by deleting from $G$, the vertices in $V_0$ and all edges to which these vertices are incident.

3. **Edge-induced subgraph**

**Def:** Let $G$ be a graph and let $E_0 \subseteq E(G)$. We define the graph $G[E_0]$ such that

\[
\begin{align*}
V(G[E_0]) &= V(G) \\
E(G[E_0]) &= E_0 \\
\forall e \in E_0 : \psi_{G[E_0]}(e) &= \psi_G(e)
\end{align*}
\]

Intuitively, $G[E_0]$ consists of all the vertices of the original graph $G$ but only the edges that belong to $E_0$. 
**Edge Subtraction**

**Def:** Let $G$ be a graph and let $E_0 \subseteq E(G)$. We define the graph $G - E_0$ as:

$$G - E_0 = G \setminus (E(G) - E_0)$$

- Note that, unlike vertex subtraction, subtracting edges does not remove vertices under any circumstances.

**Example**

![Graph Diagram]

$G$: $b$ -- $e_2$ -- $c$ -- $e_3$ -- $e_5$ -- $d$ -- $e_1$ -- $a$

$G - e_2, e_3$: $b$ -- $e_1$ -- $a$

$G - e_2, e_3, e_4$: $b$ -- $e_1$ -- $a$

$G - e_2, e_3, e_4$: $b$ -- $e_1$ -- $a$
A necessary condition for defining the graph union $G_1 \cup G_2$ of two graphs $G_1, G_2$ is that $G_1, G_2$ should not share any edges, though they may share vertices. The formal definition is:

\[
\text{Def: Let } G_1, G_2 \text{ be two graphs such that } E(G_1) \cap E(G_2) = \emptyset. \text{ We define the graph union } G = G_1 \cup G_2 \text{ such that:} \\
V(G) = V(G_1) \cup V(G_2) \\
E(G) = E(G_1) \cup E(G_2) \\
\forall e \in E(G): \psi_G(e) = \begin{cases} 
\psi_{G_1}(e), & \text{if } e \in E(G_1) \\
\psi_{G_2}(e), & \text{if } e \in E(G_2)
\end{cases}
\]

**Example**

$G_1$: \begin{align*}
&\begin{array}{ccc}
& \text{b} & \text{e2} & \text{c} \\
e_1 & \bullet & \bullet & \bullet \\
& a & e_3 & d
\end{array} \\
\end{align*}$

$G_2$: \begin{align*}
&\begin{array}{ccc}
& \text{a} & \text{e4} & \text{c} \\
e_5 & \bullet & \bullet & \bullet \\
& e_6 & e & d
\end{array} \\
\end{align*}$

$G_1 \cup G_2$: \begin{align*}
&\begin{array}{ccc}
& \text{b} & \text{e2} & \text{c} \\
e_1 & \bullet & \bullet & \bullet \\
& e_4 & e & e_3 \\
& a & e_5 & d \\
e_6 & \bullet & \bullet
\end{array} \\
\end{align*}$
This definition generalizes to the union of $n$ graphs as follows:

**Def**: Let $G_1, G_2, ..., G_n$ be graphs such that

\[ \forall k, m \in [n]: (k \neq m \implies E(G_k) \cap E(G_m) = \emptyset) \]

We define the graph $G = G_1 \cup G_2 \cup ... \cup G_n$ such that:

\[
\begin{align*}
V(G) &= U_{a \in [n]} V(G_a) = V(G_1) \cup V(G_2) \cup ... \cup V(G_n) \\
E(G) &= U_{a \in [n]} E(G_a) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n) \\
\forall e \in E(G) : (\forall k \in [n] : e \in E(G_k) \implies \psi_e(e) = \psi_{G_k}(e))
\end{align*}
\]
EXERCISES

(18) Show that the following graphs are isomorphic

(Tip: Look at the "cycles")

(19) Consider the graph $K_3 \times K_3 = G$

Draw the following:

a) $G \{a, b, d, f\}$  

f) $G - \{a, d, f\}$

b) $G \{a, d, e, f\}$  

g) $G - \{a, d, e, f\}$

c) $G \{a, b, d, e\}$  

h) $G - \{d, e, f\}$

d) $G - \{a\}$  

i) $G - \{a, c, e, f\}$

e) $G - \{a, b, d\}$
(20) In the previous exercise, let
\[ G_1 = G - \{a, c, e, f\} \]
\[ G_2 = G[\{a, b, e\}] \]
Draw \( G_1 \cup G_2 \).
[Hint: List \( V(G_1), E(G_1), V(G_2), E(G_2) \) first].

(21) In the previous exercise, show that
\[ G[\{a, d, f\}] \cup G[\{b, e\}] \neq G[\{a, d, b, e\}] \]
Connected graphs

Walks, trails, paths

- Let $G$ be a graph. A walk $w$ is a sequence of alternating vertices and edges of the form
  \[ w = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n) \]
  such that
  \[ \forall k \in [n]: \psi_G(e_k) = \{v_{k-1}, v_k\}. \]

- Features of a walk:
  a) Starting point: $s(w) = v_0$
  b) Terminal point: $t(w) = v_n$
  c) $v_k(w) = v_k$
     $e_k(w) = e_k$
  d) Vertex set: $V(w) = \{v_0, v_1, \ldots, v_n\}$
  e) Edge set: $E(w) = \{e_1, e_2, \ldots, e_n\}$
  f) Length: $l(w) = |E(w)| = n$

- The set of all walks in $G$ is denoted $W(G)$.

- A trail is a walk in which all the edges are different. A path is a walk in which all the edges and vertices are different.
Thus, for $w \in W(G)$

1) $w$ trail $\iff$
   $\forall m, n \in [l(w)]: (m \neq n \Rightarrow e_m(w) \neq e_n(w))$

2) $w$ path $\iff$
   $\exists w$ trail
   $\forall m, n \in [l(w)]: (m \neq n \Rightarrow e_m(w) \neq e_n(w))$

• We define
  $T(G) = \{ w \in W(G) | w$ is a trail $\}$
  $P(G) = \{ w \in W(G) | w$ is a path $\}$

• Let $u, v \in V(G)$ be two vertices of $G$ with $u \neq v$. Then we define
  a) Set of all trails that connect $u$ to $v$
     $T(G, u \rightarrow v) = \{ w \in T(G) | s(w) = u \land t(w) = v \}$
  b) Set of all paths that connect $u$ to $v$
     $P(G, u \rightarrow v) = \{ w \in P(G) | s(w) = u \land t(w) = v \}$

• Note that $W(G)$ is an infinite set
  (i.e. you can go back and forth between two vertices indefinitely)
  but $T(G)$ and $P(G)$ are both finite sets.
  (i.e. you will run out of combinations of distinct edges and/or vertices).
**Connected graphs**

- A graph $G$ is connected if for any two not-equal vertices $u, v \in V(G)$, there is at least one path from $u$ to $v$.

\[ G \text{ connected } \iff \forall u, v \in V(G): (u \neq v \Rightarrow |P(G, u \rightarrow v)| > 1) \]

- The following graphs are connected:
  a) Complete graph $K_n$
  b) Path graph $P_n$
  c) Cycle graph $C_n$
  d) The bipartite graph $K_{m,n}$

**Graph components**

**Thm**: Let $G$ be a graph which is not connected. Then the vertex set $V(G)$ can be partitioned into $w$ pieces $V_1, V_2, \ldots, V_w$ such that

a) $\forall m, n \in [w]: m \neq n \Rightarrow V_m \cap V_n = \emptyset$

b) $V_1 \cup V_2 \cup \ldots \cup V_w = V(G)$
c) $G[V_n]$ connected, $\forall u \neq e \exists \omega$

d) $G[V_1] \cup G[V_2] \cup \cdots \cup G[V_n] = G$

The subgraphs $G[V_1], \ldots, G[V_n]$ are called components of $G$.

- $w(G) =$ the number of components of $G$.
- Obviously:
  
  $G$ connected $\iff w(G) = 1$
  $G$ not connected $\iff w(G) > 1$.

**Bridges.**

**Thm:** For any graph $G$:

$\forall e \in E(G)$: $w(G) \leq w(G - e \in E) \leq w(G) + 1$

i.e. removing an edge may or may not increase the number of components by 1.

**Remark:** This theorem cannot be generalized to the deletion of vertices.

- Let $G$ be a graph. An edge $e \in E(G)$ is called a bridge if the deletion of $e$ increases the number of components in the resulting graph.
\[ e \in E(G) \text{ bridge} \iff w(G - e) > w(G) \]

**Example**

\[ G: \begin{array}{cccc}
  a & b & c & d \\
  \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \\
  w & z & y & x \\
\end{array} \]

The edges \( ab \) and \( bc \) are bridges.

- Let \( G \) be a connected graph. We say that:
  a) \( G \) is **weakly-linked** if it has at least one bridge
  b) \( G \) is **strongly-linked** if it has no bridges.

- Thus:
  a) \( G \) strongly-linked \( \iff \forall e \in E(G): G - e \text{ connected} \)
  b) \( G \) weakly-linked \( \iff \exists e \in E(G): G - e \text{ not connected} \).
27. Consider the following graph

a) List the components of the following graphs:
   \[ G_1 = G - \{ c \} \quad G_4 = G - \{ e \} \]
   \[ G_2 = G - \{ d \} \quad G_5 = G - \{ he, gf \} \]
   \[ G_3 = G - \{ i \} \quad G_6 = G - \{ d, f \} \]

b) What are the bridges of the graph \( G \)?

29. Consider the following graph
a) List the components of the following graphs:
\[ G_1 = G - \{ b, f, 3 \} \]
\[ G_2 = G - \{ g, 3 \} \]
\[ G_3 = G - \{ b, g, 3 \} \]
\[ G_4 = G - \{ a, h, b, g, 3 \} \]
\[ G_5 = G - \{ h, 3 \} \]
\[ G_6 = G - \{ f, 3 \} \]

b) What are the bridges of the graph \( G \)?

24) Let \( G \) be a connected graph and let \( e \in E(G) \). Show that
\[ \omega(G - \{ e \}) \leq 2 \]
The Laplacian matrix

- Let $G$ be a graph with $n = |V(G)|$ vertices:
  \[ V(G) = \{ v_1, v_2, \ldots, v_n \} \]

  The Laplacian matrix $L_G$ is defined as
  \[
  (L_G)_{ab} = \begin{cases} 
  d(v_a), & \text{if } a = b \\
  -1, & \text{if } a \neq b \text{ and } v_a \leftrightarrow v_b \\
  0, & \text{otherwise.}
  \end{cases}
  \]

- If $w(G)$ is the number of components of $G$, then the characteristic polynomial of $L_G$ has a common factor of $w(G)$ (i.e., 0 is a root with multiplicity $w(G)$).
  Thus
  \[
  \det(L_G - \lambda I) = w(G) f(\lambda)
  \]
  with $f(0) \neq 0$. 

\textbf{Graph connectivity}

\textbf{Edge connectivity} \( A(G) \)

**Def:** Let \( G \) be a graph and let \( E_0 \subseteq E(G) \). We say that \( E_0 \) is an edge cutset of \( G \) if:

\[
\begin{align*}
&\iff \{ G - E_0 \text{ not connected} \\
&\quad \forall E_1 \in \mathcal{P}(E_0) : (E_1 \neq E_0 \Rightarrow G - E_1 \text{ connected})
\end{align*}
\]

\( \bullet \) The smallest number of edges needed to construct a cutset \( E_0 \) of \( G \) is the \underline{edge-connectivity} \( A(G) \) of \( G \). More formally,

\[
A(G) = \min \{ |E_0| \mid E_0 \in \mathcal{P}(E(G)) \land E_0 \text{ edge cut-set of } G \}
\]

\textbf{Vertex connectivity} \( k(G) \)

**Def:** Let \( G \) be a graph and let \( V_0 \subseteq V(G) \). We say that \( V_0 \) is a vertex cutset of \( G \) if:

\[
\begin{align*}
&\iff \{ G - V_0 \text{ not connected} \\
&\quad \forall V_1 \in \mathcal{P}(V_0) : (V_1 \neq V_0 \Rightarrow G - V_1 \text{ connected})
\end{align*}
\]

\( \bullet \) The smallest number of vertices needed to construct a vertex cutset \( V_0 \) of \( G \) is the \underline{vertex connectivity} \( k(G) \) of \( G \).
$K(G) = \min \{ \text{Vol}(V_0 \in V(G)) : V_0 \text{ is a vertex cutset of } G \}$

1. Note that:
   - $G$ not connected $\iff$ $\chi(G) = K(G) = 0$
   - $G$ weakly linked $\iff$ $\chi(G) = 1$
   - $G$ strongly linked $\iff$ $\chi(G) > 1$

2. A property of connectivity

Recall that $\delta(G)$ is the minimum degree of $G$:

$\delta(G) = \min \{ \delta(u) : u \in V(G) \}$

It can be shown that:

**Theorem:** Let $G$ be a graph. Then:

$G$ connected $\implies K(G) \leq \chi(G) \leq \delta(G) \leq \frac{2 |E(G)|}{|V(G)|}$
EXAMPLE

Calculate the vertex connectivity \( k(G) \) and edge connectivity \( \lambda(G) \) for the following graph \( G \):

\[
\begin{align*}
g &\quad a \\
\quad &\quad \quad \downarrow \\
\quad &\quad b \\
\quad &\quad \quad \downarrow \\
\quad &\quad c \\
\quad &\quad \quad \downarrow \\
\quad &\quad d \\
\end{align*}
\]

Solution

First we note that
\[
\begin{align*}
d(a) &= d(b) = d(c) = d(d) = d(e) = d(f) = 3 \
\Rightarrow \delta(G) &= \min_{v \in V(G)} \delta(G) = 3 \Rightarrow k(G) \leq \delta(G) = 3 \Rightarrow k(G) \leq 3
\end{align*}
\]

\[
\Rightarrow k(G) = 0 \lor k(G) = 1 \lor k(G) = 2 \lor k(G) = 3.
\]

Since \( G \) connected \( \Rightarrow k(G) > 0 \).

• Try deleting one vertex

a) For \( G - a \) we have:

\[
\begin{align*}
h &\quad d \\
\quad &\quad \quad \downarrow \\
\quad &\quad e \\
\quad &\quad \quad \downarrow \\
\quad &\quad c \\
\quad &\quad \quad \downarrow \\
\quad &\quad b
\end{align*}
\]

which is connected. \( G - b \), \( G - c \) are similarly connected.
b) For $G-td_3$ we have:

which is connected, and by symmetry, $G-\tau e_3$ and $G-\tau f_3$
are also connected.

It follows from (a) and (b) that $\kappa(G) \geq 1$.

- Try deleting two vertices

  a) For $G-\tau a, b_3$ we have:

  which is still connected. By symmetry, $G-\tau b, c_3$ and $G-\tau c, a_3$ are also connected.

b) For $G-\tau a, d_3$ we have:

  which is still connected. By symmetry, $G-\tau b, e_3$ and $G-\tau c, e_3$ are also connected.

c) For $G-\tau a, e_3$ we have:

  which is still connected. By symmetry, $G-\tau a, f_3$, $G-\tau b, d_3$, $G-\tau b, f_3$, $G-\tau c, d_3$, $G-\tau c, e_3$
are also connected.
d) For $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ we have:

From (a), (b), (c), (d) it follows that $K(G) \geq 2$. Since $2 < K(G) \leq \delta(G) = 3 \Rightarrow K(G) = 3$

and since $K(G) \leq \Delta(G) < \delta(G) \Rightarrow 3 \leq \Delta(G) \leq 3$

$\Rightarrow \Delta(G) = 3$
EXERCISES

9.5 Consider the complete graph $K_a$
Let $u \in V(K_a)$. Show
a) Show that $K_a - w u = K_{a-1}$

b) Show that
$k(K_a) = \Delta(K_a) = \delta(K_a) = a - 1$

9.6 Similarly, for the complete bipartite graph $K_{a,b}$ show that
$k(K_{a,b}) = \Delta(K_{a,b}) = \delta(K_{a,b}) = \min \{a, b\}$

9.7 Show that
a) $K(P_4) = \Delta(P_4) = \delta(P_4) = 1$

b) $K(C_4) = \Delta(C_4) = \delta(C_4) = 2$

9.8 Calculate $k(G)$ and $\Delta(G)$ for the following graphs:

a)

```
 o--h--g--b
 |    |    |
|    |    |
h----e
```

b)

```
 a
|
|---
|   |
|   |
|   |
|
```

```tex
\end{align*}
```
Eulerian graphs

The Eulerian problem: Given a connected graph $G$, is there a walk that can visit every edge of the graph once and only once and return to the starting vertex at the end? If the answer is yes, we say that $G$ is an Eulerian graph and the corresponding walk is an Eulerian trail.

**Def:** Let $G$ be a connected graph. We say that $G$ Eulerian $\iff \exists w \in T(G): E(w) = E(G) \land s(w) = t(w)$

**Example**

The graph $K_{2,2}$:

```
        a
         ↖  ↘
         b   c
         |   |
         d   |
```

is Eulerian with Eulerian trail:

$w = \langle a, ac, c, cb, b, bd, d, da, a \rangle$

Euler solved the Eulerian problem by introducing the definitions for graph, vertex degree, and proving the following theorem:

**Thm:** Let $G$ be a connected graph. Then:

$G$ Eulerian $\iff \forall v \in V(G): \exists e \in N^*(v): d(v) = 2k$
EXAMPLE

Consider the graph

\[ G: \begin{array}{ccc}
& a & \\
& \text{e} & b \\
& C & \text{d} \\
& \text{d} & c
\end{array} \]

\[ d(c) = |\{ bc, cd, ce \}| = 3 \Rightarrow G \text{ not Eulerian.} \]

EXAMPLE

A connected graph with 5 vertices and 4 edges has two vertices with degree 2. Show that the graph G is not Eulerian.

Solution

We assume that \( |V(G)| = 5 \) and \( |E(G)| = 4 \) with

\[ V(G) = \{ u_1, u_2, u_3, u_4, u_5 \} \]

and \( d(u_1) = d(u_2) = 2 \). Define \( a = d(u_3) \land b = d(u_4) \land c = d(u_5) \). From the handshaking lemma:

\[
\sum_{u \in V(G)} d(u) = 2|E(G)| \Rightarrow d(u_1) + d(u_2) + d(u_3) + d(u_4) + d(u_5) = 9 + 4 \\
\Rightarrow 2 + a + b + c = 8 \\
\Rightarrow a + b + c = 4.
\]
G connected $\Rightarrow\ \forall v \in V(G) : d(v) > 0$
$\Rightarrow a > 0 \land b > 0 \land c > 0.$
$\Rightarrow a > 1 \land b > 1 \land c > 1.$

It follows that

$\alpha + b + c = 5 \iff$

$(\alpha, b, c) \in \{ (1,1,2),(1,2,1),(2,1,1) \}$

and therefore:

$a$ odd $\lor b$ odd $\lor c$ odd $\Rightarrow$

$\Rightarrow G$ not Eulerian.
EXERCISES

29. Which of the following graphs is Eulerian?

(a) \[ \begin{array}{ccc}
    & b & \\
    a & \rightarrow & c \\
    & d & \\
\end{array} \]

(b) \[ \begin{array}{ccc}
    & b & c \\
    a & \rightarrow & d \\
    e & \rightarrow & f \\
\end{array} \]

(c) \[ \begin{array}{ccc}
    & a & b \\
    \rightarrow & \rightarrow & \rightarrow \\
    c & d & e \\
\end{array} \]

(d) \[ \begin{array}{ccc}
    a & b \\
    \rightarrow & \rightarrow \\
    d & c \\
\end{array} \]

(e) \[ \begin{array}{ccc}
    a & b \\
    \rightarrow & \rightarrow \\
    \rightarrow & \rightarrow \\
\end{array} \]

(f) \[ \begin{array}{ccc}
    a & b \\
    \rightarrow & \rightarrow \\
    d & c \\
\end{array} \]

\[ k_{2,2} \]

\[ k_{3,3} \]

(g) \[ \begin{array}{ccc}
    a & b & c \\
    d & \rightarrow & \\
\end{array} \]

(h) \[ \begin{array}{ccc}
    a & b & c \\
    d & \rightarrow & e \\
    f & \rightarrow & \\
\end{array} \]
30. Show that
   a) $K_a$ Eulerian $\iff$ $a$ is odd
   b) $K_a \& b$ Eulerian $\iff$ $a$ even $\land b$ even
   c) $\forall a \in \mathbb{N}: (a \geq 2 \implies Pa$ not Eulerian$)$
   d) $\forall a \in \mathbb{N}: (a \geq 3 \implies Ga$ Eulerian$)$

31. A connected Eulerian graph has 3 vertices and 5 edges. Show that if one vertex has degree 4, then another vertex must have degree 2.

32. A connected graph with 4 edges and 4 vertices has 2 vertices of degree 2. Show that
   a) $G$ not Eulerian $\implies \exists u \in V(G): d(u) = 3.$
   b) $G$ Eulerian $\implies G$ regular.

33. Show that a connected regular graph with an odd number of vertices is always Eulerian.

34. Show that a connected regular graph with odd number of edges and whose number of vertices is a multiple of 4 is never Eulerian.
Hamiltonian graphs

Hamilton's Problem: Let G be a connected graph. Can we construct a walk that visits every vertex of the graph once and only once, without using any edge more than once, and then close the walk with a direct edge from its terminal point back to its initial point? If yes, then we say that the graph is a Hamiltonian graph, the walk is a Hamiltonian path, and the walk together with the closing edge is a Hamiltonian circuit.

![Diagram showing a Hamiltonian path and closing edge]

Recall that any walk where no edges or vertices are repeated is a path. The Hamiltonian circuit as a whole is not a path since the initial vertex is repeated once, as a terminal vertex. Thus, the reason for the distinction between the Hamiltonian path and the Hamiltonian circuit. Based on the above, we give the following definition:
Def: Let $G$ be a connected graph. We say that

$G$ Hamiltonian $\iff$

$$\exists u_1, u_2 \in V(G) : \exists e \in E(G) :$$

$$\begin{cases} u_1 \neq u_2 \land \forall w \in V(G) : \exists e \in E(G) : \Psi(e) = \sum_{i} u_i, u_{i+1} \end{cases}$$

Here, $w$ is the Hamiltonian path, $u_1$ the initial vertex, $u_2$ the terminal vertex and $e$ the closing edge.

- Note that it is not necessary for the Hamiltonian circuit to visit all the edges.

Criteria for the Hamiltonian property:

No one has successfully solved the Hamiltonian problem by proving a practical necessary and sufficient condition. We have however the following partial results:

A necessary condition:

Thm: Let $G$ be a connected graph. Then:

$G$ Hamiltonian $\Rightarrow \forall V_0 \subseteq V(G) : (V_0 \neq V(G) \Rightarrow \omega(G - V_0) < |V_0|)$

Intuitively, if the graph $G$ is Hamiltonian, then if we subtract the vertices in $V_0 \subseteq V(G)$, then the resulting
graph $G - V_0$ cannot have more components than the number of vertices in $V_0$.

- The contrapositive statement of this theorem can be used to show that a graph is not Hamiltonian.

**Corollary**: Let $G$ be a connected graph. Then $(\exists V_0 \in \mathcal{P}(V(G)) : (V_0 \neq V(G) \land \omega(G - V_0) > |V_0|)) \Rightarrow G$ not Hamiltonian

In general, proving a statement of the form $p \Rightarrow q$ also proves the contrapositive statement $\neg q \Rightarrow \neg p$. Negations can be calculated according to the following rules of Boolean logic:

<table>
<thead>
<tr>
<th>Statement</th>
<th>It's negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \in A : p(x)$</td>
<td>$\exists x \in A : \neg p(x)$</td>
</tr>
<tr>
<td>$\exists x \in A : p(x)$</td>
<td>$\forall x \in A : \neg p(x)$</td>
</tr>
<tr>
<td>$p \land q$</td>
<td>$\neg p \lor \neg q$</td>
</tr>
<tr>
<td>$p \lor q$</td>
<td>$\neg p \land \neg q$</td>
</tr>
<tr>
<td>$p \Rightarrow q$</td>
<td>$p \land \neg q$</td>
</tr>
<tr>
<td>$p \Leftrightarrow q$</td>
<td>$p \lor \neg q$</td>
</tr>
<tr>
<td>$p \Leftrightarrow q$</td>
<td>$p \Leftrightarrow q$</td>
</tr>
</tbody>
</table>
EXAMPLE

Show that the graph

\[ G: \]

\[ a \quad b \quad c \quad d \quad e \quad f \]

is not Hamiltonian.

Solution

Subtracting the vertex \( d \) gives

\[ G - \{d\} : \]

\[ a \quad b \quad c \quad e \quad f \]

It follows that

\[ G - \{d\} = G[\{a,b,c\}] \cup G[\{e,f,g\}] \Rightarrow \]

\[ w(G - \{d\}) = 2 > 1 = |\{d\}| \Rightarrow \]

\[ w(G - \{d\}) > |\{d\}| \Rightarrow \]

\[ \Rightarrow G \text{ not Hamiltonian.} \]
Ore's Theorem

**Theorem:** Let $G$ be a graph. Then:

- $G$ simple and connected
- $|V(G)| \geq 3$
- $\forall u, v \in V(G): (u, v$ not adjacent $\Rightarrow d(u) + d(v) \geq |V(G)|$

$\Rightarrow G$ is Hamiltonian

**Example**

Use Ore's Theorem to show that $G$ is Hamiltonian.

**Solution**

We note that

- $G$ is simple and connected.
- $|V(G)| = |\{a, b, c, d, e\}| = 5 \geq 3$
- $d(a) + d(d) = 2 + 3 = 5 \geq |V(G)|$
- $d(a) + d(e) = 2 + 3 = 5 \geq |V(G)|$
- $d(b) + d(c) = 3 + 3 = 6 \geq |V(G)|$

$\Rightarrow G$ is Hamiltonian.
Divac's theorem

Thm: Let $G$ be a graph. Then
\[ \begin{align*}
&G \text{ simple and connected} \\
&|V(G)| \geq 3 \implies G \text{ is Hamiltonian} \\
&S(G) \geq (1/2)|V(G)|
\end{align*} \]

Proof

Assume that
\[ \begin{align*}
&G \text{ simple and connected} \\
&|V(G)| \geq 3 \\
&S(G) \geq (1/2)|V(G)|
\end{align*} \] (1)

Let $u, v \in V(G)$ be given and assume that $u, v$ not adjacent. Then:
\[ d(u) + d(v) \geq S(G) + S(G) = 2S(G) \geq 2[(1/2)|V(G)|] = |V(G)| \implies d(u) + d(v) \geq |V(G)| \]

It follows that
\[ \forall u, v \in V(G): (u, v \text{ not adjacent} \implies d(u) + d(v) \geq |V(G)|) \] (4)

From Eq. (1), Eq. (2), Eq. (4), via Ore's theorem, it follows that $G$ is Hamiltonian.

Bipartite graphs

Thm: Let $G$ be a graph. Then
\[ \begin{align*}
&G \text{ connected and bipartite} \implies G \text{ not Hamiltonian} \\
&\exists K \in N: |V(G)| = 2K + 1
\end{align*} \]
Proof
Assume that
\{ G \text{ connected and bipartite} \}
\exists k \in \mathbb{N} : |V(G)| = 2k+1
Since, G is bipartite, we choose \( V_1 \subseteq V(G) \) and \( V_2 \subseteq V(G) \) such that
\{ \begin{align*}
V_1 \cap V_2 &= \emptyset \quad \land \quad V_1 \cup V_2 = V(G) \\
\forall e \in E(G) : \{ |\psi_G(e) \cap V_1| = 1 \ \land \ |\psi_G(e) \cap V_2| = 1 \}
\end{align*} \}
To show a contradiction, assume that G is Hamiltonian.
Then, a Hamiltonian circuit must alternate between vertices in \( V_1 \) and vertices in \( V_2 \). Because each vertex can only be visited once, it follows that
\[ |V_1| = |V_2| \Rightarrow \]
\[ \Rightarrow |V(G)| = |V_1| + |V_2| = |V_1| + |V_1| = 2|V_1| \Rightarrow \]
\[ \Rightarrow |V(G)| \text{ is even} \]
\[ \Rightarrow |V(G)| \text{ not odd} \]
which contradicts the assumption
\[ \exists k \in \mathbb{N} : |V(G)| = 2k+1 \]
It follows that G is not Hamiltonian.
EXAMPLES

Show that the following graph is not Hamiltonian:

\[ \begin{array}{c}
\text{G:} \\
\text{a} \quad \text{b} \\
\text{c} \quad \text{d} \quad \text{e}
\end{array} \]

**Solution**

Note that for \( \mathcal{V}_1 = \{a, b\} \) and \( \mathcal{V}_2 = \{c, d, e\} \):

\[
\forall e \in E(G) : \begin{cases} \\
|\psi_G(e) \cap \mathcal{V}_1| = 1 \Rightarrow G \text{ bipartite} \quad (1) \\
|\psi_G(e) \cap \mathcal{V}_2| = 1
\end{cases}
\]

Furthermore:

\[
|\mathcal{V}(G)| = \|a, b, c, d, e\| = 5 \Rightarrow |\mathcal{V}(G)|\text{ odd} \quad (2)
\]

From (1) and (2): \( G \) is not Hamiltonian.

2nd method

Consider the graph

\[ \begin{array}{c}
\text{G - \{b\}}: \\
\text{a} \\
\text{c} \quad \text{d} \quad \text{e}
\end{array} \]

Since \( G - \{b\} = G[\{a, c, d, e\}] \cup G[\{a, d\}] \Rightarrow \)

\[
\omega(G - \{b\}) = 2 > 1 = |\{b\}| \Rightarrow
\]

\[
\omega(G - \{b\}) > |\{b\}|
\]

\( \Rightarrow G \) not Hamiltonian.
EXERCISES

35) Show that the following graphs are Hamiltonian

a) \[ \begin{array}{ccc}
    a & b & c \\
    d & e & f
\end{array} \]

b) \[ \begin{array}{ccc}
    a & b \\
    c & d
\end{array} \]

c) \[ \begin{array}{ccc}
    a & b \\
    c & d
\end{array} \]

d) \[ \begin{array}{ccc}
    a & b \\
    c & d
\end{array} \]

36) Show that the following graphs are not Hamiltonian

a) \[ \begin{array}{ccc}
    a & l & f \\
    g & e & h
\end{array} \]

b) \[ \begin{array}{ccc}
    a & b \\
    e & d
\end{array} \]
37) Show that $K_a$ is Hamiltonian for all $a \geq 3$.

38) Show that
   a) $a = b \Rightarrow K_{a,b}$ Hamiltonian
   b) $a \neq b \Rightarrow K_{a,b}$ not Hamiltonian

\[ \rightarrow \text{It follows from this exercise that} \]
\[ \text{ } K_{a,b} \text{ Hamiltonian} \Leftrightarrow a = b. \]

39) Let $G$ be a graph with less than 7 vertices and vertex connectivity $\kappa(G) = 4$. Show that $G$ is Hamiltonian.

40) Show that a graph $G$ with vertex connectivity $\kappa(G) = 1$ is not Hamiltonian.

41) Show that a strongly-linked graph with 4 vertices is always Hamiltonian.