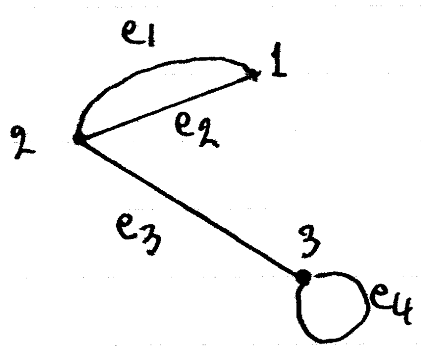
GRAPH THEORY

Graphs – Basic Terminology

- A graph G is an object that consists of
  a) A set of vertices $V(G)$
  b) A set of edges $E(G)$
  c) An incidence mapping ψ_G which maps every edge to one or two vertices, thus
  $$\psi_G : E(G) \longrightarrow P_1(V(G)) \cup P_2(V(G))$$
- Thus it is understood that the edge $e \in E(G)$ connects the vertices $\psi_G(e) = \{v_1, v_2\}$ or $\psi_G(e) = \{v\}$.

example :



$V(G) = \{1, 2, 3\}$
$E(G) = \{e_1, e_2, e_3, e_4\}$
$\psi_G(e_1) = \{1, 2\}$
$\psi_G(e_2) = \{1, 2\}$
$\psi_G(e_3) = \{2, 3\}$
$\psi_G(e_4) = \{3\} \leftarrow$ a loop.

Elementary definitions about graphs

Let G be a graph. We make the following definitions:

1) The vertices $u, v \in V(G)$ are adjacent if there is an edge that connects them.

$$u \leftrightarrow v \Leftrightarrow \exists e \in E(G) : \psi_G(e) = \{u, v\}.$$

2) The vertex $u \in V(G)$ is incident to the edge $e \in E(G)$ if the edge e connects v with itself or with another vertex.

$$u \in e \Leftrightarrow u \in \psi_G(e)$$

3) The edge $e \in E(G)$ is a loop if it connects a vertex with itself.

$$e \text{ loop} \Leftrightarrow |\psi_G(e)| = 1$$

4) The edges $e_1, e_2 \in E(G)$ are adjacent if they share at least one vertex.

$$e_1 \leftrightarrow e_2 \Leftrightarrow |\psi_G(e_1) \cap \psi_G(e_2)| \geq 1$$

Vertex Degrees

- Let $v \in V(G)$ be a vertex of G. The degree $d(v)$ of v is the number of edges $e \in E(G)$ to which v is incident, except that loops count twice.

$$\boxed{d(v) = |\{e \in E(G) \mid v \in e\}| + |\{e \in E(G) \mid v \in e \text{ and } e \text{ loop}\}|}$$

- The minimum degree $\delta(G)$ of G is:

$$\boxed{\delta(G) = \min_{u \in V(G)} d(u)}$$

- The maximum degree $\Delta(G)$ of G is

$$\boxed{\Delta(G) = \max_{u \in V(G)} d(u)}$$

- **Prop**: (The Handshaking lemma)

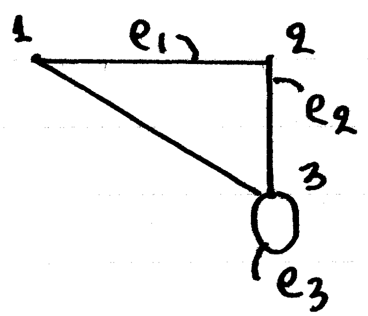
The sum of the degrees of all vertices is twice the number of edges

$$\boxed{\sum_{u \in V(G)} d(u) = 2|E(G)|}$$

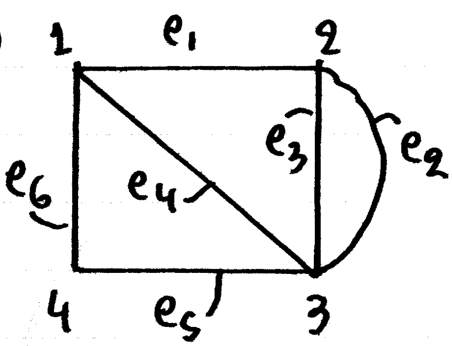
EXERCISES

① For the following graphs, list $V(G)$, $E(G)$, and the values of the incidence mapping ψ_G:

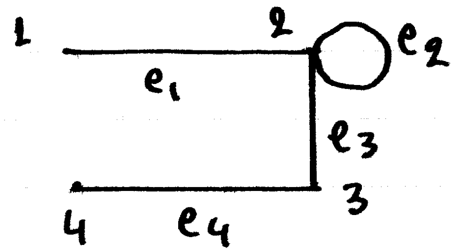
a)



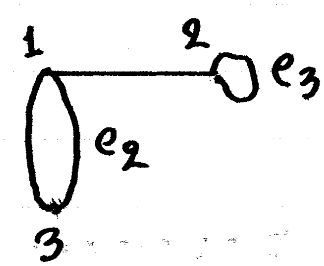
b)



c)



d)



② For the graphs of the previous exercise, list the degrees of each vertex and write $\delta(G)$ and $\Delta(G)$.

③ Show that it is not possible to create a graph with 9 vertices such that the degree of every vertex is 3.

(4) Show that it is not possible to create a graph with 7 vertices of degree 3 and 2 vertices of degree 2.

(5) Let G be a graph with 10 vertices such that

$$\delta(G) = \Delta(G) = 2$$

How many edges does G have?

(6) Let G be a graph with $|V(G)| = 8$ such that $\Delta(G) = 4$. Show that $|E(G)| < 36$.

(7) Let G be a graph such that $|V(G)| = |E(G)|$. Show that $\delta(G) < 3$

(8) Let G be a graph with $\Delta(G) = 4$. Show that

$$|E(G)| \leq 2|V(G)|$$

(9) A graph with 4 edges has a vertex with degree 4, a vertex with degree 1 and one more vertex. What is the degree of the third vertex?

Types of graphs

1) <u>Simple graphs</u>: A graph G is <u>simple</u> if it has no loops and no multiple edges.

$$\boxed{G \text{ simple} \Leftrightarrow \begin{cases} \forall e \in E(G) : |\psi_G(e)| = 2 \\ \forall e_1, e_2 \in E(G) : (\psi_G(e_1) = \psi_G(e_2) \Rightarrow e_1 = e_2) \end{cases}}$$

2) <u>Regular graphs</u>: A graph G is <u>regular</u> if all vertices have the same degree.

$$\boxed{G \text{ regular} \Leftrightarrow \forall v_1, v_2 \in V(G) : d(v_1) = d(v_2)}$$

Specifically we say that G is r-regular if all vertices have degree r.

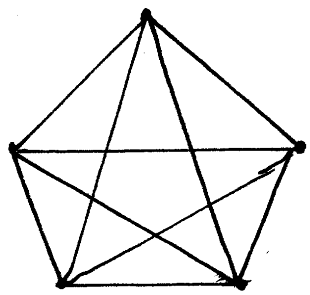
$$\boxed{G \text{ } r\text{-regular} \Leftrightarrow \forall u \in V(G) : d(u) = r}$$

3) <u>Complete graphs</u> : A graph is complete if every two vertices are connected by one edge and the graph is simple.

$$\boxed{G \text{ complete} \Leftrightarrow \begin{cases} \forall\, u,v \in V(G) : \exists\, e \in E(G) : \psi_G(e) = \{u,v\} \\ G \text{ simple} \end{cases}}$$

The complete graph with n vertices is denoted K_n.

e.g.



K_5

4) Null graphs : A null graph is a graph with no edges.

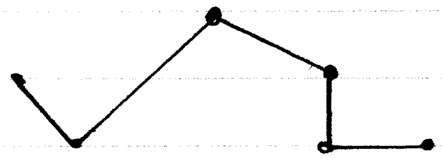
$$\boxed{G \text{ null} \Leftrightarrow E(G) = \emptyset}$$

The null graph with n vertices is denoted N_n.

5) The path graph P_n is defined as the graph with

$$\boxed{\begin{cases} V(P_n) = \{v_1, v_2, \dots, v_n\} \\ E(P_n) = \{e_1, e_2, \dots, e_{n-1}\} \\ \psi(e_k) = \{v_k, v_{k+1}\} \;, \; k = 1, 2, 3, \dots, n-1. \end{cases}}$$

e.g.

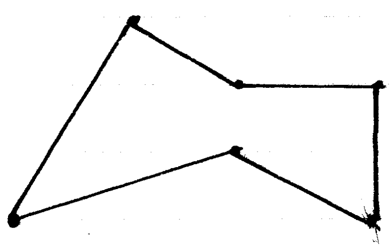


P_6

6) Cycle graphs : The cycle graph C_n is defined as the graph with

$$\boxed{\begin{cases} V(C_n) = \{v_1, v_2, \dots, v_n\} \\ E(C_n) = \{e_1, e_2, \dots, e_n\} \\ \psi(e_k) = \{v_k, v_{k+1}\} \quad , k = 1, \dots, n-1 \\ \psi(e_n) = \{v_n, v_1\} \end{cases}}$$

e.g.



C_6.

▶ Regularity

Note that

a) K_n is $(n-1)$-regular

b) N_n is 0-regular

c) C_n is 2-regular

d) P_n is not regular.

7) Bipartite graphs

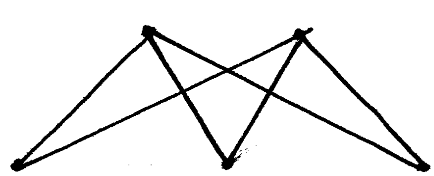
- A graph G is called <u>bipartite</u> if its vertex set $V(G)$ can be partitioned to two sets V_1 and V_2 such that every edge of G connects a vertex in V_1 with a vertex in V_2. Thus, the conditions that must be satisfied are

  a) $V(G) = V_1 \cup V_2$
  b) $V_1 \cap V_2 = \emptyset$ $\Big\}$ V_1, V_2 is a partition of $V(G)$
  c) $\forall e \in E(G): \begin{cases} |\psi_G(e) \cap V_1| = 1 \\ |\psi_G(e) \cap V_2| = 1. \end{cases}$

- The complete bipartite graph $K_{m,n}$ is a bipartite graph, which, in addition to the above conditions also satisfies

  a) $|V_1| = m$, $|V_2| = n$
  b) $\forall u \in V_1 : \forall v \in V_2 : \exists! e \in E(G) : \psi_G(e) = \{u,v\}$.

<u>example</u>



$K_{2,3}$.

EXERCISES

(10) Draw the following graphs:

a) K_4 d) $K_{1,3}$ g) P_4
b) K_5 e) $K_{2,2}$ h) C_3
c) K_6 f) $K_{3,3}$ i) C_4

(11) Which of the graphs in the previous exercise are regular?

(12) For a, b integers $a>0$ and $b>0$ evaluate the following:

a) $\delta(K_a)$ e) $\Delta(K_a)$ i) $|E(K_a)|$
b) $\delta(K_{a,b})$ f) $\Delta(K_{a,b})$ j) $|E(K_{a,b})|$
c) $\delta(P_a)$ g) $\Delta(P_a)$ k) $|E(P_a)|$
d) $\delta(C_a)$ h) $\Delta(C_a)$ l) $|E(C_a)|$

[You can check your general answers by testing them when $a=2$, $b=3$ or $a=4$, $b=3$]

(13) Show that

$K_{a,b}$ regular $\Leftrightarrow$ $a=b$

(14) Show that $K_{5,7}$ is not regular.

(15) Show that
G regular $\Leftrightarrow \delta(G) = \Delta(G)$.

(16) Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$.
If $|V_1| = a$ and $|V_2| = a+2$
show that
$$|E(G)| \leq a^2 + 2a$$

(17) Show that we cannot build a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = 4$ and $|V_2| = 3$ and $|E(G)| > 14$.

▼ Relations between graphs

- Let G, H be two graphs.
  We say that $G \cong H$ (or G is isomorphic to H) if and only if there exist two mappings
  $$f : V(G) \longrightarrow V(H)$$
  $$g : E(G) \longrightarrow E(H)$$
  such that
  a) f, g are bijections
  b) $\forall e \in E(G) : (\psi_G(e) = \{u, v\} \Leftrightarrow \psi_H(g(e)) = \{f(u), f(v)\})$

- It follows that if $G \cong H$ then G, H have the same number of edges and vertices and H can be obtained from G by relabelling the vertices of G according to the mapping f.

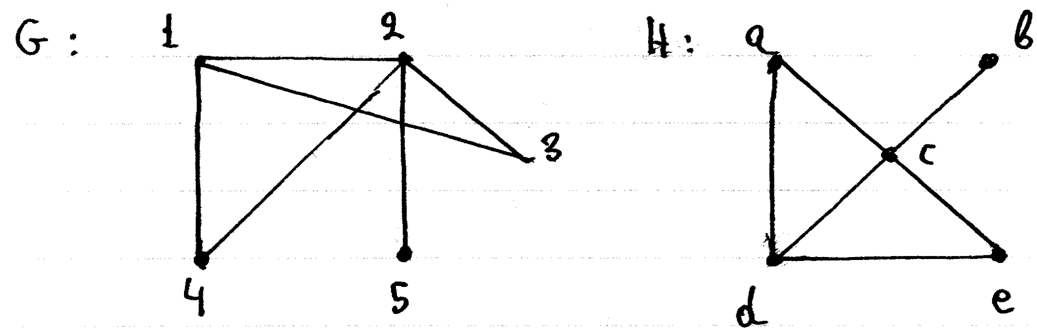
▸ Method

a) To show that $G \cong H$ it is sufficient to discover the appropriate relabelling of vertices, i.e. the mapping f.

b) To show that G, H are not isomorphic we study the degrees of the vertices.
Of course if $|E(G)| \neq |E(H)|$ or $|V(G)| \neq |V(H)|$ then the job is trivially easy.

examples

a)



$G \cong H$ because

$f(1) = d$ (both have degree 3)

$f(2) = c$ (both have degree 4)

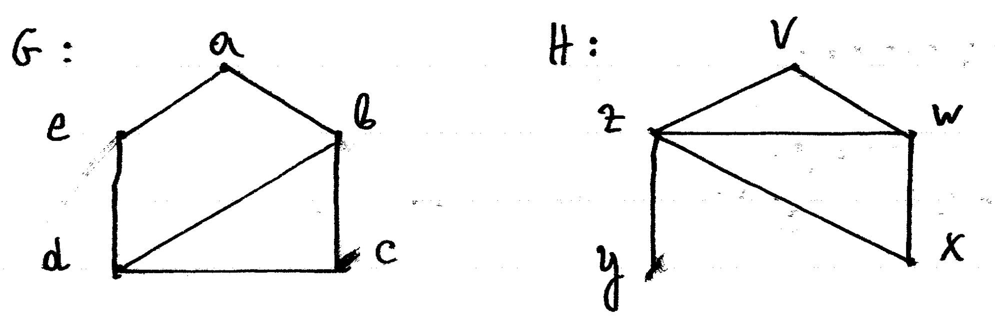
$f(3) = a$ (3 connected with 1, 2)

$f(4) = e$ (4 also connected with 1, 2)

$f(5) = b$ (both have degree 1)

is an isomorphism.

b)



Note that

$$\left.\begin{array}{l} d(y) = 1 \quad (y \in V(H)) \\ d(u) > 1, \ \forall u \in V(G) \end{array}\right\} \Rightarrow G \not\cong H.$$

▸ Properties

a) $G_1 \cong G_2 \Rightarrow G_2 \cong G_1$
b) $G_1 \cong G_2$ and $G_2 \cong G_3 \Rightarrow G_1 \cong G_3$
c) $G \cong G$.

Subgraphs

- Let G, H be two graphs. We say that G is a subgraph of H (notation $G \subseteq H$) if and only if the following conditions are satisfied:
  a) $V(G) \subseteq V(H)$
  b) $E(G) \subseteq E(H)$
  c) $\forall e \in E(G): \psi_G(e) = \psi_H(e)$.

- Thus, when $G \subseteq H$, all the edges and vertices of G are also vertices and edges of H.

- The set of all subgraphs of G is the powerset $\mathcal{P}(G)$:
$$\mathcal{P}(G) = \{H \mid H \subseteq G\}$$

▸ Properties

a) $|V(G)| = n \Rightarrow N_k \subseteq G \ , \ \forall k \leq n.$

b) $\left.\begin{array}{l} G \text{ simple} \\ |V(G)| = n \end{array}\right\} \Rightarrow G \subseteq K_n.$

Graph operations

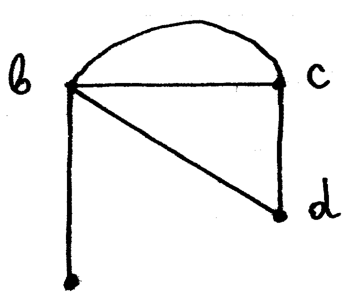
Let G be a graph.

1) Induced subgraph

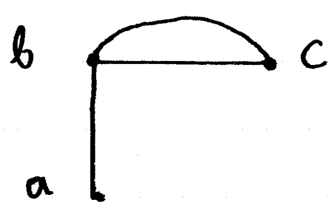
- Let $V_0 \subseteq V(G)$. The induced subgraph $G[V_0]$ is the graph that consists of the vertices in V_0 and the edges to which these vertices are incident. Thus,

$$V(G[V_0]) = V_0$$
$$E(G[V_0]) = \{ e \in E(G) \mid \psi_G(e) \subseteq V_0 \}$$
$$\forall e \in E(G[V_0]) : \psi_{G[V_0]}(e) = \psi_G(e).$$

example



a $\quad$ G



$G[\{a, b, c\}]$

2) Vertex subtraction

- Let $V_0 \subseteq V(G)$. Then $G - V_0$ is the graph obtained by deleting the vertices in V_0 and the edges to which these vertices are incident.

$$\boxed{G - V_0 = G[V(G) - V_0]}$$

3) Edge-induced subgraph

- Let $E_0 \subseteq E(G)$. The $G[E_0]$ is the subgraph of G that contains all the edges in E_0 and the vertices incident to these edges.

Thus,

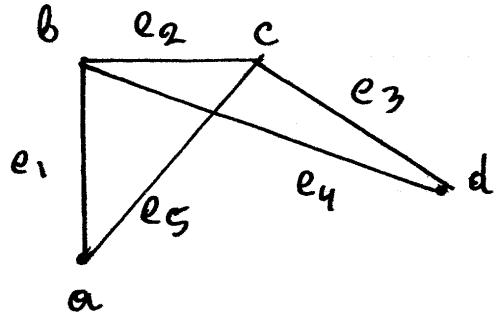
$$\boxed{\begin{array}{l} V(G[E_0]) = \bigcup\limits_{e \in E_0} \psi_G(e) \\ E(G[E_0]) = E_0 \\ \forall e \in E_0 : \psi_{G[E_0]}(e) = \psi_G(e) \end{array}}$$

4) Edge subtraction

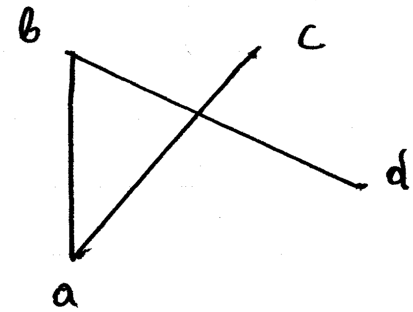
- Let $E_0 \subseteq E(G)$. The graph $G - E_0$ is the subgraph of G obtained by deleting the edges in E_0 and the vertices incident to these edges. Thus

$$\boxed{G - E_0 = G[E(G) - E_0]}$$

example



G



$G - \{e_2, e_3\}$

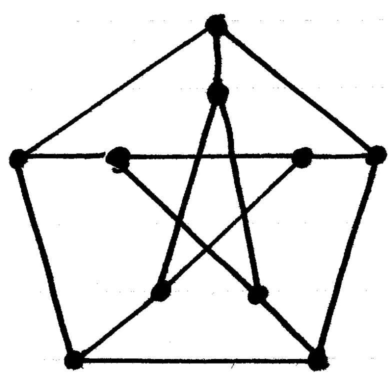
5) Graph union

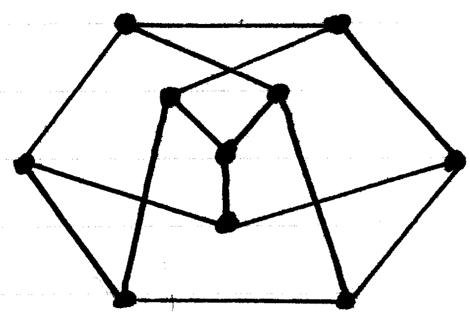
- Let $G_1 \subseteq G$ and $G_2 \subseteq G$. The union $G_1 \cup G_2$ is the subgraph of G that contains the vertices and edges of both G_1 and G_2. Thus:

$$\boxed{\begin{array}{l} V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \\ E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \\ \forall e \in E(G_1 \cup G_2):\ \psi_{G_1 \cup G_2}(e) = \psi_G(e). \end{array}}$$

EXERCISES

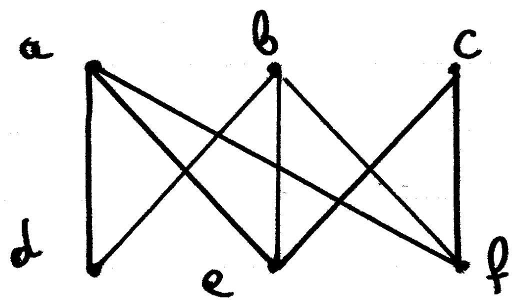
(18) Show that the following graphs are isomorphic





(Hint: Look at the "cycles")

(19) Consider the graph $K_{3,3} = G$



Draw the following:

a) $G[\{a,b,d\}]$

b) $G[\{a,d,e,f\}]$

c) $G[\{a,b,d,e\}]$

d) $G - \{a\}$

e) $G - \{a,b\}$

f) $G - \{a,d\}$

g) $G - \{c,d,e\}$

h) $G - \{d,e,f\}$

i) $G - \{a,c,e,f\}$

(20) In the previous exercise, let

$G_1 = G - \{a, c, e, f\}$

$G_2 = G[\{a, b, e\}]$

Draw $G_1 \cup G_2$.

[Hint : List $V(G_1), E(G_1), V(G_2), E(G_2)$ first].

(21) In the previous exercise show that

$G[\{a, d\}] \cup G[\{b, e\}] \neq G[\{a, d, b, e\}]$

Connected graphs

Walks, trails, paths

- Let G be a graph. A <u>walk W</u> is a sequence of alternating vertices and edges of the form
$$W = (v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n)$$
such that
$$\forall k \in [n] : \psi_G(e_k) = \{v_{k-1}, v_k\}.$$

- Features of a walk.

a) Starting point : $s(W) = v_0$
b) Terminal point : $t(W) = v_n$
c) $v_k(W) = v_k$
$e_k(W) = e_k$
d) Vertex set : $V(W) = \{v_0, v_1, \dots, v_n\}$
e) Edge set : $E(W) = \{e_1, e_2, \dots, e_n\}$
f) Length : $\ell(W) = |E(W)| = n$.

- The set of all walks in G is denoted $W(G)$.

- A <u>trail</u> is a walk in which all the edges are different. A <u>path</u> is a walk in which all the edges and vertices are different.

- Thus, for $w \in W(G)$

a) w trail $\Leftrightarrow$

$$\Leftrightarrow \forall m,n \in [\ell(w)] : (m \neq n \Rightarrow e_m(w) \neq e_n(w))$$

b) w path $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} w \text{ trail} \\ \forall m,n \in [\ell(w)] \cup \{0\} : (m \neq n \Rightarrow v_m(w) \neq v_n(w)) \end{cases}$$

- We define

$$T(G) = \{ w \in W(G) \mid w \text{ is a trail} \}$$
$$P(G) = \{ w \in W(G) \mid w \text{ is a path} \}$$

- Let $u, v \in V(G)$ be two vertices of G with $u \neq v$. Then we define

a) Set of all trails that connect u to v

$$T(G, u \to v) = \{ w \in T(G) \mid s(w) = u \wedge t(w) = v \}$$

b) Set of all paths that connect u to v

$$P(G, u \to v) = \{ w \in P(G) \mid s(w) = u \wedge t(w) = v \}.$$

- Note that $W(G)$ is an infinite set (i.e. you can go back and forth between two vertices indefinitely) but $T(G)$ and $P(G)$ are both finite sets. (i.e you will run out of combinations of distinct edges and/or vertices).

Connected graphs

- A graph G is connected if for any two not-equal vertices $u, v \in V(G)$, there is at least one path from u to v.

$$\boxed{G \text{ connected} \Leftrightarrow \forall\, u, v \in V(G) : \left(u \neq v \Rightarrow |P(G, u \to v)| \geq 1\right)}$$

- The following graphs are connected:
  a) Complete graph K_n
  b) Path graph P_n
  c) Cycle graph C_n
  d) The bipartite graph $K_{m,n}$.

Graph components

<u>Thm</u> : Let G be a graph which is <u>not</u> connected. Then the vertex set $V(G)$ can be partitioned to w pieces $V_1, V_2, \ldots, V_w$ such that

a) $\forall\, m, n \in [w] : m \neq n \Rightarrow V_m \cap V_n = \emptyset$

b) $V_1 \cup V_2 \cup \cdots \cup V_w = V(G)$

c) $G[V_n]$ connected , $\forall n \in [\omega]$
d) $G[V_1] \cup G[V_2] \cup \cdots \cup G[V_\omega] = G$

↳ The subgraphs $G[V_1], \ldots, G[V_\omega]$ are called <u>components</u> of G.

- $\omega(G)$ = the number of components of G.
- Obviously:
  G connected $\Leftrightarrow \omega(G) = 1$
  G not connected $\Leftrightarrow \omega(G) > 1$.

↳ <u>Bridges</u>.

<u>Thm</u> : For any graph G:

$\forall e \in E(G): \ \omega(G) \leqslant \omega(G - \{e\}) \leqslant \omega(G) + 1$
i.e. removing an edge may or may not increase the number of components by 1.

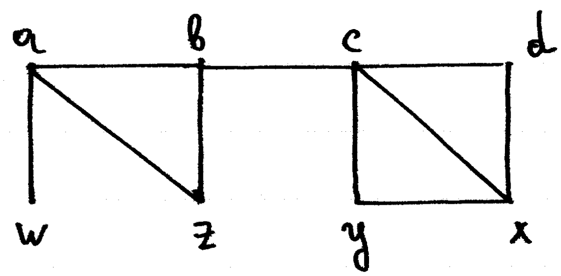
<u>Remark</u> : This theorem cannot be generalized to the deletion of vertices.

- Let G be a graph. An edge $e \in E(G)$ is called a <u>bridge</u> if the deletion of e increases the number of components in the resulting graph.

$$\boxed{e \in E(G) \text{ bridge} \Leftrightarrow \omega(G-\{e\}) > \omega(G)}$$

<u>example</u>

G:



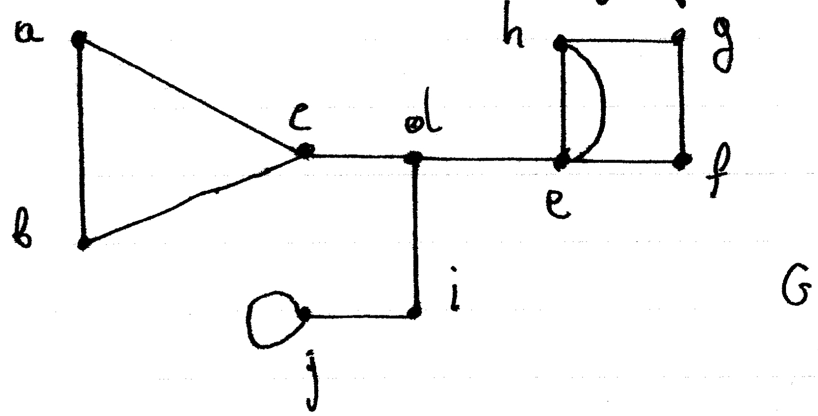
The edges aw and bc are bridges.

- Let G be a connected graph. We say that
  a) G is <u>weakly-linked</u> if it has at least one bridge
  b) G is <u>strongly-linked</u> if it has no bridges.

- Thus:
  a) G strongly-linked $\Leftrightarrow \forall e \in E(G)$: $G-\{e\}$ connected
  b) G weakly-linked $\Leftrightarrow \begin{cases} G \text{ connected} \\ \exists e \in E(G): G-\{e\} \text{ not connected.} \end{cases}$

EXERCISES

(22) Consider the following graph



a) List the components of the following graphs:

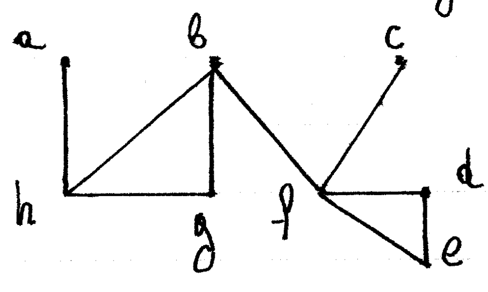
$G_1 = G - \{c\}$ $\quad$ $G_4 = G - \{e\}$

$G_2 = G - \{d\}$ $\quad$ $G_5 = G - \{he, gf\}$

$G_3 = G - \{i\}$ $\quad$ $G_6 = G - \{di\}$

b) What are the bridges of the graph G?

(23) Consider the following graph



a) List the components of the following graphs:

$G_1 = G - \{bf\}$ $\qquad$ $G_4 = G - \{ah, bg\}$

$G_2 = G - \{g\}$ $\qquad$ $G_5 = G - \{h\}$

$G_3 = G - \{b, g\}$ $\qquad$ $G_6 = G - \{f\}$

b) What are the bridges of the graph G?

(24) Let G be a connected graph and let $e \in E(G)$. Show that

$$\omega(G - \{e\}) \leq 2$$

The Laplacian matrix

- Let G be a graph with $n = |V(G)|$ vertices:

  $$V(G) = \{v_1, v_2, \ldots, v_n\}$$

  The Laplacian matrix L_G is defined as

$$\boxed{(L_G)_{ab} = \begin{cases} d(v_a) & \text{, if } a = b \\ -1 & \text{, if } a \neq b \text{ and } v_a \leftrightarrow v_b \\ 0 & \text{, otherwise.} \end{cases}}$$

- If $\omega(G)$ is the number of components of G then the characteristic polynomial of L_G has a common factor $\lambda^{\omega(G)}$ (i.e. 0 is a root with multiplicity $\omega(G)$)

  Thus

$$\boxed{\begin{array}{l} \det(L_G - \lambda I) = \lambda^{\omega(G)} f(\lambda) \\ \text{with } f(0) \neq 0 \end{array}}$$

Graph connectivity

- Let G be a connected graph

Edge connectivity $\lambda(G)$

- Let $E_0 \subseteq E(G)$. We say that E_0 is an <u>edge cutset</u> of G if and only if
  a) $G - E_0$ is not connected
  b) $\forall E_1 \subset E_0 : G - E_1$ is connected.

- The smallest number of edges needed to construct a cutset E_0 of G is the <u>edge-connectivity</u> $\lambda(G)$ of G. Thus

$$\boxed{\lambda(G) = \min\{ |E_0| \mid E_0 \subseteq E(G) \text{ is edge-cutset of } G \}}$$

Vertex connectivity $\kappa(G)$

- Let $V_0 \subseteq V(G)$. We say that V_0 is a vertex cutset of G if and only if
  a) $G - V_0$ not connected
  b) $\forall V_1 \subset V_0 : G - V_1$ connected.

- The smallest number of vertices needed to construct a vertex cutset V_0 of G is the <u>vertex-connectivity</u> $\kappa(G)$ of G. Thus

$$\boxed{\kappa(G) = \min\{ |V_0| \mid V_0 \text{ vertex cutset of } G \}}$$

↳ Note that

a) G not connected $\Leftrightarrow \lambda(G) = \kappa(G) = 0$

b) G weakly-linked $\Leftrightarrow \lambda(G) = 1$

c) G strongly-linked $\Leftrightarrow \lambda(G) > 1$

<u>A property of connectivity</u>

Recall that $\delta(G)$ is the minimum degree of G:

$$\delta(G) = \min\{ d(u) \mid u \in V(G) \}$$

It can be shown that

<u>Thm</u>: G connected $\Rightarrow \boxed{\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \frac{2|E(G)|}{|V(G)|}}$

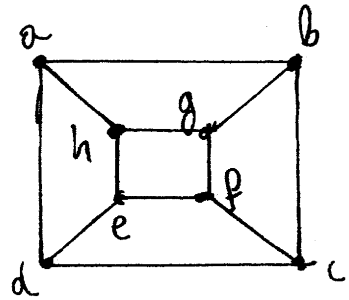
EXERCISES

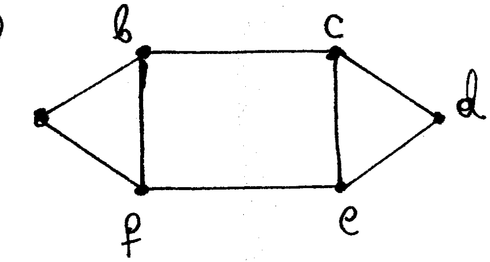
(25) Consider the complete graph K_a
Let $u \in V(K_a)$. Show
a) Show that $K_a - \{u\} = K_{a-1}$
b) Show that
$\kappa(K_a) = \lambda(K_a) = \delta(K_a) = a-1$

(26) Similarly, for the complete bipartite graph $K_{a,b}$ show that
$\kappa(K_{a,b}) = \lambda(K_{a,b}) = \delta(K_{a,b}) = \min\{a,b\}$

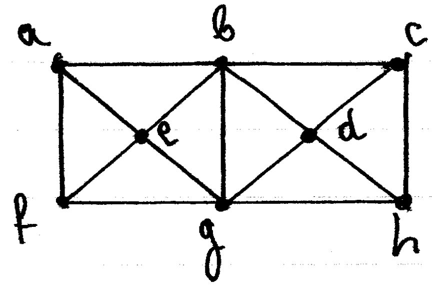
(27) Show that
a) $\kappa(P_a) = \lambda(P_a) = \delta(P_a) = 1$
b) $\kappa(C_a) = \lambda(C_a) = \delta(C_a) = 2$

(28) Calculate $\kappa(G)$ and $\lambda(G)$ for the following graphs:

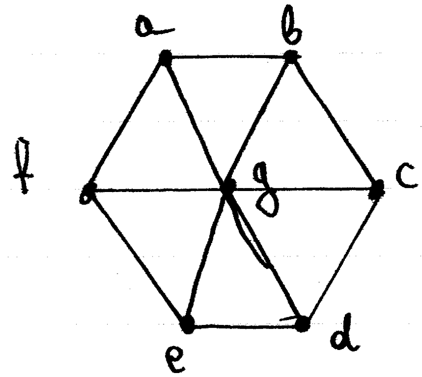
a) 

b) 

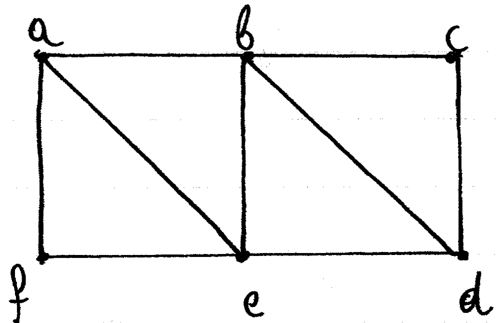
c)



d)



e)



Eulerian graphs.

Let G be a connected graph.

- Eulerian problem:
  Is there a walk that can visit every edge of a graph G only once and return to the starting point at the end?
  If the answer is yes, then we say that G is an Eulerian graph.

- Recall that a walk that visits every edge only once is a trail:

  w trail $\Leftrightarrow$
  $\forall m,n \in [l(w)] : (m \neq n \Rightarrow e_m(w) \neq e_n(w))$

  Also recall our definition of the set of all trails:

  $$T(G) = \{ w \in W(G) \mid w \text{ is a trail} \}$$

- Formal definition:

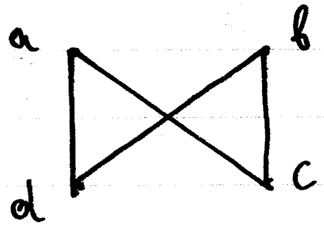
$$\boxed{G \text{ eulerian} \Leftrightarrow \exists w \in T(G) : E(w) = E(G) \wedge s(w) = t(w)}$$

Here $E(w)$ is the edge-set of the trail w.

- Obviously: $\boxed{G \text{ eulerian} \Rightarrow G \text{ connected}}$

<u>example</u>

$K_{2,2}$:



$W = acbda$ is a trail with
$E(W) = \{ac, cb, bd, da\} = E(K_{2,2})$

$\Rightarrow K_{2,2}$ Eulerian.

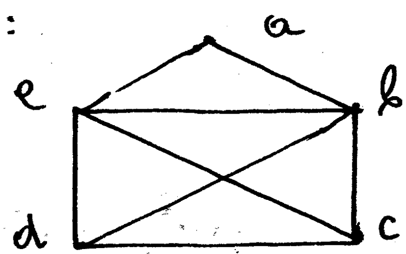
<u>Criterion</u>

A graph is Eulerian if and only if all vertices have even degree.

$$\boxed{G \text{ Eulerian} \Leftrightarrow \forall v \in V(G) : \exists k \in \mathbb{N} - \{0\} : d(v) = 2k}$$

example

$G:$



$d(c) = |\{bc, ec, dc\}| = 3 \Rightarrow G$ not Eulerian.

Partition of Eulerian graph to disjoint cycles

Let G be an connected graph.
A cycle is a trail that starts and ends at the same point. Thus:

$$\boxed{W \text{ cycle} \Leftrightarrow W \in T(G) \wedge s(W) = t(W)}$$

Let $V(W)$ be the vertex set of such a cycle. Induce a subgraph $G[V(W)]$ and consider the edge set $E(G[V(W)])$ of that subgraph.

- We say that G can be partitioned to disjoint cycles $W_1, W_2, \dots, W_n$ if and only if :

a) $W_1, W_2, \ldots, W_n$ are cycles

b) Collectively all the cycles together go through all the vertices:

$$V(W_1) \cup V(W_2) \cup \cdots \cup V(W_n) = V(G)$$

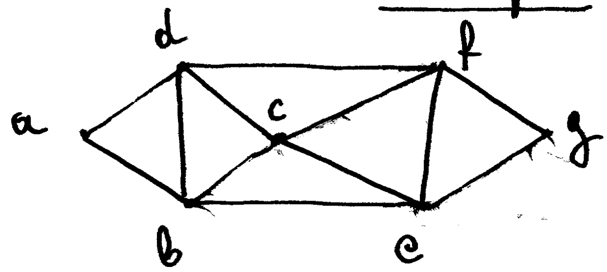
c) Any two cycles do not share an edge

$\forall\, k, m \in [n]$:

$$E(G[V(W_k)]) \cap E(G[V(W_m)]) = \emptyset.$$

__Thm__ : G Eulerian if and only if it can be partitioned to disjoint cycles.

__example__



Disjoint cycles: abcd and cegf.
trail: abcdbecfegfda

▶ To construct an Eulerian trail you must visit all disjoint cycles and return to the beginning. If the trail is not obvious then we use:

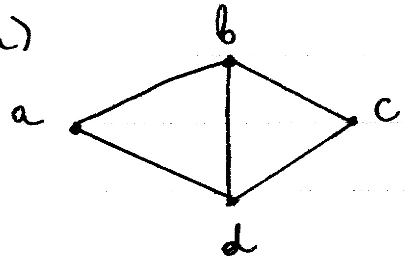
Fleury's Algorithm

1) Pick a vertex to start.
2) Pick an edge to transverse such that
   a) deleting that edge does not disconnect the graph.
   b) if no such edge exists, go ahead and choose an edge that disconnects the graph.
3) Travel over the chosen edge to the next vertex. Delete edge from graph and add edge to a sequence of edges. Thus we have a reduced graph.
4) Repeat 2-3 on the reduced graph until all edges are deleted.
5) The resulting sequence of edges is an Eulerian trail.

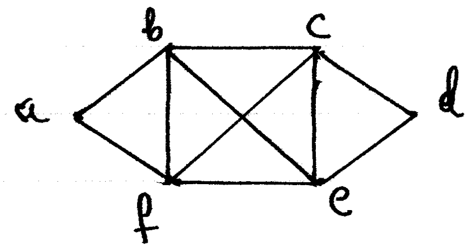
EXERCISES

(29) Which of the following graphs is Eulerian?

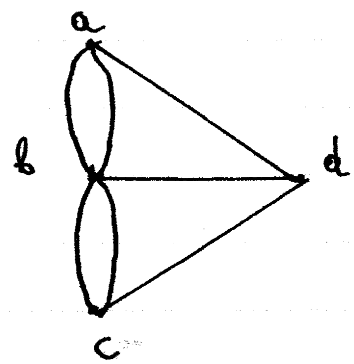
a)



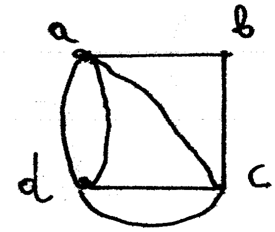
b)



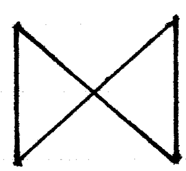
c)



d)

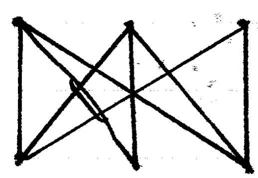


e)



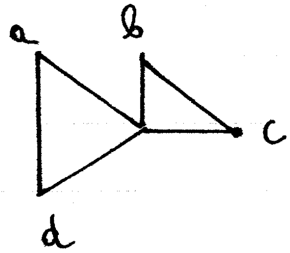
$K_{2,2}$

f)

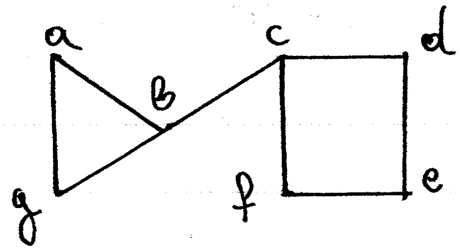


$K_{3,3}$

g)



h)



(30) Show that

a) K_a Eulerian $\Leftrightarrow$ a is odd

b) $K_{a,b}$ Eulerian $\Leftrightarrow$ a is even AND b is even

c) P_a is not Eulerian, $\forall a \geq 2$

d) C_a Eulerian, $\forall a \geq 3$

(31) An Eulerian graph has 3 vertices and 5 edges. Show that if one vertex has degree 4, then another vertex must have degree 2.

(32) A graph with 4 edges and 4 vertices has two vertices of degree 2. Show that

a) If the graph is not Eulerian then it has a vertex with degree 3

b) If the graph is Eulerian, then it is also regular.

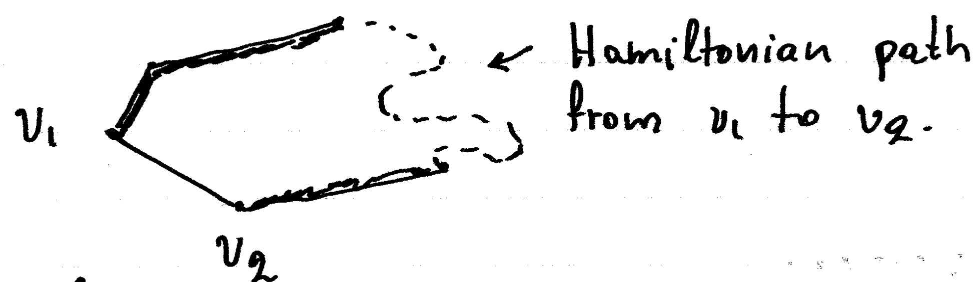
(33) Show that a regular graph with an odd number of vertices is always Eulerian.

(34) Show that a regular graph with odd number of edges and whose number of vertices is a multiple of 4 is never Eulerian.

Hamiltonian graphs

Let G be a connected graph.

- Hamilton problem:
  Can we construct a walk that visits every vertex of a graph once, without going through any edge more than once, and returns back to its starting point? If yes, then the walk is a Hamiltonian circuit and the graph G is called a Hamiltonian graph.



Hamiltonian circuit

- Recall that a path is a walk where no vertices and no edges are repeated. and that the set of all paths from v_1 to v_2 is $P(G, v_1 \to v_2)$.

- A graph G is Hamiltonian if there are two vertices v_1, v_2 such that there is

a) A direct edge from v_1 to v_2

b) A path from v_1 to v_2 that goes through all the vertices of the graph (only once).

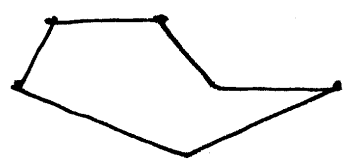
$$\boxed{\begin{array}{l} G \text{ Hamiltonian} \Leftrightarrow \\ \exists\, v_1, v_2 \in V(G) : \exists\, w \in P(G, v_1 \to v_2) : \begin{cases} v_1 \longleftrightarrow v_2 \\ V(w) = V(G) \end{cases} \end{array}}$$

The path w is called the <u>Hamiltonian path</u> and together with the direct edge it forms a <u>Hamiltonian circuit</u>.

- It is not necessary for a Hamiltonian circuit to visit all edges, but if it does then the graph is also Eulerian.

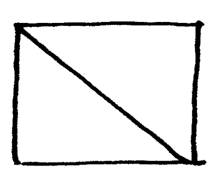
<u>examples</u>

P_2 Hamiltonian



C_6 Hamiltonian.

G:



G Hamiltonian.

- There is no general necessary and sufficient conditions for Hamiltonian graphs discovered yet.

A necessary condition

Recall that
$\omega(G)$ = number of components of G.

Thm: If G is hamiltonian, then if we remove n vertices and all edges that are incident to these vertices, then the resulting subgraph will not break into more than n components.

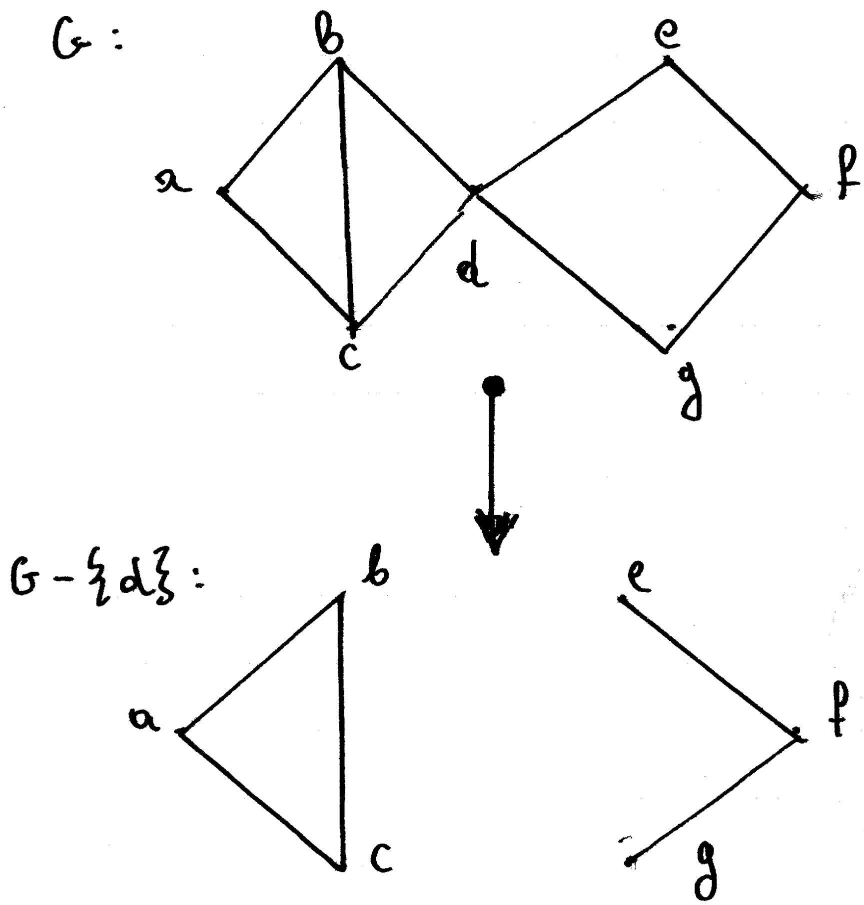
$$\boxed{G \text{ hamiltonian} \Rightarrow \forall V_0 \subset V(G): \ \omega(G - V_0) \leq |V_0|}$$

The contrapositive statement is:

$$\boxed{\exists V_0 \subset V(G): \ \omega(G - V_0) > |V_0| \Rightarrow G \text{ not hamiltonian}}$$

- We can use this statement sometimes, but not always, to show that a graph is not hamiltonian.

example



Note that $\left.\begin{array}{l} \omega(G-\{d\}) = 2 \\ |\{d\}| = 1 \end{array}\right\} \Rightarrow$

$\Rightarrow \omega(G-\{d\}) > |\{d\}| \Rightarrow$

$\Rightarrow$ G not hamiltonian.

Sufficient condition

- Ore's Theorem.

Let G be a simple connected graph with $|V(G)| \geq 3$. Then

$$\boxed{\begin{array}{l}(\forall u,v \in V(G) : (u \not\leftrightarrow v \Rightarrow d(u)+d(v) \geq |V(G)|) \Rightarrow \\ \Rightarrow G \text{ is hamiltonian.}\end{array}}$$

Recall that

$u \not\leftrightarrow v$: u,v not adjacent.

- Dirac's theorem.

$$\boxed{\left.\begin{array}{l} G \text{ simple graph} \\ |V(G)| \geq 3 \\ \delta(G) \geq (1/2)\,|V(G)| \end{array}\right\} \Rightarrow G \text{ is Hamiltonian}}$$

Proof

$\delta(G) \geq (1/2)\,|V(G)| \Rightarrow$

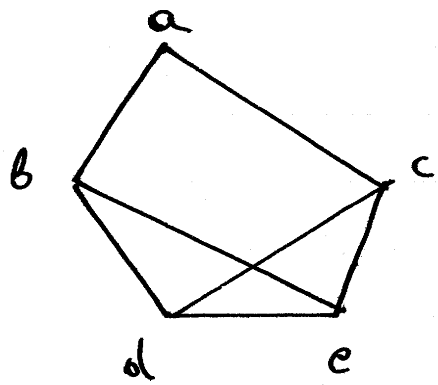
$\Rightarrow \forall u \in V(G) : \; d(u) \geq (1/2)\,|V(G)| \Rightarrow$

$\Rightarrow \forall u,v \in V(G) : \; d(u)+d(v) \geq (1/2)|V(G)| + (1/2)|V(G)|$
$= |V(G)|$

$\Rightarrow$ The cond. of Ore satisfied $\Rightarrow$ G hamiltonian. $\square$

<u>example</u>

G :

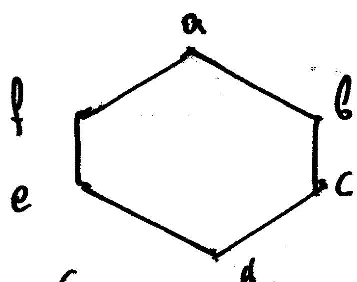


$|V(G)| = 5$

$$\left.\begin{array}{l} d(a) + d(d) = 2+3 = 5 \\ d(a) + d(e) = 2+3 = 5 \\ d(b) + d(c) = 3+3 = 6 \\ d(b) + d(e) = 3+3 = 6 \\ d(d) + d(c) = 3+3 = 6 \end{array}\right\} \begin{array}{l} \Rightarrow \text{Ore theorem applies} \\ \Rightarrow G \text{ Hamiltonian.} \end{array}$$

<u>counterexample</u>

C_6 :



$|V(C_6)| = 6$

$\forall u, v \in V(C_6)$: $d(u) + d(v) = 2 + 2 = 4 > 6 = |V(C_6)|$

$\Rightarrow$ Ore theorem does not apply.

However abcdefa is Hamiltonian circuit $\Rightarrow$

$\Rightarrow$ C_6 Hamiltonian.

On Bipartite graphs.

Thm : Let G be a graph. If

$$\boxed{\left.\begin{array}{l} G \text{ is bipartite} \\ |V(G)| = 2k+1,\ k \in \mathbb{N} \end{array}\right\} \Rightarrow G \text{ is not Hamiltonian}}$$

Proof

G bipartite $\Rightarrow$ There are $V_1, V_2 \subseteq V(G)$ such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V(G)$, and

$$\forall e \in E(G) : \begin{cases} |\psi_G(e) \cap V_1)| = 1 \\ |\psi_G(e) \cap V_2)| = 1 \end{cases}$$

Assume that G is Hamiltonian.
Then, a Hamiltonian circuit must alternate between vertices in V_1 and vertices in V_2 and visit all vertices, returning to point of origin. This requires that

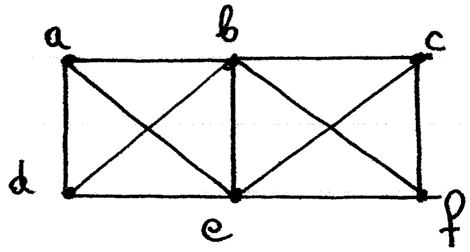
$$|V_1| = |V_2| = n \Rightarrow$$
$$\Rightarrow |V(G)| = |V_1| + |V_2| = n + n = 2n \Rightarrow$$
$$\Rightarrow |V(G)| \text{ even} \leftarrow \text{contradiction.}$$

Thus, G is not Hamiltonian. $\square$

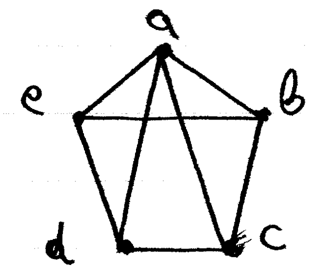
EXERCISES

(35) Show that the following graphs are Hamiltonian

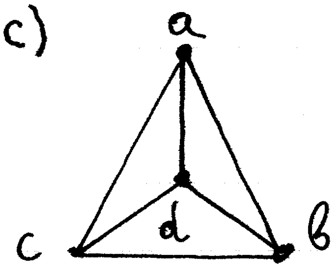
a)



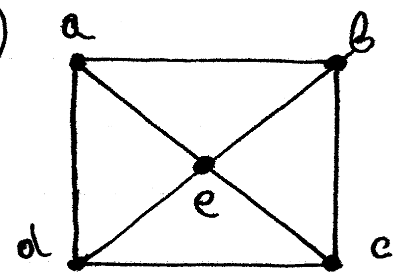
b)



c)

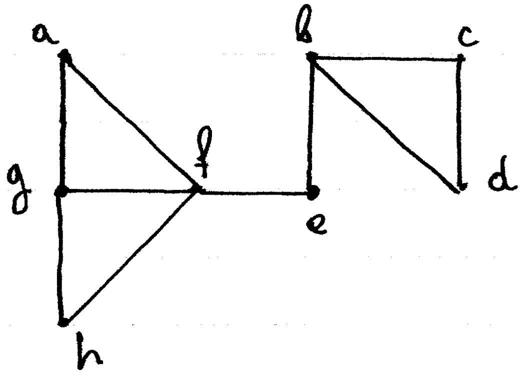


d)

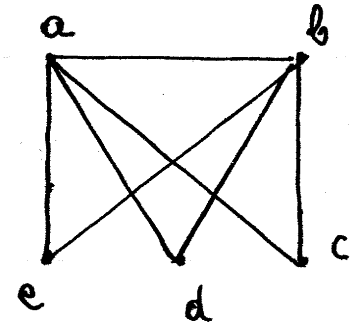


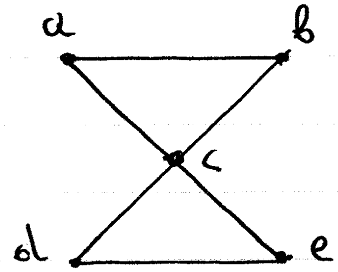
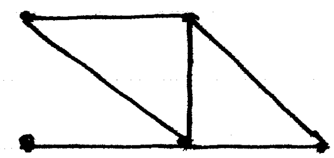
(36) Show that the following graphs are not Hamiltonian

a)



b)



c)  d) 

(37) Show that K_a is Hamiltonian for all $a \geqslant 3$.

(38) Show that
a) $a = b \Rightarrow K_{a,b}$ Hamiltonian
b) $a \neq b \Rightarrow K_{a,b}$ not Hamiltonian

↳ It follows from this exercise that $K_{a,b}$ Hamiltonian $\Leftrightarrow a = b$.

(39) Let G be a graph with less than 7 vertices and vertex connectivity $\kappa(G) = 4$. Show that G is Hamiltonian.

(40) Show that a graph G with vertex connectivity $\kappa(G) = 1$ is not Hamiltonian

(41) Show that a strongly-linked graph with 4 vertices is always Hamiltonian.

Adjacency matrix

- Let G be a graph with $n = |V(G)|$ vertices given by

$$V(G) = \{v_1, v_2, \dots, v_n\}$$

a) We define the adjacency matrix $A(G)$ by:

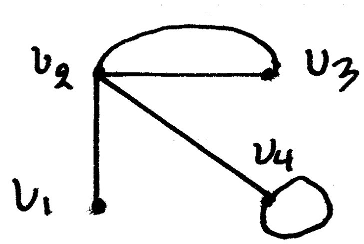
$$\boxed{[A(G)]_{ab} = |\{e \in E(G) \mid \psi_G(e) = \{v_a, v_b\}\}|}$$

Thus, $[A(G)]_{ab}$ is the number of edges that connect v_a and v_b.

b) We also define the matrix $B(G)$ as

$$\boxed{B(G) = A(G) + A^2(G) + \cdots + A^{n-1}(G)}$$

example

G : 

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Properties of adjacency matrix

1) $A(G), B(G)$ are both symmetric:

$$\boxed{\begin{array}{l} [A(G)]_{ab} = [A(G)]_{ba} \\ [B(G)]_{ab} = [B(G)]_{ba} \end{array}}$$

2) Column / Row sums of $A(G)$

$$\sum_{a=1}^{n} [A(G)]_{ab} = d(v_b)$$

$$\sum_{b=1}^{n} [A(G)]_{ab} = d(v_a)$$

3) Enumeration of walks

Recall that $W(G)$ is the set of all walks of G. Let
$$W_l(G) = \{ w \in W(G) \mid l(w) = l \}$$
be the set of all walks of G with length l.
Then:

$$\boxed{[A^k(G)]_{ab} = |\{ w \in W_k(G) \mid s(w) = v_a \wedge t(w) = v_b \}|}$$

4) Enumeration of closed walks

A closed walk $w \in W(G)$ is a walk whose starting point $s(w)$ and end point $t(w)$ coincide. It follows from the previous result that the number of closed walks with length k is given by:

$$\boxed{tr(A^k(G)) = |\{w \in W_k(G) \mid s(w) = t(w)\}|}$$

5) Criterio for connectivity

The graph G is connected if and only if all the off-diagonal elements of $B(G)$ are greater than zero.

$$\boxed{G \text{ connected} \iff [B(G)]_{ab} > 0, \ \forall a, b \in [n] : a \neq b}$$

↳ For property 4 note that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$ then

$$\boxed{tr(A^k(G)) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k} \quad \text{(why?)}$$

EXERCISES

(42) Write the incidence matrices for the following graphs:

a) K_3 d) $K_{3,3}$
b) K_4 e) C_4
c) $K_{2,3}$ f) P_4

(43) Let $A(K_3)$ be the adjacency matrix of K_3
a) Find the characteristic polynomial of $A(K_3)$
b) Show that $A^3 = 3A + 2I$
c) How many open walks does K_3 have of length 3?
d) How many closed walks does K_3 have of length 3?
e) Calculate A^5 and answer the same questions (c), (d) for walks of length 5.

(44) Let $A(K_{2,2})$ be the adjacency matrix of $K_{2,2}$
a) Find the characteristic polynomial of $A(K_{2,2})$
b) How many cycles of length 5 are there in $K_{2,2}$?

(45) Let G be a graph with adjacency matrix

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Show that G is not Eulerian.

(46) Show the following statements:

a) $\sum_{a=1}^{n} \sum_{b=1}^{n} [A(G)]_{ab} = 2|E(G)|$

b) $\left.\begin{array}{l} \sum_{a=1}^{n} [A(G)]_{ab} \geq \frac{n}{2} \\ \text{for all } b \in [n] \end{array}\right\} \Rightarrow G$ Hamiltonian

c) $\sum_{a=1}^{n} [A(G)]_{ab} \geq \lambda(G)$, $\forall b \in [n]$

The shortest path problem

- A weighted graph is a graph G in which every edge has been mapped into a real number, called the weight of the corresponding edge.
- Consequently, in a weighted graph we have a mapping
$$f : E(G) \longrightarrow \mathbb{R}$$
which maps every edge $e \in E(G)$ into a weight $f(e) \in \mathbb{R}$.
- Let $W \in P(G)$ be a path in G.
(i.e. a walk with no repeated vertices and no repeated edges).
Recall that $E(W)$ is the edge-set of W defined as the set of edges traversed by W.
- The weight $f(W)$ of the path W is defined as the sum

$$\boxed{f(W) = \sum_{e \in E(W)} f(e)}$$

- Let $a, b \in V(G)$ with $a \neq b$ be two distinct vertices of the graph G. Consider the set of paths $P(G, a \to b)$ that take you from a to b. We define the <u>distance</u> $f(a,b)$ associated with the weight function f as:

$$f(a,b) = \min_{w \in P(G, a \to b)} f(w) =$$
$$= \min_{w \in P(G, a \to b)} \left[\sum_{e \in E(w)} f(e) \right]$$

The path $w \in P(G, a \to b)$ that minimizes $f(w)$ is called the <u>shortest path</u> from a to b.

- Let us assume positive weights:

$$\forall e \in E(G) : f(e) \geqslant 0$$

Then it can be shown:

a) $\forall a, b \in E(G) : f(a,b) = f(b,a)$
b) $\forall a, b \in E(G) : f(a,b) \geqslant 0$
c) $\forall a, b, c \in E(G) : f(a,c) \leqslant f(a,b) + f(b,c)$.

Dijkstra's Algorithm

• 1 Let $u_0 = a$

$S_0 = \{u_0\}$

$L_0(u) = \begin{cases} u_0 & , \text{ if } u = u_0 \\ \infty & , \text{ if } u \neq u_0 \end{cases}$

where $u \in V(G)$.

Initialization

• 2 Assume that

$S_k = \{u_0, u_1, u_2, \dots, u_k\}$

$L_k(u) : V(G) \to \mathbb{R} \cup \{\infty\}$

have been calculated in the previous step.

If $k = |V(G)| - 1$, then stop.

Otherwise, let

• $L_{k+1}(u) = \begin{cases} L_k(u) & , u \in S_k \\ \min\{L_k(u), L_k(u_k) + f(u_k u)\}, & \text{if } u \in V(G) - S_k. \end{cases}$

(if u_k, u are not adjacent assume that

• $f(u_k u) = \infty$)

u_{k+1} = the element of $V(G) - S_k$ that minimizes $L_{k+1}(u)$

• $S_{k+1} = S_k \cup \{u_{k+1}\}$

and repeat until $k = |V(G)| - 1$.

• 3 At the conclusion of the algorithm the distance from a to ANY vertex $b \in V(G) - \{a\}$ will be given by

$$\rho(a,b) = L_{n-1}(b) \quad , \forall b \in V(G) - \{a\}$$

where $n = |V(G)|$

↳ Backtracking Algorithm

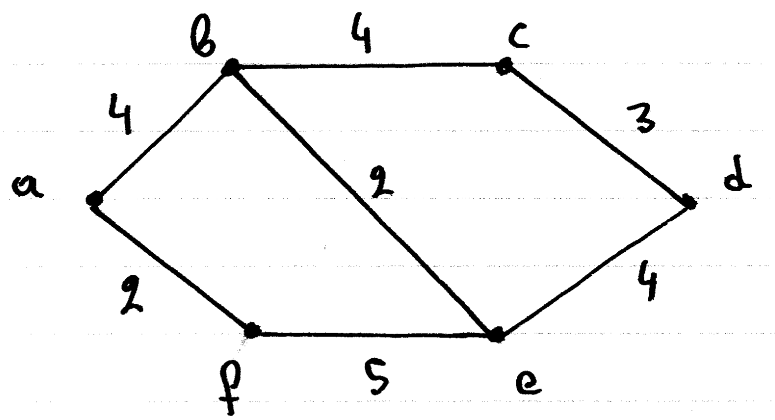
To find the shortest path from a to b we use the following backtracking algorithm:

```
add b to list
let  k = n-1
let  u = b
while  k ≥ 1
   if  L_{k-1}(u) > L_k(u)  then
        add u to list
        let  u = u_{k+1}  (listed below k-column)
   else
        let  k = k-1
   endif
endwhile
```

• The list contains the shortest path backwards.

EXAMPLE

G :



	k=0	k=1	k=2	k=3	k=4	k=5
a	0	0	0	0	0	0
b	∞	4	$\boxed{4}$	4	4	4
c	∞	∞	∞	8	$\boxed{8}$	8
d	∞	∞	∞	∞	10	$\boxed{10}$
e	∞	∞	7	$\boxed{6}$	6	6
f	∞	$\boxed{2}$	2	2	2	2
		a	f	b	e	c

$S_0 = \{a\}$

$S_1 = \{a, f\}$

$S_2 = \{a, f, b\}$

$S_3 = \{a, f, b, e\}$

$S_4 = \{a, f, b, e, c\}$

Log:

1.
   - $L_1(b) = \min\{\infty, 4\} = 4$
   - $L_1(f) = \min\{\infty, 2\} = 2$
   - exclude a – min at f

2.
   - copy a, f
   - $L_2(e) = \min\{\infty, 2+5\} = 7$
   - $L_2(b) = \min\{4, 2+\infty\} = 4$
   - exclude a, f – min b

(Log continued)

(3)
copy a, f, b
$L_3(c) = \min\{\infty, 4+4\} = 8$
$L_3(d) = \min\{\infty, 4+\infty\} = \infty$
$L_3(e) = \min\{7, 4+2\} = 6$ (! change!)
exclude a, f, b $\rightarrow$ minimum at e

(4)
copy a, f, b, e
$L_3(c) = \min\{8, 6+\infty\} = 8$
$L_3(d) = \min\{\infty, 6+4\} = 10$
exclude a, f, b, e $\longrightarrow$ minimum at c

(5)
copy a, f, b, e, c
$L_5(d) = \min\{10, 8+3\} = 10$
done

Thus $f(a,d) = L_5(d) = 10$.

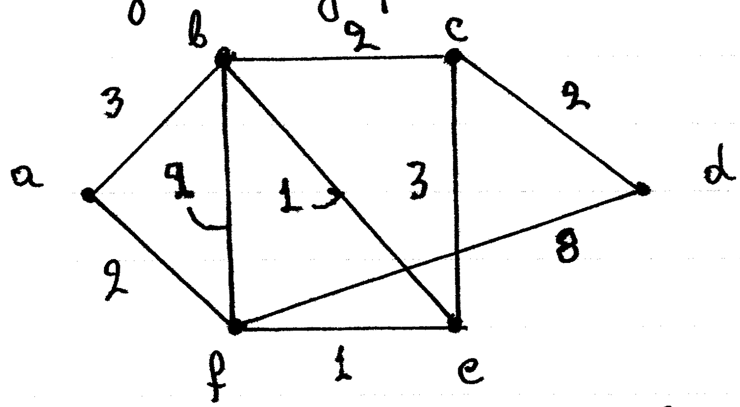
Backtrack

a	0	0	0	0	0	0	
b	∞	4	4	4	4	4	
c	∞	∞	∞	8	8	8	
d	∞	∞	∞	∞	10	10	$\boxed{d}$
e	∞	∞	7	6	6	6	
f	∞	2	2	2	2	2	
		$\boxed{a}$	f	$\boxed{b}$	$\boxed{e}$	c	

shortest path: abed.

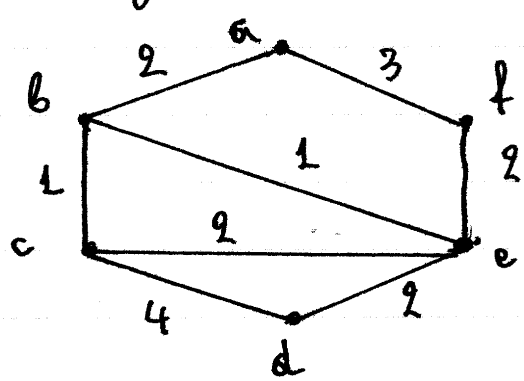
EXERCISES

(47) Apply Dijkstra's algorithm to the following weighted graph.



Find the shortest path from a to d.

(48) Similarly, for the following graph



Find the shortest path from a to d.

Trees

Cyclic vs. Acyclic graphs.

Recall that C_n is the cycle graph with n vertices. Let G be an arbitrary graph. We say that:

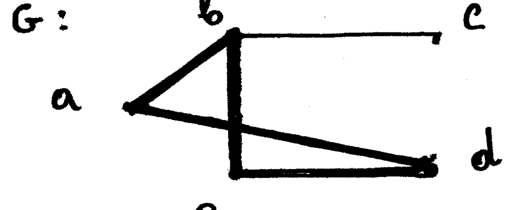
a) G is cyclic, if and only if there is a subgraph $H \subseteq G$ such that H is isomorphic to C_n for some value of n.

$$\boxed{G \text{ cyclic} \Leftrightarrow \exists H \subseteq G : \exists n \in \mathbb{N} - \{0,1\} : H \cong C_n}$$

b) G is acyclic if and only if it is not cyclic.

$$\boxed{G \text{ acyclic} \Leftrightarrow G \text{ not cyclic}}$$

EXAMPLE



G is cyclic
abeda defines a C_4 subgraph

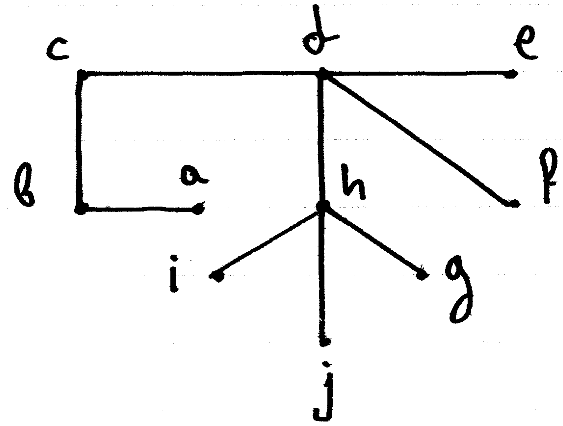
Tree definition

- Let G be a graph. We say that G is a tree, if and only if G is connected and G is acyclic.

$$\boxed{G \text{ tree} \iff \begin{cases} G \text{ connected} \\ G \text{ acyclic} \end{cases}}$$

example

G :



G is a tree.
and also bipartite with
$V_1 = \{a, c, e, f, h\}$
$V_2 = \{b, d, g, i, j\}$

Properties of trees.

1) G tree $\Rightarrow$ G simple
2) G tree $\Rightarrow$ G bipartite
- 3) G tree $\iff |E(G)| = |V(G)| - 1 \wedge G$ connected
4) G tree $\Rightarrow \forall a, b \in V(G) : |P(G, a \to b)| = 1$
(there is a unique path connecting any 2 vertices a, b)

- The unique path that connects the vertices a and b shall be denoted as p_{ab}. It follows that

$$P(G, a \longrightarrow b) = \{ p_{ab} \}$$

5) G tree $\Rightarrow \forall e \in E(G): \omega(G - \{e\}) > \omega(G)$
(i.e. removing any edge from a tree disconnects the graph, thereby increasing the number of components).

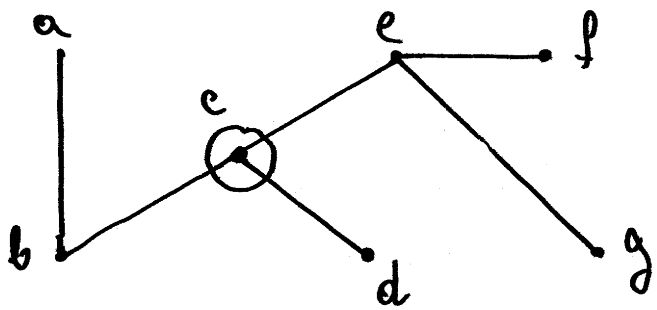
↳ Rooted trees

- Let G be a tree. Suppose we choose a vertex $a \in V(G)$ to be the root of the tree. Then, for any other vertex $b \in V(G)$ there is a unique path p_{ab} from a to b. The level l_b of the vertex b is defined as the length of the unique path p_{ab}:

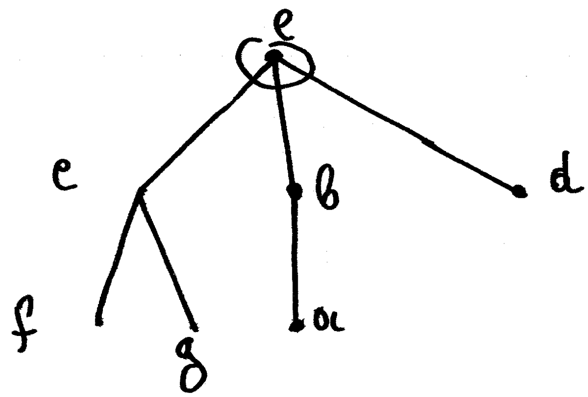
$$\boxed{l_b = l(p_{ab})}$$

Consequently a tree can be represented on the plane so that the vertices are sorted according to their order:

EXAMPLE



using e as the root:



Level 1

Level 2

Trees and connectivity

Recall that for any graph G:

$$\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \frac{2|E(G)|}{|V(G)|}$$

where

$\kappa(G)$ = vertex connectivity
$\lambda(G)$ = edge connectivity
$\delta(G)$ = minimum degree.

Assume that G is a tree.

Given that G is connected, by definition, we have

$$\kappa(G) \geq 1 \quad \text{and} \quad \lambda(G) \geq 1$$

We have also shown that

$$|V(G)| = n \Rightarrow |E(G)| = n-1$$

It follows that

$$\delta(G) \leq \frac{2|E(G)|}{|V(G)|} = \frac{2(n-1)}{n} < 2 \Rightarrow$$

$\Rightarrow \underline{\delta(G) \leq 1}$ (1) (because $\delta(G)$ is integer).

Also $\underline{\delta(G) \geq \kappa(G) \geq 1}$ (2)

From (1) and (2): $\underline{\delta(G) = 1}$

It follows that

$$1 \leq \kappa(G) \leq \delta(G) = 1 \Rightarrow \underline{\kappa(G) = 1}$$

$$1 \leq \lambda(G) \leq \delta(G) = 1 \Rightarrow \underline{\lambda(G) = 1}$$

Conclusion:

$$\boxed{G \text{ is tree} \Rightarrow \kappa(G) = \lambda(G) = \delta(G) = 1}$$

EXERCISES

(49) Let G be a tree. Show that

a) G is not Eulerian (use $\delta(G)=1$)

b) G is not Hamiltonian (use $\kappa(G)=1$)

c) There is another argument for (b) and (c) that is easier than what I suggest.

(50) A <u>forest</u> is a graph G whose components $G_1, G_2, \dots, G_n$ are trees. Show that if G is a forest, then

$$|E(G)| = |V(G)| - \omega(G)$$

(51) A saturated hydrocarbon is a molecule C_aH_b in which

a) Every C atom has 4 simple bonds

b) Every H atom has 1 simple bonds

c) No sequence of bonds forms a cycle.

Show that (a), (b), (c) imply that

$$b = 2a+2.$$

(52) In an unsaturated hydrocarbon C_aH_b we allow double and triple bonds. What is the relation between a, b if we allow d double bonds and t triple bonds?

(53) Show that K_a is not a tree, for $a \geqslant 3$
But it is a tree when $a = 2$.

(54) Show that
$K_{a,b}$ is a tree $\Leftrightarrow$ $a = 1$ or $b = 1$

(55) Show that P_a is a tree for $a \geqslant 2$
But C_a is not a tree for $a > 2$

▼ The minimum spanning tree problem

- Let G be a graph. We say that a tree T is a <u>spanning tree</u> of G if and only if

  (a) T is a tree
  (b) T is a subgraph of G
  (c) All the vertices of G are also vertices of T, and vice versa.

$$\boxed{T \text{ spanning tree of } G \iff \begin{cases} T \subseteq G \\ T \text{ is a tree} \\ V(T) = V(G) \end{cases}}$$

- The set of all spanning trees of G is denoted as

$$\boxed{\tau(G) = \{T \subseteq G \mid T \text{ spanning tree of } G\}}$$

- <u>Thm</u> : (Cayley) The complete graph K_n has n^{n-2} spanning trees.

$$\boxed{|\tau(K_n)| = n^{n-2}}$$

• <u>The problem</u> : Let $f : E(G) \to \mathbb{R}$ be a weight function that maps every edge $e \in E(G)$ to a number $f(e) \in \mathbb{R}$.
If $T \in \tau(G)$ is a spanning tree of G then the weight associated with T is given by

$$\boxed{w(T) = \sum_{e \in E(T)} f(e)}$$

A tree T_0 is a minimum spanning tree if and only if it minimizes $f(T)$.

$$\boxed{T_0 \text{ minimum spanning tree of } G \iff \forall T \in \tau(G) : f(T_0) \leq f(T).}$$

• A graph always has at least one minimum spanning tree and it is not necessarily unique.

• <u>Kruskal's Algorithm</u>

1) Choose $e_1 \in E(G)$ that minimizes $f(e_1)$
2) Assume we have chosen $e_1, e_2, \dots, e_k$.
   Choose $e_{k+1} \in E(G) - \{e_1, e_2, \dots, e_k\}$ such that

(a) $f(e_{k+1})$ is minimum

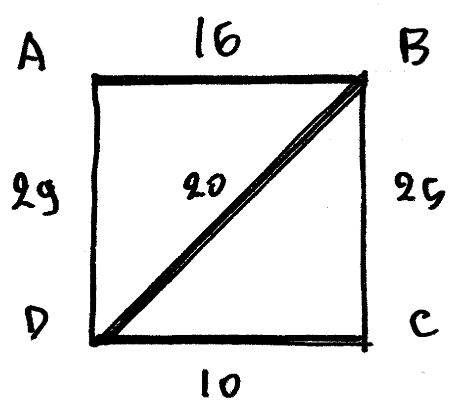
(b) The induced graph $G[\{e_1, e_2, \dots, e_{k+1}\}]$ is acyclic.

3) Repeat 2 until e_{k+1} cannot be found.

Upon completion, we have the edges
$e_1, e_2, \dots, e_n$
and the minimum spanning tree is:

$$T_0 = G[\{e_1, e_2, \dots, e_n\}]$$

example

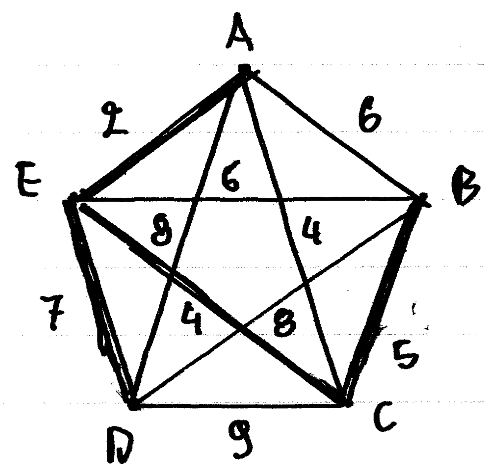


Add DC
Add AB
Add BD

Minimum spanning tree.

$$T_0 = G[\{AB, BD, DC\}]$$

EXAMPLE



Add AE
From AC, CE choose CE
Reject AC (cycle AECA)
Add BC
Reject AB (cycle AECBA)
Reject BE (cycle ECBE)
Add ED
We now have a spanning tree

Thus : $T_0 = G[\{AE, ED, EC, BC\}]$

EXERCISES

(56) Show that if G connected then

$$|E(G)| \geq |V(G)| - 1$$

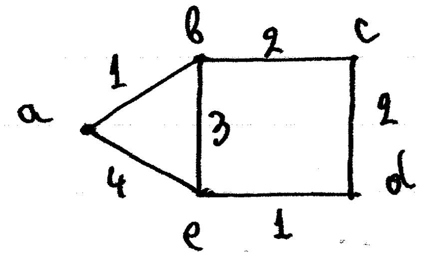
(Hint: Consider the spanning tree of G)

(57) How many spanning trees doe the following graphs have?

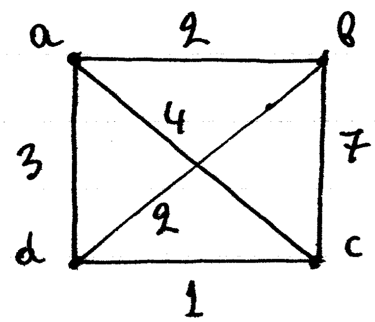
a) K_3 b) K_4 c) K_5

(58) Find the minimum spanning tree for the following graphs:

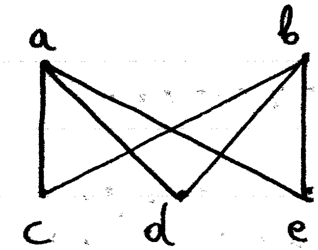
a)



b)



c)



with

$ac = 1$, $ad = 3$, $ae = 2$

$bc = 4$, $bd = 5$, $be = 3$

(59) Show that $|\tau(K_{2,n})| = n2^{n-1}$.

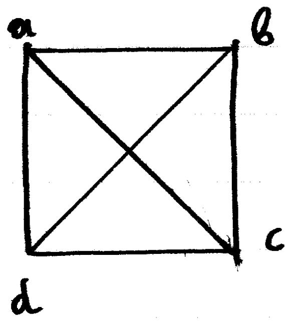
(Hint: Try $K_{2,4}$ first as an example then generalize)

Planar Graphs

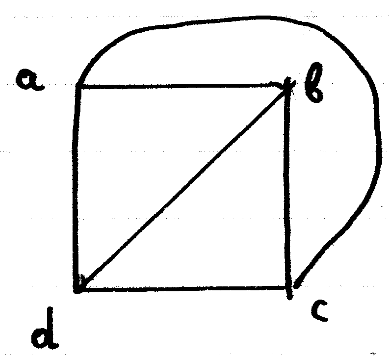
- Let G be a graph. We say that G is <u>planar</u> if and only if it can be embedded on a plane so that no two edges intersect except at the vertices.

example

K_4 is planar:

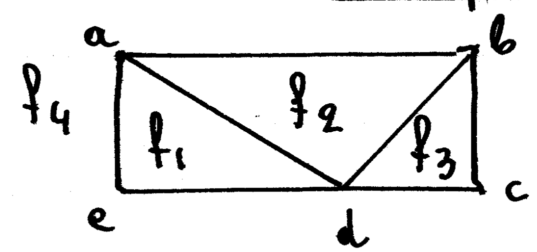


$\Rightarrow$



- A planar graph partitions the plane into regions. We call these regions <u>faces</u> and the set of all faces of G is denoted $F(G)$. Included is also the <u>infinite face</u>.

example



$F(G) = \{f_1, f_2, f_3, f_4\}$

f_4 = infinite face.

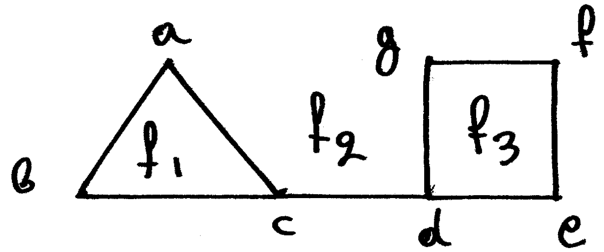
- For every edge $e \in E(G)$, on either side of the edge e there is one or two faces. We say that these faces are <u>incident</u> upon the edge e, and we define an incidence mapping:

$$f_G : E(G) \longrightarrow \mathcal{P}_1(F(G)) \cup \mathcal{P}_2(F(G))$$

such that

$$\forall e \in E(G) : f_G(e) = \{ f_0 \in F(G) \mid f_0 \text{ incident to } e \}$$

<u>example</u>



$f_G(ac) = \{f_1, f_2\}$
$f_G(de) = \{f_2, f_3\}$
$f_G(cd) = \{f_2\}$ *

- Note that if

$$e \text{ cut-edge} \Rightarrow |f_G(e)| = 1.$$

- Two faces f_1, f_2 are adjacent if there is an edge that separates them.

$$\boxed{f_1, f_2 \text{ adjacent} \Leftrightarrow \exists\, e \in E(G) : f_G(e) = \{f_1, f_2\}}$$

Dual graph

Let G be a planar graph. Using the incidence mapping f, we may define a dual graph 'G^*' as follows:
We say that G^* is the dual of G if and only if

(a) Every vertex of G^* corresponds to a face of G. Thus, there is a bijection

$$\varphi_1 : V(G^*) \longrightarrow F(G)$$

(b) Every edge of G^* corresponds to an edge of G. Thus, there is another bijection

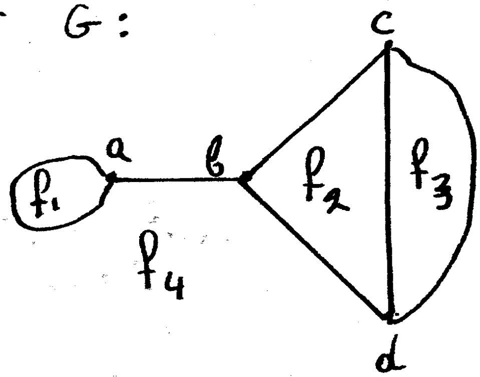
$$\varphi_2 : E(G^*) \longrightarrow E(G)$$

such that an edge e in G^* connects two vertices $v_1, v_2 \in V(G^*)$ if the edge $\varphi_2(e) \in E(G)$ connects $\varphi_1(v_1), \varphi_2(v_2) \in V(G)$
Thus the incidence function of G^* satisfies:

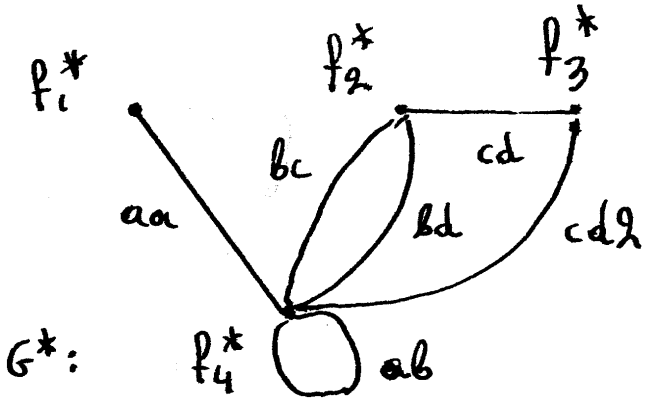
$$\boxed{\forall e \in E(G^*) : \psi_{G^*}(e) = \varphi_1^{-1}(f_G(\varphi_2(e)))}$$

EXAMPLE

For G:



the dual graph G^* is:



Note that

A loop in G	becomes	A cut-edge in G^*	(e.g. aa)
A cut-edge in G	becomes	A loop in G^*	(e.g. ab)

Degree of a face

- Let G be a planar graph with dual graph G^* given by the bijections

$$\varphi_1 : V(G^*) \longrightarrow F(G)$$
$$\varphi_2 : E(G^*) \longrightarrow E(G)$$

and let $f \in F(G)$ be a face of G. The degree of f is defined as the degree of $f^* = \varphi_1^{-1}(f)$ as a vertex in the dual graph:

$$\boxed{d_G(f) = d_{G^*}(\varphi_1^{-1}(f))}$$

EXAMPLE

In the previous example

$d_G(f_1) = 1$, $d_G(f_2) = |\{bc, cd, bd\}| = 3$

$d_G(f_3) = 2$, $d_G(f_4) = 6$ (ab counts twice)

An alternative way to define degree of f: is as follows:

- The boundary $b(f)$ of a face $f \in F(G)$ is a closed walk around the face f in which no edge is repeated except for the cut-edges which are to be repeated twice!
  (Recall that a cut-edge in G gives a loop in G^*).
  The length $l(b(f))$ is the degree of f:

$$\boxed{d_G(f) = l(b(f)), \ \forall f \in F(G)}$$

<u>EXAMPLE</u>

In the previous example:

$b(f_1) = aa \Rightarrow d_G(f_1) = 1$
$b(f_2) = bc, cd, db \Rightarrow d_G(f_2) = 3$
$b(f_3) = cd, (dc)_2 \Rightarrow d_G(f_3) = 2$
$b(f_4) = \underline{ab}, bc, (cd)_2, db, \underline{ba}, aa \Rightarrow d_G(f_4) = 6$

- We say that a planar graph G is <u>face-regular</u> with face-regularity r if and only if

$$\forall f \in F(G) : d_G(f) = r$$

Handshaking Lemma for faces.

Prop: If G is a planar graph, then

$$\boxed{\sum_{f\in F(G)} d_G(f) = 2|E(G)|}$$

Proof

$$\sum_{f\in F(G)} d_G(f) = \sum_{f\in F(G)} d_{G^*}(\varphi_1^{-1}(f)) = \qquad \text{[definition of } d_G(f)\text{]}$$

$$= \sum_{f^*\in V(G^*)} d_{G^*}(f^*) \qquad [\varphi_1^{-1} \text{ bijection}]$$

$$= 2|E(G^*)| \qquad \text{[handshaking lemma on vertices]}$$

$$= 2|E(G)| \qquad [\text{from } \varphi_2 \text{ bijection}]$$

$\square$

Properties of graphs.

1) $\boxed{\left.\begin{array}{l} G_0 \subseteq G \\ G \text{ planar} \end{array}\right\} \Rightarrow G_0 \text{ planar}}$

2) $\boxed{G \text{ tree} \Rightarrow G \text{ planar and } |F(G)| = 1}$

3) $\boxed{\left.\begin{array}{l} G \text{ planar} \\ G^* \text{ dual graph of } G \end{array}\right\} \Rightarrow G^* \text{ is planar}}$

4) Euler's Formula

$$\boxed{\left.\begin{array}{l} G \text{ planar} \\ G \text{ connected} \end{array}\right\} \Rightarrow |V(G)| - |E(G)| + |F(G)| = 2}$$

Proof

Let T be a spanning tree of G.

Then $|V(T)| = |V(G)|$

$|E(T)| = |V(T)| - 1 = |V(G)| - 1$

$|F(T)| = 1$

and therefore

$$|V(T)| - |E(T)| + |F(T)| =$$
$$= |V(G)| - (|V(G)| - 1) + 1 =$$
$$= |V(G)| - |V(G)| + 1 + 1 = 2.$$

Now consider the complete graph G and let

$$E_0 = \{e \in E(G) \mid e \notin E(T)\}$$

Removing any edge $e \in E_0$ removes a face. Removing all edges in E_0 leaves us with only one face. Therefore

$$|F(G)| - |E_0| = 1 \Rightarrow |F(G)| = |E_0| + 1$$
$$= |E_0| + |F(T)|$$

and obviously $|E(G)| = |E(T)| + |E_0|$

It follows that

$$|V(G)| - |E(G)| + |F(G)| =$$
$$= |V(T)| - (|E(T)| + |E_0|) + (|E_0| + |F(T)|)$$
$$= |V(T)| - |E(T)| - |E_0| + |E_0| + |F(T)|$$
$$= |V(T)| - |E(T)| + |F(T)| = 2. \qquad \square$$

5) Necessary condition for planarity.

$$\boxed{\left.\begin{array}{l} G \text{ planar} \\ G \text{ simple and connected} \\ |V(G)| \geq 3 \end{array}\right\} \Rightarrow |E(G)| \leq 3|V(G)| - 6}$$

<u>Proof</u>

G simple $\Rightarrow$ G has no loops or multiple edges $\Rightarrow$

$\Rightarrow \forall f \in F(G): \ d_G(f) \geq 3.$

$$\Rightarrow 2|E(G)| = \sum_{f \in F(G)} d_G(f) \geq 3|F(G)|$$

$$\Rightarrow |F(G)| \leq (2/3)|E(G)|$$

From Euler Formula

$$|E(G)| = -2 + |V(G)| + |F(G)| \leq$$
$$\leq -2 + |V(G)| + (2/3)|E(G)| \Rightarrow$$
$$\Rightarrow (1/3)|E(G)| \leq -2 + |V(G)| \Rightarrow$$
$$\Rightarrow |E(G)| \leq 3|V(G)| - 6 \qquad \square$$

6) Planarity and girth

- The girth $g(G)$ of a graph G is the length of the shortest cycle contained in the graph G.

  Thus $$\boxed{g(G) = \min\{n \mid C_n \subseteq G\}}$$

- Note that
  a) If G has a loop then $g(G) = 1$
  b) If G has a multiple edge and no loops, then $g(G) = 2$
  c) If G is simple $\Rightarrow g(G) \geq 3$.

- Also note that

$$\boxed{G \text{ planar} \Rightarrow \forall f \in F(G) : d_G(f) \geq g(G)}$$

because for any face $f \in F(G)$, the boundary $b(f)$ cannot be longer than $g(G)$.

The main result is as follows:

$$\boxed{\left.\begin{array}{l} G \text{ planar} \\ G \text{ connected} \\ G \text{ simple} \end{array}\right\} \Rightarrow |E(G)| \leqslant \frac{g(G)(|V(G)|-2)}{g(G)-2}}$$

Proof

Since
$\forall f \in F(G) : d_G(f) \geqslant g(G) \Rightarrow$

$$\Rightarrow 2|E(G)| = \sum_{f \in F(G)} d_G(f) \geqslant g(G)\,|F(G)|$$

$$\Rightarrow |F(G)| \leqslant \frac{2}{g(G)} |E(G)|.$$

Since G connected and planar, the Euler formula applies, and solving for $|E(G)|$:

$$|E(G)| = -2 + |V(G)| + |F(G)| \leqslant$$

$$\leqslant -2 + |V(G)| + \frac{2}{g(G)} |E(G)| \Rightarrow$$

$$\Rightarrow \frac{g(G)-2}{g(G)} |E(G)| \leqslant |V(G)| - 2 \Rightarrow$$

$$\Rightarrow |E(G)| \leqslant \frac{g(G)}{g(G)-2}\left(|V(G)|-2\right) \quad \square$$

- Note that with $x = g(G)$, the function

$$f(x) = \frac{x}{x-2}$$

has derivative

$$f'(x) = \frac{(x)'(x-2) - x(x-2)'}{(x-2)^2} =$$

$$= \frac{(x-2)-x}{(x-2)^2} = \frac{-2}{(x-2)^2} < 0$$

for all $x \geqslant 3$. Thus $f(x)$ is decreasing so the inequality becomes tighter as we increase the lower bound on $g(G)$. Also note that

$$f(3) = 3 \longrightarrow \text{property } 5 \quad , \text{ and}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{x-2} = \lim_{x\to+\infty} \frac{x}{x} = 1$$

Thus the coefficient cannot be smaller than 1!

APPLICATIONS

a) K_5 not planar.

Note that $|V(K_5)| = 5$ and
$\forall v \in V(K_5) : d(v) = 5 \Rightarrow$

$$\Rightarrow 2|E(K_5)| = \sum_{v \in V(K_5)} d(v) = 5|V(K_5)| =$$
$$= 5\ 4 \cdot 5 = 20 \Rightarrow |E(K_5)| = 10$$

$$3(|V(K_5)| - 2) = 3(5-2) = 3 \cdot 3 =$$
$$= 9 < 10 = |E(K_5)| \Rightarrow$$

$$\Rightarrow |E(K_5)| > 3(|V(K_5)| - 2) \Rightarrow K_5 \text{ not planar.}$$

b) $K_{3,3}$ not planar

Note that $|V(K_{3,3})| = 3 + 3 = 6$
and
$\forall v \in V(K_{3,3}) : d(v) = 3 \Rightarrow$

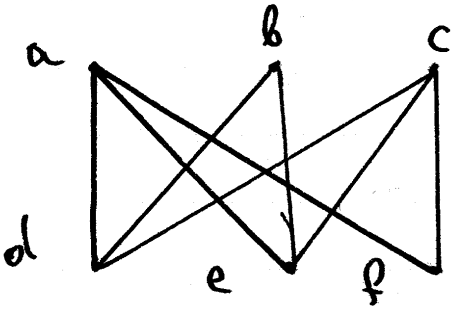
$$\Rightarrow 2|E(K_{3,3})| = \sum_{v \in V(K_{3,3})} d(v) =$$
$$= 3|V(K_{3,3})| = 3 \cdot 6 = 18$$

$\Rightarrow \quad |E(K_{3,3})| = 9.$

Unfortunately

$$3(|V(K_{3,3})|-2) = 3\cdot(6-2) = 3\cdot 4 = 12 >$$
$$> |E(K_{3,3})|$$

so we need a stronger inequality.



$V_1 = \{a, b, c\}$
$V_2 = \{d, e, f\}$

A cycle must go from V_1 to V_2. Then it must go to another vertex of V_1. We cannot return directly, without going to V_2 again. Thus a cycle must have length at least 4, thus

$$g(K_{3,3}) = 4$$

and

$$\frac{g(K_{3,3})}{g(K_{3,3})-2}\left(|V(K_{3,3})|-2\right) =$$

$$= \frac{4}{4-2}\cdot(6-2) = \frac{4\cdot 4}{2} = 8 <$$

$< 9 = |E(K_{3,3})| \Rightarrow K_{3,3}$ not planar.

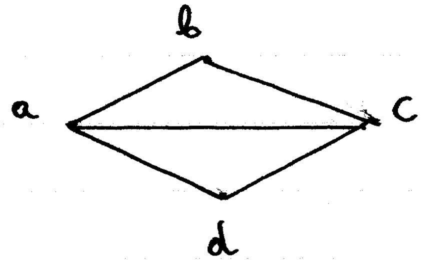
In general, to show that G is non-planar, it is sufficient to show that

$$\boxed{\begin{cases} g(G) \geq a \\ |E(G)| > \dfrac{a}{a-2}\left(|V(G)| - 2\right) \end{cases}}$$

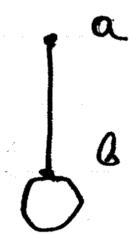
EXERCISES

(60) For the following planar graphs, identify the faces, the degree of each face and then draw the dual graph.

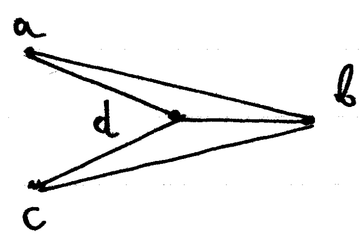
a)



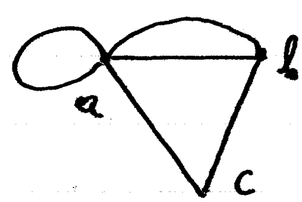
b)



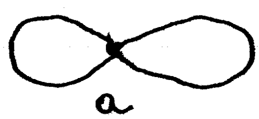
c)



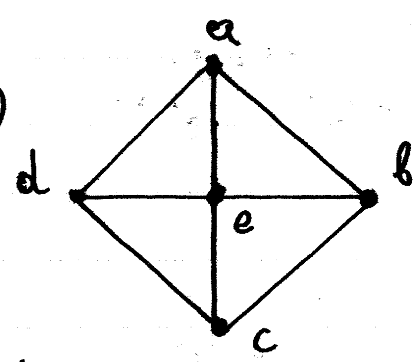
d)



e)



f)



(61) Show that if a connected planar graph is face-regular with face-regularity 4 and has 10 vertices then it must have 8 faces.

(62) Show that a face-regular planar connected graph G with face regularity r must satisfy

$$(a-2)|F(G)| = 2|V(G)|-4$$

Then show that if $|V(G)| \geq 3$ then $a \geq 3$.

(63) Consider a planar graph G which is simple, connected and regular with regularity a and face-regular with regularity b. Show that $2a+2b-ab$ divides $2ab$.
(Hint: First show that

$$(2a+2b-ab)|E(G)| = 2ab)$$

(64) Show that the following graphs are planar:

a) K_2 b) K_4
b) K_3 c) $K_{2,a}$, for $a \geq 1$

(65) Having shown that $K_{2,a}$ is planar, how many faces does it have, as a function of a?

(66) Show that

$a \geq 5 \Rightarrow K_a$ not planar
$a \geq 3 \Rightarrow K_{3,a}$ not planar
$a \geq 3$ and $b \geq 3 \Rightarrow K_{a,b}$ not planar.

(67) Show that a regular graph with regularity $r > 6$ which is also simple and connected can never be a planar graph.

(68) Show that if G is a planar connected graph which is face regular with face regularity a, then

a) $a-2$ divides $2|V(G)|-4$

b) $a=2 \Rightarrow |V(G)|=2$

c) $|V(G)| \geq 2 \Rightarrow a \geq 2$

[Hint: First show that

$$(a-2)|F(G)| = 2|V(G)|-4$$
].