LINEAR ALGEBRA

▼ Matrices – Definitions

- An $n \times m$ matrix A is a collection of nm numbers $A_{ab} \in \mathbb{R}$ (with $a \in [n]$ and $b \in [m]$) arranged in n rows and m columns as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix} \Bigg\downarrow \text{ rows } a = 1, 2, \ldots, n$$

$$\xrightarrow{\text{columns } b = 1, 2, \ldots, m}$$

> Remember:
> A_{rc} : row, column
> A_{vh} : vertical, horizontal

We also write $A = [A_{ab}]$.

A_{ab} = the element of A at row a and column b.

- $M_{nm}(\mathbb{R})$ = the set of all $n \times m$ matrices with elements from $\mathbb{R}$.
- For $n = m$, an $n \times n$ matrix is called a square matrix and we write $M_n(\mathbb{R}) = M_{nn}(\mathbb{R})$.
- Let $A, B \in M_{nm}(\mathbb{R})$ be two matrices. Then
$A = B \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = B_{ab}$.

▶ Zero matrix:

Let $A \in M_{nm}(\mathbb{R})$ be a matrix. Then
$A = \mathbb{0} \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = 0$

▶ Identity Matrix

We say that $I \in M_n(\mathbb{R})$ is an identity matrix if and only if

$$\forall a, b \in [n]: I_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$

▼ Basic operations with matrices

- Let $A, B, C \in M_{nm}(\mathbb{R})$ be given matrices, and let $\lambda \in \mathbb{R}$.
  Then, we define:
  $C = A + B \Leftrightarrow \forall a \in [n] : \forall b \in [m] : C_{ab} = A_{ab} + B_{ab}$ (addition)
  $C = \lambda A \Leftrightarrow \forall a \in [n] : \forall b \in [m] : C_{ab} = \lambda A_{ab}$. (scalar multiplication)
  We also define: $-A = (-1)A$ and $A - B = A + (-1)B$.
- Properties of matrix addition:
  $\forall A, B \in M_{nm}(\mathbb{R}) : A + B = B + A$
  $\forall A, B, C \in M_{nm}(\mathbb{R}) : (A+B)+C = A + (B+C)$
  $\forall A \in M_{nm}(\mathbb{R}) : A + \mathbf{0} = \mathbf{0} + A = A$
  $\forall A \in M_{nm}(\mathbb{R}) : \exists B \in M_{nm}(\mathbb{R}) : A + B = B + A = \mathbf{0}$
- Properties of scalar multiplication
  $\forall \lambda \in \mathbb{R} : \forall A, B \in M_{nm}(\mathbb{R}) : \lambda(A+B) = \lambda A + \lambda B$
  $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu) A$
  $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : (\lambda + \mu) A = \lambda A + \mu A$
  $\forall A \in M_{nm}(\mathbb{R}) : 1 \cdot A = A$
  $\forall \lambda \in \mathbb{R} : \lambda \mathbf{0} = \mathbf{0}$
  $\forall A \in M_{nm}(\mathbb{R}) : (-1) A = -A$

EXAMPLES

a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$.

Calculate $A+B$ and $2A-3B$.

Solution

$$A+B = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1+2 & 3+3 & 2+1 \\ 3-1 & 1+0 & 4-2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ 2 & 1 & 2 \end{bmatrix}$$

$$2A-3B = 2\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 6 & 4 \\ 6 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 9 & 3 \\ -3 & 0 & -6 \end{bmatrix} =$$

$$= \begin{bmatrix} 2-6 & 6-9 & 4-3 \\ 6-(-3) & 2-0 & 8-(-6) \end{bmatrix} = \begin{bmatrix} -4 & -3 & 1 \\ 9 & 2 & 14 \end{bmatrix}$$

b) Prove: $\forall A,B,C \in M_{nm}(\mathbb{R}) : (A+B)+C = A+(B+C)$

Solution

Let $A,B,C \in M_{nm}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then:

$$[(A+B)+C]_{ab} = (A+B)_{ab} + C_{ab} = (A_{ab} + B_{ab}) + C_{ab} =$$
$$= A_{ab} + (B_{ab} + C_{ab}) = A_{ab} + (B+C)_{ab} =$$
$$= [A+(B+C)]_{ab}.$$

It follows that

$\forall a \in [n] : \forall b \in [m] : [(A+B)+C]_{ab} = [A+(B+C)]_{ab}$

$\Rightarrow (A+B)+C = A+(B+C)$

and therefore:

$\forall A, B, C \in M_{nm}(\mathbb{R}) : (A+B)+C = A+(B+C)$.

c) Prove: $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu)A$

<u>Solution</u>

Let $\lambda, \mu \in \mathbb{R}$ and $A \in M_{nm}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then

$[\lambda(\mu A)]_{ab} = \lambda(\mu A)_{ab} = \lambda(\mu A_{ab}) = (\lambda\mu) A_{ab} = [(\lambda\mu) A]_{ab}$.

It follows that

$\forall a \in [n] : \forall b \in [m] : [\lambda(\mu A)]_{ab} = [(\lambda\mu) A]_{ab}$

$\Rightarrow \lambda(\mu A) = (\lambda\mu) A$

and therefore

$\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu) A$.

EXERCISES

① Let A, B be the matrices

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

a) Evaluate $C = 3A - 2B$

b) Solve with respect to X the equation
$2A + 3(X - B) = A + B$

② Let A, B be the matrices

$$A = \begin{bmatrix} a & -2a & c \\ 0 & -a & b \\ a+b & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2a & c \\ a & b-a & -b \\ a-b & 0 & -1 \end{bmatrix}$$

Evaluate and simplify $C = A + B$ and
$D = 2(A - B) - (A + 2B)$

③ Consider the matrix-valued functions

$$A(x) = \begin{bmatrix} 1 & x^2 \\ x & 3x \end{bmatrix}, \quad \forall x \in \mathbb{R}$$

$$B(x) = \begin{bmatrix} x-1 & 2x \\ x^2 & 1 \end{bmatrix}, \quad \forall x \in \mathbb{R}$$

a) Evaluate and simplify the function
$C(x) = 2A(2x+1) - B(x-2), \quad \forall x \in \mathbb{R}$

b) Solve with respect to $Y(x)$ the matrix equation
$3A(x) + 2(Y(x) + A(x)) = A(x+1) - B(x)$

④ Given the function

$$A(x) = \begin{bmatrix} 1 & 2x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

Show that

$A(3x) + 3A(x) = A(0) + 3A(2x), \quad \forall x \in \mathbb{R}$

⑤ Prove that

a) $\forall \lambda \in \mathbb{R} : \forall A, B \in M_{nm}(\mathbb{R}) : \lambda(A+B) = \lambda A + \lambda B$

b) $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : (\lambda + \mu)A = \lambda A + \mu A$

▼ Matrix multiplication

The product AB of two matrices A, B can be defined only when $A \in M_{nl}(\mathbb{R})$ and $B \in M_{lm}(\mathbb{R})$. That is, the number of columns of A must be equal to the number of rows of B. Then we define the product as follows:

- For $A \in M_{nl}(\mathbb{R})$ and $B \in M_{lm}(\mathbb{R})$, we define $(AB) \in M_{nm}(\mathbb{R})$ such that

$$\boxed{\forall a \in [n] : \forall b \in [m] : (AB)_{ab} = \sum_{\gamma=1}^{l} A_{a\gamma} B_{\gamma b}}$$

↳ To illustrate the definition, we consider the following special cases:

a) Row matrix X Column matrix : $A \in M_{1n}(\mathbb{R}) \wedge B \in M_{n1}(\mathbb{R})$.
Then $AB \in M_1(\mathbb{R})$ with

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} =$$

$$= \begin{bmatrix} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{bmatrix}$$

b) Product of 2×2 matrices: $A, B \in M_2(\mathbb{R})$.

$$AB = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_2 d_1 & a_1 c_2 + a_2 d_2 \\ b_1 c_1 + b_2 d_1 & b_1 c_2 + b_2 d_2 \end{bmatrix}$$

$\hookrightarrow$ From the above examples we see that the element $(AB)_{ab}$ is the product of <u>row a of matrix A</u> and <u>column b of matrix B.</u>

▶ Properties of Matrix Multiplication

$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{kl}(\mathbb{R}) : \forall C \in M_{lm}(\mathbb{R}) : (AB)C = A(BC)$

$\forall A \in M_{nk}(\mathbb{R}) : \forall B, C \in M_{km}(\mathbb{R}) : \quad A(B+C) = AB + AC$

$\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : \quad (B+C)A = BA + CA$

$\forall \lambda \in \mathbb{R} : \forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : \lambda(AB) = (\lambda A)B = A(\lambda B)$

$\forall A \in M_n(\mathbb{R}) : IA = AI = A \quad (I \in M_n(\mathbb{R})$ is the identity matrix$)$

$\forall A \in M_n(\mathbb{R}) : A\mathbb{0} = \mathbb{0}A = \mathbb{0}$

• It is not true for all matrices that $AB = BA$ (see homework for a counterexample). This creates some interesting complications.

▶ Manipulation Properties

$\forall A, B, C \in M_{nm}(\mathbb{R}) : A = B \Leftrightarrow A + C = B + C$

$\forall A, B \in M_{nk}(\mathbb{R}) : \forall C \in M_{km}(\mathbb{R}) : A = B \Rightarrow AC = BC$

$\forall C \in M_{nk}(\mathbb{R}) : \forall A, B \in M_{km}(\mathbb{R}) : A = B \Rightarrow CA = CB$

$\forall A, B, C \in M_{nm}(\mathbb{R}) : A + B = C \Leftrightarrow A = C - B$

• Note that the cancellation property $CA = CB \Rightarrow A = B$ is not true for all matrices

▶ Matrix powers

Let $A \in M_n(\mathbb{R})$ be a square matrix. We define

$$A^n = \underbrace{A \cdot A \cdot \ldots \cdot A}_{n \text{ times}}$$

$\forall a, b \in \mathbb{N} - \{0\} : \forall A \in M_n(\mathbb{R}) : A^a A^b = A^{a+b}$

$\forall a, b \in \mathbb{N} - \{0\} : \forall A \in M_n(\mathbb{R}) : (A^a)^b = A^{ab}$

EXAMPLES

a) Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. Find all $x, y \in \mathbb{R}$ such that $A^2 = xA - yI$.

Solution

We note that

$$A^2 = AA = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot2+1\cdot3 & 2\cdot1+1\cdot2 \\ 3\cdot2+2\cdot3 & 3\cdot1+2\cdot2 \end{bmatrix} =$$

$$= \begin{bmatrix} 4+3 & 2+2 \\ 6+6 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix}$$

and

$$xA - yI = x\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2x & x \\ 3x & 2x \end{bmatrix} - \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} =$$

$$= \begin{bmatrix} 2x-y & x \\ 3x & 2x-y \end{bmatrix}$$

It follows that

$$A^2 = xA - yI \Leftrightarrow \begin{bmatrix} 7 & 4 \\ 12 & 7 \end{bmatrix} = \begin{bmatrix} 2x-y & x \\ 3x & 2x-y \end{bmatrix} \Leftrightarrow \begin{cases} 2x-y=7 \\ 3x=12 \\ x=4 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2\cdot4-y=7 \\ x=4 \end{cases} \Leftrightarrow \begin{cases} 8-y=7 \\ x=4 \end{cases} \Leftrightarrow \begin{cases} y=8-7=1 \\ x=4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x=4 \wedge y=1.$$

b) Let $A, B \in M_n(\mathbb{R})$ such that $A^2 = I$ and $B^2 = B$. Show that $(2B-I)^2 = I$ and $(A+I)^2 = 2(A+I)$.

Solution

Assume that $A, B \in M_n(\mathbb{R})$ with $A^2 = I$ and $B^2 = B$. Then

$$(2B-I)^2 = (2B-I)(2B-I) = 2B(2B-I) - I(2B-I) =$$
$$= (2B)(2B) - (2B)I - I(2B) + I^2 =$$
$$= 4B^2 - 2B - 2B + I = 4B^2 - 4B + I \overset{*}{=}$$
$$= 4B - 4B + I = 0B + I = 0 + I = I.$$

and

$$(A+I)^2 = (A+I)(A+I) = A(A+I) + I(A+I) =$$
$$= AA + AI + IA + I^2 = A^2 + A + A + I =$$
$$= A^2 + 2A + I \overset{*}{=} I + 2A + I = 2A + 2I$$
$$= 2(A+I).$$

c) Prove: $\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = BA + CA$

Solution

Let $B, C \in M_{nk}(\mathbb{R})$ and $A \in M_{km}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then

$$[(B+C)A]_{ab} = \sum_{\gamma \in [k]} (B+C)_{a\gamma} A_{\gamma b} = \sum_{\gamma \in [k]} (B_{a\gamma} + C_{a\gamma}) A_{\gamma b} =$$
$$= \sum_{\gamma \in [k]} (B_{a\gamma} A_{\gamma b} + C_{a\gamma} A_{\gamma b}) =$$
$$= \sum_{\gamma \in [k]} B_{a\gamma} A_{\gamma b} + \sum_{\gamma \in [k]} C_{a\gamma} A_{\gamma b} =$$
$$= (BA)_{ab} + (CA)_{ab} = (BA + CA)_{ab}$$

It follows that

$\forall a \in [n] : \forall b \in [m] : [(B+C)A]_{ab} = (BA+CA)_{ab} \Rightarrow$

$\Rightarrow (B+C)A = BA+CA$

and therefore

$\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = BA + CA.$

d) Prove: $\forall A, B, C \in M_{nm}(\mathbb{R}) : (A = B \Rightarrow A+C = B+C)$

Solution

Let $A, B, C \in M_{nm}(\mathbb{R})$ be given and assume that $A = B$.

Then:

$A = B \Rightarrow \forall a \in [n] : \forall b \in [m] : A_{ab} = B_{ab}$ (1)

Let $a \in [n]$ and $b \in [m]$ be given. Then:

$(A+C)_{ab} = A_{ab} + C_{ab} = B_{ab} + C_{ab} = (B+C)_{ab}$

and it follows that

$\forall a \in [n] : \forall b \in [m] : (A+C)_{ab} = (B+C)_{ab}$

$\Rightarrow A+C = B+C$

and therefore we have shown that

$\forall A, B, C \in M_{nm}(\mathbb{R}) : (A = B \Rightarrow A+C = B+C)$

EXERCISES

(6) Consider the matrix

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

a) Find the unique $x, y \in \mathbb{R}$ such that $A^2 = xA + yI$

b) Use (a) to find $z, w \in \mathbb{R}$ such that $A^3 = zA + wI$.

(7) Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

evaluate and simplify

a) $C = AB$

b) $D = BA - 3B^2$

c) $E = 2A^3 - 3A + I$

(8) Given the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

evaluate and simplify $C = AB - BA$

(9) Given the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{R}$

show that: $A^2 - (a+d)A + (ad - bc)I = 0$

(10) Prove the following properties

a) $\forall A \in M_{nk}(\mathbb{R}) : \forall B, C \in M_{km}(\mathbb{R}) : A(B+C) = AB + AC$

b) $\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{k\lambda}(\mathbb{R}) : \forall C \in M_{\lambda m}(\mathbb{R}) : (AB)C = A(BC)$

(11) Rotation matrix

Let $R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$

Show that

a) $R(\vartheta_1)R(\vartheta_2) = R(\vartheta_1 + \vartheta_2)$, $\forall \vartheta_1, \vartheta_2 \in \mathbb{R}$

b) $R(\vartheta)R(-\vartheta) = I$, $\forall \vartheta \in \mathbb{R}$.

(12) For $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix}$

show that $AB \neq BA$

(13) For $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ with

$a, b \in \mathbb{R}$, show that $AB = BA$.

(14) Consider the function

$$\forall x \in \mathbb{R} : M(x) = \begin{bmatrix} 1 & 0 & x \\ -x & 1 & -x^2/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that

$\forall a, b \in \mathbb{R} : M(a)M(b) = M(a+b)$.

(15) Let $z = a+bi \in \mathbb{C}$ be a complex number, and define

$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Show that

a) $\forall z_1, z_2 \in \mathbb{C}: M(z_1+z_2) = M(z_1)+M(z_2)$

b) $\forall z_1, z_2 \in \mathbb{C}: M(z_1 z_2) = M(z_1)M(z_2)$

$\hookrightarrow$ This shows that $M(z)$ "imitates" the behaviour of complex number algebra.

(16) Let $A, B \in M_n(\mathbb{R})$. Show that

a) $AB = BA \Rightarrow (A-\lambda I)(B-\lambda I) = (B-\lambda I)(A-\lambda I), \forall \lambda \in \mathbb{R}$.

b) $(A+B)^2 = A^2+2AB+B^2 \Rightarrow AB = BA$

c) $A^2 = A \Rightarrow (A-I)^2 = I-A$

d) $AB = BA \Rightarrow A^2B^2 = B^2A^2$

e) $(B^2 = I \wedge AB = -AB) \Rightarrow AB = BA = 0$

(17) Let $A \in M_n(\mathbb{R})$ such that $A^2 = I$. Show that the matrices

$B = (1/2)(I+A)$

$C = (1/2)(I-A)$

satisfy $B^2 = B$ and $C^2 = C$.

Matrix Inverses

- Let $A \in M_n(\mathbb{R})$ be a square matrix. We say that
B inverse of $A \iff AB = BA = I$.
- We define the set of all matrices $A \in M_n(\mathbb{R})$ that have an inverse as:
$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \exists B \in M_n(\mathbb{R}): AB = BA = I\}$
We say that
A non-singular $\iff A \in GL(n, \mathbb{R})$
A singular $\iff A \notin GL(n, \mathbb{R})$
- Square matrices are not guaranteed to have an inverse.
For example, for $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have:
$$\forall x, y, z, w \in \mathbb{R}: \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
However, we will argue that if a matrix does have an inverse, it is unique:
- $$\boxed{\forall A, B, C \in M_n(\mathbb{R}): \begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C}$$

Proof

Let $A, B, C \in M_n(\mathbb{R})$ be given such that $AB = BA = I$ and $AC = CA = I$. Then
$$\begin{aligned} B &= BI && [\text{Identity matrix}] \\ &= B(AC) && [\text{Hypothesis } AC = I] \\ &= (BA)C && [\text{Associative property}] \end{aligned}$$

$= IC$ [Hypothesis $BA = I$]

$= C$ [Identity matrix]

It follows that

$$\forall A, B, C \in M_n(\mathbb{R}) : \begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C \qquad \square$$

↳ The unique inverse of A is denoted as A^{-1}, as long as it exists.

▶ Cancellation property.

$$\boxed{\begin{array}{l} \forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (CA = CB \Leftrightarrow A = B) \\ \forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (AC = BC \Leftrightarrow A = B) \end{array}}$$

Proof

We show only the first statement. Let $A, B \in M_n(\mathbb{R})$ and $C \in GL(n, \mathbb{R})$ be given such that $CA = CB$. Then

$A = IA =$ [identity matrix]

$= (C^{-1}C)A$ [C^{-1} inverse of C]

$= C^{-1}(CA)$ [associative property]

$= C^{-1}(CB)$ [hypothesis: $CA = CB$]

$= (C^{-1}C)B$ [associative property]

$= IB$ [C^{-1} inverse of C]

$= B$ [identity matrix]

It follows that
$\forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (CA = CB \Rightarrow A = B)$

- <u>Inverse of a 2x2 matrix</u>

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ be a 2x2 square matrix

a) If $D = ad - bc \neq 0$, then A is non-singular with
$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
b) If $D = ad - bc = 0$, then A is singular.

- <u>Application to 2x2 linear systems.</u>

Any 2x2 linear system given by
$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$
can be rewritten in terms of matrix algebra as:
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
and then solved using the following property:

- $\boxed{\forall A \in GL(n, \mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b)}$

Proof

Let $A \in GL(n,\mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$ be given. Then:

$$\begin{aligned} Ax = b &\Leftrightarrow A^{-1}(Ax) = A^{-1}b && [\text{cancellation property}] \\ &\Leftrightarrow (A^{-1}A)x = A^{-1}b && [\text{associative property}] \\ &\Leftrightarrow Ix = A^{-1}b && [A^{-1} \text{ inverse of } A] \\ &\Leftrightarrow x = A^{-1}b && [\text{identity matrix}] \end{aligned}$$

It follows that

$$\forall A \in GL(n,\mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b) \qquad \square$$

Consequently, if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then:

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

from which we may calculate the unique solutions for (x,y).

$\hookrightarrow$ Notation : 2×2 determinant

The expression $ad - bc$ is the 2×2 determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and we write:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

EXAMPLES

a) Use the matrix inverse to solve the system

$$\begin{cases} 2x+5y=12 \\ 3x-y=1 \end{cases}$$

Solution

$$\begin{cases} 2x+5y=12 \\ 3x-y=1 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 12 \\ 1 \end{bmatrix} = \frac{1}{2(-1)-5\cdot 3}\begin{bmatrix} -1 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 12 \\ 1 \end{bmatrix} =$$

$$= \frac{-1}{17}\begin{bmatrix} (-1)\cdot 12+(-5)\cdot 1 \\ (-3)\cdot 12+2\cdot 1 \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -12-5 \\ -36+2 \end{bmatrix} =$$

$$= \frac{-1}{17}\begin{bmatrix} -17 \\ -34 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ Thus solution set $S=\{(1,2)\}$

b) Similarly for the following parametric system:

$$\begin{cases} (a+1)x+(a-1)y=4a+2 \\ 2ax+(a-1)y=7a-1 \end{cases}$$

Solution

$$D=\begin{vmatrix} a+1 & a-1 \\ 2a & a-1 \end{vmatrix} = (a+1)(a-1)-2a(a-1) = (a-1)(a+1-2a) =$$

$$= (a-1)(1-a) = -(a-1)^2.$$

Case 1 : For $a\neq 1 \Rightarrow D\neq 0$, and therefore:

$$\begin{cases} (a+1)x+(a-1)y=4a+2 \\ 2ax+(a-1)y=7a-1 \end{cases} \Leftrightarrow \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix}^{-1} \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} a-1 & 1-a \\ -2a & a+1 \end{bmatrix} \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(4a+2) + (1-a)(7a-1) \\ -2a(4a+2) + (a+1)(7a-1) \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(4a+2-7a+1) \\ -8a^2-4a+7a^2-a+7a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(-3a+3) \\ -a^2+2a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} -3(a-1)^2 \\ -(a-1)^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\Leftrightarrow (x,y) = (3, 1)$. ; thus solution set $S = \{(3, 1)\}$.

<u>Case 2</u> : For $a = 1$; we have:

$$\begin{cases} (1+1)x + (1-1)y = 4\cdot 1+2 \\ 2\cdot 1x + (1-1)y = 7\cdot 1 - 1 \end{cases} \Leftrightarrow \begin{cases} 2x = 6 \\ 2x = 6 \end{cases} \Leftrightarrow x = 3$$

which gives as solution set:

$S = \{(x,y) \in \mathbb{R}^2 \mid x = 3\} = \{(3,y) \mid y \in \mathbb{R}\}$.

EXERCISES

(18) Use the matrix inverse to solve the following systems:

a) $\begin{cases} x+3y=4 \\ 2x+y=3 \end{cases}$ b) $\begin{cases} 2x-y=3 \\ x+2y=4 \end{cases}$ c) $\begin{cases} 3x+2y=7 \\ x+3y=10 \end{cases}$

d) $\begin{cases} 2ax+(a-3)y=a-1 \\ (a-3)x+2ay=a-a^2 \end{cases}$

e) $\begin{cases} x+(a+1)y=2 \\ (a+2)x+(1-a^2)y=5 \end{cases}$ f) $\begin{cases} ax-y=1-a \\ x-ay=a-a^2 \end{cases}$

$\hookrightarrow$ Distinguish between the values of the parameter $a \in \mathbb{R}$ where the corresponding matrix is non-singular vs. singular.

(19) Find all $a \in \mathbb{R}$ for which the matrix

$$A = \begin{bmatrix} a+3 & 2 \\ 1 & -a \end{bmatrix}$$

is non-singular.

(20) If $A, B \in M_n(\mathbb{R})$ are non-singular, show that AB is also non-singular with the inverse given by $(AB)^{-1} = B^{-1}A^{-1}$

(21) If $A, B, C \in M_n(\mathbb{R})$ and C is non-singular, then show that

a) $CA = CB \Rightarrow A = B$

b) $AC = BC \Rightarrow A = B$

(22) If $A \in M_n(\mathbb{R})$ with $A^3 = \mathbb{0}$, show that $I - A$ is non-singular with

$$(I-A)^{-1} = I + A + A^2$$

(23) If $A \in M_n(\mathbb{R})$ satisfies $A^2 + A + I = \mathbb{0}$, show that A is non-singular and $A^{-1} = A^2$.

(24) If $A, B \in M_n(\mathbb{R})$ with A being non-singular, show that

$$(A-B)A^{-1}(A+B) = (A+B)A^{-1}(A-B)$$

(25) Let $A, B \in M_n(\mathbb{R})$ with $A \neq \mathbb{0}$ and $B \neq \mathbb{0}$. Show that:

$$AB = \mathbb{0} \Rightarrow \begin{cases} A \text{ singular} \\ B \text{ singular} \end{cases}$$

Matrix Transpose

- Let $A \in M_{nm}(\mathbb{R})$ be a matrix. We define the transpose matrix $A^T \in M_{mn}(\mathbb{R})$ as:
  $\forall a \in [m] : \forall b \in [n] : (A^T)_{ab} = A_{ba}$
- Let $A \in M_n(\mathbb{R})$ be a square matrix. We say that
  A symmetric $\Leftrightarrow A^T = A \Leftrightarrow \forall a, b \in [n] : \; A^T{}_{ab} = A_{ba}$

Properties

$\forall A, B \in M_{nm}(\mathbb{R}) : \; (A+B)^T = A^T + B^T$

$\forall \lambda \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \; (\lambda A)^T = \lambda A^T$

$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T$

$\forall A \in GL(n, \mathbb{R}) : \; (A^T)^{-1} = (A^{-1})^T$

$\forall A \in M_{nm}(\mathbb{R}) : \; (A^T)^T = A$

EXAMPLES

a) Prove the property
$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : \; (AB)^T = B^T A^T$.

Solution

Let $A \in M_{nk}(\mathbb{R})$ and $B \in M_{km}(\mathbb{R})$ be given. Let $a \in [m]$ and $b \in [n]$ be given. Then:

$$[(AB)^T]_{ab} = (AB)_{ba} = \sum_{\gamma \in [k]} A_{b\gamma} B_{\gamma a} = \sum_{\gamma \in [k]} A^T_{\gamma b} B^T_{a\gamma} =$$

$$= \sum_{\gamma \in [k]} B^T_{a\gamma} A^T_{\gamma b} = (B^T A^T)_{ab}$$

It follows that

$\forall a \in [n] : \forall b \in [m] : [(AB)^T]_{ab} = (B^T A^T)_{ab}$

$\Rightarrow (AB)^T = B^T A^T$

and therefore

$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T.$ $\square$

b) Show that

$$\forall A, B \in M_n(\mathbb{R}) : \begin{cases} A, B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB \text{ symmetric.}$$

<u>Solution</u>

Let $A, B \in M_n(\mathbb{R})$ be given such that A, B symmetric and $AB = BA$. Then

$$\begin{aligned} (AB)^T &= B^T A^T && [\text{transpose of matrix product}] \\ &= BA && [\text{hypothesis: } A, B \text{ symmetric}] \\ &= AB && [\text{hypothesis: } AB = BA] \end{aligned}$$

$\Rightarrow AB$ symmetric.

It follows that

$$\forall A, B \in M_n(\mathbb{R}) : \begin{cases} A, B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB \text{ symmetric.}$$

Exercises

(26) Give proofs for all properties of the matrix transpose.

(27) Show that if $A \in M_n(\mathbb{R})$ is symmetric and non-singular, then A^{-1} is also symmetric.

(28) Let $A, B \in M_n(\mathbb{R})$. Show that
A, B, AB symmetric $\Rightarrow AB = BA$.

(29) Given $A, P \in M_n(\mathbb{R})$, show that
A symmetric $\Rightarrow B = P^T A P$ symmetric

(30) Consider the rotation matrix

$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

a) Show that $R(\vartheta)$ is non-singular with
$[R(\vartheta)]^{-1} = R(-\vartheta)$

b) Show that
$[R(\vartheta)]^T = R(-\vartheta)$

c) For what angles $\vartheta \in \mathbb{R}$ is $R(\vartheta)$ symmetric?

(31) Let $A \in M_n(\mathbb{R})$ be a square matrix.
Show that

a) $A + A^T$ symmetric

b) $A^T A$ symmetric

(32) Let $z = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and i the imaginary unit and define

$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Show that

a) $\forall z \in \mathbb{C} - \{0\} : M(1/z) = M(z)^{-1}$

b) $\forall z_1 \in \mathbb{C} : \forall z_2 \in \mathbb{C} - \{0\} : M(z_1/z_2) = M(z_1) M(z_2)^{-1}$

(33) Let $A, B \in M_n(\mathbb{R})$ be two square matrices. We say that

$$B \text{ skew-symmetric} \Leftrightarrow \forall a, b \in [n] : B_{ab} = -B_{ba}$$
$$\Leftrightarrow B^T = -B$$

Show that

$$\forall A, B \in M_n(\mathbb{R}) : \begin{cases} A \text{ symmetric} \\ B \text{ skew-symmetric} \end{cases} \Rightarrow A^2 B A^2 \text{ skew-symmetric}$$