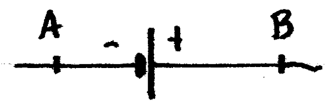
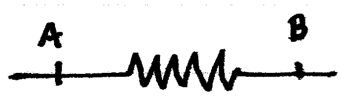
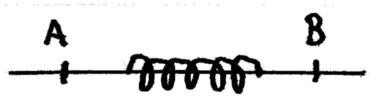
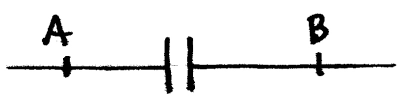
APPLICATIONS OF LINEAR SYSTEMS

▼ DC circuits

- Every circuit component is associated with a voltage drop accross that component. The circuit components we are interested are:

Name	Notation	$V_{AB} \equiv V_A - V_B$
Generator		$V_{AB} = -E$
Resistor		$V_{AB} = IR$
Inductor		$V_{AB} = L\dfrac{dI}{dt}$
Capacitor		$V_{AB} = \dfrac{1}{C}\displaystyle\int_{-\infty}^{t} I\,d\tau$
	$\longrightarrow I$	

V_{AB} = voltage drop from A to B

E = voltage of DC generator

R = resistance

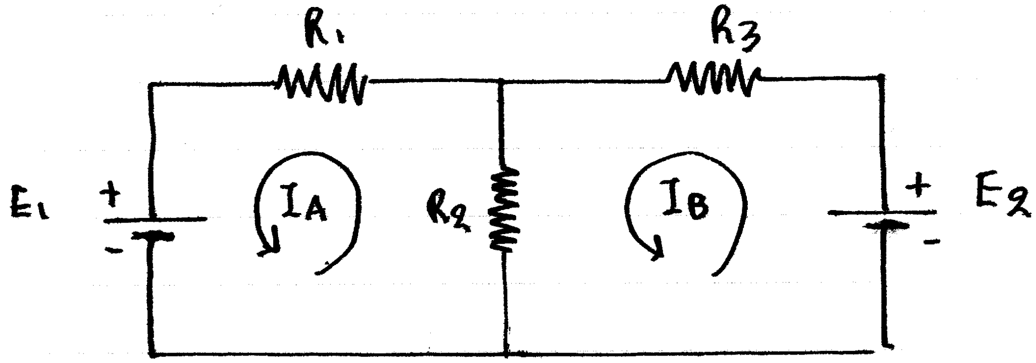
L = inductance

C = capacitance.

▶ Method: To calculate the currents around a circuit we work as follows.

- 1 Define loop currents I_A, I_B, etc. associated with each loop.
- 2 For each, the sum ~~all~~ of all voltage drops around the loop must be zero. Thus, for each loop, we have an equation.
- 3 Solve the system of equations to find the loop currents.
- 4 From the loop currents we may calculate the branch currents, voltage drops, etc.

example



$$\begin{aligned}
\text{Loop A}: & \begin{cases} +E_1 + I_A R_1 + (I_A - I_B) R_2 = 0 \\ -E_2 + I_B R_3 + (I_B - I_A) R_2 = 0 \end{cases} \Leftrightarrow \\
\text{Loop B}: &
\end{aligned}$$

$$\Leftrightarrow \begin{cases} (R_1 + R_2) I_A + (-R_2) I_B = -E_1 \\ (-R_2) I_A + (R_2 + R_3) = E_2 \end{cases}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{bmatrix}}_{A} \begin{bmatrix} I_A \\ I_B \end{bmatrix} = \begin{bmatrix} -E_1 \\ E_2 \end{bmatrix}$$

$$\det A = \begin{vmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{vmatrix} =$$

$$= (R_1 + R_2)(R_2 + R_3) - (-R_2)^2 =$$

$$= R_1 R_2 + R_1 R_3 + R_2^2 + R_2 R_3 - R_2^2 =$$

$$= R_1 R_2 + R_2 R_3 + R_3 R_1 \neq 0$$

since $R_1 > 0$, $R_2 > 0$ and $R_3 > 0$.

Thus:

$$\begin{bmatrix} I_A \\ I_B \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} R_2 + R_3 & +R_2 \\ +R_2 & R_1 + R_2 \end{bmatrix} \begin{bmatrix} -E_1 \\ E_2 \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} -E_1 (R_2 + R_3) + E_2 R_2 \\ -E_1 R_2 + E_2 (R_1 + R_2) \end{bmatrix}$$

so

$$I_A = \frac{-E_1 (R_2 + R_3) + E_2 R_2}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

$$I_B = \frac{-E_1 R_2 + E_2 (R_1 + R_2)}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

- Current through R_1 : I_A
  R_2 : $I_A - I_B$
  R_3 : I_B.

- Find the necessary and sufficient condition so there is no current through R_2.

Balance $\Leftrightarrow I_A - I_B = 0 \Leftrightarrow I_A = I_B \Leftrightarrow$

$$\Leftrightarrow -E_1(R_2 + R_3) + E_2R_2 = E_1R_2 + E_2(R_1 + R_2)$$

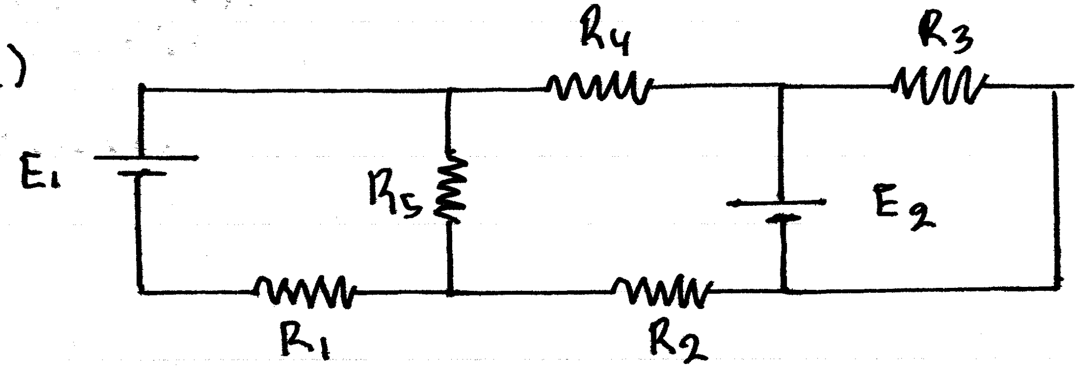
$$\Leftrightarrow E_1R_2 + E_1R_3 - E_2R_2 = E_1R_2 - E_2R_1 - E_2R_2$$

$$\Leftrightarrow E_1R_3 = -E_2R_1 \Leftrightarrow \boxed{E_1 = -E_2\frac{R_1}{R_3}}$$

EXERCISES

(1) Find the loop currents in the following circuits.

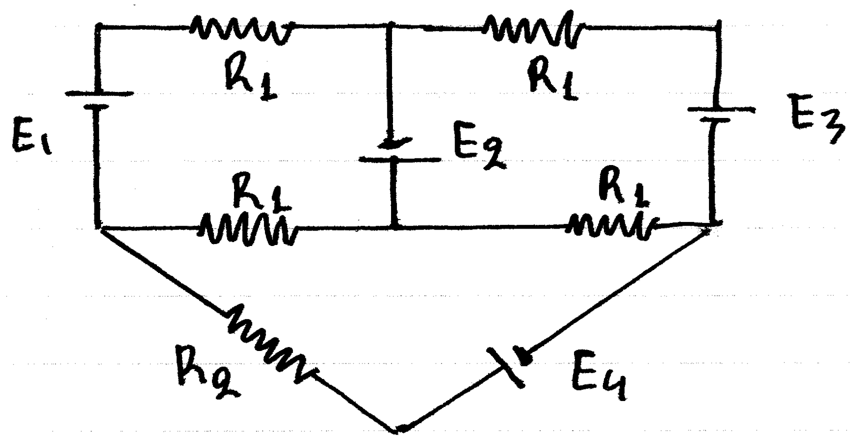
a)

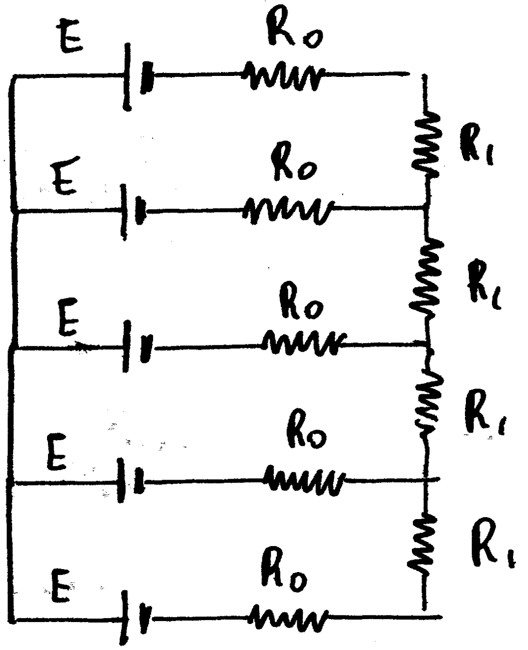


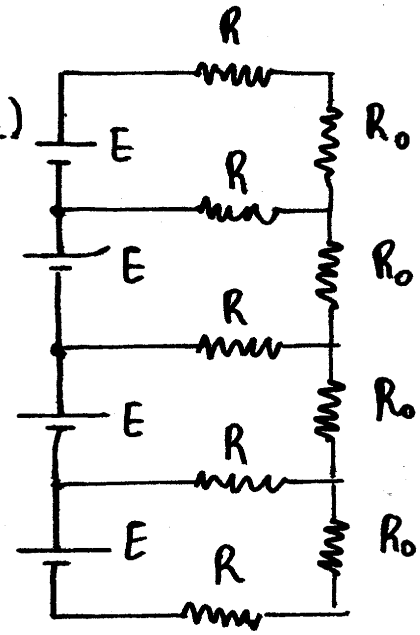
when $R_1 = R_2 = R_3 = a$

$R_4 = R_5 = b$

b)



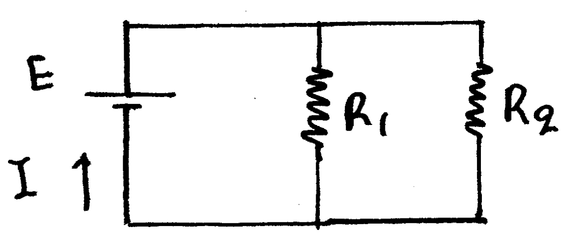
c) 

d) 

② Parallel Resistors.

Show that two resistors R_1, R_2 connected in parallel to a generator are equivalent to a single resistor R with

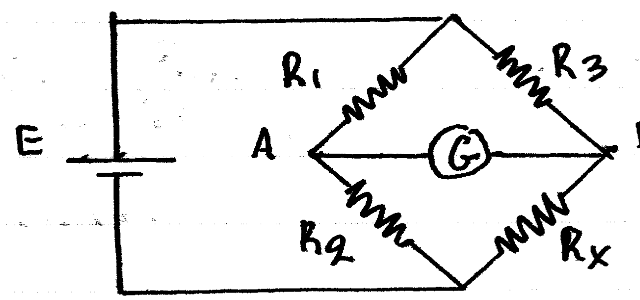
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad (1)$$



(i.e. $E = IR$ with R given by (1)).

③ Wheatstone Bridge

The Wheatstone bridge can be used to measure the unknown resistance R_X.



Ⓖ is a galvanometer measuring voltage drop V_{AB} between A and B.

Assume the Galvanometer has resistance R_g.

(a) Solve for the 3 loop currents in general.
(b) Show that when $V_{AB} = 0$ then
$$R_X = (R_2/R_1)R_3.$$

Superposition principle

- Any general DC circuit with n loops and m generators gives a linear system of equations of the form

$$\boxed{\mathcal{R}\,\mathcal{J} = P\,\mathcal{E}}$$

with $\mathcal{R}$ the resistance matrix ($\mathcal{R} \in M_{nn}(\mathbb{R})$)
P the source configuration matrix ($P \in M_{nm}(\mathbb{R})$)

and

$$\mathcal{J} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} \quad \text{and} \quad \mathcal{E} = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix}$$

- If $\mathcal{R}$ has an inverse then this system has a unique solution

$$\mathcal{J} = \mathcal{R}^{-1} P\,\mathcal{E}$$

- Thm : Let $\mathcal{J}_1$ be the solution when all generators are turned off except E_1, and similarly for $\mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_n$
Then

$$\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 + \dots + \mathcal{J}_n$$

Proof

Define

$$\mathcal{E}_1 = \begin{bmatrix} E_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathcal{E}_2 = \begin{bmatrix} 0 \\ E_2 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathcal{E}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ E_m \end{bmatrix}$$

Then

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_m$$

For each problem we get a linear system. Altogether:

$$\begin{cases} \mathcal{R}\mathcal{J}_1 = P\mathcal{E}_1 \\ \mathcal{R}\mathcal{J}_2 = P\mathcal{E}_2 \\ \vdots \\ \mathcal{R}\mathcal{J}_m = P\mathcal{E}_m \end{cases} \Rightarrow \begin{cases} \mathcal{J}_1 = \mathcal{R}^{-1}P\mathcal{E}_1 \\ \mathcal{J}_2 = \mathcal{R}^{-1}P\mathcal{E}_2 \\ \vdots \\ \mathcal{J}_m = \mathcal{R}^{-1}P\mathcal{E}_m \end{cases}$$

It follows that

$$\mathcal{J} = \mathcal{R}^{-1}P\mathcal{E} = \mathcal{R}^{-1}P(\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_m)$$

$$= \mathcal{R}^{-1}P\mathcal{E}_1 + \mathcal{R}^{-1}P\mathcal{E}_2 + \cdots + \mathcal{R}^{-1}P\mathcal{E}_m$$

$$= \mathcal{J}_1 + \mathcal{J}_2 + \cdots + \mathcal{J}_m .$$

Incidence Matrices

- Consider a circuit with n-loops. Define

  E_{ab} = the total generators shared by loop a and b

  R_{ab} = the total resistance shared by loop a and b

  I_a = the loop current for loop a

  $\delta_{ab} = \begin{cases} 1, & \text{if } a=b \\ 0, & \text{if } a \neq b. \end{cases}$

  and note that $E_{ab} = E_{ba}$ and $R_{ab} = R_{ba}$

- Can we write R_{ab} in terms of R_{ab}?
- Note that the equation for loop a reads:

$$\sum_b E_{ab} = \sum_{b \neq a} R_{ab}(I_a - I_b) + R_{aa} I_a$$

The (right-hand-side can be rewritten as:

$$\begin{aligned} RHS &= \sum_{b \neq a} R_{ab}(I_a - I_b) + R_{aa} I_a \\ &= I_a \sum_b R_{ab} - \sum_{b \neq a} R_{ab} I_b \\ &= I_a \sum_\gamma R_{a\gamma} - \sum_b R_{ab}(1-\delta_{ab}) I_b = \\ &= \sum_b \left[\delta_{ab} \sum_\gamma R_{b\gamma} \right] I_b - \sum_b R_{ab}(1-\delta_{ab}) I_b = \end{aligned}$$

$$= \sum_b \left[\delta_{ab} \sum_\gamma R_{b\gamma} - R_{ab}(1-\delta_{ab}) \right] I_b =$$

$$= \sum_b \mathcal{R}_{ab} I_b$$

therefore

$$\boxed{\mathcal{R}_{ab} = \delta_{ab} \sum_\gamma R_{b\gamma} - R_{ab}(1-\delta_{ab})}$$

<u>example</u>

For $n=2$: two loops

$$\mathcal{R}_{11} = \delta_{11} \sum_\gamma R_{1\gamma} - R_{11}(1-\cancelto{0}{\delta_{11}}) = R_{11} + R_{12}$$

$$\mathcal{R}_{12} = \cancelto{0}{\delta_{12}} \sum_\gamma R_{1\gamma} - R_{12}(1-\delta_{12}) = -R_{12}$$

$$\mathcal{R}_{21} = \cancelto{0}{\delta_{21}} \sum_\gamma R_{2\gamma} - R_{21}(1-\delta_{21}) = -R_{21} = -R_{12}$$

$$\mathcal{R}_{22} = \delta_{22} \sum_\gamma R_{2\gamma} - R_{22}(1-\cancelto{0}{\delta_{22}}) = R_{21} + R_{22}$$

$$= R_{12} + R_{22}$$

thus $$\mathcal{R} = \begin{bmatrix} R_{11} + R_{12} & -R_{12} \\ -R_{12} & R_{22} + R_{12} \end{bmatrix}$$

- It is not always true that $\mathcal{R}$ has an inverse.

  e.g. when $R_{11} = 0$ and $R_{22} = 0$
  then $\det(\mathcal{R}) = 0$

  This situation indicates a <u>short-circuit.</u>

- The incidence matrix can be of use if you want to write a computer program that can solve arbitrary circuits.

EXERCISE

(4) Derive the relation between $\mathcal{R}$ and the incidence matrix R for the general $n = 3$ circuit (3 loops). Then calculate and simplify the determinant $\det(\mathcal{R})$. Under what conditions is $\mathcal{R}$ singular?

Least-squares fit

- Consider a set of data points
  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$
  that approximately fall upon a line
  $(\ell): \; y = ax + b.$

- Want to find the best possible values for a and b.

Solution

Define
$$\begin{aligned} S_x &= x_1 + x_2 + \cdots + x_n \\ S_y &= y_1 + y_2 + \cdots + y_n \\ S_{xx} &= x_1^2 + x_2^2 + \cdots + y_n^2 \\ S_{yy} &= y_1^2 + y_2^2 + \cdots + y_n^2 \end{aligned}$$

and $S_{xy} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$

We estimate the error in the line fit by calculating

$$E(a,b) = \sum_{k=1}^{n} (y_k - (a x_k + b))^2$$

To minimize $E(a,b)$ we calculate the partial derivatives with respect to a and b :

$$\frac{\partial E(a,b)}{\partial a} = \frac{\partial}{\partial a}\sum_{k=1}^{n}\left[y_k-(ax_k+b)\right]^2 =$$

$$= \sum_{k=1}^{n}\left\{\frac{\partial}{\partial a}\left[y_k-(ax_k+b)\right]^2\right\} =$$

$$= \sum_{k=1}^{n}\left\{2\left[y_k-(ax_k+b)\right](-x_k)\right\} =$$

$$= \sum_{k=1}^{n}\left[-2x_ky_k+2ax_k^2+2bx_k\right]$$

$$= -2\sum_{k=1}^{n}x_ky_k+2a\sum_{k=1}^{n}x_k^2+2b\sum_{k=1}^{n}x_k$$

$$= -2\,\$_{xy}+2a\,\$_{xx}+2b\,\$_x$$

and

$$\frac{\partial E(a,b)}{\partial b} = \frac{\partial}{\partial b}\sum_{k=1}^{n}\left[y_k-(ax_k+b)\right]^2 =$$

$$= \sum_{k=1}^{n}\left\{\frac{\partial}{\partial b}\left[y_k-(ax_k+b)\right]^2\right\} =$$

$$= \sum_{k=1}^{n}\left\{2\left[y_k-(ax_k+b)\right](-1)\right\} =$$

$$= \sum_{k=1}^{n}\left[-2y_k+2ax_k+2b\right]$$

$$= -2\sum_{k=1}^{n}y_k+2a\sum_{k=1}^{n}x_k+2nb$$

$$= -2\,\$_y+2a\,\$_x+2nb$$

At the minimum we have

$$\begin{cases} \dfrac{\partial E(a,b)}{\partial a} = 0 \\ \dfrac{\partial E(a,b)}{\partial b} = 0 \end{cases} \Leftrightarrow \begin{cases} -2 S_{xy} + 2a S_{xx} + 2b S_x = 0 \\ -2 S_y + 2a S_x + 2n b = 0 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} S_{xx} & S_x \\ S_x & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} S_{xy} \\ S_y \end{bmatrix}$$

$$D = \begin{vmatrix} S_{xx} & S_x \\ S_x & n \end{vmatrix} = n S_{xx} - [S_x]^2$$

$$D_a = \begin{vmatrix} S_{xy} & S_x \\ S_y & n \end{vmatrix} = n S_{xy} - S_x S_y$$

$$D_b = \begin{vmatrix} S_{xx} & S_{xy} \\ S_x & S_y \end{vmatrix} = S_{xx} S_y - S_{xy} S_x$$

consequently,

$$a = \frac{n S_{xy} - S_x S_y}{n S_{xx} - [S_x]^2}$$

$$b = \frac{S_{xx} S_y - S_{xy} S_x}{n S_{xx} - [S_x]^2}$$

- Will we always have
$$D = n S_{xx} - [S_x]^2 \neq 0 \,?$$

<u>Answer:</u> We use the Lagrange identity:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2 =$$
$$= \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \begin{vmatrix} a_k & b_k \\ a_l & b_l \end{vmatrix}^2$$

It follows that

$$D = n S_{xx} - [S_x]^2 =$$
$$= (1^2+1^2+\cdots+1^2)(x_1^2+x_2^2+\cdots+x_n^2) - (1x_1+1x_2+\cdots+1x_n)^2 =$$
$$= \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \begin{vmatrix} 1 & x_k \\ 1 & x_l \end{vmatrix}^2 =$$
$$= \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} (x_k - x_l)^2 \geq 0$$

As long as at least two of $x_1, x_2, \ldots, x_n$ are not equal to each other, we see that $D > 0 \Rightarrow D \neq 0$.

- How do we know that the solution we found is really a minimum and not a maximum?

<u>Answer</u>: A sufficient condition for a minimum is that

$$\frac{\partial^2 E(a,b)}{\partial a^2} > 0 \qquad (1) \qquad , \text{and}$$

$$M(a,b) = \frac{\partial^2 E(a,b)}{\partial a^2}\frac{\partial^2 E(a,b)}{\partial b^2} - \left[\frac{\partial^2 E(a,b)}{\partial a \partial b}\right]^2$$

$$> 0 \qquad (2)$$

(1) ⟶ minimum in a direction

(2) ⟶ same curvature in all directions.

Recall that

$$\frac{\partial E(a,b)}{\partial a} = -2 S_{xy} + 2a S_{xx} + 2b S_x$$

$$\frac{\partial E(a,b)}{\partial b} = -2 S_y + 2a S_x + 2nb$$

It follows that

$$\frac{\partial^2 E(a,b)}{\partial a^2} = 2 S_{xx}, \quad \frac{\partial^2 E(a,b)}{\partial b^2} = 2n, \text{ and}$$

$$\frac{\partial^2 E(a,b)}{\partial a \partial b} = 2 S_x$$

and thus for eq. (1):

$$\frac{\partial^2 E(a,b)}{\partial a^2} = 2\$_{xx} = 2\sum_{k=1}^{n} x_k^2 > 0$$

if at least one $x_k \neq 0$.

$$M(a,b) = (2\$_{xx})(2n) - (2\$_x)^2 =$$
$$!!!\ = 4(n\$_{xx} - (\$_x)^2)$$
$$= 4D > 0 \leftarrow \text{if at least one } x_k - x_l \neq 0 \text{ for } k \neq l.$$

- Note the amusing relationship between $M(a,b)$ and the determinant D !!

EXERCISES

(5) Find the least-square fit line given the data

a) $(-1,-\lambda), (0,0), (1, \lambda+\Delta\lambda)$
b) $(0,0), (1,1), (2, 2+\varepsilon)$
c) $(1,1), (a,a), (2, 2+a)$

(6) Confirm that given two data points (x_1, y_1) and (x_2, y_2) the least-square fit line corresponds to the line that does in fact go through the two data points (i.e. you have a perfect fit). You will find

$$a = \frac{y_1 - y_2}{x_1 - x_2} \longleftarrow \text{slope}$$

$$b = y_1 - a x_1$$

which corresponds to the line

$$(\ell): \; y - y_1 = a(x - x_1)$$

(7) Suppose that your data points are not exact and you are given instead

$$(x_k \pm \sigma_{x_k}, \, y_k \pm \sigma_{y_k}) \, , \; k = 1, 2, 3, \ldots, n$$

You may still calculate a, b as usual but because your data is not exact there will be some error in your predicted a and b. So we want $a \pm \sigma_a$, and $b \pm \sigma_b$.

It can be shown that

$$\boxed{\sigma_a^2 = \sum_{k=1}^{n} \left(\frac{\partial a}{\partial x_k}\right)^2 \sigma_{x_k}^2 + \sum_{k=1}^{n} \left(\frac{\partial a}{\partial y_k}\right)^2 \sigma_{y_k}^2}$$

$$\boxed{\sigma_b^2 = \sum_{k=1}^{n} \left(\frac{\partial b}{\partial x_k}\right)^2 \sigma_{x_k}^2 + \sum_{k=1}^{n} \left(\frac{\partial b}{\partial y_k}\right)^2 \sigma_{y_k}^2}$$

Calculate the derivatives

$$\frac{\partial a}{\partial x_k}, \quad \frac{\partial a}{\partial y_k}, \quad \frac{\partial b}{\partial x_k}, \quad \frac{\partial b}{\partial y_k}$$

Obviously, given these derivatives we can write a computer program that finds σ_a and σ_b in addition to a and b from the data.