Lecture Notes on Real Analysis I

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RA 1.A: Brief introduction to Logic and Sets

BRIEF INTRODUCTION TO LOGIC AND SETS

Basic concepts

The basic concepts we wish to introduce informally are
a) Propositions
b) Sets
c) Predicates - Quantified statements.

Propositions

- A proposition p is any statement which is true or false.
- Given two propositions p, q we define the following composite propositions.

1) Conjunction $p \wedge q$: "p is true and q is true"
   - True if both p and q are true, otherwise false.
2) Disjunction: $p \vee q$: "p is true or q is true (or both)"
   - True if at least one of the two statements p or q is true, otherwise false.
3) Negation $\bar{p}$: "p is not true"
   - True if p is false. False if p is true.
4) Exclusive Disjunction $p \veebar q$: "either p or q is true (not both)"
   - True if either p or q but not both is true. Otherwise false.

5) Implication $p \Rightarrow q$: "If p is true then q is true"
True if the truth of p implies the truth of q. Note that if p is false, then we presume that $p \Rightarrow q$ is true regardless of whether q is true or false. If p is true and q is false then $p \Rightarrow q$ is false.

6) Equivalence $p \Leftrightarrow q$: "p is true if and only if q is true"
True if p and q always have the same truth value. False if p and q have opposite truth values.

Sets

- A set A is an unordered collection of elements. An element can be a number, a derived object (i.e. vectors, matrices, etc.) or another set.
- A set with a finite number of elements can be defined by listing the elements.
  e.g.: $A = \{2, 3, 6, 9, 12\}$.
- Notation: Let A, B be sets and let x be an element.
  1) $x \in A$: x belongs to A
     x is an element of A
  2) $x \notin A$: x does not belong to A
     x is not an element of A
  3) $A = B$: A and B have the same elements.
  4) $A \subseteq B$: All the elements of A belong to B

- We note that: $A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$
- <u>Special sets</u>

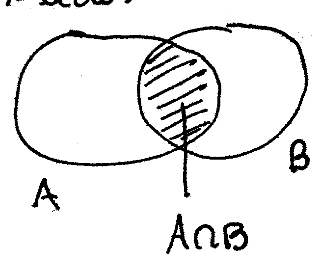
1) $\emptyset = \{\}$. The empty set.
   The empty set is the set that has no elements.
2) $\mathbb{C}$ = the set of all complex numbers
3) $\mathbb{R}$ = the set of all real numbers.
4) $\mathbb{Q}$ = the set of all rational numbers.
5) $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ = the set of all integers.
6) $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ = the set of all natural numbers.
7) For $n \in \mathbb{N}$: $[n] = \{1, 2, 3, \ldots, n\}$.

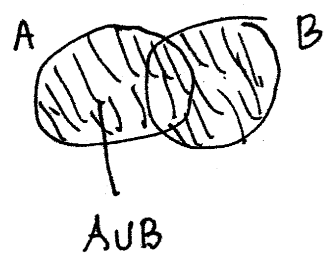
- We note that: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
- <u>Set operations</u>

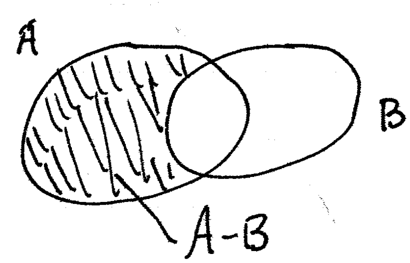
Let A, B be two sets. We define the following set operations:

1) Intersection: $A \cap B$
   $x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$
2) Union: $A \cup B$
   $x \in A \cup B \Leftrightarrow x \in A \vee x \in B$
3) Difference: $A - B$
   $x \in A - B \Leftrightarrow x \in A \wedge x \notin B$

We represent these operations with Venn Diagrams as follows:







- <u>Predicates and quantified statements</u>
- A predicate $p(x)$ is a statement about x which is true or false depending on the value of x.
- Note that x can also be an ordered collection of elements $x=(x_1,x_2,\dots,x_n)$. Then we write $p(x)$ as $p(x_1,x_2,\dots,x_n)$.
- Given a predicate $p(x)$ and a set A, we define the following quantified statements:

1) $\forall x \in A : p(x)$

   <u>For all</u> $x \in A$, $p(x)$ is satisfied.

2) $\exists x \in A : p(x)$

   <u>There is at least one</u> $x \in A$ such that $p(x)$ is satisfied.

3) $\exists! \, x \in A : p(x)$

   <u>There is a unique</u> $x \in A$ such that $p(x)$ is satisfied.

- If A is a finite set, then the above quantified statements are abbreviations for conjunction, disjunction, and exclusive disjunction: For example:

$$(\forall x \in \{a,b,c\} : p(x)) \Leftrightarrow (p(a) \wedge p(b) \wedge p(c))$$
$$(\exists x \in \{a,b,c\} : p(x)) \Leftrightarrow (p(a) \vee p(b) \vee p(c))$$
$$(\exists! \, x \in \{a,b,c\} : p(x)) \Leftrightarrow (p(a) \underline{\vee} p(b) \underline{\vee} p(c))$$

- Quantifiers can be nested to give compound quantified statements. For example:

1) $\forall x \in A : \exists y \in B : p(x,y)$

   For all $x \in A$, there is a $y \in B$, such that $p(x,y)$ is satisfied.

2) $\exists x \in A : \forall y \in B : p(x,y)$

There is an $x \in A$ such that for all $y \in B$, $p(x,y)$ is satisfied.

- Important quantified statements from algebra

$\forall a,b \in \mathbb{R} : (ab = 0 \Leftrightarrow a = 0 \lor b = 0)$

$\forall a,b \in \mathbb{R} : (a^2 + b^2 = 0 \Leftrightarrow a = 0 \land b = 0)$

$\forall a,b \in \mathbb{R} : (|a| + |b| = 0 \Leftrightarrow a = 0 \land b = 0)$

- <u>Definitions of sets</u>

There are 3 methods for defining sets:

1) <u>By listing</u>: For finite sets we can simply list the elements.

e.g.: $A = \{3, 7, 10, 12\}$

2) <u>By predicate</u>: $A = \{x \in U \mid p(x)\}$

with U a predefined set and $p(x)$ a predicate.

Belonging condition: $x \in A \Leftrightarrow (x \in U \land p(x))$

e.g.: We can use definition by predicate to define intervals:

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[n] = \{x \in \mathbb{N} \mid 1 \leq x \leq n\} = \{1, 2, \ldots, n\}$

3) <u>By mapping</u>: $A = \{\varphi(x) \mid x \in U \land p(x)\}$

with $\varphi(x)$ some <u>expression</u> of x, U a predefined set, and $p(x)$ a <u>predicate</u>.

Belonging condition: $y \in A \Leftrightarrow \exists x \in U : (\varphi(x) = y \land p(x))$

EXAMPLES

a) The set of complex numbers:

$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$.

$z \in \mathbb{C} \Leftrightarrow \exists a, b \in \mathbb{R} : z = a + bi$

b) The set of rational numbers:

$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \wedge b \in \mathbb{N} - \{0\}\}$

$x \in \mathbb{Q} \Leftrightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N} - \{0\} : x = a/b$.

c) The set of even integers

$A = \{2k \mid k \in \mathbb{Z}\}$

$x \in A \Leftrightarrow \exists k \in \mathbb{Z} : x = 2k$

d) The set of odd integers

$A = \{2k+1 \mid k \in \mathbb{Z}\}$

$x \in A \Leftrightarrow \exists k \in \mathbb{Z} : x = 2k+1$.

e) $A = \{a^2 + b^2 \mid a, b \in \mathbb{R} \wedge a + 3b < 1\}$

$x \in A \Leftrightarrow \exists a, b \in \mathbb{R} : (x = a^2 + b^2 \wedge a + 3b < 1)$

- Cartesian product

We use definition by mapping to define the cartesian product between sets.

- An ordered pair (a, b) is an ordered collection of two elements a and b. We call a and b the components of (a, b).
- We note that: $(a, b) = (c, d) \Leftrightarrow (a = c \wedge b = d)$.

- Let A, B be two sets. We define the Cartesian product
  $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$.
  We also define:
  $A^2 = A \times A = \{(a,b) \mid a \in A \wedge b \in A\}$

EXAMPLE

For $A = \{1,2,3\}$ and $B = \{5,6\}$. Calculate $A \times B$, A^2, B^2.

Solution

$A \times B = \{1,2,3\} \times \{5,6\} =$
$= \{(1,5),(1,6),(2,5),(2,6),(3,5),(3,6)\}$

$A^2 = A \times A = \{1,2,3\} \times \{1,2,3\} =$
$= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

$B^2 = B \times B = \{5,6\} \times \{5,6\} =$
$= \{(5,5),(5,6),(6,5),(6,6)\}$

The above can be generalized as follows

- An ordered n-tuplet $(x_1, x_2, \dots, x_n)$ is an ordered collection of n elements $x_1, x_2, \dots, x_n$.
- Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.
  We note that:
  $x = y \Leftrightarrow \forall a \in [n] : x_a = y_a$
- Let $A_1, A_2, \dots, A_n$ be n sets. We define:
  $A_1 \times A_2 \times \dots \times A_n = \{(x_1, x_2, \dots, x_n) \mid \forall a \in [n] : x_a \in A_a\}$
- Special case:
  $A_1 \times A_2 \times A_3 = \{(x_1, x_2, x_3) \mid x_1 \in A_1 \wedge x_2 \in A_2 \wedge x_3 \in A_3\}$.

EXERCISES

1. Let $A=[7]$, $B=\{x\in A \mid x>4\}$, and $C=\{x-1 \mid x\in B\}$.
List the elements of

   a) B b) C c) $B\cap C$ d) $B\cup C$

   e) $A-B$ f) $B-C$ g) $C-B$

2. Write out the following statements in English

   a) $\forall a\in A : \exists b\in B : (a,b)\in f$

   b) $\exists a\in A : \forall b\in B : a+b>3$

   c) $\forall a\in A : \exists b\in B : (ab>2 \wedge a+b>1)$

   d) $\forall a,b\in A : \exists c\in B : \forall d\in A : ab+bd<3$

   e) $\exists a\in A : \forall b\in B : (ab>3 \Rightarrow b>2)$

   f) $\forall a\in A : \exists b\in B : (3a>b \veebar a+b<0)$

3. Write the following statements symbolically using quantifiers.

   a) Every real number is equal to itself.

   b) There is a real number x such that $3x-1=2(x+3)$

   c) For every real number x, there is a natural number n such that $n>x$.

   d) For every real number x, there is a complex number y such that $y^2=x$.

   e) There is a real number x such that for all real numbers y we have $x+y=0$.

f) For all $\varepsilon > 0$, there is a $\delta > 0$ such that for all real numbers x, if $x_0 - \delta < x < x_0 + \delta$ then $|f(x) - a| < \varepsilon$.

g) There is a real number b such that for all natural numbers n we have $a_n < b$.

h) For all $\varepsilon > 0$, there is a natural number n_0 such that for any two natural numbers n_1 and n_2, if $n_1 > n_0$ and $n_2 > n_0$, then $|a_{n_1} - a_{n_2}| < \varepsilon$.

i) For any $M > 0$, there is a natural number n_0, such that for any other natural number n, if $n > n_0$ then $a_n > M$.

(4) Write the belonging condition $x \in A$ for the following sets, using quantifiers.

a) $A = \{x^2 + 1 \mid x \in \mathbb{Q} \wedge 2x < 1\}$

b) $A = \{3x + 1 \mid x \in \mathbb{Z} \wedge x \text{ is a prime number}\}$

c) $A = \{x \in \mathbb{R} \mid x^2 + 3x > 0\}$

d) $A = \{a^3 + b^3 + c^3 \mid a, b \in \mathbb{R} \wedge c \in \mathbb{Q} \wedge a + b + c = 0\}$

e) $A = \{x \in \mathbb{R} \mid x^2 + 2x < 0 \vee 3x + 1 > -4 + x\}$

f) $A = \{a^2 - b^2 \mid a \in \mathbb{N} \wedge b \in \mathbb{R} \wedge a + b > 5\}$

g) $A = \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z} : x = 3k\}$

h) $A = \{ab \mid a, b \in \mathbb{R} \wedge (a + b > 2 \vee a - b < -3)\}$

i) $A = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : y^2 + y = x\}$

j) $A = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R} : x < y^2 + 1\}$

k) $A = \{a + b \mid a, b \in \mathbb{R} \wedge (ab > 1 \Rightarrow a^2 + b^2 > 2)\}$

l) $A = \{abc \mid a, b, c \in \mathbb{R} \wedge (a + b > 2 \,\underline{\vee}\, a - c < 3)\}$

m) $A = \{2a + 3b \mid a, b \in \mathbb{R} \wedge ab > 1 \wedge a - b < 0\}$

(5) List the elements for the following cartesian products

a) $A \times B$ with $A = \{2, 3, 4\}$ and $B = \{7, 8\}$

b) $A \times B$ with $A = \{1\}$ and $B = \{3, 9\}$

c) $A \times B$ with $A = \{3\}$ and $B = \{5\}$

d) $[2] \times [3]$

e) $A \times B$ with $A = [5] - [2]$ and $B = [2] \cap [4]$

f) $A \times B \times C$ with $A = [3] - \{1\}$, $B = [3] \cap [6]$, and $C = [2]$.

g) $A \times B \times C$ with $A = \{2\}$, $B = [2]$, $C = [4] - [2]$.

RA 1.B: Brief introduction to Proofs

BRIEF INTRODUCTION TO PROOF

▼ Negation and contrapositive of statements

- Let P, Q be compound statements. We say that $P \equiv Q$ (P and Q are equivalent) if and only if the compound statement $P \Leftrightarrow Q$ is always true, regardless of the truth value of the constituent statemends that compose P and Q.
- The following equivalences can be used to negate compound statements:

$\overline{p \wedge q} \equiv \bar{p} \vee \bar{q}$	$\overline{p \veebar q} \equiv p \Leftrightarrow q$
$\overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$	$\overline{p \Leftrightarrow q} \equiv p \veebar q$
$\overline{p \Rightarrow q} \equiv p \wedge \bar{q}$	

- Quantifred statements can be negated by the following rules

$\overline{\forall x \in A : p(x)} \equiv \exists x \in A : \overline{p(x)}$
$\overline{\exists x \in A : p(x)} \equiv \forall x \in A : \overline{p(x)}$

- Every statement of the form $P \Rightarrow Q$ is equivalent to the contrapositive statement $\bar{Q} \Rightarrow \bar{P}$. Consequently any proof of $P \Rightarrow Q$ also proves $\bar{Q} \Rightarrow \bar{P}$. The converse statement $Q \Rightarrow P$ is NOT equivalent to $P \Rightarrow Q$ and requires separate proof.

- We note that since

$$(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

the contrapositive statement of $P \Leftrightarrow Q$ is $\overline{P} \Leftrightarrow \overline{Q}$.

EXAMPLES

a) Write the negation of the definition of the limit from calculus

$$\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)$$

Solution

$$\lim_{x \to x_0} f(x) \neq \ell \Leftrightarrow$$

$$\Leftrightarrow \exists \varepsilon > 0 : \overline{\exists \delta > 0 : \forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \overline{\forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : \overline{(0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : (0 < |x - x_0| < \delta \wedge \overline{|f(x) - \ell| < \varepsilon})$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : (0 < |x - x_0| < \delta \wedge |f(x) - \ell| \geq \varepsilon)$$

b) The contrapositive to the statement
$\forall a,b \in \mathbb{R}: (ab=0 \Rightarrow a=0 \vee b=0)$
is given by:

$$\forall a,b\in\mathbb{R}: (\overline{a=0 \vee b=0} \Rightarrow \overline{ab=0}) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (\overline{a=0} \wedge \overline{b=0} \Rightarrow ab\neq 0) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (a\neq 0 \wedge b\neq 0 \Rightarrow ab\neq 0).$$

c) The contrapositive to the statement
$\forall a,b\in\mathbb{R}: (a^2+b^2=0 \Rightarrow a=0 \wedge b=0)$
is given by:

$$\forall a,b\in\mathbb{R}: (\overline{a=0 \wedge b=0} \Rightarrow \overline{a^2+b^2=0}) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (\overline{a=0} \vee \overline{b=0} \Rightarrow a^2+b^2\neq 0)$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (a\neq 0 \vee b\neq 0 \Rightarrow a^2+b^2\neq 0).$$

EXERCISES

① Write the negation of all the statements from Exercises 2 and 3 [Brief Introduction to Logic and Sets] both in terms of quantified statement notation and in English.

② Write the non-belonging condition $x \notin A$ for the sets given in Exercise 4 [Brief Introduction to Logic and Sets] both in terms of quantified statement notation and in English.

③ Write the contrapositive of the following statements, both in terms of quantified statement notation and in English.

a) $\forall a \in \mathbb{R}: a \geq 3 \Rightarrow a > 5$

b) $\forall a, b \in \mathbb{R}: |a| + |b| = 0 \Rightarrow (a = 0 \wedge b = 0)$

c) $\forall a, b \in \mathbb{R}: a^2 = b^2 \Leftrightarrow (a = b \vee a = -b)$

d) $\forall a, b, c, d \in \mathbb{R}: (a < b \wedge c < d) \Rightarrow a + c < b + d$

e) $\forall a, b, c \in \mathbb{R}: (a > 0 \wedge b > c > 0) \Rightarrow ab > ac$

(Hint: $b > c > 0$ is equivalent to $b > c \wedge c > 0$)

f) $\forall a, b, c \in \mathbb{R}: a^3 + b^3 + c^3 = 3abc \Rightarrow (a + b + c = 0 \vee a = b = c)$

(Hint: $a = b = c$ is equivalent to $a = b \wedge b = c$)

Methodology for writing proofs

Proving implications

(1) To prove $\boxed{p \Rightarrow q}$

- **Direct Method**

  Assume p is true.
  [Prove q]

- **Contrapositive Method**

  We will show that $\bar{q} \Rightarrow \bar{p}$
  Assume $\bar{q}$ is true.
  [Prove $\bar{p}$]
  It follows that $p \Rightarrow q$

- **Contradiction Method**

  Assume p is true.
  To derive a contradiction, assume $\bar{q}$.
  [Prove r, using $p \wedge \bar{q}$]
  [Prove $\bar{r}$] ← Contradiction.
  It follows that q is true.

(2) To prove $\boxed{p \Leftrightarrow q}$

($\Rightarrow$): Assume p is true [Prove q]	($\Leftarrow$): Assume q is true [Prove p]

Proofs involving sets

Let A, B be two sets.

(1) To prove $\boxed{A \subseteq B}$

[We prove $x \in A \Rightarrow x \in B$]

(2) To prove $\boxed{A = B}$

[We prove $x \in A \Rightarrow x \in B$]
It follows that $A \subseteq B$ (1)
[We prove $x \in B \Rightarrow x \in A$]
It follows that $B \subseteq A$ (2)
From (1) and (2): $A = B$.

- For proofs involving sets, we recall that

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$
$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$
$$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$$
$$x \in \{x \in A \mid p(x)\} \Leftrightarrow x \in A \wedge p(x)$$
$$x \in \{\varphi(x) \mid x \in A \wedge p(x)\} \Leftrightarrow \exists y \in A : (\varphi(y) = x \wedge p(y))$$

Proofs involving identities

Let a, b be two expressions.
To prove $a = b$.

- Direct Method

$$a = \cdots = \cdots = \\ = \cdots = \cdots = b$$

- Indirect Method

$$a = \cdots = \cdots = c \quad (1)$$
$$b = \cdots = \cdots = c \quad (2)$$

From (1) and (2): $a = b$.

Proofs involving quantified statements

1 To prove $\boxed{\forall x \in A : p(x)}$

Let $x \in A$ be given.
[Prove $p(x)$]
It follows that $\forall x \in A : p(x)$.

2 To prove $\boxed{\exists x \in A : p(x)}$

- 1st method

[Define an $x \in A$]
[Prove that $p(x)$ is true]
It follows that $\exists x \in A : p(x)$

(Note that x can be indirectly defined by deducing a statement of the form $\exists x \in B : r(x)$ via a theorem or by constructing it from other variables that have been indirectly defined via existential statements)

▸ <u>2nd method</u>

$p(x) \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftarrow x \in S$

Choose an $x \in S$. Show that $x \in A \wedge p(x)$.

It follows that $\exists x \in A : p(x)$.

RA 1.1: Structure of the set of real numbers

STRUCTURE OF THE SET OF REAL NUMBERS

Preliminaries

Let A be a set and $p(x)$ a statement about x. We will use the following notation for quantified statements:

- Universal quantifier
  $\forall x \in A : p(x)$
  "For all $x \in A$, $p(x)$ is satisfied"
- Existential quantifier
  $\exists x \in A : p(x)$
  "There exists at least one $x \in A$ such that $p(x)$ is satisfied"
- Unique existential quantifier
  $\exists!\, x \in A : p(x)$
  "There exists a unique $x \in A$ such that $p(x)$ is satisfied"

We define $\mathbb{R}$ to be the set of real numbers. Although there are several constructions of $\mathbb{R}$ from the set of natural numbers, we sidestep the construction, and assume that $\mathbb{R}$ exists and satisfies 3 axioms

1) The field axioms
2) The axiom of order
3) The axiom of completeness

All properties of real numbers are then derived as a consequence of these axioms.

The field axiom

Axiom: (Field axiom)
The set $\mathbb{R}$ is endowed with two operations: addition ("+") and multiplication ("·") such that:

1) $\forall x, y \in \mathbb{R} : (x+y = y+x \wedge xy = yx)$ Commutative
2) $\forall x, y, z \in \mathbb{R} : \begin{cases} (x+y)+z = x+(y+z) \\ (xy)z = x(yz) \end{cases}$ Associative
3) $\forall x \in \mathbb{R} : \begin{cases} x+0 = 0+x = x \\ 1x = x1 = x \end{cases}$ Neutral elements
4) $\forall x \in \mathbb{R} : \exists y \in \mathbb{R} : x+y = y+x = 0$ Additive inverse
5) $\forall x \in \mathbb{R} - \{0\} : \exists y \in \mathbb{R} : xy = yx = 1$ Multiplicative inverse
6) $\forall x, y, z \in \mathbb{R} : x(y+z) = xy + xz$ Distributive

Remark
An immediate consequence of the field axiom is that

$$\begin{cases} (\mathbb{R}, +) \text{ abelian group} \\ (\mathbb{R} - \{0\}, \cdot) \text{ abelian group} \end{cases}$$

It follows that the $y \in \mathbb{R}$ claimed to exist in items 3, 4 has to be unique (see my Linear Algebra Lecture notes for more details). As a result, items 3, 4 can be strengthened to read:

$$\forall x \in \mathbb{R} : \exists! \, y \in \mathbb{R} : x+y = y+x = 0$$
$$\forall x \in \mathbb{R} - \{0\} : \exists! \, y \in \mathbb{R} : xy = yx = 1$$

and that leads to the following notation:

<u>notation</u> :

Let $x \in \mathbb{R}$ be given. Then we introduce the following notation.

a) $-x$ is the unique number such that $x+(-x)=(-x)+x=0$

b) If $x \neq 0$, then $x^{-1} = 1/x$ is the unique number such that $xx^{-1} = x^{-1}x = 1$

c) Subtraction: $\forall x, y \in \mathbb{R}: x-y = x+(-y)$

d) Division: $\forall x \in \mathbb{R}: \forall y \in \mathbb{R}-\{0\}: x/y = xy^{-1}$

e) 0 is the zero element

f) 1 is the unit element.

Immediate consequences of the field axioms

① $\boxed{\text{Uniqueness of zero element} \quad (\forall x \in \mathbb{R}: x+z = z+x = x) \Rightarrow z=0}$

<u>Proof</u>

Assume that $\forall x \in \mathbb{R}: x+z = z+x = x$. Then:

$$\begin{aligned} z &= 0+z && [0 \text{ zero element}] \\ &= 0 && [\text{hypothesis}]. \end{aligned}$$

② $\boxed{\text{Uniqueness of unit number} \quad (\forall x \in \mathbb{R}: xz = zx = x) \Rightarrow z = 1}$

<u>Proof</u>

Assume that $\forall x \in \mathbb{R}: xz = zx = x$. Then

$$\begin{aligned} z &= 1z && [1 \text{ unit element}] \\ &= 1 && [\text{hypothesis}]. \end{aligned}$$

③ **Addition cancellation law**
$\forall x,y,z \in \mathbb{R} : (x+z = y+z \Leftrightarrow x = y)$

Proof

Let $x,y,z \in \mathbb{R}$ be given.

($\Rightarrow$) : Assume that $x+z = y+z$. Then:

$$
\begin{aligned}
x &= x+0 && [\text{zero element}] \\
&= x+[z+(-z)] && [-z \text{ inverse of } z] \\
&= (x+z)+(-z) && [\text{associative}] \\
&= (y+z)+(-z) && [\text{hypothesis}] \\
&= y+[z+(-z)] && [\text{associative}] \\
&= y+0 && [-z \text{ inverse of } z] \\
&= y && [\text{zero element}]
\end{aligned}
$$

($\Leftarrow$) : Assume that $x=y$. Then, it immediately follows that $x+z=y+z$. □

④ **Multiplication cancellation law**
a) $\forall x,y,z \in \mathbb{R} : (x=y \Rightarrow xz = yz)$
b) $\forall x,y,z \in \mathbb{R} : \begin{cases} xz = yz \\ z \neq 0 \end{cases} \Rightarrow x=y$

Proof

Let $x,y,z \in \mathbb{R}$ be given.

a) Assume that $x=y$. It immediately follows that $xz=yz$.

b) Assume that $xz=yz$ and $z\neq 0$. Then:

$z\neq 0 \Rightarrow \exists z' \in \mathbb{R} : zz' = z'z = 1$.

Choose $z' \in \mathbb{R}$ such that $zz' = z'z = 1$. Then:

$$\begin{aligned} x &= x1 && [\text{unit element}] \\ &= x(zz') && [z' \text{ inverse of } z] \\ &= (xz)z' && [\text{associative}] \\ &= (yz)z' && [\text{hypothesis}] \\ &= y(zz') && [\text{associative}] \\ &= y1 && [z' \text{ inverse of } z] \\ &= y && [\text{unit element}] \end{aligned}$$

□

(5) **Multification law**
$\forall x \in \mathbb{R}: \; 0x = x0 = 0$

Proof

Let $x \in \mathbb{R}$ be given. Choose some $y \in \mathbb{R}$. Then:

$$\begin{aligned} xy + 0 &= xy && [\text{zero element}] \\ &= x(y+0) && [\text{zero element}] \\ &= xy + x0 && [\text{distributive}] \\ \Rightarrow 0 &= x0 && [\text{addition cancellation law}] \end{aligned}$$

It follows, via commutative, that $0x = x0 = 0$.

(6) **Law of signs**
$\forall x, y \in \mathbb{R}: \; (-x)y = x(-y) = -xy$

Proof

Let $x, y \in \mathbb{R}$ be given. Then:

$$
\begin{aligned}
xy + (-x)y &= [x + (-x)]y && \text{[distributive]}\\
&= 0y && \text{[$-x$ inverse of x]}\\
&= 0 \Rightarrow && \text{[nullification]}
\end{aligned}
$$

$\Rightarrow (-x)y$ additive inverse of xy

$\Rightarrow -xy = (-x)y.$

and

$$
\begin{aligned}
xy + x(-y) &= x[y + (-y)] && \text{[distributive]}\\
&= x0 && \text{[$-y$ inverse of y]}\\
&= 0 && \text{[nullification]}
\end{aligned}
$$

$\Rightarrow x(-y)$ additive inverse of xy

$\Rightarrow -xy = x(-y).$

We conclude that $(-x)y = x(-y) = -xy$.

$\mapsto$ The following are immediate corollaries of the law of signs:

$\forall x \in \mathbb{R} : -(-x) = x$

$\forall x, y \in \mathbb{R} : (-x)(-y) = xy.$

7 $\boxed{\begin{array}{l}\text{Zero product property.}\\ \forall x,y \in \mathbb{R} : (xy = 0 \Leftrightarrow (x = 0 \lor y = 0))\end{array}}$

Proof

Let $x, y \in \mathbb{R}$ be given.

$(\Rightarrow)$: Assume that $xy = 0$. We distinguish between the following cases.

Case 1: Assume that $x=0$. Then

$$x=0 \Rightarrow \underline{x=0 \lor y=0}.$$

Case 2: Assume that $x \neq 0$. Then

$$\begin{aligned} xy &= 0 && \text{[hypothesis]} \\ &= x0 && \text{[zero element]} \\ \Rightarrow y &= 0 && \text{[via } x\neq 0\text{, cancellation law]} \\ \Rightarrow & \underline{x=0 \lor y=0} \end{aligned}$$

In both cases we get $x=0 \lor y=0$.

($\Leftarrow$): Assume that $x=0 \lor y=0$. We distinguish between the following cases:

Case 1: Assume that $x=0$. Then $xy = 0y = 0$

Case 2: Assume that $y=0$. Then $xy = x0 = 0$

In both cases we find that $xy=0$.

⑧ **Adding/Multiplying equations**

$$\forall x,y,a,b\in\mathbb{R}: \begin{cases} x=y \\ a=b \end{cases} \Rightarrow \begin{cases} x+a = y+b \\ ax = by \end{cases}$$

The proof is trivial.

⑨ **Sum of squares**

$$\forall a,b\in\mathbb{R}: (a^2+b^2=0 \Leftrightarrow (a=0 \land b=0))$$

$$\forall a_1,\dots,a_n\in\mathbb{R}: (a_1^2+a_2^2+\dots+a_n^2=0 \Leftrightarrow \forall k\in\{1,2,\dots,n\}: a_k=0)$$

The proof requires the axiom of order.

THEORY QUESTIONS

(1) State the field axiom of $\mathbb{R}$.

(2) Show the following properties of real numbers using the field axiom:

a) $\forall x,y,z \in \mathbb{R}: (x+z = y+z \Leftrightarrow x=y)$

b) $\forall x,y,z \in \mathbb{R}: \begin{cases} xz = yz \\ z \neq 0 \end{cases} \Rightarrow x = y$

(3) Show the following using the field axioms and the cancellation law.

a) $\forall x \in \mathbb{R}: (0x = x0 = 0)$ (nullification law)

(4) Show the following using the field axioms, the cancellation law and the nullification law

a) $\forall x,y \in \mathbb{R}: (-x)y = x(-y) = -xy$

b) $\forall x,y \in \mathbb{R}: xy = 0 \Leftrightarrow (x=0 \vee y=0)$

EXERCISE

(5) Use proof by induction to show that

$$\forall x_1, x_2, \dots, x_n \in \mathbb{R}: (x_1 x_2 \cdots x_n = 0 \Leftrightarrow$$
$$\Leftrightarrow (x_1 = 0 \vee x_2 = 0 \vee \dots \vee x_n = 0)).$$

Integer powers

We define the following number sets:

a) Set of natural numbers

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

$\mathbb{N}^* = \mathbb{N} - \{0\}$

b) Set of integers

$\mathbb{Z} = \{x, -x \mid x \in \mathbb{N}\} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

$\mathbb{Z}^* = \mathbb{Z} - \{0\}.$

Then we define integer powers as follows

Def: Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then we define

$$a^n = \begin{cases} 1 & , \text{ if } n = 0 \\ a^{n-1} a & , \text{ if } n \in \mathbb{N}^* \end{cases}$$

$$a^{-n} = \frac{1}{a^n} \quad , \text{ if } a \neq 0$$

Using proof by induction on $\mathbb{Z}$, we can show that:

Prop:

a) $\forall a \in \mathbb{R} - \{0\} : \forall x, y \in \mathbb{Z} : a^x a^y = a^{x+y}$

b) $\forall a \in \mathbb{R} - \{0\} : \forall x, y \in \mathbb{Z} : (a^x)^y = a^{xy}$

c) $\forall a, b \in \mathbb{R} - \{0\} : \forall x \in \mathbb{Z} : (ab)^x = a^x b^x$

Proof of (a)

Let $a \in \mathbb{R} - \{0\}$ and $x, \in \mathbb{Z}$ be given.

For $y = 0$:

$a^x a^y = a^x a^0 = a^x 1 = a^x = a^{x+0} = a^{x+y}$

For $y = n$, we assume that $a^x a^n = a^{x+n}$.

For $y = n+1$, it follows that:

$$a^x a^y = a^x a^{n+1} = a^x (a^n a) = (a^x a^n) a =$$
$$= a^{x+n} a = a^{x+n+1} = a^{x+y}$$

For $y = n-1$, it follows that

$$a^x a^y = a^x a^{n-1} = \frac{(a^x a^{n-1})a}{a} = \frac{a^x (a^{n-1} a)}{a} =$$
$$= \frac{a^x a^n}{a} = \frac{a^{x+n}}{a} = a^{x+n-1} = a^{x+y}$$

Statement (a) follows by induction, for all $y \in \mathbb{Z}$.

Proof of (b), (c) ▸ $\boxed{\text{Homework.}}$ You can use (a) to prove (b) and (c), again via induction.

THEORY QUESTIONS

(6) Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. State the definition of a^n and a^{-n}.
(7) Show that $\forall a \in \mathbb{R} - \{0\} : \forall x, y \in \mathbb{Z} : a^x a^y = a^{x+y}$.

EXERCISES

(8) Show, using proof by induction on $\mathbb{Z}$, the following:
a) $\forall a \in \mathbb{R} - \{0\} : \forall x, y \in \mathbb{Z} : (a^x)^y = a^{xy}$
b) $\forall a, b \in \mathbb{R} - \{0\} : \forall x \in \mathbb{Z} : (ab)^x = a^x b^x$
(9) Note that by definition $0^0 = 1$. Explain why we cannot define 0^{-1} if we wish the main properties of powers (see questions 7,8) to be satisfied. Generalize the argument for 0^{-x} for all $x \in \mathbb{N}^*$.

The order axiom

Let $\mathcal{P}(\mathbb{R})$ be the set of all subsets of $\mathbb{R}$ such that
$A \in \mathcal{P}(\mathbb{R}) \Leftrightarrow A \subseteq \mathbb{R}$.

Axiom : (order axiom)

$$\boxed{\exists!\ \mathbb{R}_+^* \in \mathcal{P}(\mathbb{R}) : \begin{cases} \forall x, y \in \mathbb{R}_+^* : (x+y) \in \mathbb{R}_+^* \wedge xy \in \mathbb{R}_+^* \\ \forall x \in \mathbb{R} \quad : x = 0 \veebar x \in \mathbb{R}_+^* \veebar -x \in \mathbb{R}_+^* \end{cases}}$$

Here $\veebar$ represents an exclusive "or"
$\mathbb{R}_+^*$ is the set of strictly positive numbers.
We now define the relations "$<$", "$>$", "$\leq$", "$\geq$".

Def. (Inequalities)
Let $x, y \in \mathbb{R}$ be given. Then

$$x < y \Leftrightarrow (y - x) \in \mathbb{R}_+^*$$
$$x > y \Leftrightarrow (x - y) \in \mathbb{R}_+^*$$
$$x \leq y \Leftrightarrow (x < y \vee x = y)$$
$$x \geq y \Leftrightarrow (x > y \vee x = y).$$

notation : We write:

$\mathbb{R}_+^* = \{x \in \mathbb{R} \mid x > 0\}$ $\qquad$ $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$

$\mathbb{R}_-^* = \{x \in \mathbb{R} \mid x < 0\}$ $\qquad$ $\mathbb{R}_- = \{x \in \mathbb{R} \mid x < 0\}$

$\mathbb{R}^* = \mathbb{R} - \{0\}$

- The following statements are immediate consequences of the order axiom

a) $\forall x, y \in \mathbb{R}: (x > 0 \land y > 0) \Rightarrow (x+y > 0 \land xy > 0)$
b) $\forall x, y \in \mathbb{R}: (x < 0 \land y < 0) \Rightarrow (x+y < 0 \land xy > 0)$
c) $\forall x, y \in \mathbb{R}: (x = y \veebar x < y \veebar x > y)$

→ <u>Equisigned and heterosigned numbers</u>

<u>Def</u>: Let $x, y \in \mathbb{R}$ be given. Then:
x, y equisigned $\Leftrightarrow x, y \in \mathbb{R}_+^* \lor x, y \in \mathbb{R}_-^*$
x, y heterosigned $\Leftrightarrow (x \in \mathbb{R}_+^* \land y \in \mathbb{R}_-^*) \lor (x \in \mathbb{R}_-^* \land y \in \mathbb{R}_+^*)$

We now show that:

<u>Thm</u>: (<u>Law of signs</u>)
$\forall x, y \in \mathbb{R}: x, y$ equisigned $\Leftrightarrow xy > 0$
$\forall x, y \in \mathbb{R}: x, y$ heterosigned $\Leftrightarrow xy < 0$

Immediate consequences of the law of signs:
a) $\forall a \in \mathbb{R}: (a \neq 0 \Rightarrow a^2 > 0)$
b) $1 > 0 \longleftarrow$ Show that $1 \neq 0$. Then $1 = 1 \cdot 1 = 1^2 > 0$.
c) $\forall a \in \mathbb{R}: (a \neq 0 \Rightarrow a, 1/a$ equisigned$)$.
→ $\forall a \in \mathbb{R}: (a > 0 \Rightarrow 1/a > 0)$
→ $\forall a \in \mathbb{R}: (a < 0 \Rightarrow 1/a < 0)$
d) $\forall x, y \in \mathbb{R}: (xy > 0 \Rightarrow x/y > 0)$
$\forall x, y \in \mathbb{R}: (xy < 0 \Rightarrow x/y < 0)$

Proof of law of signs

($\Rightarrow$) :

Assume that x, y equisigned. Then:

$$\underline{x, y \text{ equisigned}} \Rightarrow (x>0 \wedge y>0) \vee (x<0 \wedge y<0)$$
$$\Rightarrow xy>0 \vee (-x>0 \wedge -y>0)$$
$$\Rightarrow xy>0 \vee (-x)(-y)>0$$
$$\Rightarrow xy>0 \vee xy>0 \Rightarrow \underline{xy>0}.$$

Assume that x, y heterosigned. Then

$$\underline{x, y \text{ heterosigned}} \Rightarrow (x>0 \wedge y<0) \vee (x<0 \wedge y>0)$$
$$\Rightarrow (x>0 \wedge -y>0) \vee (-x>0 \wedge y>0)$$
$$\Rightarrow x(-y)>0 \vee (-x)y>0$$
$$\Rightarrow -xy>0 \vee -xy>0 \Rightarrow -xy>0 \Rightarrow \underline{xy<0}.$$

($\Leftarrow$)

We note that

$$xy \neq 0 \Rightarrow x \neq 0 \wedge y \neq 0$$
$$\Rightarrow (x>0 \,\underline{\vee}\, x<0) \wedge (y>0 \,\underline{\vee}\, y<0)$$
$$\Rightarrow x, y \text{ equisigned} \,\underline{\vee}\, x, y \text{ heterosigned}$$

and it follows that

$$\underline{xy>0} \Rightarrow xy \neq 0 \wedge \overline{xy<0}$$
$$\Rightarrow \begin{cases} x, y \text{ equisigned} \,\underline{\vee}\, x, y \text{ heterosigned} \\ x, y \text{ not heterosigned} \end{cases}$$
$$\Rightarrow \underline{x, y \text{ equisigned}}$$

and

$$\underline{xy<0} \Rightarrow xy \neq 0 \wedge \overline{xy>0}$$
$$\Rightarrow \begin{cases} x, y \text{ equisigned} \,\underline{\vee}\, x, y \text{ heterosigned} \\ x, y \text{ not equisigned} \end{cases}$$
$$\Rightarrow \underline{x, y \text{ heterosigned}}$$

$\square$

In this argument we used the contrapositive of
$\forall a,b \in \mathbb{R} : (ab = 0 \Leftrightarrow (a=0 \vee b=0))$
which is given by
$\forall a,b \in \mathbb{R} : (ab \neq 0 \Leftrightarrow (a \neq 0 \wedge b \neq 0))$

Transitive property.

Prop : $\boxed{\forall x,y,z \in \mathbb{R} : ((x>y \wedge y>z) \Rightarrow x>z)}$

Proof

Let $x,y,z \in \mathbb{R}$ be given and assume that $x>y \wedge y>z$.
Then:

$$\begin{cases} x>y \\ y>z \end{cases} \Rightarrow \begin{cases} x-y>0 \\ y-z>0 \end{cases} \Rightarrow (x-y)+(y-z)>0$$
$$\Rightarrow x-z>0 \Rightarrow x>z. \qquad \square$$

Immediate corollaries of the transitive property is:
$\forall x,y,z \in \mathbb{R} : ((x<y \wedge y<z) \Rightarrow x<z)$
$\forall x \in \mathbb{R}^*_+ : \forall y \in \mathbb{R}^*_- : x>y$

Order and operations on $\mathbb{R}$.

① $\boxed{\forall x,y,z \in \mathbb{R} : (x>y \Leftrightarrow x+z>y+z)}$

Proof

Let $x,y,z \in \mathbb{R}$ be given. Then

$$\underline{x+z > y+z} \Leftrightarrow (x+z)-(y+z) > 0$$
$$\Leftrightarrow x+z-y-z > 0$$
$$\Leftrightarrow x-y > 0 \Leftrightarrow \underline{x > y} \qquad \square$$

② $\boxed{\forall a,x,y \in \mathbb{R} : ((x>y \land a>0) \Rightarrow ax > ay)}$

<u>Proof</u>

Let $a,x,y \in \mathbb{R}$ be given and assume that $x>y \land a>0$. Then:

$$\begin{cases} x>y \\ a>0 \end{cases} \Rightarrow \begin{cases} x-y>0 \\ a>0 \end{cases} \Rightarrow a(x-y)>0 \Rightarrow ax-ay>0$$
$$\Rightarrow ax > ay. \qquad \square$$

③ $\boxed{\forall a,x,y \in \mathbb{R} : ((x>y \land a<0) \Rightarrow ax < ay)}$

<u>Proof</u>

Let $a,x,y \in \mathbb{R}$ be given and assume that $x>y \land a<0$. Then:

$$\begin{cases} x>y \\ a<0 \end{cases} \Rightarrow \begin{cases} x-y>0 \\ a<0 \end{cases} \Rightarrow a(x-y)<0 \Rightarrow ax-ay<0$$
$$\Rightarrow ax < ay. \qquad \square$$

④ $\boxed{\forall a,b,x,y \in \mathbb{R} : ((x>y \land a>b) \Rightarrow x+a > y+b)}$

<u>Proof</u>

Let $a, b, x, y \in \mathbb{R}$ be given and assume that $x > y \wedge a > b$.
Then:

$$\begin{cases} x > y \\ a > b \end{cases} \Rightarrow \begin{cases} x - y > 0 \\ a - b > 0 \end{cases} \Rightarrow (x-y) + (a-b) > 0 \Rightarrow$$

$$\Rightarrow (x+a) - (y+b) > 0 \Rightarrow x + a > y + b. \qquad \square$$

⑤ $$\boxed{\forall a, b, x, y \in \mathbb{R} : \left(\begin{cases} x > a > 0 \\ y > b > 0 \end{cases} \Rightarrow xy > ab \right)}$$

<u>Proof</u>

Let $a, b, x, y \in \mathbb{R}$ be given and assume that $x > a > 0$ and $y > b > 0$. Then, we note that

$$xy - ab = xy - ay + ay - ab = (x-a)y + (y-b)a \qquad (1)$$

and it follows that

$$\begin{cases} x > a > 0 \\ y > b > 0 \end{cases} \Rightarrow \begin{cases} x - a > 0 \wedge y > 0 \\ y - b > 0 \wedge a > 0 \end{cases} \Rightarrow \begin{cases} (x-a)y > 0 \\ (y-b)a > 0 \end{cases}$$

$$\Rightarrow (x-a)y + (y-b)a > 0$$

$$\Rightarrow xy - ab > 0 \qquad \text{[via Eq. (1)]}$$

$$\Rightarrow xy > ab \qquad \square$$

↳ Statements ④ and ⑤ show that

a) We can always add two inequalities that have the same direction

b) We can always multiply two inequalities that have the same direction if both sides on both inequalities are positive.

c) Using the method of induction, we can show that

$$\boxed{\begin{array}{l} \forall x,y \in \mathbb{R}_+ : \forall n \in \mathbb{N}^* : (x > y \Leftrightarrow x^n > y^n) \\ \forall x,y \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : (x > y \Leftrightarrow x^{-n} < y^{-n}) \end{array}}$$

6 $\boxed{\forall a,b \in \mathbb{R} : a^2+b^2=0 \Leftrightarrow a=0 \land b=0}$

<u>Proof</u>

($\Rightarrow$): Assume that $a^2+b^2=0$. To show that $a=0$, assume that $a \neq 0$, in order to show a contradiction. It follows that

$$\begin{cases} a \neq 0 \\ b^2 \geq 0 \end{cases} \Rightarrow \begin{cases} a^2 > 0 \\ b^2 \geq 0 \end{cases} \Rightarrow a^2+b^2 \geq a^2 > 0 \Rightarrow$$

$$\Rightarrow a^2+b^2 > 0 \leftarrow \text{contradiction with hypothesis}$$

It follows that $a=0$. Similarly, we show that $b=0$. We conclude that $\underline{a=0 \land b=0}$.

($\Leftarrow$): Assume that $a=0 \land b=0$. Then $a^2+b^2=0^2+0^2=0$.

THEORY QUESTIONS

(10) State the order axiom of $\mathbb{R}$

(11) Let $x, y \in \mathbb{R}$ be given. State the definition for

a) x, y equisigned

b) x, y heterosigned

(12) Use the order axiom and the law of signs to show that

a) $\forall x, y, z \in \mathbb{R} : ((x > y \wedge y > z) \Rightarrow x > z)$

b) $\forall x, y, z \in \mathbb{R} : (x > y \Leftrightarrow x + z > y + z)$

c) $\forall a, x, y \in \mathbb{R} : ((x > y \wedge a > 0) \Rightarrow ax > ay)$

d) $\forall a, x, y \in \mathbb{R} : ((x > y \wedge a < 0) \Rightarrow ax < ay)$

e) $\forall a, b, x, y \in \mathbb{R} : ((x > y \wedge a > b) \Rightarrow x + a > y + b)$

f) $\forall a, b, x, y \in \mathbb{R} : \left(\begin{cases} x > a > 0 \\ y > b > 0 \end{cases} \Rightarrow xy > ab \right)$

g) $\forall a, b \in \mathbb{R} : (a^2 + b^2 = 0 \Leftrightarrow a = 0 \wedge b = 0)$

EXERCISE

(12a) Use the properties ~~defined~~ derived from the order axiom to prove that

a) $\forall x, y \in \mathbb{R}_+ : \forall n \in \mathbb{N}^* : (x > y \Leftrightarrow x^n > y^n)$

b) $\forall x, y \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : (x > y \Leftrightarrow x^{-n} < y^{-n})$

The Bernoulli inequality

$$\boxed{\forall a\in\mathbb{R} : \forall n\in\mathbb{N} : (a>-1 \Rightarrow (1+a)^n \geq 1+na)}$$

Proof

Let $a\in\mathbb{R}$ and $n\in\mathbb{N}$ be given and assume that $a>-1$. Then, we have $1+a>0$. We use proof by induction on $n\in\mathbb{N}$.

For $n=0$:

$$\begin{cases} (1+a)^n = (1+a)^0 = 1 \\ 1+na = 1+0a = 1 \end{cases} \Rightarrow (1+a)^n \geq 1+na$$

For $n=k$, assume that $(1+a)^k \geq 1+ka$.

For $n=k+1$, we will show that $(1+a)^{k+1} \geq 1+(k+1)a$

We have:

$$\begin{cases} (1+a)^k \geq 1+ka \\ 1+a>0 \end{cases} \Rightarrow (1+a)^k(1+a) \geq (1+ka)(1+a) \Rightarrow$$

$$\begin{aligned} \Rightarrow (1+a)^{k+1} &= (1+a)^k(1+a) \\ &\geq (1+ka)(1+a) \\ &= 1+a+ka+ka^2 \\ &= 1+(k+1)a+ka^2 \\ &\geq 1+(k+1)a \Rightarrow (1+a)^{k+1} \geq 1+(k+1)a \end{aligned}$$

By induction, we conclude that

$\forall a\in\mathbb{R} : \forall n\in\mathbb{N} : (a>-1 \Rightarrow (1+a)^n \geq 1+na)$ $\square$

EXERCISES

(13) Use the Bernoulli inequality or proof by induction to show that

a) $\forall n \in \mathbb{N}^* : 5^n \geq 1+4n$

b) $\forall n \in \mathbb{N}^* : 3^n \geq 2^n (n+1)$

c) $\forall n \in \mathbb{N}^* : (1+1/n)^n \geqslant 2$

d) $\forall n \in \mathbb{N}^* : \left(\dfrac{2n}{n+1}\right)^n \geqslant \dfrac{n+1}{2}$

e) $\forall n \in \mathbb{N}^* : 2^{n+2} > 2n+5$

f) $\forall n \in \mathbb{N}^* : 3^{2n} > 2^{2n+1}$

g) $\forall n \in \mathbb{N}^* : (n \geqslant 4 \Rightarrow 3^{n-1} > n^2)$

h) $\forall n \in \mathbb{N}^* : (n \geqslant 10 \Rightarrow 2^n > n^3)$

i) $\forall a,b \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : (n \geqslant 2 \Rightarrow (a+b)^n > a^n + na^{n-1}b)$.

Intervals and absolute values

Def: Let $a, b \in \mathbb{R}$ be given. We define:

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$	$(a,+\infty) = \{x \in \mathbb{R} \mid x > a\}$
$(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$	$[a,+\infty) = \{x \in \mathbb{R} \mid x \geq a\}$
$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$	$(-\infty,b) = \{x \in \mathbb{R} \mid x < b\}$
$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$	$(-\infty,b] = \{x \in \mathbb{R} \mid x \leq b\}$

Def: Let $x \in \mathbb{R}$ be given. We define the absolute value $|x|$ such that

$$|x| = \begin{cases} x & , \text{ if } x \in [0,+\infty) \\ -x & , \text{ if } x \in (-\infty, 0) \end{cases}$$

The following are immediate consequences of the absolute value definition:

a) $\forall x \in \mathbb{R} : (x = 0 \Leftrightarrow |x| = 0)$

b) $\forall x \in \mathbb{R} : (|x| \geq x \land |x| \geq -x)$

c) $\forall x \in \mathbb{R} : |x| = \max\{x, -x\}$

d) $\forall x \in \mathbb{R} : -|x| \leq x \leq |x|$

e) $\forall x \in \mathbb{R} : |-x| = |x|$

f) $\forall x \in \mathbb{R} : |x| \geq 0$

g) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N} : |x|^{2n} = x^{2n}$

Properties of the absolute value

(1) Let $x \in \mathbb{R}$ and $p \in (0, +\infty)$ be given. Then
a) $|x| < p \Leftrightarrow x \in (-p, p)$
b) $|x| > p \Leftrightarrow x \in (-\infty, -p) \cup (p, +\infty)$
c) $|x| = p \Leftrightarrow (x = p \vee x = -p)$.

Proof

Let $x \in \mathbb{R}$ and $p \in (0, +\infty)$ be given.

a) $\underline{|x| < p} \Leftrightarrow |x|^2 < p^2$ [via $|x| \geq 0$ and $p > 0$]

$\Leftrightarrow x^2 < p^2 \Leftrightarrow x^2 - p^2 < 0 \Leftrightarrow (x-p)(x+p) < 0$

$\Leftrightarrow x-p,\ x+p$ heterosigned

$$\Leftrightarrow \begin{cases} x - p < 0 \\ x + p > 0 \end{cases} \Leftrightarrow \begin{cases} x < p \\ x > -p \end{cases} \Leftrightarrow -p < x < p$$

$\Leftrightarrow \underline{x \in (-p, p)}$

b) $\underline{|x| > p} \Leftrightarrow |x|^2 > p^2$ [via $|x| \geq 0$ and $p > 0$]

$\Leftrightarrow x^2 > p^2 \Leftrightarrow x^2 - p^2 > 0 \Leftrightarrow (x-p)(x+p) > 0$

$\Leftrightarrow x-p,\ x+p$ equisigned

$$\Leftrightarrow \begin{cases} x - p > 0 \\ x + p > 0 \end{cases} \vee \begin{cases} x - p < 0 \\ x + p < 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x > p \\ x > -p \end{cases} \vee \begin{cases} x < p \\ x < -p \end{cases} \Leftrightarrow$$

$\Leftrightarrow x > p \vee x < -p \Leftrightarrow x \in (p, +\infty) \vee x \in (-\infty, -p)$

$\Leftrightarrow \underline{x \in (-\infty, -p) \cup (p, +\infty)}$

c) $\underline{|x| = p} \Leftrightarrow |x|^2 = p^2 \Leftrightarrow x^2 = p^2 \Leftrightarrow x^2 - p^2 = 0$
$\Leftrightarrow (x-p)(x+p) = 0 \Leftrightarrow x-p=0 \vee x+p=0$
$\Leftrightarrow \underline{x = p \vee x = -p.}$

② $\boxed{\forall x,y \in \mathbb{R}: ||x|-|y|| \leq |x+y| \leq |x|+|y|}$

Proof

Let $x,y \in \mathbb{R}$ be given. Then:

$$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases} \Rightarrow -(|x|+|y|) \leq x+y \leq |x|+|y|$$

$\Rightarrow |x+y| \leq |x|+|y|$ [via ①]

We conclude that $\forall x,y \in \mathbb{R}: |x+y| \leq |x|+|y|$.

Furthermore, we have:

$|x| = |x+y-y| \leq |x+y| + |-y| = |x+y| + |y| \Rightarrow$
$\Rightarrow |x+y| \geq |x|-|y|$ (1)

and

$|y| = |y+x-x| \leq |y+x| + |-x| = |x+y| + |x| \Rightarrow$
$\Rightarrow |x+y| \geq |y|-|x|$ (2)

From Eq. (1) and Eq. (2):

$|x+y| \geq \max\{|x|-|y|, |y|-|x|\} =$
$= \max\{|x|-|y|, -(|x|-|y|)\} = ||x|-|y||$
$\Rightarrow ||x|-|y|| \leq |x+y|.$

We conclude that

$\forall x,y \in \mathbb{R}: ||x|-|y|| \leq |x+y| \leq |x|+|y|.$ $\square$

③ $\boxed{\forall x,y\in\mathbb{R}: |xy| = |x||y|}$

Proof

Let $x,y\in\mathbb{R}$ be given. Then,

$|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 = (|x||y|)^2 \Rightarrow$

$\Rightarrow |xy| = |x||y|$ [via $|xy|\geq 0 \land |x||y|\geq 0$]. □

↳ Immediate consequences of these properties are the following statements

a) $\forall x\in\mathbb{R}: \forall y\in\mathbb{R}^*: \left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$

b) $\forall x_1,x_2,\ldots,x_n\in\mathbb{R}: |x_1+x_2+\cdots+x_n| \leq |x_1|+|x_2|+\cdots+|x_n|$

c) $\forall x_1,x_2,\ldots,x_n\in\mathbb{R}: |x_1x_2\cdots x_n| = |x_1||x_2|\cdots|x_n|$

④ $\boxed{\forall x,y\in\mathbb{R}: (|x|+|y|=0 \Leftrightarrow (x=0\land y=0))}$

Proof

Let $x,y\in\mathbb{R}$ be given.

($\Rightarrow$): Assume that $|x|+|y|=0$. Then, we have:

$$\begin{cases} x\leq |x|\leq |x|+|y| = 0 \\ x\geq -|x| \geq -|x|-|y| = -(|x|+|y|) = -0 = 0 \end{cases} \Rightarrow$$

$\Rightarrow x\leq 0 \land x\geq 0 \Rightarrow x=0$.

Similarly, we show that $y=0$. We conclude that $x=0\land y=0$.

($\Leftarrow$): Assume that $x=0\land y=0$. Then

$|x|+|y| = |0|+|0| = 0+0 = 0$. □

An immediate consequence of (4) is the following statement:

$$\forall x_1, x_2, \dots, x_n \in \mathbb{R}: \left(|x_1| + |x_2| + \dots + |x_n| = 0 \iff \right.$$
$$\left. \iff (x_1 = 0 \land x_2 = 0 \land \dots \land x_n = 0) \right)$$

THEORY QUESTIONS

(14) Let $x \in \mathbb{R}$ be given. State the definition of $|x|$.

(15) Show that:

a) $\forall x \in \mathbb{R} : \forall p \in (0, +\infty) : (|x| < p \Leftrightarrow x \in (-p, p))$

b) $\forall x \in \mathbb{R} : \forall p \in (0, +\infty) : (|x| > p \Leftrightarrow x \in (-\infty, -p) \cup (p, +\infty))$

c) $\forall x \in \mathbb{R} : \forall p \in (0, +\infty) : (|x| = p \Leftrightarrow (x = p \vee x = -p))$

d) $\forall x, y \in \mathbb{R} : ||x| - |y|| \leq |x+y| \leq |x| + |y|$

e) $\forall x, y \in \mathbb{R} : |xy| = |x||y|$

f) $\forall x, y \in \mathbb{R} : (|x| + |y| = 0 \Leftrightarrow (x = 0 \wedge y = 0))$

EXERCISES

(16) Let $a, b, x, y \in \mathbb{R}$ be given. Show that

a) $x, y \in (a, b) \Rightarrow |x - y| < |a - b|$

b) $a < x < b \Rightarrow ||a - x| - |b - x|| = |a + b - 2x|$

c) $x < a < b \ \vee \ a < b < x \Rightarrow ||a - x| - |b - x|| = b - a$

d) $a < x < 1 \Rightarrow ||x - 1| + |x - a|| \geq |1 - x| - |a - x|$

(17) Show that

a) $\forall a, b \in \mathbb{R}^* : \left(\left| \dfrac{a|b| + b|a|}{ab} \right| = 2 \Rightarrow a, b \text{ equisigned} \right)$

b) $\forall a, b \in \mathbb{R} : (||a| - |b|| = |a + b| \Rightarrow ab \leq 0)$

(18) Show that

a) $\forall x, y \in \mathbb{R}: \max\{x, y\} = (1/2)(x+y+|x-y|)$

b) $\forall x, y \in \mathbb{R}: \min\{x, y\} = (1/2)(x+y-|x-y|)$

c) $\forall x, y, z \in \mathbb{R}: \max\{x, y, z\} = (1/4)(2x+y+z+|y-z|+$
$+|2x-y-z-|y-z||)$

d) $\forall x, y, z \in \mathbb{R}: \min\{x, y, z\} = (1/4)(2x+y+z-|y-z|-$
$-|2x-y-z+|y-z|)$

(19) Show that

a) $(a<b \land |x-a|<|x-b|) \Rightarrow x<(1/2)(a+b)$

b) $\begin{cases} |x-x_0|<\varepsilon/2 \\ |y-y_0|<\varepsilon/2 \end{cases} \Rightarrow \begin{cases} |x+y-(x_0+y_0)|<\varepsilon \\ |x-y-(x_0+y_0)|<\varepsilon \end{cases}$

c) $\begin{cases} |x-x_0| < \min\left\{1, \dfrac{\varepsilon}{2(y_0+1)}\right\} \\ |y-y_0| < \dfrac{\varepsilon}{2(y_0+1)} \end{cases} \Rightarrow |xy-x_0y_0|<\varepsilon$

d) $\begin{cases} |y-y_0| < (1/2)\min\{|y_0|, \varepsilon|y_0|^2\} \\ y_0 \neq 0 \end{cases} \Rightarrow \begin{cases} y \neq 0 \\ |1/y - 1/y_0| < \varepsilon. \end{cases}$

▼ Axiom of completeness and well-ordering principle

We begin with the following definition:

Def: Let $a \in \mathbb{R}$ be given and let S be a set such that $S \subseteq \mathbb{R} \wedge S \neq \emptyset$. Then:

a) a upper bound of $S \Leftrightarrow \forall x \in S: x \leq a$

b) a lower bound of $S \Leftrightarrow \forall x \in S: x \geq a$

c) S upper bounded $\Leftrightarrow \exists b \in \mathbb{R}$: b upper bound of S

d) S lower bounded $\Leftrightarrow \exists b \in \mathbb{R}$: b lower bound of S

e) $a = \max S \Leftrightarrow a \in S \wedge (a$ upper bound of $S)$

f) $a = \min S \Leftrightarrow a \in S \wedge (a$ lower bound of $S)$

g) $a = \sup S \Leftrightarrow \begin{cases} a \text{ upper bound of } S \\ \forall \varepsilon \in (0, +\infty): a - \varepsilon \text{ not upper bound of } S \end{cases}$

h) $a = \inf S \Leftrightarrow \begin{cases} a \text{ lower bound of } S \\ \forall \varepsilon \in (0, +\infty): a + \varepsilon \text{ not lower bound of } S \end{cases}$

- $\sup S$ is the "least upper bound" of S, if it exists.
- $\inf S$ is the "greatest lower bound" of S, if it exists.

EXAMPLES

a) $\inf (a, +\infty) = a$, but $\min (a, +\infty)$ undefined

b) $\sup (-\infty, b) = b$, but $\max (-\infty, b)$ undefined

c) $\inf [a, b] = \min [a, b] = a$

d) $\sup [a, b] = \max [a, b] = b$

Well-ordering principle

We introduce the following notation:

a) Let $n \in \mathbb{N}$. We define
$[n] = \{x \in \mathbb{N} \mid 1 \leq x \leq n\} = \{1, 2, \dots, n\}$
and note that $[0] = \emptyset$.

b) Let A, B be two sets. We define $\mathrm{Map}(A, B)$ as the set of all mappings $\varphi : A \to B$.

Def: Let S be a set. We say that
S finite $\Longleftrightarrow \exists n \in \mathbb{N} : \exists \varphi \in \mathrm{Map}(S, [n]) : \varphi$ bijection

Axiom: Let S be a set. Then
$$\begin{cases} S \subseteq \mathbb{R} \\ S \text{ finite} \end{cases} \Longrightarrow \begin{cases} \exists a \in S : \max S = a \\ \exists a \in S : \min S = a \end{cases}$$

The well-ordering principle is a fundamental axiom of set theory, although it can also be derived from the "axiom of choice".

Axiom of completeness

Axiom: a) $\begin{cases} \emptyset \neq S \subseteq \mathbb{R} \\ S \text{ upper bounded} \end{cases} \Longrightarrow \exists a \in \mathbb{R} : \sup S = a$

b) $\begin{cases} \emptyset \neq S \subseteq \mathbb{R} \\ S \text{ lower bounded} \end{cases} \Longrightarrow \exists a \in \mathbb{R} : \inf S = a$

Consequences of the completeness axiom

(1) **Thm**: (Archimedes theorem)
$\forall x \in \mathbb{R} : \exists n \in \mathbb{N}^* : n > x$

Proof

To show a contradiction, assume that the negation of the claim, which reads:

$\exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : n \leq x$

is satisfied. Choose some $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}^* : n \leq x$.

Then, we have:

$$\begin{cases} x \text{ upper bound of } \mathbb{N} \\ \emptyset \neq \mathbb{N} \subseteq \mathbb{R} \end{cases} \Rightarrow \begin{cases} \mathbb{N} \text{ upper bounded} \\ \emptyset \neq \mathbb{N} \subseteq \mathbb{R} \end{cases} \Rightarrow$$

$$\Rightarrow \exists b \in \mathbb{R} : \sup \mathbb{N} = b$$

Choose $b \in \mathbb{N}$ such that $\sup \mathbb{N} = b$. Then:

$$b - 1 < b = \sup \mathbb{N} \Rightarrow b - 1 < \sup \mathbb{N} \Rightarrow$$

$$\Rightarrow \underline{b - 1 \text{ not upper bound of } \mathbb{N}}$$

$$\Rightarrow \cancel{\forall} n_0 \in \mathbb{N} : n_0 \cancel{\leq} b - 1$$

$$\Rightarrow \exists n_0 \in \mathbb{N} : n_0 > b - 1$$

Choose an $n_0 \in \mathbb{N}$ such that $n_0 > b - 1$. It follows that

$n_0 + 1 > (b-1) + 1 = b = \sup \mathbb{N} \Rightarrow n_0 + 1 > \sup \mathbb{N}$ (1)

and $n_0 + 1 \in \mathbb{N} \Rightarrow n_0 + 1 \leq \sup \mathbb{N}$ (2)

Eq. (1) and Eq. (2) contradict. We conclude that

$\forall x \in \mathbb{R} : \exists n \in \mathbb{N}^* : n > x$.

② **Thm:** (Approximation property)

$$\boxed{\begin{array}{l} \left\{\begin{array}{l} A \text{ upper bounded} \\ \emptyset \neq A \subseteq \mathbb{R} \end{array}\right. \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists a \in A : \sup A - \varepsilon \leq a \\ \left\{\begin{array}{l} A \text{ lower bounded} \\ \emptyset \neq A \subseteq \mathbb{R} \end{array}\right. \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists a \in A : \inf A + \varepsilon \geq a \end{array}}$$

Proof

a) Assume that A upper bounded and $\emptyset \neq A \subseteq \mathbb{R}$. Let $\varepsilon \in (0,+\infty)$ be given. To show a contradiction, assume that $\forall a \in A : \sup A - \varepsilon > a$. Then, we have:

$(\forall a \in A : a < \sup A - \varepsilon) \Rightarrow \sup A - \varepsilon$ upper bound on A

$\Rightarrow \sup A \leq \sup A - \varepsilon \Rightarrow -\varepsilon \geq 0 \Rightarrow \varepsilon \leq 0 \leftarrow$ Contradiction.

It follows that $\exists a \in A : \sup A - \varepsilon \leq a$.

b) Assume that A lower bounded and $\emptyset \neq A \subseteq \mathbb{R}$. Let $\varepsilon \in (0,+\infty)$ be given. To show a contradiction, assume that $\forall a \in A : \inf A + \varepsilon < a$. Then, we have:

$(\forall a \in A : a > \inf A + \varepsilon) \Rightarrow \inf A + \varepsilon$ lower bound of A

$\Rightarrow \inf A \geq \inf A + \varepsilon \Rightarrow \varepsilon \leq 0 \longleftarrow$ Contradiction

It follows that $\exists a \in A : \inf A + \varepsilon \geq 0$. □

Characterization of intervals

The following characterization of intervals is used later in differential calculus. We begin with the definition:

Def: Let I be a set with $\emptyset \neq I \subseteq \mathbb{R}$. Then
I interval $\Leftrightarrow \exists a, b \in \mathbb{R} : (I = [a,b] \vee I = [a,b) \vee I = (a,b] \vee$
$\vee I = (a,b) \vee I = [a,+\infty) \vee I = (a,+\infty) \vee I = (-\infty,b) \vee I = (-\infty,b])$

Now we derive the following equivalent characterization

Thm: Let I be a set with $\emptyset \neq I \subseteq \mathbb{R}$ Then:
I interval $\Leftrightarrow \forall a,b \in I : (a < b \Rightarrow [a,b] \subseteq I)$

Proof

($\Rightarrow$): Easy to show but tedious. (Homework).

($\Leftarrow$): Assume that $\forall a,b \in I : (a < b \Rightarrow [a,b] \subseteq I)$.

Since $I \neq \emptyset$, choose some $t \in I$ and define $A = [t,+\infty) \cap I$ and $B = (-\infty,t] \cap I$, and note that

$$A \cup B = [I \cap [t,+\infty)] \cup [I \cap (-\infty,t]] = I \cap [[t,+\infty) \cup (-\infty,t]]$$
$$= I \cap \mathbb{R} = I$$

and

$$A = I \cap [t,+\infty) \subseteq [t,+\infty) \Rightarrow (\forall x \in A : x \in [t,+\infty)) \Rightarrow$$
$$\Rightarrow \forall x \in A : t \leq x$$

We distinguish between the following cases:

<u>Case 1:</u> Assume that A is not upper bounded. We obviously have: $\underline{A = [t, +\infty) \cap I \subseteq [t, +\infty)}$. (1)

We will now show that $[t, +\infty) \subseteq A$. Let $x \in [t, +\infty)$ be given. It follows that $t \leq x$. Furthermore, since

A not upper bounded $\Rightarrow \exists \xi \in A : x \leq \xi$

Choose $\xi \in A$ such that $x \leq \xi$. Then, we have:

$$\begin{cases} t, \xi \in A \\ t \leq x \leq \xi \end{cases} \Rightarrow \begin{cases} t, \xi \in I \\ x \in [t, \xi] \end{cases} \Rightarrow x \in I \quad \text{[via hyp]}.$$

and: $x \in I \wedge x \in [t, +\infty) \Rightarrow x \in I \cap [t, +\infty) \Rightarrow x \in A$.

It follows that $(\forall x \in [t, +\infty) : x \in A) \Rightarrow \underline{[t, +\infty) \subseteq A}$. (2)

From Eq.(1) and Eq.(2): $A = [t, +\infty)$.

<u>Case 2:</u> Assume that A is upper bounded. Then, we can define $p = \sup A$, and it follows that:

$$\begin{aligned} p = \sup A &\Rightarrow p \text{ upper bound of } A \Rightarrow \\ &\Rightarrow \forall x \in A : t \leq x \leq p \quad \text{[via previous result } t \leq x] \\ &\Rightarrow (\forall x \in A : x \in [t, p]) \Rightarrow \underline{A \subseteq [t, p]} \quad (3) \end{aligned}$$

We will now show that $[t, p) \subseteq A$.

Let $x \in [t, p)$ be given. By the approximation property:

$$\forall \varepsilon \in (0, +\infty) : \exists \xi \in A : \sup A - \varepsilon < \xi$$

$$\Rightarrow \exists \xi \in A : \sup A - (p - x) < \xi \quad [\text{via } \varepsilon = p - x > 0]$$

Choose some $\xi \in A$ such that $\sup A - (p - x) < \xi$. Then, we have:

$$x = p - (p - x) = \sup A - (p - x) < \xi \Rightarrow \underline{x < \xi}$$

therefore:

$$\begin{cases} t \leqslant x \leqslant s \\ t, s \in A \end{cases} \Rightarrow \begin{cases} x \in [t, s] \\ t, s \in I \end{cases} \Rightarrow x \in I.$$

and it follows that

$$\begin{cases} x \in I \\ x \in [t, p) \end{cases} \Rightarrow \begin{cases} x \in I \\ x \in [t, +\infty) \end{cases} \Rightarrow x \in I \cap [t, +\infty) \Rightarrow x \in A.$$

We have thus shown that

$$(\forall x \in [t, p) : x \in A) \Rightarrow [t, p) \subseteq A.$$

We conclude that

$$[t, p) \subseteq A \subseteq [t, p] \Rightarrow A = [t, p) \vee A = [t, p].$$

In both cases, we find that

$$\exists p \in \mathbb{R} : (A = [t, p) \vee A = [t, p] \vee A = [t, +\infty))$$

Similarly, we can show that

$$\exists q \in \mathbb{R} : (B = (q, t] \vee B = [q, t] \vee B = (-\infty, t]).$$

Since $I = A \cup B$, it immediately follows that I follows one of the 8 interval forms. $\square$

THEORY QUESTIONS

(20) State the axiom of completeness.

(21) State the definition for the following statements

a) p upper bound of A

b) p lower bound of A

c) A upper bounded

d) A lower bounded

e) $p = \sup(A)$

f) $p = \inf(A)$.

(22) Use quantifier algebra to write out the detailed definition for the following negated statements.

a) A not upper bounded

b) A not lower bounded

c) $p \neq \sup(A)$

d) $p \neq \inf(A)$

(23) State and prove the Archimedes theorem:

$(\forall x \in \mathbb{R}_+^* : \exists n \in \mathbb{N}^* : n > x)$.

EXERCISE - PROJECT

(24) Let I be a set with $\varnothing \neq I \subseteq \mathbb{R}$. Write the complete proof of the theorem:

I interval $\Leftrightarrow \forall a, b \in I : (a < b \Rightarrow [a, b] \subseteq I)$.

EXERCISES

(25) Let A, B be sets such that $\emptyset \neq A \subseteq B \subseteq \mathbb{R}$.
Show that

a) $\sup(A) \leqslant \sup(B)$
b) $\inf(A) \geqslant \inf(B)$

(26) Let A, B be sets such that $\emptyset \neq A \subseteq \mathbb{R}$ and $\emptyset \neq B \subseteq \mathbb{R}$ and consider the definitions

$$A+B = \{x+y \mid x \in A \wedge y \in B\}$$
$$-B = \{-y \mid y \in B\}$$

Show that

a) $\sup(A+B) = \sup(A) + \sup(B)$
b) $\inf(-A) = -\sup(A)$
c) $\sup(-A) = -\inf(A)$
d) $\sup(A+(-B)) = \sup(A) - \inf(B)$.

Rational and real numbers

> **Def**: Let $x \in \mathbb{R}$ be given. We say that
> $$x \text{ rational} \iff \exists a \in \mathbb{Z} : \exists b \in \mathbb{N}^* : x = a/b$$

notation: The set of all rational numbers is denoted as
$$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \wedge b \in \mathbb{N}^*\} \text{ and } \mathbb{Q}^* = \mathbb{Q} - \{0\}$$
We also define:
$$\mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geq 0\} \qquad \mathbb{Q}_- = \{x \in \mathbb{Q} \mid x \leq 0\}$$
$$\mathbb{Q}_+^* = \{x \in \mathbb{Q} \mid x > 0\} \qquad \mathbb{Q}_-^* = \{x \in \mathbb{Q} \mid x < 0\}$$
→ From the axiom of completeness we can show that
$$\forall x \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : \exists y \in \mathbb{R}_+^* : y^n = x$$
The unique $y \in \mathbb{R}_+^*$ such that $y^n = x$ is denoted as $\sqrt[n]{x}$. We can also write $\sqrt{x} = \sqrt[2]{x}$. We can then argue that $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$. The details are as follows:

Lemma: $\boxed{\forall a, b \in \mathbb{R} : \forall n \in \mathbb{N}^* : (0 < a < b \Rightarrow b^n - a^n < n(b-a) b^{n-1})}$

Proof

Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}^*$ be given and assume that $0 < a < b$.
Then, we have
$$b^n - a^n = (b-a) \sum_{k=0}^{n-1} (b^k a^{n-1-k})$$
$$< (b-a) \sum_{k=0}^{n-1} (b^k b^{n-1-k}) \quad [\text{via } b - a > 0 \wedge n-1-k \geq 0]$$
$$= (b-a) \sum_{k=0}^{n-1} b^{n-1} = (b-a)(n b^{n-1}) \Rightarrow b^n - a^n < n(b-a) b^{n-1} \quad \square$$

<u>Thm</u> : $\boxed{\forall x \in \mathbb{R}_+^* : \forall n \in \mathbb{N}^* : \exists y \in \mathbb{R}_+^* : y^n = x}$

<u>Proof</u>

Let $x \in \mathbb{R}_+^*$ and $n \in \mathbb{N}^*$ be given. To construct $y \in \mathbb{R}_+^*$, we define

$$S = \{t \in \mathbb{R}_+^* : t^n \leq x\}$$

▸ We claim that $S \neq \emptyset$.

Let $p = x/(x+1)$. Then, it follows that:

$$\begin{cases} 0 < p < x \\ 0 < p < 1 \end{cases} \Rightarrow \begin{cases} 0 < p < x \\ 0 < p^{n-1} < 1 \end{cases} \Rightarrow 0 < p^n < x \Rightarrow$$

$$\Rightarrow p \in \mathbb{R}_+^* \wedge p^n < x \Rightarrow p \in S.$$

and therefore $S \neq \emptyset$. This proves the claim.

▸ Claim: S is upper bounded.

It is sufficient to show that: $\exists M \in \mathbb{R} : \forall t \in S : t \leq M$.

Choose $\underline{M = x+1}$. Let $\underline{t \in S}$ be given. Then, we have:

$$M = x+1 \Rightarrow \begin{cases} 1 < M \\ 0 < x < M \end{cases} \Rightarrow \begin{cases} 1 < M^{n-1} \\ 0 < x < M \end{cases} \Rightarrow x < M^n$$

and it follows that

$$t \in S \Rightarrow t^n < x < M^n \Rightarrow \begin{cases} t^n < M^n \\ t > 0 \wedge M > 0 \end{cases} \Rightarrow \underline{t < M}$$

We have thus shown that

$(\exists M \in \mathbb{R} : \forall t \in S : t \leq M) \Rightarrow S$ upper bounded.

and this proves the claim.

▸ Since $0 \neq S \subseteq \mathbb{R} \wedge S$ upper bounded, via the axiom of completeness, we choose $y = \sup S$. We will now show that $y^n = x$.

To show a contradiction, assume that $y^n \neq x$. Then, we distinguish between the following cases:

<u>Case 1</u>: Assume that $y^n < x$. Choose $h \in \left(0, \min\left\{1, \dfrac{x-y^n}{n(y+1)^{n-1}}\right\}\right)$ noting that the interval is not empty because $x - y^n > 0$. It follows that:

$$\begin{aligned}
(y+h)^n &= [(y+h)^n - y^n] + y^n \\
&< n[(y+h)-y](y+h)^{n-1} + y^n \qquad [\text{via Lemma}] \\
&= nh(y+h)^{n-1} + y^n \\
&< n\frac{x-y^n}{n(y+1)^{n-1}}(y+h)^{n-1} + y^n \qquad \left[\text{via } h < \frac{x-y^n}{n(y+1)^{n-1}}\right] \\
&= (x-y^n)\frac{(y+h)^{n-1}}{(y+1)^{n-1}} + y^n \\
&< (x-y^n)\frac{(y+1)^{n-1}}{(y+1)^{n-1}} + y^n \qquad [\text{via } h < 1] \\
&= (x-y^n) + y^n = x \Rightarrow (y+h)^n < x \Rightarrow
\end{aligned}$$

$\Rightarrow y+h \in S \Rightarrow y+h \leq \sup S \Rightarrow y+h \leq y \Rightarrow \underline{h \leq 0}$ which is a contradiction, since h was chosen to satisfy $h > 0$. Therefore, this case does not materialize.

<u>Case 2</u>: Assume that $y^n > x$. We now define

$$h = \frac{y^n - x}{ny^{n-1}}$$

and note that $0 < h < y^n/(ny^{n-1}) = y$

$0 < h < y^n/(ny^{n-1}) = y \Rightarrow 0 < y-h < y \Rightarrow$

$$\begin{aligned}
\Rightarrow y^n - (y-x)^n &< n[y-(y-h)]y^{n-1} \qquad [\text{via Lemma}] \\
&= nhy^{n-1} = n\frac{y^n - x}{ny^{n-1}}y^{n-1} = y^n - x \Rightarrow
\end{aligned}$$

$\Rightarrow y^n-(y-h)^n < y^n - x \Rightarrow -(y-h)^n < -x \Rightarrow (y-h)^n > x$
$\Rightarrow (\forall t \in S: t^n < x < (y-h)^n) \Rightarrow (\forall t \in S: t < y-h)$
$\Rightarrow y-h$ upper bound of $S \Rightarrow y-h \geq \sup S \Rightarrow$
$\Rightarrow y-h \geq y \Rightarrow -h \geq 0 \Rightarrow h \leq 0$
which is a contradiction, since h was chosen to satisfy $h>0$. Therefore Case 2 does not materialize.
From the above argument, we conclude that $y^n = x$.
We have thus proved the theorem □

$\boxed{\underline{\text{Thm:}} \quad \sqrt{2} \notin \mathbb{Q} \quad \text{(Hipassos of Metapontum)}}$

<u>Proof</u>

To show a contradiction, assume that $\sqrt{2} \in \mathbb{Q}$. Then, we have:

$$\sqrt{2} \in \mathbb{Q} \Rightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N}^* : \sqrt{2} = a/b$$

Choose some $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$ such that $\sqrt{2} = a/b$ so that the ratio a/b has no further simplifications. It follows that

$$a = b\sqrt{2} \Rightarrow a^2 = (b\sqrt{2})^2 = 2b^2 \Rightarrow a^2 \text{ even} \Rightarrow a \text{ even}$$
$$\Rightarrow \exists \lambda \in \mathbb{Z} : a = 2\lambda.$$

Choose $\lambda \in \mathbb{Z}$ such that $a = 2\lambda$. Then, we have:

$$2b^2 = a^2 = (2\lambda)^2 = 4\lambda^2 \Rightarrow b^2 = 2\lambda^2 \Rightarrow b^2 \text{ even} \Rightarrow$$
$$\Rightarrow b \text{ even.}$$

This is a contradiction, because (a even $\wedge$ b even) implies that the fraction a/b can be simplified in contradiction with our choice above. We conclude that $\sqrt{2} \notin \mathbb{Q}$. □

THEORY QUESTIONS

(27) Show that $\sqrt{2} \notin \mathbb{Q}$.

(28) Write the definition of "x is rational" using quantifier notation.

EXERCISES

(29) Show that

a) $\sqrt{3} \notin \mathbb{Q}$

b) $\sqrt{6} \notin \mathbb{Q}$

c) $\sqrt{2}+\sqrt{3} \notin \mathbb{Q}$ (Hint: use (a) and (b))

(30) Let $a, b \in \mathbb{Q}_+^*$ be given. Show that:

$$(\sqrt{a} \notin \mathbb{Q} \wedge a \neq b) \Rightarrow (\sqrt{a}-\sqrt{b}) \notin \mathbb{Q}$$

(31) Let $a, b, c, d \in \mathbb{Q}$ with $b>0$ and $d>0$ and $\sqrt{b} \notin \mathbb{Q}$ and $\sqrt{d} \notin \mathbb{Q}$. Show that

a) $a+\sqrt{b} = c+\sqrt{d} \Leftrightarrow (a=c \wedge b=d)$

b) $a-\sqrt{b} = c-\sqrt{d} \Leftrightarrow (a=c \wedge b=d)$

(Hint: for exercise 31 use the result from exercise 30).

RA 1.2: Limits of sequences and nets

SEQUENCES AND NETS

▼ Sequences and nets - definitions

Def: A sequence (a_n) is a mapping $a: \mathbb{N} \longrightarrow \mathbb{R}$ or $a: \mathbb{N}^* \longrightarrow \mathbb{R}$.

▶ The net is a generalization of the sequence definition that allows us to define limits and study their properties in a general way which specializes to
a) Limits of sequences
b) Limits of functions
c) Limits of partitions (used to define integrals).

Def: A directed set $(D, <)$ consists of a set D and a relation "$<$" such that

$$\begin{cases} \forall x \in D: \ x < x \\ \forall x, y, z \in D: \ ((x < y \land y < z) \Rightarrow x < z) \\ \forall x, y \in D: \exists z \in D: \ (x < z \land y < z) \end{cases}$$

Def: A net (a_w) is a mapping $a: D \longrightarrow \mathbb{R}$ where $(D, <)$ is a directed set

▶ Note that $(\mathbb{N}, \leq)$ and $(\mathbb{N}^*, \leq)$ are directed sets, so a sequence is a special case of a net. Thus all definitions given on nets also apply to sequences.

Basic properties of nets

Def: Let (a_n) be a net on $(D,<)$. We say that

a) (a_n) increasing $\iff \forall p,q\in D : (p<q \Rightarrow a_p \leq a_q)$

b) (a_n) decreasing $\iff \forall p,q\in D : (p<q \Rightarrow a_p \geq a_q)$

c) (a_n) upper bounded $\iff \exists b\in\mathbb{R} : \exists n_0\in D : \forall n\in D : (n\geq n_0 \Rightarrow a_n \leq b)$

d) (a_n) lower bounded $\iff \exists b\in\mathbb{R} : \exists n_0\in D : \forall n\in D : (n\geq n_0 \Rightarrow a_n \geq b)$

e) (a_n) bounded $\iff \begin{cases} (a_n) \text{ lower bounded} \\ (a_n) \text{ upper bounded} \end{cases}$

f) (a_n) negatively upper bounded $\iff$
$\iff \exists b\in\mathbb{R} : \exists n_0\in D : \forall n\in D : (n\geq n_0 \Rightarrow a_n \leq b < 0)$

g) (a_n) positively " lower bounded $\iff$
$\iff \exists b\in\mathbb{R} : \exists n_0\in D : \forall n\in D : (n\geq n_0 \Rightarrow a_n \geq b > 0)$

▸ For sequences, some of these definitions simplify as follows:

(a_n) increasing $\iff \forall n\in\mathbb{N}^* : a_{n+1} \geq a_n$
(a_n) decreasing $\iff \forall n\in\mathbb{N}^* : a_{n+1} \leq a_n$
(a_n) upper bounded $\iff \exists b\in\mathbb{R} : \forall n\in\mathbb{N}^* : a_n \leq b$
(a_n) lower bounded $\iff \exists b\in\mathbb{R} : \forall n\in\mathbb{N}^* : a_n \geq b$
(a_n) negatively upper bounded $\iff \exists b\in\mathbb{R} : \forall n\in\mathbb{N}^* : a_n \leq b < 0$
(a_n) positively lower bounded $\iff \exists b\in\mathbb{R} : \forall n\in\mathbb{N}^* : a_n \geq b > 0$

The following proposition is a convenient criterion for showing that (a_n) is bounded:

Prop: Let (a_n) be a net on $(D, <)$. Then, we have:
(a_n) bounded $\Leftrightarrow \exists b \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| \leq b)$

Proof

($\Rightarrow$): Assume that (a_n) bounded. Then, we have:

$$(a_n) \text{ bounded} \Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b) \\ \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \geq b) \end{cases}$$

Choose $b_1, b_2 \in \mathbb{R}$ and $n_1, n_2 \in D$ such that:

$$\begin{cases} \forall n \in D : (n > n_1 \Rightarrow a_n \leq b_1) \\ \forall n \in D : (n > n_2 \Rightarrow a_n \geq b_2) \end{cases}$$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1 \wedge n_0 > n_2$. Let $\underline{b = \max\{|b_1|, |b_2|\} > 0}$

We will show that $\forall n \in D : (n > n_0 \Rightarrow |a_n| \leq b)$.

Let $\underline{n \in D}$ be given and assume that $\underline{n > n_0}$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n \leq b_1 \leq |b_1| \leq \max\{|b_1|, |b_2|\} = b \\ a_n \geq b_2 \geq -|b_2| \geq -\max\{|b_1|, |b_2|\} = -b \end{cases}$$

$$\Rightarrow -b \leq a_n \leq b \Rightarrow \underline{|a_n| \leq b}.$$

We have thus shown that

$$\exists b \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| \leq b). \quad (1)$$

($\Leftarrow$): Assume that Eq. (1) is satisfied. Choose $b \in (0, +\infty)$ and $n_0 \in D$ such that

$\forall n \in D : (n > n_0 \Rightarrow |a_n| \leq B)$

Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$n > n_0 \Rightarrow |a_n| \leq B \Rightarrow -B \leq a_n \leq B \Rightarrow a_n \leq B \wedge a_n \geq -B$.

We have thus shown that

$$\begin{cases} \forall n \in D : (n > n_0 \Rightarrow a_n \leq B) \\ \forall n \in D : (n > n_0 \Rightarrow a_n \geq -B) \end{cases} \Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ lower bounded} \end{cases}$$

$\Rightarrow (a_n)$ bounded. □

→ Note that for sequences, the proposition simplifies to

$$\boxed{(a_n) \text{ bounded} \Leftrightarrow \exists B \in (0, +\infty) : \forall n \in \mathbb{N}^k : (|a_n| \leq B)}$$

EXAMPLES

a) Show that (a_n) given by

$$\forall n \in \mathbb{N}^*: \ a_n = \frac{3n^2 - 4n\cos(n^2+1) + 3}{4n^2+3n-2}$$

is bounded.

Solution

Let $n \in \mathbb{N}^*$ be given. Then we have:

$$|a_n| = \left| \frac{3n^2 - 4n\cos(n^2+1)+3}{4n^2+3n-2} \right| = \frac{|3n^2 - 4n\cos(n^2+1)+3|}{4n^2+3n-2}$$

$$\leq \frac{|3n^2| + |4n||\cos(n^2+1)| + 3}{4n^2+3n-2} \leq \frac{3n^2+4n+3}{4n^2+3n-2} \leq$$

$$\leq \frac{3n^2+4n^2+3n^2}{4n^2+3n-2} \leq \frac{10n^2}{4n^2} = \frac{10}{4} \Rightarrow |a_n| \leq 10/4.$$

We have thus shown that:

$(\forall n \in \mathbb{N}^* : |a_n| \leq 10/4) \Rightarrow (a_n)$ bounded. $\square$

b) Show that (a_n) given by

$$\forall n \in \mathbb{N}^*: \ a_n = 3n^2 - n - \cos(n^2 - n)$$

is <u>not</u> bounded.

Solution

To show a contradiction, assume that (a_n) is not bounded.

Then, we have:

(a_n) not bounded $\Rightarrow \exists p \in (0, +\infty) : \forall n \in \mathbb{N}^* : |a_n| \leq p$.

Choose a $p \in (0,+\infty)$ such that $\forall n \in \mathbb{N}^* : |a_n| \leq p$.

Let $\underline{n \in \mathbb{N}^*}$ be given. Then, we have:

$$p \geq |a_n| = |3^{n^2-n} - \cos(n^2-n)| \geq ||3^{n^2-n}| - |\cos(n^2-n)||$$
$$\geq |3^{n^2-n}| - |\cos(n^2-n)| \geq 3^{n^2-n} - 1 = (1+2)^{n^2-n} - 1$$
$$\geq 1 + 2(n^2-n) - 1 = 2n^2 - n = n(2n-1) \geq 2n-1 \Rightarrow$$
$$\Rightarrow p \geq 2n - 1 \Rightarrow 2n \leq p-1 \Rightarrow \underline{n \leq (p-1)/2}.$$

We have thus shown that

$\forall n \in \mathbb{N}^*: n \leq (p-1)/2$

which is a contradiction with the Archimedes theorem

We conclude that (a_n) not bounded. $\square$

THEORY QUESTIONS

① Let (a_n) be a net on $(D,<)$. Write the definitions for the following statements:

a) (a_n) increasing

b) (a_n) decreasing

c) (a_n) upper bounded

d) (a_n) lower bounded

e) (a_n) bounded

f) (a_n) negatively upper bounded

g) (a_n) positively lower bounded.

② Let (a_n) be a net on $(D,<)$. Show that:

(a_n) bounded $\Leftrightarrow \exists B\in(0,+\infty):\exists n_0\in D:\forall n\in D:(n>n_0\Rightarrow|a_n|\leq B)$

EXERCISES

③ Show that the following sequences are bounded

a) $a_n=\dfrac{n}{n^2+8}$

b) $a_n=\dfrac{n\cos n+\sin n}{n^2}$

c) $a_n=\dfrac{1}{n}\sin\left(\dfrac{\pi n}{10}\right)$

d) $a_n=\dfrac{5\sin(3n)}{4n}$

e) $a_n=\dfrac{4n+5}{5^n}$

f) $a_n=\dfrac{3n^2-1}{\sqrt{4^n}}$

④ Show that the following sequences are not bounded

a) $a_n=\dfrac{4n^2+1}{5n}$

b) $a_n=-4n^2+3n+1$

c) $a_n=\dfrac{2n^2+5}{3n+n\sin n}$

d) $a_n=(-2)^{n+1}+(-2)^n+2$.

(5) Let (a_n) and (b_n) be two sequences. Show that:

a) $\begin{cases} (a_n) \text{ bounded} \\ \forall n \in \mathbb{N}^*: b_n = a_n/n \end{cases} \Rightarrow (b_n) \text{ bounded}$

b) $\begin{cases} (a_n), (b_n) \text{ increasing} \\ \forall n \in \mathbb{N}^*: c_n = a_n + b_n \end{cases} \Rightarrow (c_n) \text{ increasing}$

c) $\begin{cases} (a_n), (b_n) \text{ bounded} \\ \forall n \in \mathbb{N}^*: c_n = a_n (a_n + b_n)^2 \end{cases} \Rightarrow (c_n) \text{ bounded}$

Definition of limit of nets and sequences

Def: Let (a_n) be a net on $(D,\leq)$ and let $l\in\mathbb{R}$.
We say that

$\lim a_n = l \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in D: \forall n\in D: (n\geq n_0 \Rightarrow |a_n - l|<\varepsilon)$

$\lim a_n = +\infty \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in D: \forall n\in D: (n\geq n_0 \Rightarrow a_n > 1/\varepsilon)$

$\lim a_n = -\infty \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in D: \forall n\in D: (n\geq n_0 \Rightarrow a_n < -1/\varepsilon)$

(a_n) convergent $\Leftrightarrow \exists l\in\mathbb{R}: \lim a_n = l$

(a_n) divergent $\Leftrightarrow \begin{cases} (a_n) \text{ not convergent} \\ \lim a_n \neq +\infty \wedge \lim a_n \neq -\infty \end{cases}$

- Note that when (a_n) is a sequence, we introduce the notation
  $[n_0] = \{x\in\mathbb{N} \mid 1\leq x\leq n_0\} = \{1,2,\ldots,n_0\}$
  and note that the limit definitions simplify as follows:

$\lim\limits_{n\in\mathbb{N}^*} a_n = l \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: |a_n - l|<\varepsilon$

$\lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: a_n > 1/\varepsilon$

$\lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \Leftrightarrow \forall \varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: a_n < -1/\varepsilon$

Zero sequences and nets

Let (a_n) be a net on $(D,<)$. We recall the definition
$\lim a_n = 0 \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$
The following is an immediate consequence of this definition:
$\lim a_n = 0 \Leftrightarrow \lim(-a_n) = 0 \Leftrightarrow \lim |a_n| = 0$

Properties of zero nets

Let $(a_n), (b_n)$ be nets on $(D,<)$. We show the following properties:

① $\boxed{\lim a_n = 0 \Rightarrow (a_n) \text{ bounded}}$

Proof

Assume that $\lim a_n = 0$. Then, we have:

$$\lim a_n = 0 \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$$
$$\Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < 1) \quad [\text{via } \varepsilon = 1]$$
$$\Rightarrow (a_n) \text{ bounded} \qquad \square$$

② $$\boxed{\begin{cases} \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| \leq |b_n|) \\ \lim b_n = 0 \end{cases} \Rightarrow \lim a_n = 0}$$

Proof

From hypothesis, choose $n_1 \in D$ such that
$\forall n \in D : (n > n_1 \Rightarrow |a_n| \leq |b_n|)$.

We note that

$\lim b_n = 0 \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |b_n| < \varepsilon)$

Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow |b_n| < \varepsilon)$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $\underline{n \in D}$ be given and assume that $\underline{n > n_0}$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |a_n| \leq |b_n| \\ |b_n| < \varepsilon \end{cases} \Rightarrow \underline{|a_n| < \varepsilon}$$

We have thus shown that:

$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$

$\Rightarrow \lim a_n = 0$ □

③ $$\boxed{\begin{cases} \lim a_n = 0 \\ (b_n) \text{ bounded} \end{cases} \Rightarrow \lim (a_n b_n) = 0}$$

<u>Proof</u>

We have:

(b_n) bounded $\Rightarrow \exists p \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |b_n| \leq p)$

Choose $p \in (0,+\infty)$ and $n_1 \in D$ such that

$\forall n \in D : (n > n_1 \Rightarrow |b_n| \leq p)$

We also have:

$\lim a_n = 0 \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$

Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Since $\varepsilon/p > 0$, choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow |a_n| < \varepsilon/p)$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $\underline{n \in D}$ be given and assume that $\underline{n > n_0}$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |b_n| \leq p \\ |a_n| < \varepsilon/p \end{cases} \Rightarrow$$

$$\Rightarrow |a_n b_n| = |a_n||b_n| \leq |a_n| p < (\varepsilon/p)p = \varepsilon$$

$$\Rightarrow \underline{|a_n b_n| < \varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n b_n| < \varepsilon)$$

$$\Rightarrow \lim(a_n b_n) = 0 \qquad \square$$

▶ Immediate consequence of (3) is the statement

$$\boxed{\lim a_n = 0 \Rightarrow \forall \lambda \in \mathbb{R} : \lim(\lambda a_n) = 0}$$

(4) $\boxed{\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \begin{cases} \lim(a_n + b_n) = 0 \\ \lim(a_n b_n) = 0 \end{cases}}$

<u>Proof</u>

a) We have:

$$\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \begin{cases} \lim a_n = 0 \\ (b_n) \text{ bounded} \end{cases} \Rightarrow \lim(a_n b_n) = 0$$

b) We have:

$$\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \begin{cases} \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon) \\ \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |b_n| < \varepsilon) \end{cases}$$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $n_1 \in D$ and $n_2 \in D$ such that

$$\begin{cases} \forall n \in D : (n > n_1 \Rightarrow |a_n| < \varepsilon/2) \\ \forall n \in D : (n > n_2 \Rightarrow |b_n| < \varepsilon/2) \end{cases}$$

Choose $n_0 \in D$ such that $n_0 \geq n_1$ and $n_0 \geq n_2$. Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |a_n| < \varepsilon/2 \\ |b_n| < \varepsilon/2 \end{cases} \Rightarrow$$

$$\Rightarrow |a_n + b_n| \leq |a_n| + |b_n| < \varepsilon/2 + |b_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\Rightarrow |a_n + b_n| < \varepsilon$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow |a_n + b_n| < \varepsilon)$$

$$\Rightarrow \lim (a_n + b_n) = 0$$

$\square$

$$\boxed{5 \; \begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n \geq n_0 \Rightarrow a_n \geq 0) \end{cases} \Rightarrow \forall k \in \mathbb{N}^* : \lim \sqrt[k]{a_n} = 0}$$

Proof: Homework

EXAMPLE

a) Use the limit definition to show that

$$\lim_{n\in\mathbb{N}^*} \frac{\sin(n)+\cos(n)}{n^2+1} = 0$$

Solution

Define $\forall n\in\mathbb{N}^*: a_n = \dfrac{\sin(n)+\cos(n)}{n^2+1}$

Let $\varepsilon\in(0,+\infty)$ be given. We have:

$$|a_n| = \left|\frac{\sin(n)+\cos(n)}{n^2+1}\right| = \frac{|\sin(n)+\cos(n)|}{n^2+1} \leqslant$$

$$\leqslant \frac{|\sin(n)|+|\cos(n)|}{n^2+1} \leqslant \frac{1+1}{n^2+1} = \frac{2}{n^2+1}$$

$$\leqslant \frac{2}{n^2} < \frac{1}{\varepsilon} \Leftrightarrow n^2 > \varepsilon \Leftrightarrow n > \sqrt{\varepsilon}. \quad (1)$$

Choose $n_0\in\mathbb{N}^*$ such that $n_0 > \sqrt{\varepsilon}$, via the Archimedes theorem. Let $n\in\mathbb{N}^*-[n_0]$ be given. Then, we have:

$n > n_0 \Rightarrow n > \sqrt{\varepsilon} \Rightarrow |a_n| < \varepsilon$ [via Eq. (1)]

We have thus shown that

$\forall\varepsilon\in(0,+\infty): \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: |a_n| < \varepsilon$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = 0.$$

$\square$

THEORY QUESTIONS

(6) Prove the following properties, with (a_n), (b_n) nets on $(D,<)$.

a) $\lim a_n = 0 \Rightarrow (a_n)$ bounded

b) $\begin{cases} \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| \leq |b_n|) \\ \lim b_n = 0 \end{cases} \Rightarrow \lim a_n = 0$

c) $\begin{cases} \lim a_n = 0 \\ (b_n) \text{ bounded} \end{cases} \Rightarrow \lim (a_n b_n) = 0$

d) $\begin{cases} \lim a_n = 0 \\ \lim b_n = 0 \end{cases} \Rightarrow \lim (a_n + b_n) = 0$

e) $\begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \geq 0) \end{cases} \Rightarrow \forall k \in \mathbb{N}^* : \lim \sqrt[k]{a_n} = 0$

EXERCISES

(7) Use the limit definition to show that $\lim a_n = 0$ for the following sequences:

a) $a_n = \dfrac{(-1)^n}{(n+1)^2}$ b) $a_n = \dfrac{1+\sqrt{n}}{n^3}$

c) $a_n = \dfrac{5}{3n^2-1}$ d) $a_n = \dfrac{\sin(n) - \cos(n)}{n+4}$

e) $a_n = \dfrac{n-1}{n^2+1}$ f) $a_n = \dfrac{(-1)^n}{3^n}$

g) $a_n = \dfrac{n^2+5n-1}{n^3+n+3}$ h) $a_n = \dfrac{\sin(2n) + 4\cos(3n)}{n+3}$

(8) Let $(a_n), (b_n)$ be sequences such that

$$\begin{cases} \forall n \in \mathbb{N}^* : (a_n > 0 \wedge b_n > 0) \\ \lim\limits_{n\in\mathbb{N}^*} a_n = 0 \wedge \lim\limits_{n\in\mathbb{N}^*} b_n = 0 \end{cases}$$

Show that $\lim\limits_{n\in\mathbb{N}^*} \dfrac{a_n^2 + b_n^2}{a_n + b_n} = 0$

using the limit properties.
(Hint: Use the Cauchy identity $x^2 + y^2 = (x+y)^2 - 2xy$.)

(9) Given the sequence

$$\forall n \in \mathbb{N}^* : a_n = \frac{5n}{6n+7}$$

show via the limit definition that $\lim\limits_{n\in\mathbb{N}^*} a_n \neq 0$.

(10) Let $(a_n), (b_n)$ be sequences such that

$$\begin{cases} \forall n \in \mathbb{N}^* : (a_n > 0 \wedge b_n > 0) \\ \lim\limits_{n\in\mathbb{N}^*} a_n = 0 \wedge \lim\limits_{n\in\mathbb{N}^*} b_n = 0 \end{cases}$$

Show that $\lim\limits_{n\in\mathbb{N}^*} \dfrac{a_n^3 + b_n^3}{a_n + b_n} = 0$

using the limit properties.
(Hint: Use the Cauchy identity $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$)

Basic zero sequences

The limits of the following sequences can be used as theorems for other exercises.

(1) $\boxed{\forall p \in (0,+\infty) : \lim_{n\in\mathbb{N}^*} \frac{1}{n^p} = 0}$

Proof

Define $\forall n\in\mathbb{N}^*: a_n = 1/n^p$. Let $\varepsilon \in (0,+\infty)$ be given.
We note that

$$|a_n| < \varepsilon \Leftrightarrow |1/n^p| < \varepsilon \Leftrightarrow 1/n^p < \varepsilon \Leftrightarrow n^p > 1/\varepsilon \Leftrightarrow$$
$$\Leftrightarrow n > (1/\varepsilon)^{1/p}$$

Choose $n_0 \in \mathbb{N}^*$ such that $n_0 > (1/\varepsilon)^{1/p}$, via the Archimedes theorem. Let $n \in \mathbb{N}^* - [n_0]$ be given. Then, we have:

$$n > n_0 \Rightarrow n > (1/\varepsilon)^{1/p} \Rightarrow |a_n| < \varepsilon$$

We have thus shown that

$$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n| < \varepsilon$$
$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \frac{1}{n^p} = 0 \qquad \square$$

(2) $\boxed{\begin{cases} \forall n\in\mathbb{N}^* : a_n = p^n \\ |p| < 1 \end{cases} \Rightarrow \lim_{n\in\mathbb{N}^*} a_n = 0}$

Proof

We distinguish between the following cases.

Case 1 : Assume that $p = 0$. Then; we have:

$$(\forall n \in \mathbb{N}^+ : a_n = 0^n = 0) \Rightarrow \lim_{n\in\mathbb{N}^+} a_n = 0.$$

<u>Case 2</u>: Assume that $p \neq 0$ and $|p| < 1$. Then $1/|p| > 1$, and we can choose $b \in (0, +\infty)$ such that $1/|p| = 1 + b$. Let $n \in \mathbb{N}^*$ be given. Then, we have:

$1/|p|^n = (1+b)^n \geq 1 + nb > nb > 0 \Rightarrow 1/|p|^n > nb > 0 \Rightarrow$

$\Rightarrow |a_n| = |p^n| = |p|^n < 1/(nb) = |1/(nb)| \Rightarrow |a_n| < |1/(nb)|$

and conclude that

$$\forall n \in \mathbb{N}^* : |a_n| < |1/(nb)|. \qquad (1)$$

We also have:

$$\lim_{n \in \mathbb{N}^*} \frac{1}{n} = 0 \Rightarrow \lim_{n \in \mathbb{N}^*} \frac{1}{nb} = 0 \qquad (2)$$

From Eq. (1) and Eq. (2): $\lim_{n \in \mathbb{N}^*} a_n = 0$ $\square$

THEORY QUESTIONS

(11) Show that

a) $\forall p \in (0,+\infty): \lim\limits_{n\in\mathbb{N}^*} \dfrac{1}{n^p} = 0$

b) $\begin{cases} \forall n\in\mathbb{N}^*: a_n = p^n \\ |p|<1 \end{cases} \Rightarrow \lim\limits_{n\in\mathbb{N}^*} a_n = 0$

EXERCISES

(12) Use the limit properties to show that $\lim\limits_{n\in\mathbb{N}^*} a_n = 0$ for the following sequences:

a) $a_n = \dfrac{4n^2+3}{n^3}$

b) $a_n = \dfrac{5+\cos(n)}{3n^4}$

c) $a_n = \dfrac{n}{(-3)^n(n^2+2)}$

d) $a_n = \dfrac{n!}{n^n}$

e) $a_n = \dfrac{1^2+2^2+\cdots+n^2}{n^4+5n+2}$

f) $a_n = \dfrac{1^3+2^3+\cdots+n^3}{3n^5+2n}$

g) $a_n = \sum\limits_{a=1}^{n} \dfrac{\sin(a)}{a^2+1}$

Convergent nets and sequences

Let (a_n) be a net on $(D,<)$ and recall the definitions

$$\boxed{\begin{array}{l}\lim a_n = \ell \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n - \ell| < \varepsilon) \\ (a_n) \text{ convergent} \Leftrightarrow \exists \ell \in \mathbb{R} : \lim a_n = \ell\end{array}}$$

When (a_n) is a sequence, the limit definition simplifies to

$$\boxed{\lim_{n\in\mathbb{N}^*} a_n = \ell \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon}$$

Uniqueness of convergent limit

Thm: Let $(a_n), (b_n)$ be nets on $(D,<)$ and let $\ell_1, \ell_2 \in \mathbb{R}$. Then, we have

$$\begin{cases} \lim a_n = \ell_1 \\ \lim b_n = \ell_2 \end{cases} \Rightarrow \ell_1 = \ell_2$$

Proof

To show a contradiction, assume that $\ell_1 \neq \ell_2$. Since,

$$\begin{cases} \lim a_n = \ell_1 \\ \lim b_n = \ell_2 \end{cases} \Rightarrow \begin{cases} \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n - \ell_1| < \varepsilon) \\ \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n - \ell_2| < \varepsilon) \end{cases}$$

For $\varepsilon = |\ell_1 - \ell_2|/2$, choose $n_1, n_2 \in D$ such that

$$\begin{cases} \forall n \in D : (n > n_1 \Rightarrow |a_n - \ell_1| < |\ell_1 - \ell_2|/2) \\ \forall n \in D : (n > n_2 \Rightarrow |a_n - \ell_2| < |\ell_1 - \ell_2|/2) \end{cases}$$

Choose $n \in D$ such that $n > n_1$ and $n > n_2$. It follows that
$$\begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} |a_n - l_1| < |l_1 - l_2|/2 \\ |a_n - l_2| < |l_1 - l_2|/2 \end{cases} \Rightarrow$$
$$\Rightarrow |l_1 - l_2| = |(a_n - l_2) - (a_n - l_1)| \leq |a_n - l_2| + |a_n - l_1|$$
$$< |l_1 - l_2|/2 + |l_1 - l_2|/2 = |l_1 - l_2| \Rightarrow$$
$$\Rightarrow |l_1 - l_2| < |l_1 - l_2|$$
which is a contradiction. We conclude that $l_1 = l_2$ □

General properties

Let (a_n) be a net on $(D, <)$. We have the following general properties:

① $\boxed{(a_n) \text{ convergent} \Rightarrow (a_n) \text{ bounded}}$

Proof

Choose $l \in \mathbb{R}$ such that $\lim a_n = l$. Then, we have:
$$\lim a_n = l \Rightarrow \lim (a_n - l) = 0 \Rightarrow (a_n - l) \text{ bounded}$$
$$\Rightarrow \exists b \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n - l| \leq b)$$
Choose $b \in (0, +\infty)$ and $n_0 \in D$ such that:
$$\forall n \in D : (n > n_0 \Rightarrow |a_n - l| \leq b)$$
Let $\underline{n \in D}$ be given and assume that $\underline{n > n_0}$. Then, we have:
$$n > n_0 \Rightarrow |a_n - l| \leq b \Rightarrow -b \leq a_n - l \leq b \Rightarrow$$
$$\Rightarrow \underline{-b + l \leq a_n \leq b + l}$$
We have thus shown that:

$\forall n \in D: (n > n_0 \Rightarrow -b+\ell \leq a_n \leq b+\ell)$

$\Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow (a_n) \text{ bounded}$ □

(2) $\boxed{\lim a_n = \ell \neq 0 \Rightarrow \exists n_0 \in D: \forall n \in D: (n > n_0 \Rightarrow a_n, \ell \text{ equisigned})}$

Proof

We have:

$\lim a_n = \ell \Rightarrow \lim(a_n - \ell) = 0 \Rightarrow$

$\Rightarrow \exists n_0 \in D: \forall n \in D: (n > n_0 \Rightarrow |a_n - \ell| < |\ell|/2)$

using $\varepsilon = |\ell|/2$. Choose $n_0 \in D$ such that

$\forall n \in D: (n > n_0 \Rightarrow |a_n - \ell| < |\ell|/2)$

Let $n \in D$ be given and assume that $n > n_0$. It follows that

$n > n_0 \Rightarrow |a_n - \ell| < |\ell|/2 \Rightarrow -|\ell|/2 < a_n - \ell < |\ell|/2 \Rightarrow$

$\Rightarrow \ell - |\ell|/2 < a_n < \ell + |\ell|/2.$

We distinguish between the following cases.

Case 1: Assume that $\ell > 0$. Then, we have:

$a_n > \ell - |\ell|/2 = \ell - \ell/2 = \ell/2 > 0 \Rightarrow a_n > \ell/2 > 0 \Rightarrow$

$\Rightarrow$ a_n, ℓ equisigned.

Case 2: Assume that $\ell < 0$. Then, we have:

$a_n < \ell + |\ell|/2 = \ell - \ell/2 = \ell/2 < 0 \Rightarrow a_n < \ell/2 < 0 \Rightarrow$

$\Rightarrow$ a_n, ℓ equisigned

We have thus shown that

$\exists n_0 \in D: \forall n \in D: (n > n_0 \Rightarrow a_n, \ell \text{ equisigned})$ □

A corollary of property 2 is the following statement:

$$\boxed{\lim a_n = \ell \neq 0 \Rightarrow \exists n_0 \in D : \forall n \in D : \left(n > n_0 \Rightarrow \frac{|\ell|}{2} < |a_n| < \frac{3|\ell|}{2}\right)}$$

③ $$\boxed{\begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} \Rightarrow \lim a_n \geq 0}$$

Proof

To show a contradiction, assume that $\lim a_n < 0$. Then, via property 2 we have:

$\lim a_n < 0 \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n < 0)$

Choose $n_1 \in D$ such that

$\forall n \in D : (n > n_1 \Rightarrow a_n < 0)$

From the hypothesis, choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow a_n > 0)$

Choose $n \in D$ such that $n > n_1$ and $n > n_2$. Then, we have:

$$\begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n < 0 \\ a_n > 0 \end{cases}$$

which is a contradiction. We conclude that $\lim a_n \geq 0$. □

EXAMPLE

Use the limit definition to show that $\lim_{n \in \mathbb{N}^*} \dfrac{n^2+3n-1}{2n^2+n+1} = \dfrac{1}{2}$

Solution

Define $\forall n \in \mathbb{N}^* : a_n = \dfrac{n^2+3n-1}{2n^2+n+1}$. Let $\varepsilon \in (0,+\infty)$ be given.

Then, we have:

$$|a_n - 1/2| = \left| \frac{n^2+3n-1}{2n^2+n+1} - \frac{1}{2} \right| = \left| \frac{2(n^2+3n-1)-(2n^2+n+1)}{2(2n^2+n+1)} \right|$$

$$= \frac{|2n^2+6n-2-2n^2-n-1|}{2(2n^2+n+1)} = \frac{|(2-2)n^2+(6-1)n+(-2-1)|}{2(2n^2+n+1)}$$

$$= \frac{|5n-3|}{2(2n^2+n+1)} \leq \frac{|5n|+|3|}{2(2n^2+n+1)} = \frac{5n+3}{2(2n^2+n+1)}$$

$$< \frac{5n+3}{4n^2} \leq \frac{5n+3n}{4n^2} = \frac{8n}{4n^2} = \frac{2}{n} < \varepsilon \iff$$

$$\iff n/2 > 1/\varepsilon \iff n > 2/\varepsilon$$

Via the Archimedes theorem, choose $n_0 \in \mathbb{N}^*$ such that $n_0 > 2/\varepsilon$. Let $n \in \mathbb{N}^* - [n_0]$ be given. Then, we have:

$n > n_0 \Rightarrow n > 2/\varepsilon \Rightarrow |a_n - 1/2| < \varepsilon$

We have thus shown that

$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - 1/2| < \varepsilon$

$\Rightarrow \lim_{n \in \mathbb{N}^+} a_n = 1/2$ $\square$

THEORY QUESTIONS

(13) Show that if $(a_n), (b_n)$ are nets on $(D, <)$:

a) $\begin{cases} \lim a_n = l_1 \\ \lim a_n = l_2 \end{cases} \Rightarrow l_1 = l_2$

b) (a_n) convergent $\Rightarrow$ (a_n) bounded

c) $\lim a_n = l \neq 0 \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n, l$ equisigned$)$

d) $\begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} \Rightarrow \lim a_n \geq 0$

EXERCISES

(14) Use the limit definition to show that

a) $\lim\limits_{n \in \mathbb{N}^*} \dfrac{2n}{3n+1} = \dfrac{2}{3}$ b) $\lim\limits_{n \in \mathbb{N}^*} \dfrac{n^2 - n}{(n+1)^2} = 1$

c) $\lim\limits_{n \in \mathbb{N}^*} \dfrac{2n^3}{n^3 - 1} = 2$ d) $\lim\limits_{n \in \mathbb{N}^*} \left(3 + \dfrac{1}{n}\right)^2 = 9$

e) $\lim\limits_{n \in \mathbb{N}^*} \left[\left(\dfrac{3}{4}\right)^n - 1\right] = -1$ f) $\lim\limits_{n \in \mathbb{N}^*} \left(1 + \dfrac{1}{n}\right)^5 = 1$

Limits and operations

① $\boxed{(a_n),(b_n) \text{ convergent} \Rightarrow \begin{cases} \lim(a_n+b_n)=\lim a_n+\lim b_n \\ \lim(a_nb_n)=\lim a_n \lim b_n \end{cases}}$

Proof

Since $(a_n),(b_n)$ convergent, we define $a=\lim a_n$ and $b=\lim b_n$.

a) It follows that

$$\begin{cases} \lim(a_n-a)=0 \\ \lim(b_n-b)=0 \end{cases} \Rightarrow$$

$$\Rightarrow \lim[(a_n+b_n)-(a+b)]=\lim[(a_n-a)+(b_n-b)]=0$$

$$\Rightarrow \lim(a_n+b_n)=a+b=\lim a_n+\lim b_n.$$

b) Since

$$\forall n\in\mathbb{N}^*: a_nb_n-ab=a_nb_n-ab_n+ab_n-ab=$$
$$=(a_n-a)b_n+a(b_n-b) \qquad (1)$$

and

$$\lim(b_n-b)=0 \Rightarrow \lim[a(b_n-b)]=0 \qquad (2)$$

and

$$\begin{cases} \lim(a_n-a)=0 \\ b_n \text{ convergent} \end{cases} \Rightarrow \begin{cases} \lim(a_n-a)=0 \\ b_n \text{ bounded} \end{cases} \Rightarrow$$

$$\Rightarrow \lim[(a_n-a)b_n]=0 \qquad (3)$$

it follows from Eq.(1), Eq.(2), Eq.(3) that

$$\lim(a_nb_n-ab)=0 \Rightarrow \lim(a_nb_n)=ab=\lim a_n \lim b_n \quad \square$$

$$\boxed{② \begin{cases} (a_n) \text{ convergent} \\ \lim a_n \neq 0 \end{cases} \Rightarrow \lim \frac{1}{a_n} = \frac{1}{\lim a_n}}$$

Proof

Define $a = \lim a_n$. We note that:

$\lim a_n \neq 0 \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a|/2 < |a_n| < 3|a|/2)$

Let $n \in D$ be given and assume that $n > n_0$. Note that since $a \neq 0$, we have:

$|a_n| > |a|/2 > 0 \Rightarrow 1/|a_n| < 2/|a|$

and therefore

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a - a_n}{a_n a}\right| = \frac{|a - a_n|}{|a_n||a|} < \frac{2|a - a_n|}{|a|^2}$$

We have thus shown that

$$\forall n \in D : \left(n > n_0 \Rightarrow \left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2|a - a_n|}{|a|^2}\right) \quad (1)$$

We also have:

$$\lim a_n = a \Rightarrow \lim (a_n - a) = 0 \Rightarrow \lim |a_n - a| = 0 \Rightarrow$$

$$\Rightarrow \lim \frac{2|a - a_n|}{|a|^2} = 0 \quad (2)$$

From Eq. (1) and Eq. (2):

$$\lim \left(\frac{1}{a_n} - \frac{1}{a}\right) = 0 \Rightarrow \lim \frac{1}{a_n} = \frac{1}{a} = \frac{1}{\lim a_n} \quad \square$$

→ The following result is an immediate consequence of the limit operation properties

Prop: Let $(a_n), (b_n)$ be nets on $(D, <)$. Then:

$$\begin{cases} (a_n), (b_n) \text{ converges} \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n) \end{cases} \Rightarrow \lim a_n \leq \lim b_n$$

Proof

Choose $n_0 \in D$ such that

$\forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n)$

Then, we have:

$$\begin{cases} (a_n), (b_n) \text{ convergent} \\ \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n) \end{cases} \Rightarrow \begin{cases} (a_n - b_n) \text{ convergent} \\ \forall n \in D : (n > n_0 \Rightarrow a_n - b_n \leq 0) \end{cases}$$

$$\Rightarrow \lim (a_n - b_n) \leq 0 \Rightarrow \lim a_n - \lim b_n \leq 0$$

$$\Rightarrow \lim a_n \leq \lim b_n \qquad \square$$

EXAMPLES

Use the limit properties to evaluate the following limits:

a) $a_n = \dfrac{n^2+3n-1}{3n^2+5n+2} \leftarrow \lim\limits_{n\in\mathbb{N}^*} a_n$

Solution

$$a_n = \frac{n^2+3n-1}{3n^2+5n+2} = \frac{n^2(1+3n^{-1}-n^{-2})}{n^2(3+5n^{-1}+2n^{-2})} =$$

$$= \frac{1+3n^{-1}-n^{-2}}{3+5n^{-1}+2n^{-2}}, \quad \forall n\in\mathbb{N}^* \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \frac{1+3n^{-1}-n^{-2}}{3+5n^{-1}+2n^{-2}} =$$

$$= \frac{1+3\cdot 0-0}{3+5\cdot 0+2\cdot 0} = \frac{1}{3}$$

b) $a_n = \dfrac{2^{n+1}+3^{2n}}{9^n+5^{n+1}}, \quad \forall n\in\mathbb{N}^*$

Solution

$$a_n = \frac{2^{n+1}+3^{2n}}{9^n+5^{n+1}} = \frac{2\cdot 2^n+9^n}{9^n+5\cdot 5^n} = \frac{9^n[2(2/9)^n+1]}{9^n[1+5(5/9)^n]}$$

$$= \frac{2(2/9)^n+1}{1+5(5/9)^n}, \quad \forall n\in\mathbb{N}^* \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} \frac{2(2/9)^n+1}{1+5(5/9)^n} = \frac{2\cdot 0+1}{1+5\cdot 0} =$$

$$= \frac{0+1}{1+0} = 1.$$

THEORY QUESTIONS

(15) Let $(a_n), (b_n)$ be nets on $(D, <)$. Show that

a) $(a_n), (b_n)$ convergent $\Rightarrow \lim(a_n + b_n) = \lim a_n + \lim b_n$

b) $(a_n), (b_n)$ convergent $\Rightarrow \lim(a_n b_n) = \lim a_n \lim b_n$

c) $\begin{cases} (a_n) \text{ convergent} \\ \lim a_n \neq 0 \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = \dfrac{1}{\lim a_n}$

EXERCISES

(16) Use the limit properties to evaluate the limit $\lim\limits_{n \in \mathbb{N}^*} a_n$ for the following sequences

a) $a_n = \left(1 + \dfrac{4}{n^2} - \dfrac{5}{n^3}\right)^9$

b) $a_n = \dfrac{2n^3 + 4n^2 - 2}{3n^3 + 6n - 5}$

c) $a_n = \dfrac{n^2 + 2n + 3}{3n^3 + n^2 - 1}$

d) $a_n = \dfrac{2^n + 5^n}{4^n + 7^n}$

e) $a_n = \dfrac{5 + 3^n + 5^{n+1}}{7 + 2^n + 5^{n+4}}$

f) $a_n = \dfrac{2 \cdot 5^n - 3^{2n}}{6 + 4^{2n+1}}$

▼ Squeeze theorem and n-root limits

Thm : (Squeeze theorem)
Let $(a_n), (b_n), (c_n)$ be nets on $(D, <)$. Then, we have:

$$\begin{cases} \exists n_0 \in D : \forall n \in D : (n \succ n_0 \Rightarrow a_n \leq b_n \leq c_n) \\ \lim a_n = \lim c_n = \ell \end{cases} \Rightarrow \lim b_n = \ell$$

Proof

Choose $n_0 \in D$ such that: $\forall n \in D : (n \succ n_0 \Rightarrow a_n \leq b_n \leq c_n)$

Let $n \in D$ be given and assume that $n \succ n_0$. Then, we have:

$$a_n \leq b_n \leq c_n \Rightarrow 0 \leq b_n - a_n \leq c_n - a_n \Rightarrow$$
$$\Rightarrow |b_n - a_n| \leq c_n - a_n \leq |c_n - a_n|$$

We have thus shown that

$$\forall n \in D : (n \succ n_0 \Rightarrow |b_n - a_n| \leq |c_n - a_n| \qquad (1)$$

We also have:

$$\lim (c_n - a_n) = \lim c_n - \lim a_n = \ell - \ell = 0 \qquad (2)$$

From Eq.(1) and Eq.(2), it follows that

$$\lim (b_n - a_n) = 0 \Rightarrow$$
$$\Rightarrow \lim b_n = \lim [(b_n - a_n) + a_n] = \lim (b_n - a_n) + \lim a_n$$
$$= 0 + \ell = \ell$$

□

• → Results on n-root limits

① $\boxed{\lim_{n \in \mathbb{N}^*} \sqrt[n]{n} = 1}$

Proof

Since $(\forall n\in\mathbb{N}^*: n\geq 1) \Rightarrow (\forall n\in\mathbb{N}^*: \sqrt[n]{n}\geq 1)$, we define a sequence (p_n) with $\forall n\in\mathbb{N}^*: p_n\geq 0$ such that
$\forall n\in\mathbb{N}^*: \sqrt[n]{n} = (1+p_n)^2$

Let $n\in\mathbb{N}^*$ be given. Then, we have:

$$\sqrt[n]{n} = (1+p_n)^2 \Rightarrow n = (1+p_n)^{2n} \Rightarrow$$
$$\Rightarrow \sqrt{n} = (1+p_n)^n \geq 1+np_n > np_n \Rightarrow \sqrt{n} > np_n$$
$$\Rightarrow |p_n| = p_n < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = \left|\frac{1}{\sqrt{n}}\right| \Rightarrow$$
$$\Rightarrow \underline{|p_n| < |1/\sqrt{n}|.}$$

We have thus shown that

$$\begin{cases} \forall n\in\mathbb{N}^*: |p_n| < |1/\sqrt{n}| \\ \lim_{n\in\mathbb{N}^*} (1/\sqrt{n}) = 0 \end{cases} \Rightarrow \lim_{n\in\mathbb{N}^*} p_n = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} (\sqrt[n]{n} - 1) = \lim_{n\in\mathbb{N}^*} [(1+p_n)^2 - 1] = \lim_{n\in\mathbb{N}^*} (1+2p_n+p_n^2 - 1)$$
$$= \lim_{n\in\mathbb{N}^*} (2p_n + p_n^2) = 2\lim_{n\in\mathbb{N}^*} p_n + (\lim_{n\in\mathbb{N}^*} p_n)^2$$
$$= 2\cdot 0 + 0 = 0 \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{n} = 1. \qquad \square$$

2 $\boxed{\forall a\in(0,+\infty): \lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = 1}$

<u>Proof</u>

Let $a\in(0,+\infty)$ be given. We distinguish between the following cases.

<u>Case 1:</u> Assume that $a=1$. Then, we have:

$$\lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = \lim_{n\in\mathbb{N}^*} \sqrt[n]{1} = \lim_{n\in\mathbb{N}^*} 1 = 1$$

Case 2: Assume that $a>1$. Then, we have

$\forall n\in\mathbb{N}^* : \sqrt[n]{a} > 1$

and we can therefore define a sequence (p_n) such that

$\forall n\in\mathbb{N}^* : (\sqrt[n]{a} = 1+p_n \land p_n > 0)$

Let $n\in\mathbb{N}^*$ be given. Then, we have:

$$\sqrt[n]{a} = 1+p_n \Rightarrow a = (1+p_n)^n \geq 1+np_n > np_n \Rightarrow$$
$$\Rightarrow |p_n| = p_n < a/n = |a/n| \Rightarrow \underline{|p_n| < |a/n|}$$

We have thus shown that

$$\begin{cases} \forall n\in\mathbb{N}^* : |p_n| < |a/n| \\ \lim_{n\in\mathbb{N}^*} (a/n) = 0 \end{cases} \Rightarrow \lim_{n\in\mathbb{N}^*} p_n = 0 \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = \lim_{n\in\mathbb{N}^*} (1+p_n) = 1 + \lim_{n\in\mathbb{N}^*} p_n = 1+0 = 1$$

Case 3: Assume that $0<a<1$. Then, we have:

$$0<a<1 \Rightarrow 1/a > 1 \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{1/a} = 1 \quad \text{[via case 2]}$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a} = \lim_{n\in\mathbb{N}^*} \frac{1}{\sqrt[n]{1/a}} = \frac{1}{\lim_{n\in\mathbb{N}^*} \sqrt[n]{1/a}}$$

$$= \frac{1}{1} = 1 \qquad \square$$

③ $\boxed{\lim_{n\in\mathbb{N}^*} a_n = a \in (0,+\infty) \Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a_n} = 1}$

<u>Proof</u>

Since

$$\lim_{n\in\mathbb{N}^*} a_n = a > 0 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n, a \text{ equisigned}$$
$$\Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n > 0 \qquad (1)$$

and

$$\lim_{n\in\mathbb{N}^*} a_n = a > 0 \Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : \frac{|a|}{2} < |a_n| < \frac{3|a|}{2} \qquad (2)$$

Via Eq. (1), choose $n_1 \in \mathbb{N}^*$ such that
$\forall n \in \mathbb{N}^* - [n_1] : a_n > 0$

Via Eq. (2) choose $n_2 \in \mathbb{N}^*$ such that
$\forall n \in \mathbb{N}^* - [n_2] : |a|/2 < |a_n| < 3|a|/2$

Define $n_0 = \max\{n_1, n_2\}$. Let $\underline{n \in \mathbb{N}^* - [n_0]}$ be given.
Then, we have:

$$\begin{cases} |a|/2 < |a_n| < 3|a|/2 \\ a_n > 0 \end{cases} \Rightarrow |a|/2 < a_n < 3|a|/2 \Rightarrow$$
$$\Rightarrow \underline{\sqrt[n]{|a|/2} < \sqrt[n]{a_n} < \sqrt[n]{3|a|/2}}$$

We have thus shown that

$$\begin{cases} \forall n \in \mathbb{N}^* - [n_0] : \sqrt[n]{|a|/2} < \sqrt[n]{a_n} < \sqrt[n]{3|a|/2} \\ \lim_{n\in\mathbb{N}^*} \sqrt[n]{|a|/2} = \lim_{n\in\mathbb{N}^*} \sqrt[n]{3|a|/2} = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \lim_{n\in\mathbb{N}^*} \sqrt[n]{a_n} = 1 \qquad \square$$

THEORY QUESTIONS

(17) Let $(a_n), (b_n), (c_n)$ be nets on $(D, <)$. Show that

$$\begin{cases} \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n \leq c_n) \\ \lim a_n = \lim c_n = \ell \end{cases} \Rightarrow \lim b_n = \ell$$

(18) Show that:

a) $\lim\limits_{n \in \mathbb{N}} \sqrt[n]{n} = 1$ b) $\forall a \in (0, +\infty) : \lim\limits_{n \in \mathbb{N}^*} \sqrt[n]{a} = 1$

c) $\lim\limits_{n \in \mathbb{N}^*} a_n = a > 0 \Rightarrow \lim\limits_{n \in \mathbb{N}^*} \sqrt[n]{a_n} = 1$

EXERCISES

(19) Use limit properties to evaluate the limit of the following sequences

a) $a_n = \sqrt[n]{n^2+1}$ b) $a_n = \sqrt[n]{2n^3 - n + 5}$

c) $a_n = \sqrt[n]{2^n + 3^n + 5^n}$ d) $a_n = \sqrt[n]{3 + 1/n}$

e) $a_n = \sqrt[n]{\dfrac{7n+1}{3n+2}}$ f) $a_n = \sqrt[n]{\dfrac{5^{n+1}+2}{5^n + 4^n}}$

(20) Use the squeeze theorem and limit properties to evaluate the limit of the following sequences

a) $a_n = \dfrac{n^3}{n^4+1} + \dfrac{n^3}{n^4+2} + \dots + \dfrac{n^3}{n^4+n}$

b) $a_n = \dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \dots + \dfrac{n}{n^2+n}$

c) $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}$

d) $a_n = \frac{\sin(1)}{n^2+1} + \frac{\sin(2)}{n^2+2} + \dots + \frac{\sin(n)}{n^2+n}$

e) $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$

Convergent recursive sequences

Thm: Let (a_n) be a net on $(D,<)$. Then, we have:

$$\begin{cases} (a_n) \text{ increasing} \\ (a_n) \text{ upper bounded} \end{cases} \Rightarrow \begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D : \lim a_n = \sup\{a_n \mid n \in D \wedge n > n_0\} \end{cases}$$

Proof

Since,

(a_n) upper bounded $\Rightarrow \exists n_0 \in D : \exists b \in \mathbb{R} : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b)$

choose $n_0 \in D$ and $b \in \mathbb{R}$ such that

$\forall n \in D : (n > n_0 : a_n \leq b)$

It follows that the set $A = \{a_n \mid n \in D \wedge n > n_0\}$ is upper bounded, so by the axiom of completeness we can define $x = \sup A$. We will now show that $\lim a_n = x$.

Let $\varepsilon \in (0, +\infty)$ be given. From the approximation theorem choose $n_1 \in D$ such that $n_1 > n_0$ and $x - \varepsilon < a_{n_1} \leq x$.

Let $n \in D$ be given and assume that $n > n_1$. It follows that

$$n > n_1 \Rightarrow \begin{cases} x - \varepsilon < a_{n_1} \leq x \\ a_{n_1} \leq a_n \leq \sup A = x \end{cases} \Rightarrow$$

$$\Rightarrow x - \varepsilon < a_{n_1} \leq a_n \leq x < x + \varepsilon$$

$$\Rightarrow x - \varepsilon < a_n < x + \varepsilon \Rightarrow$$

$$\Rightarrow -\varepsilon < a_n - x < \varepsilon \Rightarrow |a_n - x| < \varepsilon$$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_1 \in D : \forall n \in D : (n > n_1 \Rightarrow |a_n - x| < \varepsilon)$

$\Rightarrow \lim a_n = x \Rightarrow (a_n)$ convergent $\square$

Similarly, we can show that

Thm: Let (a_n) be a net on $(D,<)$. Then, we have:

$$\begin{cases} (a_n) \text{ decreasing} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow \begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D: \lim a_n = \inf\{a_n \mid n \in D \wedge n > n_0\} \end{cases}$$

Note that when (a_n) is a sequence, we define:

$$\inf a_n = \inf\{a_n \mid n \in \mathbb{N}^*\}$$
$$\sup a_n = \sup\{a_n \mid n \in \mathbb{N}^*\}$$

and the previous theorems simplify to the following statements:

$$\begin{cases} (a_n) \text{ increasing} \\ (a_n) \text{ upper bounded} \end{cases} \Rightarrow \begin{cases} (a_n) \text{ convergent} \\ \lim\limits_{n \in \mathbb{N}^*} a_n = \sup a_n \end{cases}$$

$$\begin{cases} (a_n) \text{ decreasing} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow \begin{cases} (a_n) \text{ convergent} \\ \lim\limits_{n \in \mathbb{N}^*} a_n = \inf a_n \end{cases}$$

EXAMPLE

Evaluate the limit of the sequence (a_n) defined recursively by:

$$\begin{cases} a_1 = 5 \\ \forall n \in \mathbb{N}^* : a_{n+1} = \dfrac{2(a_n - 12)}{a_n - 8} \end{cases}$$

Solution

We note that

$$\forall n \in \mathbb{N}^* : a_{n+1} - a_n = \frac{2(a_n - 12)}{a_n - 8} - a_n = \frac{2(a_n - 12) - a_n(a_n - 8)}{a_n - 8}$$

$$= \frac{2a_n - 24 - a_n^2 + 8a_n}{a_n - 8} = \frac{-a_n^2 + 10a_n - 24}{a_n - 8} =$$

$$= \frac{-(a_n^2 - 10a_n + 24)}{a_n - 8} = \frac{-(a_n - 6)(a_n - 4)}{a_n - 8}$$

- We need to compare (a_n) with $4, 6, 8$.
- We claim that $\forall n \in \mathbb{N}^* : a_n < 6$.

For $n = 1$, $a_1 = 5 < 6$. For $n = k$, assume that $a_k < 6$.
For $n = k+1$, we will show that $a_{k+1} < 6$. We have:

$$a_{k+1} - 6 = \frac{2(a_k - 12)}{a_k - 8} - 6 = \frac{2(a_k - 12) - 6(a_k - 8)}{a_k - 8}$$

$$= \frac{2a_k - 24 - 6a_k + 48}{a_k - 8} = \frac{-4a_k + 24}{a_k - 8} =$$

$$= \frac{-4(a_k - 6)}{a_k - 8}$$

and therefore,

$$a_k < 6 \Rightarrow \begin{cases} a_k - 6 < 0 \\ a_k - 8 < 0 \end{cases} \Rightarrow a_{k+1} - 6 < 0 \Rightarrow a_{k+1} < 6$$

We have thus shown the claim

▸ We claim that $\forall n \in \mathbb{N}^*: a_n > 4$.

For $n=1$, we have $a_1 = 5 > 4$. For $n=k$, assume that $a_k > 4$. For $n=k+1$, we will show that $a_{k+1} > 4$. We have:

$$a_{k+1} - 4 = \frac{2(a_k-12)}{a_k-8} - 4 = \frac{2(a_k-12)-4(a_k-8)}{a_k-8} =$$

$$= \frac{2a_k - 24 - 4a_k + 32}{a_k - 8} = \frac{-2a_k+8}{a_k-8} = \frac{-2(a_k-4)}{a_k-8}$$

and therefore

$$4 < a_k < 6 \Rightarrow \begin{cases} a_k - 4 > 0 \\ a_k - 8 < 0 \end{cases} \Rightarrow a_{k+1} - 4 > 0 \Rightarrow a_{k+1} > 4.$$

We have thus shown the claim.

We conclude that

$$\forall n \in \mathbb{N}^*: 4 < a_n < 6 \Rightarrow \begin{cases} \forall n \in \mathbb{N}^*: a_{n+1} - a_n < 0 \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} (a_n) \text{ decreasing} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow (a_n) \text{ convergent}$$

$$\Rightarrow \exists x \in \mathbb{R}: \lim_{n \in \mathbb{N}^*} a_n = x$$

Choose $x \in \mathbb{R}$ such that $\lim_{n \in \mathbb{N}^*} a_n = x$. Then, we have:

$$x = \lim_{n \in \mathbb{N}^*} a_{n+1} = \lim_{n \in \mathbb{N}^*} \frac{2(a_n-12)}{a_n-8} = \frac{2(x-12)}{x-8} \Leftrightarrow$$

$$\Leftrightarrow x(x-8) = 2(x-12) \Leftrightarrow x^2 - 8x = 2x - 24 \Leftrightarrow$$
$$\Leftrightarrow x^2 - 8x - 2x + 24 = 0 \Leftrightarrow x^2 - 10x + 24 = 0$$
$$\Leftrightarrow (x-6)(x-4) = 0 \Leftrightarrow x - 6 = 0 \vee x - 4 = 0$$
$$\Leftrightarrow x = 6 \vee x = 4 \Leftrightarrow x \in \{4, 6\}.$$

Since

$$\begin{cases} a_1 = 5 \\ (a_n) \text{ decreasing} \end{cases} \Rightarrow (\forall n \in \mathbb{N}^* : a_n \leq 5) \Rightarrow$$
$$\Rightarrow x = \lim_{n \in \mathbb{N}^*} a_n \leq 5 \Rightarrow x \neq 6.$$

we conclude that $\lim_{n \in \mathbb{N}^*} a_n = x = 4$ □

THEORY QUESTIONS

(21) Let (a_n) be a net on $(D,<)$. Show that

$$\begin{cases} (a_n) \text{ increasing} \\ (a_n) \text{ upper bounded} \end{cases} \Rightarrow \begin{cases} (a_n) \text{ convergent} \\ \lim a_n = \sup\{a_n \mid n \in D \land n > n_0\}, \exists n_0 \in D. \end{cases}$$

EXERCISES

(22) Let (a_n) be a net on $(D,<)$. Write the proof for the statement

$$\begin{cases} (a_n) \text{ decreasing} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow \begin{cases} (a_n) \text{ convergent} \\ \exists n_0 \in D: \lim a_n = \inf\{a_n \mid n \in D \land n > n_0\}. \end{cases}$$

(23) Show that the following sequences are convergent and evaluate their limit:

a) $\begin{cases} a_1 = 1 \\ \forall n \in \mathbb{N}^*: a_{n+1} = \sqrt{1+a_n} \end{cases}$

b) $\begin{cases} a_1 = 3 \\ a_{n+1} = (3a_n - 4)/5 \end{cases}$

c) $\begin{cases} a_1 = 2 \\ a_{n+1} = (2a_n - 3)/4 \end{cases}$

d) $\begin{cases} a_1 = 1/4 \\ a_{n+1} = (1/2)a_n^2 + (1/8) \end{cases}$

e) $\begin{cases} a_1 = 1 \\ a_{n+1} = (1/3)a_n + 2 \end{cases}$

f) $\begin{cases} a_1 = 0 \\ a_{n+1} = (3a_n + 1)/4 \end{cases}$

g) $\begin{cases} a_1 = 3 \\ a_{n+1} = (a_n^2 + 4)/5 \end{cases}$

h) $\begin{cases} a_1 = 2 \\ a_{n+1} = \sqrt{1+2a_n} - 1 \end{cases}$

i) $\begin{cases} a_1 = 2 \\ a_{n+1} = \sqrt{a_n + 6} \end{cases}$

j) $\begin{cases} a_1 = 3 \\ a_{n+1} = (1/2)(a_n + 2/a_n) \end{cases}$

▼ Nested intervals

Nested intervals of rational numbers can be used to approximate and define real numbers.

Def: Let $([a_n, b_n]): [a_1, b_1], [a_2, b_2], \dots$ be a sequence of closed intervals. We say that

$$([a_n, b_n]) \text{ nested} \Leftrightarrow \begin{cases} \forall n \in \mathbb{N}^* : [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \\ \lim\limits_{n \in \mathbb{N}^*} (a_n - b_n) = 0 \end{cases}$$

We show that every nested interval sequence $([a_n, b_n])$ has at least one common element $x \in \mathbb{R}$.

Thm: $([a_n, b_n])$ nested $\Rightarrow \exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : x \in [a_n, b_n]$

Proof

$([a_n, b_n])$ nested $\Rightarrow \forall n \in \mathbb{N}^* : [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

$\Rightarrow \forall n \in \mathbb{N}^* : a_1 \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b_1$

$$\Rightarrow \begin{cases} (a_n) \text{ increasing} \\ (a_n) \text{ upper bounded} \\ (b_n) \text{ decreasing} \\ (b_n) \text{ lower bounded} \end{cases} \Rightarrow \begin{cases} (a_n), (b_n) \text{ convergent} \\ \lim\limits_{n \in \mathbb{N}^*} a_n = \sup a_n \\ \lim\limits_{n \in \mathbb{N}^*} b_n = \inf b_n \end{cases}$$

Choose $x = \lim\limits_{n \in \mathbb{N}^*} a_n = \sup a_n$ and note that

$$\inf b_n = \lim_{n \in \mathbb{N}^*} b_n = \lim_{n \in \mathbb{N}^*} [a_n - (a_n - b_n)] =$$

$$= \lim_{n\in\mathbb{N}^*} a_n - \lim_{n\in\mathbb{N}^*} (a_n - b_n) = x - 0 = x$$

It follows that

$$\begin{cases} x = \sup a_n \\ x = \inf b_n \end{cases} \Rightarrow \begin{cases} \forall n\in\mathbb{N}^* : a_n \leq x \\ \forall n\in\mathbb{N}^* : b_n \geq x \end{cases} \to \forall n\in\mathbb{N}^* : a_n \leq x \leq b_n$$

$$\Rightarrow \forall n\in\mathbb{N}^* : x \in [a_n, b_n]$$

We have thus shown that $\exists x\in\mathbb{R} : \forall n\in\mathbb{N}^* : x\in[a_n, b_n]$. □

▶ We will now show that this element is unique:

Thm: $\begin{cases} ([a_n, b_n]) \text{ nested} \\ x_1, x_2 \in \bigcap_{n\in\mathbb{N}^*} [a_n, b_n] \end{cases} \Rightarrow x_1 = x_2$

Proof

To show a contradiction, assume that $x_1 \neq x_2$, and with no loss of generality assume that $x_1 < x_2$. Then, we have:

$([a_n, b_n])$ nested $\Rightarrow \lim_{n\in\mathbb{N}^*} (a_n - b_n) = 0 \Rightarrow$

$\Rightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - b_n| < \varepsilon$

$\Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - b_n| < |x_2 - x_1|$

Choose $n_0 \in \mathbb{N}^*$ such that

$$\forall n\in\mathbb{N}^* - [n_0] : |a_n - b_n| < |x_2 - x_1| \qquad (1)$$

Choose an $n \in \mathbb{N}^* - [n_0]$. Then:

$$x_1, x_2 \in [a_n, b_n] \Rightarrow |a_n - b_n| \geq |x_2 - x_1| \qquad (2)$$

Eq. (1) and Eq. (2) contradict. It follows that $x_1 = x_2$ □

Let $\mathrm{Seq}(A)$ be the set of all mappings $a : \mathbb{N}^* \to A$.
Thus $\mathrm{Seq}(\mathbb{Q})$ is the set of all rational sequences.

We will now show that every real number can be approximated using nested intervals with rational endpoints

Thm: $\forall x \in \mathbb{R} : \exists a, b \in \mathrm{Seq}(\mathbb{Q}) : \begin{cases} ([a_n, b_n]) \text{ nested} \\ \{x\} = \bigcap\limits_{n \in \mathbb{N}^*} [a_n, b_n] \end{cases}$

<u>Proof</u>

Let $x \in \mathbb{R}$ be given.

- Construction of $[a_1, b_1]$: From Archimedes theorem, choose $a_1, b_1 \in \mathbb{Z}$ such that $b_1 > x$ and $a_1 > -x$.
It follows that: $a_1 < x < b_1 \Rightarrow x \in [a_1, b_1]$.
- Assume that $[a_k, b_k]$ has been constructed. To construct $[a_{k+1}, b_{k+1}]$ we define:

$$a_{k+1} = \begin{cases} (1/2)(a_k + b_k), & \text{if } x > (1/2)(a_k + b_k) \\ a_k, & \text{if } x \leq (1/2)(a_k + b_k) \end{cases}$$

$$b_{k+1} = \begin{cases} b_k, & \text{if } x > (1/2)(a_k + b_k) \\ (1/2)(a_k + b_k), & \text{if } x \leq (1/2)(a_k + b_k) \end{cases}$$

By construction, we have

$$\forall n \in \mathbb{N}^* : \begin{cases} [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \\ b_{n+1} - a_{n+1} = (1/2)(b_n - a_n) \end{cases} \Rightarrow$$

$$\Rightarrow \forall n \in \mathbb{N}^* : \begin{cases} [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \Rightarrow ([a_n, b_n]) \text{ nested.} \\ \lim\limits_{n \in \mathbb{N}^*} (a_n - b_n) = 0 \end{cases}$$

We also have:

$$\left\{\begin{array}{l} x\in[a_1,b_1] \\ \forall n\in\mathbb{N}^*: (x\in[a_n,b_n]\Rightarrow x\in[a_{n+1},b_{n+1}]) \end{array}\right. \Rightarrow$$

$$\Rightarrow \forall n\in\mathbb{N}^*: x\in[a_n,b_n]$$

$$\Rightarrow \{x\} = \bigcap_{n\in\mathbb{N}^*}[a_n,b_n] \qquad \square$$

THEORY QUESTIONS

(24) Let $([a_n, b_n])$ be a sequence of intervals. State the necessary and sufficient conditions for the statement: "$([a_n, b_n])$ nested".

(25) Prove the following theorems

a) $([a_n, b_n])$ nested $\Rightarrow \exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : x \in [a_n, b_n]$

b) $\begin{cases} ([a_n, b_n]) \text{ nested} \\ x_1, x_2 \in \bigcap\limits_{n \in \mathbb{N}^*} [a_n, b_n] \end{cases} \Rightarrow x_1 = x_2$

c) $\forall x \in \mathbb{R} : \exists a, b \in \mathrm{Seq}(\mathbb{Q}) : \begin{cases} ([a_n, b_n]) \text{ nested} \\ \{x\} = \bigcap\limits_{n \in \mathbb{N}^*} [a_n, b_n] \end{cases}$

EXERCISES

(26) Let $([a_n, b_n])$ and $([c_n, d_n])$ be nested interval sequences such that

$\{x\} = \bigcap\limits_{n \in \mathbb{N}^*} [a_n, b_n]$ and $\{y\} = \bigcap\limits_{n \in \mathbb{N}^*} [c_n, d_n]$

Show that.

a) $\{x+y\} = \bigcap\limits_{n \in \mathbb{N}^*} [a_n + c_n, b_n + d_n]$

b) $\{xy\} = \bigcap\limits_{n \in \mathbb{N}^*} [a_n c_n, b_n d_n]$

(27) Let $([a_n, b_n])$ be a sequence of intervals such that $\lim_{n\in\mathbb{N}^*} (a_n - b_n) = 0$. Explain why it is not possible to show that

$\exists x \in \mathbb{R} : \forall n \in \mathbb{N}^* : x \in [a_n, b_n]$

without the additional assumption that

$\forall n \in \mathbb{N}^* : [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

↳ Hint: Construct a counterexample $([a_n, b_n])$ such that
$\bigcap_{n\in\mathbb{N}^*} [a_n, b_n] = \varnothing \land \lim_{n\in\mathbb{N}^*} (a_n - b_n) = 0$

Hint 2: Drift, drift, drift away.
gently down the stream...

Cauchy sequences

Def: Let (a_n) be a sequence. We say that
(a_n) Cauchy $\Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}|$
$: |a_{n_1} - a_{n_2}| < \varepsilon$

▸ Our main result is that:

(a_n) convergent $\Leftrightarrow$ (a_n) Cauchy.

We can therefore use the negation of the definition above to show that a sequence (a_n) is not convergent:
(a_n) not convergent $\Leftrightarrow$ (a_n) not Cauchy $\Leftrightarrow$
$\Leftrightarrow \exists \varepsilon \in (0,+\infty) : \forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geq \varepsilon$
The details are given in the following:

Properties of Cauchy sequences

① $\boxed{(a_n) \text{ Cauchy} \Rightarrow (a_n) \text{ Bounded}}$

Proof

Assume that (a_n) Cauchy. Then, we have:
(a_n) Cauchy $\Rightarrow$
$\Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < \varepsilon$
$\Rightarrow \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < 1$
Choose $n_0 \in \mathbb{N}^*$ such that

$\forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < 1$

Let $n \in \mathbb{N}^* - [n_0]$ be given. Then, we have:
$|a_n| - |a_{n_0+1}| \leq ||a_n| - |a_{n_0+1}|| \leq |a_n - a_{n_0+1}| < 1 \Rightarrow$
$\Rightarrow |a_n| < 1 + |a_{n_0+1}|$.

Choose $b = 1 + |a_{n_0+1}|$. We have thus shown that
$(\exists b \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n| \leq b) \Rightarrow$
$\Rightarrow$ (a_n) bounded $\square$

(2) $\boxed{(a_n) \text{ convergent} \Rightarrow (a_n) \text{ Cauchy.}}$

Proof

Assume that (a_n) convergent. Define $\ell = \lim_{n \in \mathbb{N}^*} a_n$.
It follows that
$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon$
Let $\varepsilon \in (0,+\infty)$ be given. Choose $n_0 \in \mathbb{N}^*$ such that
$\forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon/2$.
Let $n_1, n_2 \in \mathbb{N}^* - [n_0]$ be given. Then, we have:

$$\begin{cases} n_1 > n_0 \\ n_2 > n_0 \end{cases} \Rightarrow \begin{cases} |a_{n_1} - \ell| < \varepsilon/2 \\ |a_{n_2} - \ell| < \varepsilon/2 \end{cases} \Rightarrow$$

$$\Rightarrow |a_{n_1} - a_{n_2}| = |(a_{n_1} - \ell) - (a_{n_2} - \ell)| \leq |a_{n_1} - \ell| + |a_{n_2} - \ell|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow$$
$$\Rightarrow |a_{n_1} - a_{n_2}| < \varepsilon$$

We have thus shown that
$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| < \varepsilon$
$\Rightarrow$ (a_n) Cauchy. $\square$

(3) $\boxed{(a_n) \text{ Cauchy} \Rightarrow (a_n) \text{ convergent}}$

Proof

Assume that (a_n) Cauchy. We will construct a nested $([b_n, c_n])$ interval sequence such that

$$\begin{cases} \forall k\in\mathbb{N}^*: \exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: a_n\in[b_k,c_k] \\ \bigcap\limits_{k\in\mathbb{N}^*}[b_k,c_k]=\{\lim\limits_{n\in\mathbb{N}^*} a_n\} \end{cases}$$

▶ <u>Construction of $[b_1,c_1]$</u>: Since

(a_n) Cauchy $\Rightarrow$ (a_n) bounded $\Rightarrow$

$\Rightarrow \exists b_1, c_1\in\mathbb{R}: \forall n\in\mathbb{N}^*: a_n\in[b_1,c_1]$

Choose $b_1, c_1\in\mathbb{R}$ such that $\forall n\in\mathbb{N}^*: a_n\in[b_1,c_1]$.

We have thus constructed $[b_1,c_1]$.

▶ Assume that $[a_k, b_k]$ has been constructed such that

$\exists n_0\in\mathbb{N}^*: \forall n\in\mathbb{N}^*-[n_0]: a_n\in[b_k,c_k]$

▶ <u>Construction of $[b_{k+1}, c_{k+1}]$</u>:

Choose $p_1\in\mathbb{N}^*$ such that $\forall n\in\mathbb{N}^*-[p_1]: a_n\in[b_k,c_k]$.

Since, (a_n) Cauchy $\Rightarrow$

$\Rightarrow \exists n_0\in\mathbb{N}^*: \forall n_1,n_2\in\mathbb{N}^*-[n_0]: |a_{n_1}-a_{n_2}|<|b_k-c_k|/4$

choose $p_2\in\mathbb{N}^*$ such that

$\forall n_1,n_2\in\mathbb{N}^*-[p_2]: |a_{n_1}-a_{n_2}|<|b_k-c_k|/4$

Define $n_0=\max\{p_1,p_2\}+1$ and choose

$$\begin{cases} b_{k+1}=\max\{b_k,\ a_{n_0}-|b_k-c_k|/4\} \\ c_{k+1}=\min\{c_k,\ a_{n_0}+|b_k-c_k|/4\} \end{cases}$$

thus constructing $[b_{k+1}, c_{k+1}]$.

▶ <u>Claim</u>: $\forall n\in\mathbb{N}^*-[n_0]: a_n\in[b_{k+1},c_{k+1}]$

Let $n\in\mathbb{N}^*-[n_0]$ be given. Then, we have:

$n>n_0 \Rightarrow n>p_1 \Rightarrow a_n\in[b_k,c_k]$

and

$$n > n_0 \Rightarrow \begin{cases} n > p_2 \\ n_0 > p_2 \end{cases} \Rightarrow |a_n - a_{n_0}| < |b_k - c_k|/4 \Rightarrow$$

$$\Rightarrow -|b_k - c_k|/4 < a_n - a_{n_0} < |b_k - c_k|/4$$

$$\Rightarrow a_{n_0} - |b_k - c_k|/4 < a_n < a_{n_0} + |b_k - c_k|/4$$

and therefore

$$\max\{b_k, a_{n_0} - |b_k - c_k|/4\} < a_n < \min\{c_k, a_{n_0} + |b_k - c_k|/4\}$$

$$\Rightarrow b_{k+1} < a_n < c_{k+1} \Rightarrow a_n \in [b_{k+1}, c_{k+1}].$$

and this proves the claim.

▸ Claim: $[b_{k+1}, c_{k+1}] \subseteq [b_k, c_k]$.

We have:

$$a_{n_0} \in [b_k, c_k] \Rightarrow b_k \leq a_{n_0} \leq c_k \Rightarrow \begin{cases} b_k < a_{n_0} + |b_k - c_k|/4 \\ a_{n_0} - |b_k - c_k|/4 < c_k \end{cases}$$

$$\Rightarrow \max\{b_k, a_{n_0} - |b_k - c_k|/4\} < \min\{c_k, a_{n_0} + |b_k - c_k|/4\}$$

$$\Rightarrow b_{k+1} < c_{k+1}$$

noting that all other pairwise combinations in the definition of b_{k+1} and c_{k+1} also satisfy the same inequality

We also note that, by definition, we have $b_{k+1} \geq b_k$ and $c_{k+1} \leq c_k$. It follows that

$$b_k \leq b_{k+1} \leq c_{k+1} \leq c_k \Rightarrow [b_{k+1}, c_{k+1}] \subseteq [b_k, c_k]$$

thus proving the claim.

▸ <u>Claim</u>: $\lim_{n \in \mathbb{N}^*} (b_n - c_n) = 0$

We note that:

$$b_{k+1} = \max\{b_k, a_{n_0} - |b_k - c_k|/4\} \geq a_{n_0} - |b_k - c_k|/4 \Rightarrow$$
$$\Rightarrow -b_{k+1} \leq -a_{n_0} + |b_k - c_k|/4$$

and

$$c_{k+1} = \min\{c_k, a_{n_0} + |b_k - c_k|/4\} \leq a_{n_0} + |b_k - c_k|/4$$

and therefore

$$c_{k+1} - b_{k+1} \leq [-a_{n_0} + |b_k - c_k|/4] + [a_{n_0} + |b_k - c_k|/4]$$
$$= |b_k - c_k|/2 \Rightarrow |b_{k+1} - c_{k+1}| \leq |b_k - c_k|/2.$$

We have thus shown that

$$\forall n \in \mathbb{N}^* : |b_{n+1} - c_{n+1}| \leq |b_n - c_n|/2$$
$$\Rightarrow \forall n \in \mathbb{N}^* : |b_{n+1} - c_{n+1}| \leq |b_1 - c_1|/2^n$$
$$\Rightarrow \lim_{n \in \mathbb{N}^*} (b_n - c_n) = 0$$

- We conclude from the above that $([b_n, c_n])$ is nested.
- Define $\{\ell\} = \bigcap_{n \in \mathbb{N}^*} [b_n, c_n]$. We will now show that

$$\lim_{n \in \mathbb{N}^*} a_n = \ell.$$

Let $\varepsilon \in (0, +\infty)$ be given. Since $\lim_{n \in \mathbb{N}^*} (b_n - c_n) = 0$, choose $k \in \mathbb{N}^*$ such that $|b_k - c_k| < \varepsilon$. Choose $n_0 \in \mathbb{N}^*$ such that $\forall n \in \mathbb{N}^* - [n_0] : a_n \in [b_k, c_k]$. Let $n \in \mathbb{N}^* - [n_0]$ be given. It follows that

$$\begin{cases} a_n \in [b_k, c_k] \\ \ell \in [b_k, c_k] \end{cases} \Rightarrow |a_n - \ell| \leq |b_k - c_k| < \varepsilon \Rightarrow |a_n - \ell| < \varepsilon$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : |a_n - \ell| < \varepsilon$$
$$\Rightarrow \lim_{n \in \mathbb{N}^*} a_n = \ell \Rightarrow (a_n) \text{ convergent.} \qquad \square$$

Methodology

We can use the contrapositive of property 2 to show that a sequence (a_n) is not convergent by proving the statement
$\exists \varepsilon \in (0,+\infty) : \forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geq \varepsilon$

EXAMPLE

Show that the sequence (a_n) with
$$\forall n \in \mathbb{N}^* : a_n = \frac{(-1)^n n}{n+2}$$
is not convergent.

Solution

Let $n_0 \in \mathbb{N}^*$ be given. Via the Archimedes theorem, choose $n_1 = 2k > n_0$ and $n_2 = 2k+1 > n_0$ with $k \in \mathbb{N}^*$. Then, we have:

$$|a_{n_1} - a_{n_2}| = \left| \frac{(-1)^{2k}(2k)}{2k+2} - \frac{(-1)^{2k+1}(2k+1)}{(2k+1)+2} \right| =$$
$$= \left| \frac{k}{k+1} + \frac{2k+1}{2k+3} \right| = \frac{k}{k+1} + \frac{2k+1}{2k+3} \geq$$
$$\geq \frac{k}{k+1} \geq \frac{k}{k+k} = \frac{k}{2k} = \frac{1}{2} \Rightarrow$$
$$\Rightarrow \underline{|a_{n_1} - a_{n_2}| \geq 1/2}$$

We have thus shown that:

$\forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geq 1/2$
$\Rightarrow \exists \varepsilon \in (0, +\infty) : \forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geq \varepsilon$
$\Rightarrow (a_n)$ not Cauchy $\Rightarrow (a_n)$ not convergent. $\square$

THEORY QUESTIONS

(28) State the definition for
a) (a_n) Cauchy.
b) (a_n) not Cauchy

(29) Let (a_n) be a sequence. Prove that
a) (a_n) Cauchy $\Rightarrow$ (a_n) bounded
b) (a_n) convergent $\Rightarrow$ (a_n) Cauchy.

(30)* Let (a_n) be a sequence. Prove that
a) (a_n) Cauchy $\Rightarrow$ (a_n) convergent.
(optional).

EXERCISES

(31) Let (a_n), (b_n) be two sequences. Use the Cauchy sequence definition to show that
a) $(a_n),(b_n)$ Cauchy $\Rightarrow$ (a_n+b_n) Cauchy
b) $(a_n),(b_n)$ Cauchy $\Rightarrow$ $(a_n b_n)$ Cauchy
c) (a_n) Cauchy $\Rightarrow$ $(|a_n|)$ Cauchy.

(32) Show that the following sequences are not convergent.

a) $a_n = \dfrac{1+(-1)^n}{2}$ b) $a_n = \sin(n\pi/2)$

c) $a_n = \dfrac{(-1)^n(n+2)}{3n}$ d) $a_n = (-1)^n(3n+2)$

e) $a_n = \dfrac{2(-2)^n + 2^n}{(-2)^n - 3\cdot 2^{n-1}}$ f) $a_n = \dfrac{n^2 + (-1)^n n^2}{n+1}$

g) $a_n = \dfrac{n\cos(3n\pi/4)}{n+1}$

▼ Sequences / nets with limit going to infinity.

Let (a_n) be a net on $(D, <)$. We recall the following definitions:

$$\boxed{\begin{aligned} &\lim a_n = +\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon) \\ &\lim a_n = -\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n < -1/\varepsilon) \end{aligned}}$$

An immediate consequence of these definitions (via choosing $\varepsilon = 1$) are the following statements:

$$\boxed{\begin{aligned} &\lim a_n = +\infty \Rightarrow (a_n) \text{ lower bounded} \\ &\lim a_n = -\infty \Rightarrow (a_n) \text{ upper bounded} \\ &\lim a_n = \pm\infty \Leftrightarrow \lim (-a_n) = \mp\infty \\ &(\lim a_n = +\infty \vee \lim a_n = -\infty) \Rightarrow \lim |a_n| = +\infty \end{aligned}}$$

Note that when (a_n) is a sequence, these definitions simplify to:

$$\boxed{\begin{aligned} &\lim_{n \in \mathbb{N}^*} a_n = +\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n > 1/\varepsilon \\ &\lim_{n \in \mathbb{N}^*} a_n = -\infty \Leftrightarrow \forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* - [n_0] : a_n < -1/\varepsilon \end{aligned}}$$

We also note that when (a_n) is a sequence, then:

$$\boxed{\lim_{n \in \mathbb{N}^*} a_n = \pm\infty \Leftrightarrow \lim_{n \in \mathbb{N}^*} a_{n+k} = \pm\infty}$$

→ <u>Uniqueness</u>: To establish uniqueness, we first prove the following statements:

(1) $\boxed{\lim a_n = +\infty \Rightarrow (a_n) \text{ not upper bounded}}$

<u>Proof</u>

To show a contradiction, assume that (a_n) upper bounded.

Then, we have:

(a_n) bounded $\Rightarrow \exists b \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leqslant b)$

Choose $b \in (0,+\infty)$ and $n_1 \in D$ such that

$\forall n \in D : (n > n_1 \Rightarrow a_n \leqslant b)$

We also have:

$\lim a_n = +\infty \Rightarrow \forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

$\Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > b)$.

Choose $n_2 \in D$ such that

$\forall n \in D : (n > n_2 \Rightarrow a_n > b)$

Choose $n_0 \in D$ such that $n_0 > n_1$ and $n_0 > n_2$. Then, we have

$$\begin{cases} n_0 > n_1 \\ n_0 > n_2 \end{cases} \Rightarrow \begin{cases} a_{n_0} \leqslant b \\ a_{n_0} > b \end{cases}$$

which is a contradiction. We conclude that (a_n) not upper bounded. □

▸ An immediate consequence of property 1 is:

(2) $\boxed{\lim a_n = -\infty \Rightarrow (a_n) \text{ not lower bounded}}$

Uniqueness of the limit is established by noting that

$\lim a_n = +\infty \Rightarrow (a_n)$ not upper bounded $\Rightarrow \lim a_n \neq -\infty$

$\lim a_n = -\infty \Rightarrow (a_n)$ not lower bounded $\Rightarrow \lim a_n \neq +\infty$

and

$\lim a_n = +\infty \vee \lim a_n = -\infty \Rightarrow (a_n)$ not bounded $\Rightarrow$
$\Rightarrow (a_n)$ not convergent $\Rightarrow \forall \ell \in \mathbb{R} : \lim a_n \neq \ell.$
and
$\lim a_n = \ell \Rightarrow (a_n)$ convergent $\Rightarrow (a_n)$ bounded $\Rightarrow$

$$\Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ lower bounded} \end{cases} \Rightarrow \begin{cases} \lim a_n \neq +\infty \\ \lim a_n \neq -\infty \end{cases}$$

We conclude that if $\lim a_n$ exists, it has a unique evaluation in the set $\mathbb{R} \cup \{+\infty, -\infty\}$.

Properties of nets with limit to infinity.

Let $(a_n), (b_n)$ be nets on $(D, <)$. Then, we have:

(1) If $\lim a_n = +\infty$, then:
a) b_n lower bounded $\Rightarrow \lim (a_n + b_n) = +\infty$
b) b_n positively lower bounded $\Rightarrow \lim (a_n b_n) = +\infty$
c) b_n negatively upper bounded $\Rightarrow \lim (a_n b_n) = -\infty$

Proof

a) Assume that $\lim a_n = +\infty$ and b_n lower bounded. Then,
b_n lower bounded $\Rightarrow \exists b \in \mathbb{R} : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow b_n \geq b)$
Choose $b \in \mathbb{R}$ and $n_1 \in D$ such that $\forall n \in D : (n > n_1 \Rightarrow b_n \geq b)$
Let $\varepsilon \in (0, +\infty)$ be given. Then, we have
$\lim a_n = +\infty \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon - b)$
Choose $n_2 \in D$ such that $\forall n \in D : (n > n_2 \Rightarrow a_n > 1/\varepsilon - b)$

Choose $n_0 \in D$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $n \in D$ be given and assume that $n > n_0$. Then we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} b_n \geq b \\ a_n > 1/\varepsilon - b \end{cases} \Rightarrow \underline{a_n + b_n > 1/\varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n + b_n > 1/\varepsilon)$$

$$\Rightarrow \lim (a_n + b_n) = +\infty$$

b) Assume that $\lim a_n = +\infty$ and (b_n) positively lower bounded. Then, we have:

b_n positively lower bounded $\Rightarrow$

$$\Rightarrow \exists b \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow b_n \geq b)$$

Choose $b \in (0, +\infty)$ and $n_1 \in D$ such that

$$\forall n \in D : (n > n_1 \Rightarrow b_n \geq b)$$

Let $\varepsilon \in (0, +\infty)$ be given. Since,

$$\lim a_n = +\infty \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/(\varepsilon b))$$

choose $n_2 \in D$ such that

$$\forall n \in D : (n > n_2 \Rightarrow a_n > 1/(\varepsilon b))$$

Choose $n_0 \in D$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $n \in D$ be given and assume that $n > n_0$. Then, we have

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n > 1/(\varepsilon b) > 0 \\ b_n \geq b > 0 \end{cases} \Rightarrow$$

$$\Rightarrow a_n b_n \geq a_n b > [1/(\varepsilon b)] b = 1/\varepsilon \Rightarrow \underline{a_n b_n > 1/\varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n b_n > 1/\varepsilon)$$

$$\Rightarrow \lim (a_n b_n) = +\infty$$

c) Homework.

Similarly we can show that

(2) $\boxed{\begin{array}{l}\text{If } \lim a_n = -\infty \text{ , then} \\ \text{a) } (b_n) \text{ upper bounded} \Rightarrow \lim(a_n + b_n) = -\infty \\ \text{b) } (b_n) \text{ negatively upper bounded} \Rightarrow \lim(a_n b_n) = +\infty \\ \text{c) } (b_n) \text{ positively lower bounded} \Rightarrow \lim(a_n b_n) = -\infty\end{array}}$

The proof is to simply apply property 1 on the nets $-a_n$ and $-b_n$.

(3) $\boxed{\begin{cases} \lim a_n \in \{+\infty, -\infty\} \\ \forall n \in D : a_n \neq 0 \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = 0}$

<u>Proof</u>

We distinguish between the following cases.

<u>Case 1</u>: Assume that $\lim a_n = +\infty$. Then, we have:

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

Let $\underline{\varepsilon \in (0, +\infty)}$ be given. Choose $\underline{n_0 \in D}$ such that

$\forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

Let $\underline{n \in D}$ be given and assume that $\underline{n > n_0}$. Then, we have:

$n > n_0 \Rightarrow a_n > 1/\varepsilon > 0 \Rightarrow 0 < 1/a_n < \varepsilon \Rightarrow \underline{|1/a_n| < \varepsilon}$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |1/a_n| < \varepsilon)$

$\Rightarrow \lim (1/a_n) = 0.$

<u>Case 2</u>: Assume that $\lim a_n = -\infty$. Then, we have:

$$\lim a_n = -\infty \Rightarrow \lim(-a_n) = +\infty \Rightarrow \lim \frac{1}{-a_n} = 0 \Rightarrow$$

$$\Rightarrow \lim \frac{1}{a_n} = 0 \qquad \square$$

$$\boxed{\begin{array}{ll} \text{④ a)} \begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} \Rightarrow \lim \frac{1}{a_n} = +\infty \\ \text{b)} \begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n < 0) \end{cases} \Rightarrow \lim \frac{1}{a_n} = -\infty \end{array}}$$

Proof

a) Let $\varepsilon \in (0, +\infty)$ be given. Choose $n_1 \in D$ such that
$\forall n \in D : (n > n_1 \Rightarrow a_n > 0)$

Since $\lim a_n = 0 \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow |a_n| < \varepsilon)$, choose $n_2 \in D$ such that $\forall n \in D : (n > n_2 \Rightarrow |a_n| < \varepsilon)$. Choose $n_0 \in D$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $n \in D$ be given and assume that $n > n_0$. Then, we have:

$$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n > 0 \\ |a_n| < \varepsilon \end{cases} \Rightarrow 0 < a_n < \varepsilon \Rightarrow$$

$$\Rightarrow \frac{1}{a_n} > \frac{1}{\varepsilon}$$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow 1/a_n > 1/\varepsilon)$

$\Rightarrow \lim(1/a_n) = +\infty$

b) Homework! $\square$

(5) $\begin{cases} \lim a_n = +\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n) \end{cases} \Rightarrow \lim b_n = +\infty$

$\begin{cases} \lim a_n = -\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \geq b_n) \end{cases} \Rightarrow \lim b_n = -\infty$

Proof

a) Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Choose $n_1 \in D$ such that
$\forall n \in D : (n > n_1 \Rightarrow a_n \leq b_n)$

Since $\lim a_n = +\infty \Rightarrow \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$
choose $n_2 \in D$ such that
$\forall n \in D : (n > n_2 \Rightarrow a_n > 1/\varepsilon)$

Choose $\underline{n_0 \in D}$ such that $n_0 > n_1$ and $n_0 > n_2$. Let $\underline{n \in D}$ be given and assume that $n > n_0$. Then, we have:

$n > n_0 \Rightarrow \begin{cases} n > n_1 \\ n > n_2 \end{cases} \Rightarrow \begin{cases} a_n \leq b_n \\ a_n > 1/\varepsilon \end{cases} \Rightarrow \underline{b_n > 1/\varepsilon}$

We have thus shown that
$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow b_n > 1/\varepsilon)$
$\Rightarrow \lim b_n = +\infty$

b) Homework

THEORY QUESTIONS

(33) Let (a_n), (b_n) be nets. Prove that

a) $\lim a_n = +\infty \Rightarrow (a_n)$ not upper bounded

b) $\begin{cases} \lim a_n = +\infty \\ (b_n) \text{ lower bounded} \end{cases} \Rightarrow \lim (a_n + b_n) = +\infty$

c) $\begin{cases} \lim a_n = +\infty \\ b_n \text{ positively lower bounded} \end{cases} \Rightarrow \lim (a_n b_n) = +\infty$

d) $\begin{cases} \lim a_n \in \{+\infty, -\infty\} \\ \forall n \in D : a_n \neq 0 \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = 0$

e) $\begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n > 0) \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = +\infty$

f) $\begin{cases} \lim a_n = +\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \leq b_n) \end{cases} \Rightarrow \lim b_n = +\infty$

EXERCISES

(34) Use the limit definition to write complete proofs for the following statements

a) $\lim a_n = -\infty \Rightarrow (a_n)$ not lower bounded

b) $\begin{cases} \lim a_n = +\infty \\ b_n \text{ negatively upper bounded} \end{cases} \Rightarrow \lim (a_n b_n) = -\infty$

c) $\begin{cases} \lim a_n = 0 \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n < 0) \end{cases} \Rightarrow \lim \dfrac{1}{a_n} = -\infty$

d) $\begin{cases} \lim a_n = -\infty \\ \exists n_0 \in D : \forall n \in D : (n > n_0 \Rightarrow a_n \geq b_n) \end{cases} \Rightarrow \lim b_n = -\infty$

Basic sequence limits

(1) $\boxed{\forall p \in (0,+\infty) : \lim_{n\in\mathbb{N}^*} n^p = +\infty}$

Proof

Let $p \in (0,+\infty)$ be given and define $\forall n \in \mathbb{N}^* : a_n = n^p$.
Let $\varepsilon \in (0,+\infty)$ be given. Then, we have:
$a_n > 1/\varepsilon \Leftrightarrow n^p > 1/\varepsilon > 0 \Leftrightarrow n > (1/\varepsilon)^{1/p}$
Via the Archimedes theorem, choose $n_0 \in \mathbb{N}^*$ such that $n_0 > (1/\varepsilon)^{1/p}$. Let $n \in \mathbb{N}^*$ be given and assume that $n > n_0$. Then, we have
$n > n_0 \Rightarrow n > (1/\varepsilon)^{1/p} \Rightarrow a_n > 1/\varepsilon$.
We have thus shown that
$\forall \varepsilon \in (0,+\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$
$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} n^p = +\infty$ □

- Note that combining this result with Property 3 immediately gives the following result:

(2) $\boxed{\forall p \in (-\infty,0) : \lim_{n\in\mathbb{N}^*} n^p = 0}$

(3) $\boxed{\forall a \in (1,+\infty) : \lim_{n\in\mathbb{N}^*} a^n = +\infty}$

Proof

Let $a \in (1, +\infty)$ be given and define $p \in (0, +\infty)$ such that $a = 1+p$. Let $\varepsilon \in (0, +\infty)$ be given. Then, we have:
$a^n = (1+p)^n \geq 1 + np > np > 1/\varepsilon \Leftarrow n > 1/(\varepsilon p)$.
Via the Archimedes theorem, choose $n_0 \in \mathbb{N}^*$ such that $n_0 > 1/(\varepsilon p)$. Let $n \in \mathbb{N}^*$ be given and assume that $n > n_0$. Then, we have:
$n > n_0 \Rightarrow n > 1/(\varepsilon p) \Rightarrow a^n > 1/\varepsilon$
We have thus shown than
$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > n_0 \Rightarrow a^n > 1/\varepsilon)$
$\Rightarrow \lim\limits_{n \in \mathbb{N}^*} a^n = +\infty$

We conclude that $\forall a \in (1, +\infty) : \lim\limits_{n \in \mathbb{N}^*} a^n = +\infty$ $\square$

$\to$ Result 1 can be combined with the following result, derived from the limit properties, to find the limits of polynomial and rational sequences

$$\boxed{\begin{aligned} &\forall n \in \mathbb{N}^* : c_n = \frac{b_p n^p + b_{p-1} n^{p-1} + \cdots + b_1 n + b_0}{a_q n^q + a_{q-1} n^{q-1} + \cdots + a_1 n + a_0} \Rightarrow \\ &\Rightarrow \lim_{n \in \mathbb{N}^*} c_n = \lim_{n \in \mathbb{N}^*} \frac{b_p n^p}{a_q n^q} \end{aligned}}$$

For $q = 0$ and $a_0 = 1$, this result reduces to a polynomial sequence as well.

EXAMPLES

a) Use the limit definition to show that $\lim_{n\in\mathbb{N}^*} a_n = +\infty$ for $\forall n\in\mathbb{N}^* : a_n = \sin(3n) + 3n^2 - n$

Solution

Let $\varepsilon \in (0, +\infty)$ be given. Then, we have:

$a_n = \sin(3n) + 3n^2 - n \geq -1 + 3n^2 - n \geq -n^2 + 3n^2 - n^2$
$= n^2 > 1/\varepsilon > 0 \Leftrightarrow n > 1/\sqrt{\varepsilon}$.

Via the Archimedes theorem, choose $n_0 \in \mathbb{N}^*$ such that $n_0 > 1/\sqrt{\varepsilon}$. Let $n \in \mathbb{N}^*$ be given and assume that $n > n_0$. Then, we have:

$n > n_0 \Rightarrow n > 1/\sqrt{\varepsilon} \Rightarrow a_n > 1/\varepsilon$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$

$\Rightarrow \lim_{n\in\mathbb{N}^*} a_n = +\infty$ □

b) Use the limit definition to show that $\lim_{n\in\mathbb{N}^*} a_n = +\infty$ for $\forall n\in\mathbb{N}^* : a_n = n^5 + 3n^3 - 9n^2$

Solution

Let $\varepsilon \in (0, +\infty)$ be given. Then, we have:

$a_n = n^5 + 3n^3 - 9n^2 = n^5 + 3n^2(n-3)$.

If we restrict $n > 3$, then we have:

$a_n = n^5 + 3n^2(n-3) \geq n^5 + 3n^2 \geq n^5 > 1/\varepsilon > 0 \Leftrightarrow$
$\Leftrightarrow n > (1/\varepsilon)^{1/5}$

Via the Archimedes theorem, choose $n_0 \in \mathbb{N}^*$ such that $n_0 > \max\{3, (1/\varepsilon)^{1/5}\}$. Let $\underline{n \in \mathbb{N}^*}$ be given and assume that $\underline{n > n_0}$. Then, we have:

$$n > n_0 \Rightarrow n > \max\{3, (1/\varepsilon)^{1/5}\} \Rightarrow$$

$$\Rightarrow \begin{cases} n > 3 \\ n > (1/\varepsilon)^{1/5} \end{cases} \Rightarrow \underline{a_n > 1/\varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > n_0 \Rightarrow a_n > 1/\varepsilon)$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} a_n = +\infty$$

□

THEORY QUESTIONS

(35) Prove that

a) $\forall p \in (0, +\infty): \lim\limits_{n \in \mathbb{N}^*} n^p = +\infty$

b) $\forall a \in (1, +\infty): \lim\limits_{n \in \mathbb{N}^*} a^n = +\infty$

(36) Use the limit definition to show that $\lim\limits_{n \in \mathbb{N}^*} a_n = +\infty$ for the following sequences

a) $a_n = 3n^3 + n^2 + 5n + 1$

b) $a_n = n^5 + n^4 - 10n^3$

c) $a_n = \dfrac{n^3 + \sin(2n)}{n^2 + 1}$

d) $a_n = n 3^n + n^2$

e) $a_n = 3^n + 6^n + n\cos(5n)$

f) $a_n = (5n + \sin(5n)) 3^n$

(37) Use the limit definition to show that $\lim\limits_{n \in \mathbb{N}^*} a_n = -\infty$ for the following sequences

a) $a_n = (1 - 2n)^3$

b) $a_n = \sin(5n) - 3n^2$

c) $a_n = -n^2 - n + 1$

d) $a_n = -5n^3 + n - 2$

e) $a_n = \dfrac{4 - n^3}{4n}$

f) $a_n = \cos(3n) - n^2 5^n$

g) $a_n = n\sin(3n) - 2^n - 5^n$

h) $a_n = 3^n - 5^n + \cos(2n)$

i) $a_n = (\cos(3n) - 3n) 7^n$

(38) Use the properties of the limit to evaluate the limit of the following sequences

a) $a_n = 3n^3 - 5n^2 + 2n - 1$

b) $a_n = \dfrac{-2n^3 + 3n - 5}{n^2 - 3n - 2}$

c) $a_n = 2^n + 3^n - 5^n$

d) $a_n = (3 + \cos(5n))(2^n - 5^n)$

e) $a_n = (\sin(9n) - 3)(2^n - 5^n + 3^n + \cos(3n))$

f) $a_n = \sum_{k=1}^{n} \dfrac{n^2}{n^2 + k^2}$

g) $a_n = \dfrac{7^n - 3^n}{5^n + 2^n}$

h) $a_n = \dfrac{4^n + 3^n - 6^n}{5^n + 2^n}$

i) $a_n = \sum_{k=1}^{n} k^n$

j) $a_n = \sum_{k=1}^{n} \dfrac{n^3 + \sin(n)}{n^3 + k^2}$

k) $a_n = \sum_{k=1}^{n} \dfrac{n5^n + \cos(5n)}{n2^n + k}$

l) $a_n = \sum_{k=1}^{n} \dfrac{n^2 + \cos(2n)}{(n+k)^2}$

m) $a_n = \dfrac{1}{(n+1)^2} \sum_{k=1}^{n} (k^2 + \cos(k) + \sin(k))$

Divergent sequences

In order to show that a sequence (a_n) (or more generally any net) is divergent (i.e. that $\lim a_n$ does not exist), we have to show that

$$\begin{cases} (a_n) \text{ not convergent} \\ \lim a_n \neq +\infty \wedge \lim a_n \neq -\infty \end{cases}$$

To do that, it is helpful to use the following results:

$$\boxed{\begin{array}{l} (a_n) \text{ not bounded} \Rightarrow (a_n) \text{ not convergent} \\ (|a_n|) \text{ not convergent} \Rightarrow (a_n) \text{ not convergent} \\ \lim a_n = +\infty \vee \lim a_n = -\infty \Rightarrow (a_n) \text{ not bounded} \\ (a_n) \text{ not Cauchy} \Rightarrow (a_n) \text{ not convergent.} \end{array}}$$

Specifically for a sequence (a_n), we can also use the following results:

$$\boxed{\begin{array}{l} \lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \Rightarrow \begin{cases} (a_n) \text{ lower bounded} \\ (a_n) \text{ not upper bounded} \end{cases} \\ \lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \Rightarrow \begin{cases} (a_n) \text{ upper bounded} \\ (a_n) \text{ not lower bounded} \end{cases} \\ (a_{pn+k}) \text{ not upper bounded} \Rightarrow (a_n) \text{ not upper bounded} \\ (a_{pn+k}) \text{ not lower bounded} \Rightarrow (a_n) \text{ not lower bounded} \end{array}}$$

with $p, k \in \mathbb{N}^*$

EXAMPLES

a) Show that $\forall n \in \mathbb{N}^*: a_n = (-1)^n(2n+3)$ is divergent

Solution

Since

$|a_n| = |(-1)^n(2n+3)| = |2n+3| = 2n+3 > 2n, \forall n \in \mathbb{N}^*$

and

$\lim\limits_{n\in\mathbb{N}^*} (2n) = +\infty$

it follows that

$\lim\limits_{n\in\mathbb{N}^*} |a_n| = +\infty \Rightarrow (|a_n|)$ not bounded $\Rightarrow$

$\Rightarrow (|a_n|)$ not convergent $\Rightarrow$

$\Rightarrow$ $\underline{(a_n) \text{ not convergent}}$

To show that $\lim\limits_{n\in\mathbb{N}^*} a_n \neq +\infty$, assume that $\lim\limits_{n\in\mathbb{N}^*} a_n = +\infty$ in order to show a contradiction. Then, we have:

$\lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \Rightarrow$ $\underline{(a_n) \text{ lower bounded}}$

and

$a_{2n+1} = (-1)^{2n+1}(2(2n+1)+3) = -(4n+2+3) = -4n-5, \forall n \in \mathbb{N}^*$

$\Rightarrow \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} = \lim\limits_{n\in\mathbb{N}^*} (-4n-5) = \lim\limits_{n\in\mathbb{N}^*} (-4n) = -\infty \Rightarrow$

$\Rightarrow (a_{2n+1})$ not lower bounded $\Rightarrow$

$\Rightarrow$ $\underline{(a_n) \text{ not lower bounded}}$

which is a contradiction. It follows that $\underline{\underline{\lim\limits_{n\in\mathbb{N}^*} a_n \neq +\infty}}$

To show that $\lim\limits_{n\in\mathbb{N}^*} a_n \neq -\infty$, we assume that $\lim a_n = -\infty$ in order to show a contradiction. Then, we have

$\lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \Rightarrow (a_n)$ upper Bounded

and

$a_{2n} = (-1)^{2n}(2(2n)+3) = 4n+3,\ \forall n\in\mathbb{N}^* \Rightarrow$

$\Rightarrow \lim\limits_{n\in\mathbb{N}^*} a_{2n} = \lim\limits_{n\in\mathbb{N}^*}(4n+3) = \lim\limits_{n\in\mathbb{N}^*} 4n = +\infty \Rightarrow$

$\Rightarrow (a_{2n})$ not upper bounded

$\Rightarrow (a_n)$ not upper Bounded

It follows that $\underline{\underline{\lim\limits_{n\in\mathbb{N}^*} a_n \neq -\infty}}$, and we conclude that

$$\left.\begin{cases} (a_n) \text{ not convergent} \\ \lim\limits_{n\in\mathbb{N}^*} a_n \neq +\infty \wedge \lim\limits_{n\in\mathbb{N}^*} a_n \neq -\infty \end{cases}\right\} \Rightarrow (a_n) \text{ divergent.}$$

b) Show that $\forall n \in \mathbb{N}^* : a_n = \dfrac{(-1)^n (3n)}{n+1}$ is divergent

Solution

Since

$$|a_n| = \left| \frac{(-1)^n (3n)}{n+1} \right| = \frac{3n}{n+1} \leqslant \frac{3n}{n} = 3, \ \forall n \in \mathbb{N}^*$$

$\Rightarrow (a_n)$ bounded $\Rightarrow \lim\limits_{n\in\mathbb{N}^*} a_n \neq +\infty \wedge \lim\limits_{n\in\mathbb{N}^*} a_n \neq -\infty$

it is sufficient to show that

$\exists \varepsilon \in (0,+\infty) : \forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| \geqslant \varepsilon$

Choose $\varepsilon = 1$. Let $n_0 \in \mathbb{N}^*$ be given. Since:

$$\lim_{n\in\mathbb{N}^*} a_{2n} = \lim_{n\in\mathbb{N}^*} \frac{(-1)^{2n} 3(2n)}{2n+1} = \lim_{n\in\mathbb{N}^*} \frac{6n}{2n+1} =$$

$$= \lim_{n\in\mathbb{N}^*} \frac{6n}{2n} = \frac{6}{2} = 3 \Rightarrow$$

$\Rightarrow \exists p \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > p \Rightarrow |a_{2n} - 3| < 1)$

and

$$\lim_{n\in\mathbb{N}^*} a_{2n+1} = \lim_{n\in\mathbb{N}^*} \frac{(-1)^{2n+1} 3(2n+1)}{(2n+1)+1} = \lim_{n\in\mathbb{N}^*} \frac{-6n-3}{2n+2}$$

$$= \lim_{n\in\mathbb{N}^*} \frac{-6n}{2n} = \frac{-6}{2} = -3 \Rightarrow$$

$\Rightarrow \exists q \in \mathbb{N}^* : \forall n \in \mathbb{N}^* : (n > q \Rightarrow |a_{2n+1} - (-3)| < 1)$

Choose $p, q \in \mathbb{N}^*$ such that

$$\begin{cases} \forall n \in \mathbb{N}^* : (n > p \Rightarrow |a_{2n} - 3| < 1) \\ \forall n \in \mathbb{N}^* : (n > q \Rightarrow |a_{2n+1} + 3| < 1) \end{cases}$$

Choose $n_1 = \max\{n_0, p\} + 1$ and $n_2 = \max\{n_0, q\} + 1$

which implies that $n_1, n_2 \in \mathbb{N}^* - [n_0]$. Then, we have:

$$\begin{cases} n_1 > p \\ n_2 > q \end{cases} \Rightarrow \begin{cases} |a_{n_1} - 3| < 1 \\ |a_{n_2} + 3| < 1 \end{cases} \Rightarrow \begin{cases} -1 < a_{n_1} - 3 < 1 \\ -1 < a_{n_2} + 3 < 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 2 < a_{n_1} < 4 \\ -4 < a_{n_2} < -2 \end{cases} \Rightarrow a_{n_2} < -2 < 2 < a_{n_1}$$

$$\Rightarrow |a_{n_1} - a_{n_2}| > |2 - (-2)| = 4$$

$$\Rightarrow |a_{n_1} - a_{n_2}| > 4$$

We have thus shown that

$$\forall n_0 \in \mathbb{N}^* : \exists n_1, n_2 \in \mathbb{N}^* - [n_0] : |a_{n_1} - a_{n_2}| > 4$$

$\Rightarrow (a_n)$ not Cauchy $\Rightarrow$ <u>(a_n) not convergent</u>

We conclude that

$$\begin{cases} (a_n) \text{ not convergent} \\ \lim_{n \in \mathbb{N}^*} a_n \neq +\infty \wedge \lim_{n \in \mathbb{N}^*} a_n \neq -\infty \end{cases} \Rightarrow (a_n) \text{ divergent.}$$

EXERCISES

(39) Show that the following sequences are divergent.

a) $a_n = \sin(n\pi/7)$ b) $a_n = \sin(n\pi/4) + \cos(n\pi/2)$

c) $a_n = \dfrac{(-1)^n n^2}{3n+1}$ d) $a_n = (-1)^n 7^n$

e) $a_n = \dfrac{(-1)^n 7^n}{7^n + 5^n}$ f) $a_n = \dfrac{6(-1)^n 3^n}{3^n + 2^n}$

g) $a_n = (-1)^n \sqrt[n]{2^n + 3^n}$ h) $a_n = (-1)^n \sqrt[n]{n^2 + 3n + 2}$

(40) Let (a_n) be a sequence. Show that

a) $\lim\limits_{n\in\mathbb{N}^*} a_n = l \in \mathbb{R} \iff \lim\limits_{n\in\mathbb{N}^*} a_{2n} = l \wedge \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} = l$

b) $\lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \iff \lim\limits_{n\in\mathbb{N}^*} a_{2n} = +\infty \wedge \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} = +\infty$

c) $\lim\limits_{n\in\mathbb{N}^*} a_{2n} = l_1 \in \mathbb{R} \wedge \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} = l_2 \in \mathbb{R} \wedge l_1 \neq l_2 \Rightarrow$

$\Rightarrow (a_n)$ divergent

d) $\begin{cases} (a_{2n+1}) \text{ convergent} \\ \lim\limits_{n\in\mathbb{N}^*} a_{2n} \in \{+\infty, -\infty\} \end{cases} \Rightarrow (a_n)$ divergent

e) $\begin{cases} (a_{2n}) \text{ convergent} \\ \lim\limits_{n\in\mathbb{N}^*} a_{2n+1} \in \{+\infty, -\infty\} \end{cases} \Rightarrow (a_n)$ divergent.

RA 1.3: Limits of functions

LIMITS OF FUNCTIONS

▼ Weierstrass limit definition

Let $f: A \to \mathbb{R}$ be a function with domain $\operatorname{dom}(f) = A \subseteq \mathbb{R}$. In order to define $\lim_{x \to \sigma} f(x) = L$, we begin with the following notation:

Notation

a) The neighborhood $N(\sigma, \delta)$ is defined as

$$N(\sigma,\delta) = \begin{cases} (x_0-\delta, x_0) \cup (x_0, x_0+\delta) &, \text{if } \sigma = x_0 \\ (x_0-\delta, x_0) &, \text{if } \sigma = x_0^- \\ (x_0, x_0+\delta) &, \text{if } \sigma = x_0^+ \\ (1/\delta, +\infty) &, \text{if } \sigma = +\infty \\ (-\infty, -1/\delta) &, \text{if } \sigma = -\infty \end{cases}$$

b) The interval $I(L, \varepsilon)$ is defined as

$$I(L,\varepsilon) = \begin{cases} (\ell-\varepsilon, \ell+\varepsilon) &, \text{if } L = \ell \in \mathbb{R} \\ (1/\varepsilon, +\infty) &, \text{if } L = +\infty \\ (-\infty, -1/\varepsilon) &, \text{if } L = -\infty \end{cases}$$

Note that the corresponding belonging conditions are:

$x \in N(x_0, \delta) \iff 0 < |x - x_0| < \delta$

$x \in N(x_0^-, \delta) \iff x_0 - \delta < x < x_0$

$x \in N(x_0^+, \delta) \iff x_0 < x < x_0 + \delta$

$x \in N(+\infty, \delta) \iff x > 1/\delta$

$x \in N(-\infty, \delta) \iff x < -1/\delta$

$y \in I(\ell, \varepsilon) \iff |y - \ell| < \varepsilon$

$y \in I(+\infty, \varepsilon) \iff y > 1/\varepsilon$

$y \in I(-\infty, \varepsilon) \iff y < -1/\varepsilon$

Remarks

a) It is easy to show that

$$\begin{cases} \forall \delta_1, \delta_2 \in (0,+\infty) : (\delta_1 < \delta_2 \Rightarrow N(\sigma,\delta_1) \subseteq N(\sigma,\delta_2)) \\ \forall \varepsilon_1, \varepsilon_2 \in (0,+\infty) : (\varepsilon_1 < \varepsilon_2 \Rightarrow I(L,\varepsilon_1) \subseteq I(L,\varepsilon_2)) \end{cases}$$

thus, decreasing δ or ε tends to make the neighborhood $N(\sigma,\delta)$ or interval $I(L,\varepsilon)$ tighter.

b) We can also show that

$$\begin{cases} \forall \delta_1, \delta_2 \in (0,+\infty) : N(\sigma,\delta_1) \cap N(\sigma,\delta_2) = N(\sigma, \min\{\delta_1,\delta_2\}) \\ \forall \varepsilon_1, \varepsilon_2 \in (0,+\infty) : I(L,\varepsilon_1) \cap I(L,\varepsilon_2) = I(L, \min\{\varepsilon_1,\varepsilon_2\}) \end{cases}$$

c) Relationship between neighborhoods and intervals:

$$\begin{cases} N(x_0,\delta) = I(x_0,\delta) - \{x_0\} \\ N(+\infty,\delta) = I(+\infty,\delta) \\ N(-\infty,\delta) = I(-\infty,\delta) \end{cases}$$

▸ We now give the following definitions:

Def: Let $A \subseteq \mathbb{R}$ be a set. We say that

$$\sigma \text{ limit point of } A \Leftrightarrow \forall \delta \in (0,+\infty) : N(\sigma,\delta) \cap A \neq \varnothing$$

▸ interpretation: σ is a limit point of A if and only if regardless of how much we "squeeze" $N(\sigma,\delta)$ by decreasing δ, it always overlaps with A.

Def: Let $f: A \to \mathbb{R}$ be a function, σ a limit point of A, and $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then

$$\lim_{x \to \sigma} f(x) = L \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(\sigma,\delta) \Rightarrow f(x) \in I(L,\varepsilon))$$

- interpretation: In the above definition:
  ε = how close we want $f(x)$ to be to L
  δ = how close x must be brought to σ so that $f(x)$ will be as close to L as required by our choice of ε.

  Thus, as we choose smaller ε, it should always be possible to find a smaller δ that works.
- For each choice of σ and L we get a corresponding Weierstrass limit definition (5 choices for σ, 3 choices for L, thus 15 possible definitions). For example, for $\sigma = x_0 \in \mathbb{R}$ and $L = \ell \in \mathbb{R}$, we have:

$$\lim_{x\to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow \Rightarrow |f(x) - \ell| < \varepsilon)$$

- Note the following immediate consequences of the limit definition

$$\boxed{\begin{aligned} &\lim_{x\to\sigma} (f(x) - \ell) = 0 \Leftrightarrow \lim_{x\to\sigma} f(x) = \ell \\ &\lim_{x\to\sigma} f(x) = \ell \Leftrightarrow \lim_{x\to\sigma} [-f(x)] = -\ell \\ &\lim_{x\to\sigma} f(x) = \pm\infty \Leftrightarrow \lim_{x\to\sigma} [-f(x)] = \mp\infty \end{aligned}}$$

Def: Let $f: A \to \mathbb{R}$ be a function, σ a limit point of A. We say that:

$$\lim_{x\to\sigma} f(x) \text{ does not exist} \Leftrightarrow \begin{cases} \forall \ell \in \mathbb{R} : \lim_{x\to\sigma} f(x) \neq \ell \\ \lim_{x\to\sigma} f(x) \neq +\infty \wedge \lim_{x\to\sigma} f(x) \neq -\infty \end{cases}$$

EXAMPLES

a) Use the limit definition to show that

$$\lim_{x \to -\infty} \frac{3x^2+2x-1}{x^2+2x+9} = 3$$

Solution

Define $\forall x \in \mathbb{R}: f(x) = \dfrac{3x^2+2x-1}{x^2+2x+9}$, and note that:

$$f(x) - 3 = \frac{3x^2+2x-1}{x^2+2x+9} - 3 = \frac{(3x^2+2x-1) - 3(x^2+2x+9)}{x^2+2x+9}$$

$$= \frac{3x^2+2x-1-3x^2-6x-27}{x^2+2x+9} =$$

$$= \frac{(3-3)x^2+(2-6)x+(-1-27)}{x^2+2x+9} =$$

$$= \frac{-4x-28}{x^2+2x+9} = \frac{-2(2x+9)}{x^2+2x+9}$$

with $\begin{cases} \forall x \in \mathbb{R}: x^2+2x+9 = (x+1)^2+8 > 0 \\ \forall x \in (-\infty, -5): 2x+9 < 0 \end{cases}$

Restrict the domain of f to $A = (-\infty, -5)$.

Let $\varepsilon \in (0, +\infty)$ be given. We have; for all $x \in (-\infty, -5)$:

$$|f(x) - 3| = \left| \frac{-2(2x+9)}{x^2+2x+9} \right| = \frac{-2(2x+9)}{x^2+2x+9} \leq \frac{-4x}{x^2+2x+9}$$

$$\leq \frac{-4x}{x^2+2x} = \frac{-4x}{x(x+2)} = \frac{-4}{x+2} < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow -4 > \varepsilon(x+2) \qquad [\text{via } x+2 < 0]$$

$$\Leftrightarrow \varepsilon x + 2\varepsilon < -4 \Leftrightarrow \varepsilon x < -4 - 2\varepsilon \Leftrightarrow \quad x < \frac{-4-2\varepsilon}{\varepsilon}$$

Choose $\underline{\delta = \varepsilon/(2\varepsilon+4) > 0}$. Let $\underline{x \in A}$ be given and assume that $\underline{x \in N(-\infty, \delta)}$. Then, we have:

$$x \in N(-\infty, \delta) \Rightarrow x \in (-\infty, -1/\delta) \Rightarrow x < -1/\delta \Rightarrow$$
$$\Rightarrow x < \frac{-4-2\varepsilon}{\varepsilon} \Rightarrow \underline{|f(x) - 3| < \varepsilon}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(-\infty, \delta) \Rightarrow |f(x) - 3| < \varepsilon)$$
$$\Rightarrow \lim_{x \to -\infty} f(x) = 3$$

b) Use the limit definition to show that

$$\lim_{x \to 1} (x^2+2x+3) = 6$$

<u>Solution</u>

Define $f(x) = x^2+2x+3$, $\forall x \in \mathbb{R}$. Then, we have:

$$f(x)-6 = (x^2+2x+3)-6 = x^2+2x+3-6 = x^2+2x-3$$
$$= (x+3)(x-1), \forall x \in \mathbb{R}.$$

Restrict the domain of f to $A = (0,1) \cup (1,2)$. Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Then, for all $x \in A$, we have:

$$|f(x)-6| = |(x+3)(x-1)| = |x+3||x-1| = |(x-1)+4||x-1|$$
$$\leq [|x-1|+4]|x-1| < [1+4]|x-1| = 5$$
$$= 5|x-1| < \varepsilon \iff |x-1| < \varepsilon/5.$$

Choose $\underline{\delta = \varepsilon/5}$. Let $\underline{x \in A}$ be given and assume that $\underline{x \in N(1,\delta)}$. Then, we have:

$$x \in N(1,\delta) \Rightarrow 0 < |x-1| < \delta \Rightarrow 0 < |x-1| < \varepsilon/5 \Rightarrow$$
$$\Rightarrow \underline{|f(x)-6| < \varepsilon}.$$

We have thus shown that

$$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(1,\delta) \Rightarrow |f(x)-6| < \varepsilon)$$
$$\Rightarrow \lim_{x \to 1} (x^2+2x+3) = 6$$

THEORY QUESTIONS

① State the definition of the neighborhood $N(\sigma,\delta)$ for ~~an~~ arbitrary σ and the definition of the statement:

σ limit point of A

with $A \subseteq \mathbb{R}$, using quantifier notation.

② State the definition of the neighborhood $N(\sigma,\delta)$ and interval $I(L,\varepsilon)$ for an arbitrary choice of σ and L, and then state the general definition of the statement $\lim_{x\to\sigma} f(x) = L$, using quantifier notation.

③ State the definition of the statement

$\lim_{x\to\sigma} f(x)$ does not exist

for an arbitrary choice of σ.

④ Use quantifiers to write the specific definitions for the following statements, without using the neighborhood / interval notation

a) $\lim_{x\to x_0} f(x) = \ell \in \mathbb{R}$

b) $\lim_{x\to x_0^-} f(x) = -\infty$

c) $\lim_{x\to x_0^+} f(x) = +\infty$

d) $\lim_{x\to -\infty} f(x) = \ell \in \mathbb{R}$

e) $\lim_{x\to +\infty} f(x) = -\infty$

f) $\lim_{x\to -\infty} f(x) = +\infty$

g) $\lim_{x \to x_0^+} f(x) = \ell \in \mathbb{R}$ h) $\lim_{x \to 0} f(x) = 0$

EXERCISES

⑤ Use the neighborhood definition and proof by cases to show that

a) $\forall \delta_1, \delta_2 \in (0, +\infty): (\delta_1 < \delta_2 \Rightarrow N(\sigma, \delta_1) \subseteq N(\sigma, \delta_2))$

b) $\forall \delta_1, \delta_2 \in (0, +\infty): N(\sigma, \delta_1) \cap N(\sigma, \delta_2) = N(\sigma, \min\{\delta_1, \delta_2\})$

$\hookrightarrow$ Use (a) to prove (b).

⑥ Use the limit definition to show that:

a) $\lim_{x \to -\infty} \frac{2x-3}{x-5} = 2$ b) $\lim_{x \to +\infty} \frac{x^2}{x-2} = +\infty$

c) $\lim_{x \to +\infty} \frac{\sin(3x)}{x+1} = 0$ d) $\lim_{x \to +\infty} \frac{2x^2+3x+1}{3x^2+x+5} = \frac{2}{3}$

e) $\lim_{x \to 1} \frac{(x+1)(x+3)}{x^2+2x+5} = 1$ f) $\lim_{x \to 3^-} \frac{x^3}{2x+1} = \frac{27}{7}$

g) $\lim_{x \to 2^+} \frac{3x+1}{x-2} = +\infty$ h) $\lim_{x \to 3^-} \frac{5x+2}{x-3} = -\infty$

i) $\lim_{x \to -1^+} \frac{x^3}{2x+2} = -\infty$ j) $\lim_{x \to -\infty} \frac{2x\cos x}{x^2-2} = 0$

k) $\lim_{x \to +\infty} \frac{3x^2\cos x}{x^3+2} = 0$ l) $\lim_{x \to 0} x^3 \sin(1/x) = 0$

m) $\lim_{x \to +\infty} \frac{\cos x}{x^3} = 0$

(7) Use the limit definition to show that

a) $\lim\limits_{x \to -\infty} \dfrac{2x\cos x}{x^2-2} = 0$

b) $\lim\limits_{x \to +\infty} \dfrac{3x^2\cos x}{x^3+2} = 0$

c) $\lim\limits_{x \to 0} x^3 \sin(1/x) = 0$

d) $\lim\limits_{x \to +\infty} \dfrac{\cos x}{x^3} = 0$

Relation between side limits $x \to x_0^{\pm}$ and $x \to x_0$

Thm: Let $f: A \to \mathbb{R}$ be a function, let x_0 be a limit point of A, and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, we have

$$\lim_{x \to x_0} f(x) = L \Longleftrightarrow \lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L$$

Proof

($\Rightarrow$): Assume that $\lim_{x \to x_0} f(x) = L$. Then, we have:

$$\lim_{x \to x_0} f(x) = L \Rightarrow$$
$$\Rightarrow \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow f(x) \in I(L, \varepsilon))$$

Let $\underline{\varepsilon \in (0, +\infty)}$ be given. Choose $\underline{\delta \in (0, +\infty)}$ such that $\forall x \in A : (x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

Let $\underline{x \in A}$ be given. Then, we have:

$$\underline{x \in (x_0, x_0 + \delta)} \Rightarrow x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow$$
$$\Rightarrow \underline{f(x) \in I(L, \varepsilon)}$$

and

$$\underline{x \in (x_0 - \delta, x_0)} \Rightarrow x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta) \Rightarrow$$
$$\Rightarrow \underline{f(x) \in I(L, \varepsilon)}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : \begin{cases} x \in (x_0 - \delta, x_0) \Rightarrow f(x) \in I(L, \varepsilon) \\ x \in (x_0, x_0 + \delta) \Rightarrow f(x) \in I(L, \varepsilon) \end{cases}$$

$$\Rightarrow \lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L.$$

$(\Rightarrow)$: Assume that $\lim_{x\to x_0^+} f(x)=L \wedge \lim_{x\to x_0^-} f(x)=L$.

Let $\underline{\varepsilon\in(0,+\infty)}$ be given. Then, we have:

$$\begin{cases}\lim_{x\to x_0^+} f(x)=L \\ \lim_{x\to x_0^-} f(x)=L\end{cases} \Rightarrow \begin{cases}\exists\delta\in(0,+\infty):\forall x\in A:(x\in(x_0-\delta,x_0)\Rightarrow f(x)\in I(L,\varepsilon)) \\ \exists\delta\in(0,+\infty):\forall x\in A:(x\in(x_0,x_0+\delta)\Rightarrow f(x)\in I(L,\varepsilon))\end{cases}$$

Choose $\delta_1,\delta_2\in(0,+\infty)$ such that

$$\begin{cases}\forall x\in A:(x\in(x_0-\delta_1,x_0)\Rightarrow f(x)\in I(L,\varepsilon)) \\ \forall x\in A:(x\in(x_0,x_0+\delta_2)\Rightarrow f(x)\in I(L,\varepsilon))\end{cases}$$

Choose $\underline{\delta=\min\{\delta_1,\delta_2\}}$. Let $\underline{x\in A}$ be given and assume that $\underline{x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta)}$. Then, we have:

$$\begin{aligned} x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta) &\Rightarrow x\in(x_0-\delta,x_0)\vee x\in(x_0,x_0+\delta) \\ &\Rightarrow x\in(x_0-\delta_1,x_0)\vee x\in(x_0,x_0+\delta_2) \\ &\Rightarrow f(x)\in I(L,\varepsilon)\vee f(x)\in I(L,\varepsilon) \\ &\Rightarrow \underline{f(x)\in I(L,\varepsilon)} \end{aligned}$$

We have thus shown that

$$\forall\varepsilon\in(0,+\infty):\exists\delta\in(0,+\infty):\forall x\in A:(x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta)\Rightarrow$$
$$\Rightarrow f(x)\in I(L,\varepsilon))$$

$$\Rightarrow \lim_{x\to x_0} f(x)=L$$ □

▸ An immediate consequence of this result is the following statement:

$$\boxed{\lim_{x\to x_0^+} f(x)=L_1 \wedge \lim_{x\to x_0^-} f(x)=L_2 \wedge L_1\neq L_2 \Rightarrow \lim_{x\to x_0} f(x) \text{ does not exist}}$$

→ Methodology: To show that $\lim_{x\to\sigma} f(x) = L$ by definition

•1 Investigate $f(x) \in I(L,\varepsilon)$ and, if needed, restrict the domain A of f to $A = A_0 \cap N(\sigma,\delta)$ for an appropriate δ with A_0 the widest possible domain.

•2 Let $\underline{\varepsilon \in (0,+\infty)}$ be given. Derive an equivalence

$$f(x) \in \$ \subseteq I(L,\varepsilon) \Leftrightarrow x \in N(\sigma, g(\varepsilon))$$

•3 Choose $\underline{\delta = g(\varepsilon)}$. Let $\underline{x \in A}$ be given and assume that $\underline{x \in N(\sigma,\delta)}$. Then, we have

$$x \in N(\sigma,\delta) \Rightarrow x \in N(\sigma, g(\varepsilon)) \Rightarrow f(x) \in \$ \Rightarrow$$
$$\Rightarrow \underline{f(x) \in I(L,\varepsilon)}.$$

•4 We have thus shown that

$$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(\sigma,\delta) \Rightarrow f(x) \in I(L,\varepsilon))$$
$$\Rightarrow \lim_{x\to\sigma} f(x) = L.$$

EXAMPLES

a) Use the limit definition to show that for

$$f(x) = \begin{cases} 3x, & \text{if } x \in [1, +\infty) \\ -x+4, & \text{if } x \in (-\infty, 1) \end{cases}$$

we have $\lim\limits_{x \to 1} f(x) = 3$

Solution

• Limit $x \to 1^+$: Restrict the domain of f to $A = [1, +\infty)$.
Let $\varepsilon \in (0, +\infty)$ be given. Then for all $x \in A$, we have
$|f(x) - 3| = |3x - 3| = |3(x-1)| = 3|x-1| < \varepsilon \Leftrightarrow |x-1| < \varepsilon/3.$
Choose $\delta = \varepsilon/3$. Let $x \in A$ be given and assume that $x \in N(1^+, \delta)$. Then, we have:

$$x \in N(1^+, \delta) \Rightarrow x \in N(1, \delta) \Rightarrow 0 < |x-1| < \delta \Rightarrow$$
$$\Rightarrow |x-1| < \varepsilon/3 \Rightarrow |f(x) - 3| < \varepsilon.$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(1^+, \delta) \Rightarrow |f(x) - 3| < \varepsilon)$$
$$\Rightarrow \lim_{x \to 1^+} f(x) = 3.$$

• Limit $x \to 1^-$: Restrict the domain of f to $A = (-\infty, 1)$.
Let $\varepsilon \in (0, +\infty)$ be given. Then, for all $x \in A$, we have
$|f(x) - 3| = |(-x+4) - 3| = |-x+1| = |x-1|.$
Choose $\delta = \varepsilon$. Let $x \in A$ be given and assume that $x \in N(1^-, \delta)$. Then, we have:

$$x \in N(1^-, \delta) \Rightarrow x \in N(1, \delta) \Rightarrow 0 < |x-1| < \delta \Rightarrow$$
$$\Rightarrow |x-1| < \varepsilon \Rightarrow |f(x) - 3| < \varepsilon$$

We have thus shown that

$$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in A : (x \in N(1^-,\delta) \Rightarrow |f(x)-3| < \varepsilon)$$

$$\Rightarrow \lim_{x \to 1^-} f(x) = 3.$$

- From the above, we conclude that
  $$\lim_{x \to 1^+} f(x) = 3 \land \lim_{x \to 1^-} f(x) = 3 \Rightarrow \lim_{x \to 1} f(x) = 3.$$

THEORY QUESTIONS

(8) Let $f: A \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be a limit point of A and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Show that

a) $\lim_{x \to x_0} f(x) = L \Rightarrow \left(\lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L\right)$

b) $\left(\lim_{x \to x_0^+} f(x) = L \wedge \lim_{x \to x_0^-} f(x) = L\right) \Rightarrow \lim_{x \to x_0} f(x) = L$

EXERCISES

(9) Let $f: A \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ limit point of A and let $L_1, L_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$. Show that

$$\left\{\begin{array}{l} \lim_{x \to x_0^+} f(x) = L_1 \\ \lim_{x \to x_0^-} f(x) = L_2 \\ L_1 \neq L_2 \end{array}\right. \Rightarrow \lim_{x \to x_0} f(x) \text{ does not exist.}$$

(10) Use the limit definition, in conjunction with side limits, to show that

a) $\lim_{x \to 2} f(x) = 7$ with $f(x) = \begin{cases} 3x+1, & \text{if } x \in [2, +\infty) \\ 4x-1, & \text{if } x \in (-\infty, 2) \end{cases}$

b) $\lim_{x \to 1} f(x) = 2$ with $f(x) = \begin{cases} x^2+x, & \text{if } x \in (-\infty, 1) \\ 2x^3, & \text{if } x \in (1, +\infty) \end{cases}$

c) $\lim_{x\to -1} f(x)$ does not exist, with

$$f(x)=\begin{cases} 3x^2 & , \text{ if } x\in(-\infty,-1) \\ 3x & , \text{ if } x\in(-1,+\infty) \end{cases}$$

d) $\lim_{x\to 3} f(x)$ does not exist, with

$$f(x)=\begin{cases} x(x+2) & , \text{ if } x\in(3,+\infty) \\ x^2-2 & , \text{ if } x\in(-\infty,3). \end{cases}$$

▼ Function limits as net limits

Function limits are a special case of a net limit, and as such they inherit all the properties that we have previously established on convergent nets. The connection between the two concepts is established by the following theorem:

Thm: Let $f: A \to \mathbb{R}$ be a function, let σ be a limit point of A, let $\delta_0 \in (0, +\infty)$, and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, define $(D, <_\sigma)$ such that

$$\begin{cases} D = N(\sigma, \delta_0) \cap A \\ \|x\|_\sigma = \inf\{\delta \in (0, +\infty) \mid x \in N(\sigma, \delta) \cap A\} \\ \forall x_1, x_2 \in D : (x_1 <_\sigma x_2 \Leftrightarrow \|x_1\|_\sigma \geq \|x_2\|_\sigma) \end{cases}$$

Then, we have:

$$\begin{cases} (D, <_\sigma) \text{ is a directed set} \\ \lim_{x \to \sigma} f(x) = L \Leftrightarrow \lim_{x \in D} f(x) = L \end{cases}$$

→ Note that $\|x\|_\sigma$ represents how close x is to the limit point σ. Also, $x_1 <_\sigma x_2$ is the statement that x_2 is closer to the limit point σ than x_1.

Proof

Since σ limit point of A, it follows that $D = N(\sigma, \delta_0) \cap A \neq \emptyset$.
Define $S(x) = \{\delta \in (0, +\infty) \mid x \in N(\sigma, \delta) \cap A\}$.

▶ We claim that $\|x\|_\sigma = \inf S(x)$ is well-defined for all $x \in D$.

Let $x \in D$ be given. Then, we have:

$S(x) = \{\delta \in (0,+\infty) \mid x \in N(\sigma,\delta)\} \subseteq (0,+\infty) \subseteq \mathbb{R} \Rightarrow S(x) \subseteq \mathbb{R}$ (1)

and

$x \in D \Rightarrow x \in N(\sigma, \delta_0) \cap A \Rightarrow \delta_0 \in S(x) \Rightarrow S(x) \neq \emptyset$ (2)

and

$S(x) \subseteq (0,+\infty) \Rightarrow \forall \delta \in S(x) : \delta \in (0,+\infty)$
$\Rightarrow \forall \delta \in S(x) : \delta \geq 0$
$\Rightarrow S(x)$ lower bounded (3)

From Eq. (1), Eq. (2), Eq. (3) via the axiom of completeness it follows that $\|x\|_\sigma = \inf S(x)$ is well-defined.
This proves the claim

▶ We will show that $(D, <_\sigma)$ is a directed set.

• <u>$<_\sigma$ reflective property.</u>

Let <u>$x \in D$</u> be given. Then $\|x\|_\sigma \geq \|x\|_\sigma \Rightarrow$ <u>$x <_\sigma x$</u>. It follows that:

$\forall x \in D : x <_\sigma x$

• <u>$<_\sigma$ transitive property.</u>

Let <u>$x, y, z \in D$</u> be given and assume that <u>$x <_\sigma y$</u> and <u>$y <_\sigma z$</u>. Then, we have:

$$\begin{cases} x <_\sigma y \\ y <_\sigma z \end{cases} \Rightarrow \begin{cases} \|x\|_\sigma \geq \|y\|_\sigma \\ \|y\|_\sigma \geq \|z\|_\sigma \end{cases} \Rightarrow \|x\|_\sigma \geq \|z\|_\sigma \Rightarrow \underline{x <_\sigma z}$$

We have thus shown that

$\forall x, y, z \in D : ((x <_\sigma y \wedge y <_\sigma z) \Rightarrow x <_\sigma z)$

• <u>$<_\sigma$ refinement property</u>

Let <u>$x, y \in D$</u> be given. Since σ is a limit point of A, choose $z \in N(\sigma, \min\{\|x\|_\sigma, \|y\|_\sigma\}) \cap A$. Then, we have:

$$\|z\|_\sigma \leq \min\{\|x\|_\sigma, \|y\|_\sigma\} \Rightarrow \begin{cases} \|z\|_\sigma \leq \|x\|_\sigma \\ \|z\|_\sigma \leq \|y\|_\sigma \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \underline{x <_\sigma z} \\ \underline{y <_\sigma z} \end{cases}$$

We have thus shown that

$\forall x, y \in D : \exists z \in D : (x <_\sigma z \wedge y <_\sigma z)$

From the above, we conclude that $(D, <_\sigma)$ is a directed set.

▸ We will show that $\lim_{x \to \sigma} f(x) = L \Leftrightarrow \lim_{x \in D} f(x) = L$.

$(\Rightarrow)$: Assume that $\lim_{x \to \sigma} f(x) = L$. It follows that

$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta \in (0, +\infty)$ such that

$\forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

Since σ limit point of A, we can choose

$n_0 \in N(\sigma, \min\{\delta_0, \delta\}) \cap A \subseteq D. \Rightarrow \underline{n_0 \in D}$.

and note that $\|n_0\|_\sigma \leq \min\{\delta_0, \delta\}$. Let $\underline{n \in D}$ be given and assume that $\underline{n >_\sigma n_0}$. Then, we have:

$$n >_\sigma n_0 \Rightarrow \|n\|_\sigma \leq \|n_0\|_\sigma \leq \min\{\delta_0, \delta\} \leq \delta \Rightarrow$$

$$\Rightarrow \|n\|_\sigma \leq \delta \Rightarrow n \in N(\sigma, \delta) \cap A \Rightarrow \underline{f(n) \in I(L, \varepsilon)}.$$

We have thus shown that

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n >_\sigma n_0 \Rightarrow f(n) \in I(L, \varepsilon))$

$\Rightarrow \lim_{n \in D} f(n) = L$.

$(\Leftarrow)$: Assume that $\lim_{n \in D} f(n) = L$. Then, we have:

$\forall \varepsilon \in (0, +\infty) : \exists n_0 \in D : \forall n \in D : (n >_\sigma n_0 \Rightarrow f(n) \in I(L, \varepsilon))$

Let $\varepsilon \in (0, +\infty)$ be given. Chose $n_0 \in D$ such that

$\forall n \in D : (n >_\sigma n_0 \Rightarrow f(n) \in I(L, \varepsilon))$

Choose $\delta = \|n_0\|_\sigma \in (0, +\infty)$. Let $x \in A$ be given and assume that $x \in N(\sigma, \delta)$. Then, we have:

$$\begin{cases} x \in A \\ x \in N(\sigma, \delta) \end{cases} \Rightarrow x \in N(\sigma, \delta) \cap A \Rightarrow \|x\|_\sigma \leq \delta = \|n_0\|_\sigma$$

$$\Rightarrow \|x\|_\sigma \leq \|n_0\|_\sigma \Rightarrow x >_\sigma n_0 \Rightarrow f(x) \in I(L, \varepsilon).$$

We have thus shown that:

$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (x \in N(\sigma, \delta) \Rightarrow f(x) \in I(L, \varepsilon))$

$\Rightarrow \lim_{x \to \sigma} f(x) = L.$

This concludes the proof $\square$

Properties of limits of functions

Since limits of functions are special cases of net limits, the following properties of function limits are immediately obtained:

1. → Uniqueness

$\boxed{\begin{array}{l}\text{Let } f: A \to \mathbb{R} \text{ with } \sigma \text{ limit point of } A \text{ and let} \\ L_1, L_2 \in \mathbb{R} \cup \{+\infty, -\infty\}. \text{ Then, we have:} \\ \left(\lim\limits_{x\to\sigma} f(x) = L_1 \wedge \lim\limits_{x\to\sigma} f(x) = L_2\right) \Rightarrow L_1 = L_2.\end{array}}$

2. → Functions with finite limits

Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and let σ be a limit point of both A and B and assume that:

$$\lim_{x\to\sigma} f(x) = l_1 \in \mathbb{R} \wedge \lim_{x\to\sigma} g(x) = l_2 \in \mathbb{R}$$

Then, we have:

a) $\lim\limits_{x\to\sigma} [f(x) + g(x)] = l_1 + l_2$

b) $\lim\limits_{x\to\sigma} [f(x) g(x)] = l_1 l_2$

c) $\forall a \in \mathbb{R}: \lim\limits_{x\to\sigma} [af(x)] = a l_1$

d) $l_2 \neq 0 \Rightarrow \lim\limits_{x\to\sigma} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{l_1}{l_2}$

e) $\lim\limits_{x\to\sigma} |f(x)| = |l_1|$

f) $l_1 > 0 \Rightarrow \forall k \in \mathbb{N}^*: \lim\limits_{x\to\sigma} \sqrt[k]{f(x)} = \sqrt[k]{l_1}$

3 → Functions with limits going to infinity.

Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and let σ be a limit point of A and B. Let $\delta \in (0, +\infty)$ and $a \in \mathbb{R}$. Then, we have:

a) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a \\ \lim\limits_{x \to \sigma} f(x) = +\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) + g(x)] = +\infty$

b) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a \\ \lim\limits_{x \to \sigma} f(x) = -\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) + g(x)] = -\infty$

c) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) > a > 0 \\ \lim\limits_{x \to \sigma} f(x) = \pm\infty, \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) g(x)] = \pm\infty$

d) $\begin{cases} \forall x \in N(\sigma, \delta) \cap B : g(x) < a < 0 \\ \lim\limits_{x \to \sigma} f(x) = \pm\infty \end{cases} \Rightarrow \lim\limits_{x \to \sigma} [f(x) g(x)] = \mp\infty$

When the limit of $g(x)$ is also known, then this result can be combined with the following statements:

$\lim\limits_{x \to \sigma} g(x) > 0 \lor \lim\limits_{x \to \sigma} g(x) = +\infty \Rightarrow \exists \delta \in (0, +\infty) : \exists a \in \mathbb{R} : \forall x \in N(\sigma, \delta) \cap B :$
$: g(x) > a > 0$

$\lim\limits_{x \to \sigma} g(x) < 0 \lor \lim\limits_{x \to \sigma} g(x) = -\infty \Rightarrow \exists \delta \in (0, +\infty) : \exists a \in \mathbb{R} : \forall x \in N(\sigma, \delta) \cap B :$
$: g(x) < a < 0$

giving several deductions that are summarized in the tables given below:

$f(x)\downarrow \setminus g(x)\to$	a	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	?
$-\infty$	$-\infty$	?	$-\infty$

$$\lim_{x\to\sigma}[f(x)+g(x)]$$

$f(x)\downarrow\ g(x)\to$	0	$p>0$	$n<0$	$+\infty$	$-\infty$
$+\infty$	?	$+\infty$	$-\infty$	$+\infty$	$-\infty$
$-\infty$	?	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$\lim_{x\to\sigma}[f(x)g(x)]$$

↳ The "?" correspond to indeterminate forms. It means that the limit cannot be determined without more information, and the limit may or may not exist.

9 → <u>Limit forms $k/0$, $1/(\pm\infty)$.</u>

Let $f: A\to\mathbb{R}$ and $\delta\in(0,+\infty)$ and let σ be a limit point of A.

a) $\begin{cases}\forall x\in N(\sigma,\delta)\cap A: f(x)>0\\ \lim_{x\to\sigma} f(x)=0\end{cases} \Rightarrow \lim_{x\to\sigma}\dfrac{1}{f(x)}=+\infty$

b) $\begin{cases}\forall x\in N(\sigma,\delta)\cap A: f(x)<0\\ \lim_{x\to\sigma} f(x)=0\end{cases} \Rightarrow \lim_{x\to\sigma}\dfrac{1}{f(x)}=-\infty$

c) $\lim_{x\to\sigma} f(x)\in\{+\infty,-\infty\} \Rightarrow \lim_{x\to\sigma}\dfrac{1}{f(x)}=0$

Immediate consequences of limit properties

The following results are immediate consequences of the limit properties

(1) Monomial function

$$\forall x_0 \in \mathbb{R} : \forall k \in \mathbb{N}^* : \lim_{x \to x_0} x^k = x_0^k$$

$$\forall k \in \mathbb{N}^* : \lim_{x \to +\infty} x^k = +\infty \qquad \forall k \in \mathbb{N}^* : \lim_{x \to -\infty} x^{2k+1} = -\infty$$

$$\forall k \in \mathbb{N}^* : \lim_{x \to -\infty} x^{2k} = +\infty \qquad \forall k \in \mathbb{N}^* : \lim_{x \to \pm\infty} x^{-k} = 0$$

(2) Polynomial function

Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.
Then, we have:

a) $\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} f(x) = f(x_0)$

b) $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n$

③ → Rational function

Let $P : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$ with

$$\begin{cases} P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \\ Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0 \end{cases}$$

Then, we have:

a) $\forall x_0 \in \mathbb{R} : \left(Q(x_0) \neq 0 \Rightarrow \lim\limits_{x \to x_0} \dfrac{P(x)}{Q(x)} = \dfrac{P(x_0)}{Q(x_0)} \right)$

b) $\lim\limits_{x \to \pm\infty} \dfrac{P(x)}{Q(x)} = \lim\limits_{x \to \pm\infty} \dfrac{a_n x^n}{b_m x^m}$

④ → Rational k/0 limits

$\forall a \in \mathbb{R} : \lim\limits_{x \to a^+} \dfrac{1}{x-a} = +\infty$

$\forall a \in \mathbb{R} : \lim\limits_{x \to a^-} \dfrac{1}{x-a} = -\infty$

$\forall a \in \mathbb{R} : \lim\limits_{x \to a} \dfrac{1}{x-a} = +\infty$

EXAMPLES

a) Let $f(x) = \dfrac{ax^3 + x - 2}{(a-1)x^2 + x + 1}$. Use the properties of limits to calculate $\lim\limits_{x \to +\infty} f(x)$

Solution

We distinguish between the following cases:

Case 1: Assume that $a \in \mathbb{R} - \{0,1\}$. Then, we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{ax^3 + x - 2}{(a-1)x^2 + x + 1} = \lim_{x\to+\infty} \frac{ax^3}{(a-1)x^2} =$$

$$= \frac{a}{a-1} \lim_{x\to+\infty} x = \frac{a}{a-1}(+\infty)$$

We use a sign table for the sign of $a/(a-1)$:

a		0		1	
a	$-$	0	$+$		$+$
$a-1$	$-$		$-$	0	$+$
	$+$	0	$-$	$\Vert$	$+$

to conclude that:

$$\lim_{x\to+\infty} f(x) = \begin{cases} +\infty, & \text{if } a \in (-\infty,0) \cup (1,+\infty) \\ -\infty, & \text{if } a \in (0,1) \end{cases}$$

Case 2: Assume that $a = 0$. Then, we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x-2}{-x^2 + x + 1} = \lim_{x\to+\infty} \frac{x}{-x^2} =$$

$$= \lim_{x\to+\infty} \frac{-1}{x} = 0$$

<u>Case 3</u>: Assume that $a=1$. Then, we have:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3+x-2}{x+1} = \lim_{x\to+\infty} \frac{x^3}{x} = \lim_{x\to+\infty} x^2$$

$$= +\infty .$$

We conclude that:

$$\lim_{x\to+\infty} f(x) = \begin{cases} +\infty & , \text{ if } a\in(-\infty,0)\cup[1,+\infty) \\ -\infty & , \text{ if } a\in(0,1) \\ 0 & , \text{ if } a=0 \end{cases}$$

b) Use the limit properties to calculate $\lim_{x \to -\infty} f(x)$ for $f(x) = \sqrt{x^2 - 5x + 6} + ax$, for all $a \in \mathbb{R}$.

<u>Solution</u>

Since

$$\lim_{x \to -\infty} (x^2 - 5x + 6) = \lim_{x \to -\infty} x^2 = +\infty \Rightarrow$$

$$\Rightarrow \exists \mu \in (0, +\infty) : \forall x \in (-\infty, -\mu) : x^2 - 5x + 6 > 0$$

it follows that $-\infty$ is a limit point of the domain of f for all $a \in \mathbb{R}$. Choose $\mu \in (0, +\infty)$ such that $\forall x \in (-\infty, -\mu) : x^2 - 5x + 6 > 0$.

Then, we have:

$$\forall x \in (-\infty, -\mu) : f(x) = \sqrt{x^2 - 5x + 6} + ax =$$

$$= \sqrt{x^2 (1 - 5x^{-1} + 6x^{-2})} + ax =$$

$$= |x| \sqrt{1 - 5x^{-1} + 6x^{-2}} + ax =$$

$$= -x \sqrt{1 - 5x^{-1} + 6x^{-2}} + ax = \quad [\text{via } x < -\mu < 0]$$

$$= x(a - \sqrt{1 - 5x^{-1} + 6x^{-2}}) \equiv x g(x)$$

where we define $g(x) = a - \sqrt{1 - 5x^{-1} + 6x^{-2}}$

and

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} [a - \sqrt{1 - 5x^{-1} + 6x^{-2}}] = a - \sqrt{1 - 0 + 0}$$

$$= a - 1$$

We distinguish between the following cases:

<u>Case 1:</u> Assume that $a \in (1, +\infty)$. Then, we have:

$$\lim_{x \to -\infty} x = -\infty \wedge \lim_{x \to -\infty} g(x) = a - 1 > 0 \Rightarrow \lim_{x \to -\infty} f(x) = -\infty$$

Case 2: Assume that $a \in (-\infty, 1)$. Then, we have:

$\lim\limits_{x\to-\infty} x = -\infty \wedge \lim\limits_{x\to-\infty} g(x) = a-1 < 0 \Rightarrow \lim\limits_{x\to+\infty} f(x) = +\infty$

Case 3: Assume that $a = 1$. Then, we have:

$$f(x) = \sqrt{x^2-5x+6} + x = \frac{(\sqrt{x^2-5x+6})^2 - x^2}{\sqrt{x^2-5x+6} - x} =$$

$$= \frac{x^2-5x+6-x^2}{|x|\sqrt{1-5x^{-1}+6x^{-2}} - x} = \frac{-5x+6}{-x\sqrt{1-5x^{-1}+6x^{-2}} - x}$$

$$= \frac{-5+6x^{-1}}{-\sqrt{1-5x^{-1}+6x^{-2}} - 1}, \quad \forall x \in (-\infty, -\rho)$$

$$\Rightarrow \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-5+6x^{-1}}{-\sqrt{1-5x^{-1}+6x^{-2}} - 1}$$

$$= \frac{-5+0}{-\sqrt{1-0+0} - 1} = \frac{-5}{-1-1} = \frac{5}{2}$$

From all of the above, we conclude that

$$\lim_{x\to-\infty} f(x) = \begin{cases} -\infty, & \text{if } a \in (1, +\infty) \\ +\infty, & \text{if } a \in (-\infty, 1) \\ 5/2, & \text{if } a = 1 \end{cases}$$

THEORY QUESTION

(11) Let $f: A \to \mathbb{R}$ and let σ be a limit point of A and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. State the construction of the limit statement $\lim_{x \to \sigma} f(x) = L$ in terms of a directed set $(D, <)$ and the corresponding directed set limits.

EXERCISES

(12) Let $f: \mathbb{R} \to \mathbb{R}$ such that

$\forall x \in \mathbb{R}: f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

Show that: $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n$

(13) Let $p: \mathbb{R} \to \mathbb{R}$ and $q: \mathbb{R} \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}: \begin{cases} p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \end{cases}$$

Show that: $\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \lim_{x \to \pm\infty} \frac{a_n x^n}{b_m x^m}$

(14) Use the limit properties to evaluate the following limits for all $a \in \mathbb{R}$.

a) $\lim_{x \to +\infty} f(x)$ with $f(x) = \dfrac{(a^2-1)x^2 + ax}{(a-1)x^2 + (a+1)x + 3}$

b) $\lim_{x \to -\infty} f(x)$ with $f(x) = \dfrac{(a-1)x^3 + (a+1)x}{(a+1)x^3 + (a-1)x}$

c) $\lim\limits_{x\to+\infty} f(x)$ with $f(x) = \dfrac{ax^3+(a+1)x^2-ax+3}{(a+1)x^2+2ax+1}$

d) $\lim\limits_{x\to-\infty} f(x)$ with $f(x) = \dfrac{x^2}{x+a} - \dfrac{x^2}{x-a}$

(15) Consider the function $f: A \to \mathbb{R}$ with $f(x) = \sqrt{a-x^2-2x} - x$ with $a \in \mathbb{R}$. Show that:
$\forall a \in \mathbb{R}: \lim\limits_{x\to+\infty} f(x)$ not well-defined.

↳ We say that for $f: A \to \mathbb{R}$, the limit $\lim\limits_{x\to\sigma} f(x)$ is <u>not well-defined</u> $\Leftrightarrow$ σ not a limit point of A.

(16) Find the set $S \subseteq \mathbb{R}$ of all $a \in \mathbb{R}$ for which the following limits are well-defined, and then evaluate the limit in terms of the parameter $a \in S$.

a) $\lim\limits_{x\to+\infty} f(x)$ with $f(x) = x\left(\sqrt{ax^2+6x+3} - x\right)$

b) $\lim\limits_{x\to-\infty} f(x)$ with $f(x) = \sqrt{ax^2+2x-1} - \sqrt{x^2+1}$

c) $\lim\limits_{x\to+\infty} f(x)$ with $f(x) = \sqrt{x^2-4x+a} - x$

d) $\lim\limits_{x\to-\infty} f(x)$ with $f(x) = |x|\left[\sqrt{ax^2+2x+1} - x\right]$

(17) Use the limit properties to show that

a) $\lim\limits_{x\to+\infty} \left[\sqrt{4x^2-3x+1} - ax + b\right] = 1/4 \Leftrightarrow (a,b) = (2,1)$

b) $\lim\limits_{x\to-\infty} \left[\sqrt[3]{x^3+1} - ax - b\right] = 0 \Leftrightarrow (a,b) = (1,0)$

▼ Limit composition theorem

Def: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$. We define the composition $f \circ g: G \to \mathbb{R}$ such that

$$\begin{cases} G = \{x \in B \mid g(x) \in A\} \\ \forall x \in G: (f \circ g)(x) = f(g(x)) \end{cases}$$

Note that the belonging condition for the domain G of $f \circ g$ is given by

$$\boxed{x \in G \Longleftrightarrow \begin{cases} x \in B \\ g(x) \in A \end{cases}}$$

The following theorems make it possible to calculate the limit of the composition $f \circ g$:

Thm: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: G \to \mathbb{R}$. Let σ be a limit point of G. Then, we have:

$$\begin{cases} \lim_{x \to \sigma} g(x) = a \in \mathbb{R} \\ \lim_{x \to a} f(x) = f(a) \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = f(a)$$

Proof

Let $\varepsilon \in (0, +\infty)$ be given. Since

$\lim_{x \to a} f(x) = f(a) \Rightarrow \exists \delta \in (0, +\infty): \forall x \in A: (0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

We choose $\delta_1 \in (0, +\infty)$ such that
$\forall x \in A : (0 < |x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \varepsilon)$
Since:
$\lim_{x \to \sigma} g(x) = a \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$

choose $\underline{\delta \in (0, +\infty)}$ such that
$\forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$
Let $\underline{x \in G}$ be given and assume that $\underline{x \in N(\sigma, \delta)}$. We will show that $|f(g(x)) - f(a)| < \varepsilon$. We have:

$$\begin{cases} x \in G \\ x \in N(\sigma, \delta) \end{cases} \Rightarrow \begin{cases} x \in B \\ x \in N(\sigma, \delta) \end{cases} \quad [\text{via } G \subseteq B]$$
$$\Rightarrow |g(x) - a| < \delta_1$$

We need the stronger condition $0 < |g(x) - a| < \delta_1$, so we distinguish between the following cases.

Case 1: Assume that $g(x) = a$. Then, we have
$|f(g(x)) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon \Rightarrow \underline{|f(g(x)) - f(a)| < \varepsilon}.$

Case 2: Assume that $g(x) \neq a$. Then, we have:

$$\begin{cases} 0 < |g(x) - a| < \delta_1 \\ g(x) \in A \end{cases} \Rightarrow \underline{|f(g(x)) - f(a)| < \varepsilon}$$

We have thus shown that
$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in G : (x \in N(\sigma, \delta) \Rightarrow |f(g(x)) - f(a)| < \varepsilon)$
$\Rightarrow \lim_{x \to \sigma} f(g(x)) = f(a).$ $\square$

If we replace the condition $\lim_{x \to a} f(x) = f(a)$ (compare with definition of continuity, given later) with the more general statement $\lim_{x \to a} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\}$, then we need an additional assumption to ensure the composition theorem still works:

Prop: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: C \to \mathbb{R}$, and let σ be a limit point of C. Then, we have:

$$\begin{cases} \lim_{x \to \sigma} g(x) = a \in \mathbb{R} \\ \lim_{x \to a} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \exists \delta \in (0, +\infty) : \forall x \in B \cap N(\sigma, \delta) : g(x) \neq a \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = L$$

Proof

Let $\varepsilon \in (0, +\infty)$ be given. By hypothesis, choose $\delta_0 \in (0, +\infty)$ such that

$\forall x \in B \cap N(\sigma, \delta_0) : g(x) \neq a$.

Since:

$$\lim_{x \to a} f(x) = L \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in A : (0 < |x - a| < \delta \Rightarrow f(x) \in I(L, \varepsilon))$$

choose $\delta_1 \in (0, +\infty)$ such that

$\forall x \in A : (0 < |x - a| < \delta_1 \Rightarrow f(x) \in I(L, \varepsilon))$

Since:

$$\lim_{x \to \sigma} g(x) = a \Rightarrow \exists \delta \in (0, +\infty) : \forall x \in B : (x \in N(\sigma, \delta) \Rightarrow |g(x) - a| < \delta_1)$$

choose $\delta_2 \in (0, +\infty)$ such that

$\forall x \in B : (x \in N(\sigma, \delta_2) \Rightarrow |g(x) - a| < \delta_1)$

Choose $\underline{\delta \in (0, +\infty)}$ such that $\delta = \min\{\delta_0, \delta_2\}$. Let $\underline{x \in G}$ be given and assume that $\underline{x \in N(\sigma, \delta)}$. Then we have

$$x \in N(\sigma, \delta) \cap G \Rightarrow \begin{cases} x \in N(\sigma, \delta_0) \cap B \\ x \in N(\sigma, \delta_2) \cap G \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} g(x) \neq a \\ |g(x) - a| < \delta_1 \\ x \in G \end{cases} \Rightarrow \begin{cases} 0 < |g(x) - a| < \delta_1 \\ g(x) \in A \end{cases} \Rightarrow$$

$$\Rightarrow \underline{f(g(x)) \in I(L, \varepsilon)}$$

We have thus shown that

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in G : (x \in N(\sigma, \delta) \Rightarrow f(g(x)) \in I(L, \varepsilon))$$

$$\Rightarrow \lim_{x \to \sigma} f(g(x)) = L. \qquad \square$$

For the case $\lim_{x \to \sigma} g(x) \in \{+\infty, -\infty\}$, the previous proof can be modified to show the following statement:

Prop: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and $f \circ g : G \to \mathbb{R}$, and let σ be a limit point of G. Then, we have:

a) $$\begin{cases} \lim_{x \to \sigma} g(x) = +\infty \\ \lim_{x \to +\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = L$$

b) $$\begin{cases} \lim_{x \to \sigma} g(x) = -\infty \\ \lim_{x \to -\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim_{x \to \sigma} f(g(x)) = L$$

Proof : $\boxed{\text{Homework.}}$

→ The following statements are immediate consequences of the limit composition theorems:

1) $\lim\limits_{x \to +\infty} f(x) = L \Rightarrow \lim\limits_{n \in \mathbb{N}^*} f(n) = L$
2) $\lim\limits_{x \to x_0} f(x) = L \Leftrightarrow \lim\limits_{h \to 0} f(x_0 + h) = L$
3) $\lim\limits_{x \to x_0} f(x) = L \Leftrightarrow \lim\limits_{h \to 1} f(x_0 h) = L$

→ When the function g is a sequence with $g : \mathbb{N}^* \to \mathbb{R}$ and we choose $\sigma = +\infty$, then the composition theorems give the following statements:

1) $\left\{ \begin{array}{l} \lim\limits_{n \in \mathbb{N}^*} a_n = x_0 \in \mathbb{R} \\ \lim\limits_{x \to x_0} f(x) = f(x_0) \end{array} \right. \Rightarrow \lim\limits_{n \in \mathbb{N}^*} f(a_n) = f(x_0)$
2) $\left\{ \begin{array}{l} \lim\limits_{n \in \mathbb{N}^*} a_n = x_0 \in \mathbb{R} \\ \lim\limits_{x \to x_0} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \forall n \in \mathbb{N}^* : a_n \neq x_0 \end{array} \right. \Rightarrow \lim\limits_{n \in \mathbb{N}^*} f(a_n) = L$

3) $\begin{cases} \lim\limits_{n\in\mathbb{N}^*} a_n = +\infty \\ \lim\limits_{x\to+\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim\limits_{n\in\mathbb{N}^*} f(a_n) = L$
4) $\begin{cases} \lim\limits_{n\in\mathbb{N}^*} a_n = -\infty \\ \lim\limits_{x\to-\infty} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \end{cases} \Rightarrow \lim\limits_{n\in\mathbb{N}^*} f(a_n) = L$

THEORY QUESTIONS

(18) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$. State the domain and definition of the function composition $f \circ g$.

(19) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: C \to \mathbb{R}$ and let σ be a limit point of C. Prove that

a) $\lim\limits_{x \to \sigma} g(x) = a \in \mathbb{R} \wedge \lim\limits_{x \to a} f(x) = f(a) \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = f(a)$

b) $\begin{cases} \lim\limits_{x \to \sigma} g(x) = a \in \mathbb{R} \\ \lim\limits_{x \to a} f(x) = L \in \mathbb{R} \cup \{+\infty, -\infty\} \\ \exists \delta \in (0, +\infty) : \forall x \in B \cap N(\sigma, \delta) : g(x) \neq a \end{cases} \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = L$

EXERCISES

(20) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: C \to \mathbb{R}$, let σ be a limit point of C and let $L \in \mathbb{R} \cup \{+\infty, -\infty\}$.
Prove that:

a) $\lim\limits_{x \to \sigma} g(x) = +\infty \wedge \lim\limits_{x \to +\infty} f(x) = L \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = L$

b) $\lim\limits_{x \to \sigma} g(x) = -\infty \wedge \lim\limits_{x \to -\infty} f(x) = L \Rightarrow \lim\limits_{x \to \sigma} f(g(x)) = L$

(21) Let $f: A \to \mathbb{R}$ with $A = (0, +\infty)$. Use the composition theorem to show that

a) $\lim_{x \to +\infty} f(x) = L \iff \lim_{x \to a^+} f\left(\frac{1}{x-a}\right) = L$

b) $\lim_{x \to -\infty} f(x) = L \iff \lim_{x \to a^-} f\left(\frac{1}{x-a}\right) = L$

$\hookrightarrow$ Note that you have to show both the "$\Rightarrow$" and "$\Leftarrow$" statements using separate arguments.

Trigonometric limits

Limits of trigonometric functions are established via the inequality:

$$\boxed{\forall x \in (-\pi/2, 0) \cup (0, \pi/2): |\sin x| < |x| < |\tan x|}$$

via the limit definition, as follows:

Basic trigonometric limits

① $$\boxed{\lim_{x \to x_0} \sin x = \sin x_0 \quad , \forall x_0 \in \mathbb{R}}$$

Proof

Let $x_0 \in \mathbb{R}$ be given. Let $\varepsilon \in (0, +\infty)$ be given. Choose $\delta = \min\{\varepsilon, \pi/2\}$. Let $x \in \mathbb{R}$ be given and assume that $x \in N(x_0, \delta)$. Then, we have:

$$|\sin x - \sin x_0| = \left| 2\sin\left(\frac{x - x_0}{2}\right)\cos\left(\frac{x + x_0}{2}\right)\right| =$$

$$= 2\left|\sin\left(\frac{x - x_0}{2}\right)\right| \cdot \left|\cos\left(\frac{x + x_0}{2}\right)\right|$$

$$\leq 2\left|\sin\left(\frac{x - x_0}{2}\right)\right| \leq 2\left|\frac{x - x_0}{2}\right| =$$

$$= 2 \cdot \frac{|x - x_0|}{2} = |x - x_0| \Rightarrow |\sin x - \sin x_0| \leq |x - x_0|.$$

and it follows that

$$x \in N(x_0, \delta) \Rightarrow 0 < |x - x_0| < \delta \leqslant \varepsilon \Rightarrow |x - x_0| < \varepsilon \Rightarrow$$
$$\Rightarrow \underline{|\sin x - \sin x_0| < \varepsilon}$$

We have thus shown that:

$$\forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in \mathbb{R} : (x \in N(x_0, \delta) \Rightarrow |\sin x - \sin x_0| < \varepsilon)$$
$$\Rightarrow \lim_{x \to x_0} \sin x = \sin x_0$$

(2) $\boxed{\forall x_0 \in \mathbb{R} : \lim_{x \to x_0} \cos x = \cos x_0}$

<u>Proof</u>

Let <u>$x_0 \in \mathbb{R}$</u> be given. Let <u>$\varepsilon \in (0, +\infty)$</u> be given. Choose <u>$\delta = \min\{\varepsilon, \pi/2\}$</u>. Let <u>$x \in \mathbb{R}$</u> be given and assume that <u>$x \in N(x_0, \delta)$</u>. Then, we have:

$$|\cos x - \cos x_0| = \left| 2 \sin\left(\frac{x + x_0}{2}\right) \sin\left(\frac{x_0 - x}{2}\right) \right| =$$
$$= 2 \left| \sin\left(\frac{x + x_0}{2}\right) \right| \cdot \left| \sin\left(\frac{x_0 - x}{2}\right) \right|$$
$$\leqslant 2 \left| \sin\left(\frac{x_0 - x}{2}\right) \right| \leqslant 2 \left| \frac{x_0 - x}{2} \right| =$$
$$= 2 \frac{|x - x_0|}{2} = |x - x_0|$$

and it follows that

$$x \in N(x_0, \delta) \Rightarrow 0 < |x - x_0| < \delta \leqslant \varepsilon \Rightarrow |x - x_0| < \varepsilon \Rightarrow$$
$$\Rightarrow \underline{|\cos x - \cos x_0| < \varepsilon}$$

We have thus shown that

$\forall \varepsilon \in (0,+\infty) : \exists \delta \in (0,+\infty) : \forall x \in \mathbb{R} : (x \in N(\sigma,\delta) \Rightarrow |\cos x - \cos x_0| < \varepsilon)$

$\Rightarrow \lim_{x \to x_0} \cos x = \cos x_0$ $\square$

▸ Using the limit properties, from the previous two result we immediately obtain:

(3)	$\forall x_0 \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} : \lim_{x \to x_0} \tan x = \tan x_0$
(4)	$\forall x_0 \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} : \lim_{x \to x_0} \cot x = \cot x_0$

→ Upgrades via limit composition theorem

Using the composition theorem, these results can be upgraded to obtain:

(1)	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x \to \sigma} \sin(g(x)) = \sin a$
(2)	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} \Rightarrow \lim_{x \to \sigma} \cos(g(x)) = \cos a$
(3)	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \to \sigma} \tan(g(x)) = \tan a$
(4)	$\lim_{x \to \sigma} g(x) = a \in \mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\} \Rightarrow \lim_{x \to \sigma} \cot(g(x)) = \cot a$

0/0 trigonometric limits

The squeeze theorem for function limits follows from the squeeze theorem for convergent nets, and it reads:

Thm: Let $f: A \to \mathbb{R}$ and $g_1: A \to \mathbb{R}$ and $g_2: A \to \mathbb{R}$ and let σ be a limit point of A. Then, we have:

$$\begin{cases} \forall x \in A \cap N(\sigma, \delta): g_1(x) \leq f(x) \leq g_2(x) \\ \lim_{x \to \sigma} g_1(x) = \lim_{x \to \sigma} g_2(x) = \ell \in \mathbb{R} \end{cases} \Rightarrow \lim_{x \to \sigma} f(x) = \ell.$$

We now use the squeeze theorem to calculate the following limits:

① $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

Proof

Define $\forall x \in \mathbb{R}^*: f(x) = (\sin x)/x$. Let $x \in (-\pi/2, 0) \cup (0, \pi/2)$ be given. Then, we have:

$\sin x, x$ equisigned $\Rightarrow x \sin x > 0 \Rightarrow \frac{\sin x}{x} > 0$

and therefore:

$$f(x) = \frac{\sin x}{x} = \left| \frac{\sin x}{x} \right| = \frac{|\sin x|}{|x|} \leq \frac{|x|}{|x|} = 1$$

and

$$f(x) = \frac{\sin x}{x} = \left| \frac{\sin x}{x} \right| = \frac{|\sin x|}{|x|} \geq \frac{|\sin x|}{|\tan x|} = \left| \frac{\sin x}{\tan x} \right|$$

$$= \left| \frac{\sin x}{(\sin x)/(\cos x)} \cdot \right| = \left| \frac{1}{1/(\cos x)} \right| = |\cos x|$$

We have thus shown that

$$\forall x \in (-\pi/2, 0) \cup (0, \pi/2) : |\cos x| \leq f(x) \leq 1 \qquad (1)$$

It follows that

$$\lim_{x \to 0} \cos x = \cos 0 = 1 \Rightarrow \lim_{x \to 0} |\cos x| = |1| = 1 \Rightarrow$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 \quad [\text{via Eq. } (1)] \quad \square$$

An immediate corollary is:

② $$\boxed{\lim_{x \to 0} \frac{\tan x}{x} = 1}$$

Using the composition theorem, these results can be upgraded to give:

$$\boxed{\begin{cases} \forall x \in N(\sigma, \delta) \cap \mathrm{dom}(g) : g(x) \neq 0 \\ \lim\limits_{x \to \sigma} g(x) = 0 \end{cases} \Rightarrow \lim_{x \to \sigma} \frac{\sin(g(x))}{g(x)} = \lim_{x \to \sigma} \frac{\tan(g(x))}{g(x)} = 1}$$

An immediate consequence of these generalizations is that:

$\forall a \in \mathbb{R}^*$:	$\lim\limits_{x \to 0} \frac{\sin(ax)}{ax} = \lim\limits_{x \to 0} \frac{\tan(ax)}{ax} = 1$
$\forall a \in \mathbb{R}$:	$\lim\limits_{x \to a} \frac{\sin(x-a)}{x-a} = \lim\limits_{x \to a} \frac{\tan(x-a)}{x-a} = 1$

THEORY QUESTIONS

(22) Prove the following statements

a) $\forall x_0 \in \mathbb{R}: \lim_{x \to x_0} \sin x = \sin x_0$

b) $\forall x_0 \in \mathbb{R}: \lim_{x \to x_0} \cos x = \cos x_0$

c) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

EXERCISES

(23) Use the limit properties to evaluate the following limits (WITHOUT use of the De L'Hospital theorem).

a) $\lim_{x \to +\infty} \frac{\sin x}{x} = 0$

b) $\lim_{x \to 0} \frac{\cos x - \cos(5x)}{x \sin x} = 12$

c) $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{\sin(5x)} = \frac{1}{20}$

d) $\lim_{x \to 0^+} \frac{2x - \sin x}{\sqrt{1 - \cos x}} = \sqrt{2}$

e) $\lim_{x \to 0} \frac{1 - \cos x}{x \sin x} = \frac{1}{2}$

f) $\lim_{x \to \pi/2} \frac{1 - \sin x}{\sqrt{1 + \cos(2x)}} = 0$

g) $\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos(2x)}}{\sin^2 x} = \frac{\sqrt{2}}{2}$

h) $\lim_{x \to 0} \frac{1 - \cos^3 x}{x \sin(2x)} = \frac{3}{4}$

i) $\lim_{x \to 0} \left[\frac{2}{\sin^2 x} - \frac{1}{1 - \cos x} \right] = \frac{1}{2}$

j) $\lim_{x \to 0} \frac{\sqrt{\cos x} - 1}{x^2} = \frac{-1}{4}$

k) $\forall n \in \mathbb{N}^*: \lim_{x \to 0} \frac{1 - \cos^n x}{x^2} = \frac{n}{2}$

Trigonometric limits with $x \to \pm\infty$

These limits usually do not exist and we can show that using proof by contradiction as follows:

- 1 To show a contradiction, assume that $\lim_{x \to +\infty} f(x) = L$.
- 2 Define sequences $(a_n), (b_n)$ such that
  $\lim_{n \in \mathbb{N}^*} f(a_n) = L_1 \wedge \lim_{n \in \mathbb{N}^*} f(b_n) = L_2 \wedge L_1 \neq L_2$
- 3 Use the composition theorem to show that $L_1 = L_2 = L$ and thus derive a contradiction.

EXAMPLE

Show that $\lim_{x \to +\infty} \sin x$ does not exist.

Solution

To show a contradiction, assume that $\lim_{x \to +\infty} \sin x = L$ with $L \in \mathbb{R} \cup \{+\infty, -\infty\}$. Define $(a_n), (b_n)$ such that

$$\forall n \in \mathbb{N}^* : (a_n = 2n\pi \wedge b_n = 2n\pi + \pi/4)$$

Since

$$\lim_{n \in \mathbb{N}^*} a_n = +\infty \wedge \lim_{n \in \mathbb{N}^*} b_n = +\infty \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = \lim_{n \in \mathbb{N}^*} f(b_n) = L$$

We also have:

$$l_1 = \lim_{n \in \mathbb{N}^*} f(a_n) = \lim_{n \in \mathbb{N}^*} f(2n\pi) = \lim_{n \in \mathbb{N}^*} \sin(2n\pi) =$$

$$= \lim_{n \in \mathbb{N}^*} \sin 0 = \lim_{n \in \mathbb{N}^*} 0 = 0$$

$$l_2 = \lim_{n\in\mathbb{N}^*} f(b_n) = \lim_{n\in\mathbb{N}^*} \sin(2n\pi + \pi/4) = \lim_{n\in\mathbb{N}^*} \sin(\pi/4)$$

$$= \sin(\pi/4) = \sqrt{2}/2$$

It follows that $\lim_{n\in\mathbb{N}^*} a_n \neq \lim_{n\in\mathbb{N}^*} b_n$, which is a contradiction

We conclude that $\lim_{x\to+\infty} \sin x$ does not exist. $\square$

EXERCISES

(24) Show that the following limits do not exist

a) $\lim\limits_{x \to 0} \sin(1/x)$

b) $\lim\limits_{x \to 1} \cos\left(\dfrac{1}{x-1}\right)$

c) $\lim\limits_{x \to \pi/2} \dfrac{x - \pi/2}{\sqrt{1 - \sin x}}$

d) $\lim\limits_{x \to 0} \left[\dfrac{1}{x} \sin\left(\dfrac{1}{x}\right)\right]$

e) $\lim\limits_{x \to +\infty} [2\cos(3x) - 1]$

f) $\lim\limits_{x \to +\infty} \sqrt{3 + \cos(x/2)}$

g) $\lim\limits_{x \to +\infty} \tan x$

h) $\lim\limits_{x \to \pi} [2x \tan(x/2) + 3]$.

RA 1.4: Continuity

FUNCTION CONTINUITY

▼ Definition of a continuous function

Function continuity is defined at a point $x_0 \in \mathbb{R}$ and over a subset $S \subseteq \mathbb{R}$ as follows:

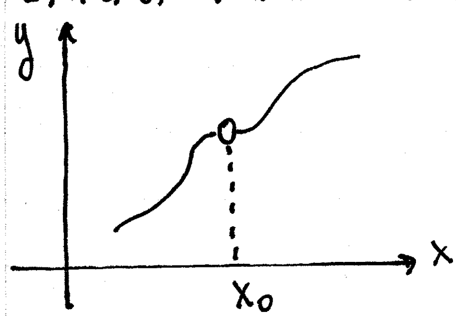
> **Def:** Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. We say that:
>
> a) f continuous at $x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)$
>
> b) f continuous at $S \Leftrightarrow \forall x_0 \in S : \lim_{x \to x_0} f(x) = f(x_0)$

Note that, via the limit definition, the definition of continuity over a set S can be rewritten as follows:

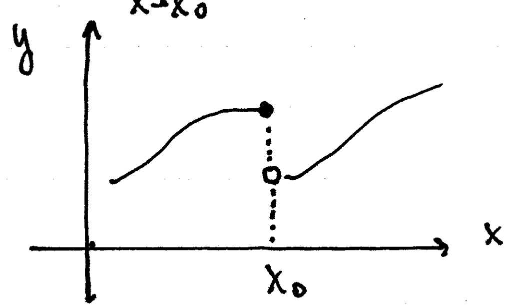
$$f \text{ continuous on } S \Leftrightarrow \forall x_0 \in S : \lim_{x \to x_0} f(x) = f(x_0) \Leftrightarrow$$
$$\Leftrightarrow \forall x_0 \in S : \forall \varepsilon \in (0, +\infty) : \exists \delta \in (0, +\infty) : \forall x \in A : (0 < |x - x_0| < \delta \Rightarrow$$
$$\Rightarrow |f(x) - f(x_0)| < \varepsilon)$$

▸ There are three ways continuity at a point x_0 may fail:

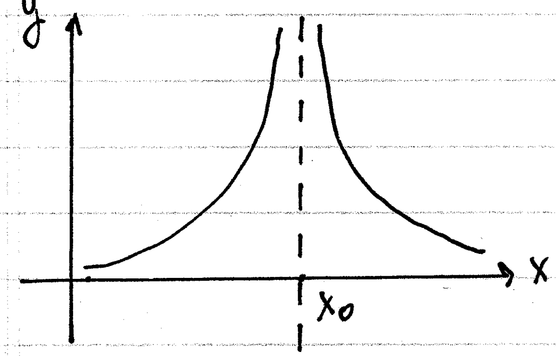
1) $f(x_0)$ is not defined



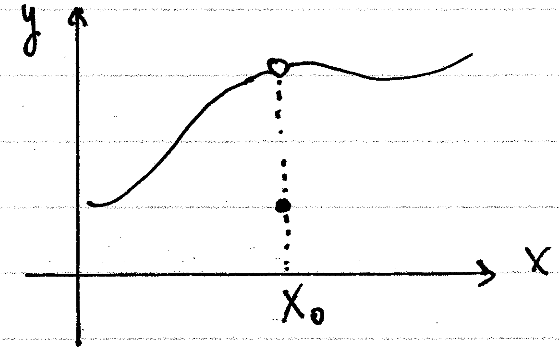
2) $\lim_{x \to x_0} f(x)$ does not exist



3) $\lim_{x \to x_0} f(x) \in \{+\infty, -\infty\}$



4) $\lim_{x \to x_0} f(x) \neq f(x_0)$



Continuity of basic functions

Let $\mathbb{R}[x]$ be the set of all polynomials functions $p: \mathbb{R} \to \mathbb{R}$ such that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad \forall x \in \mathbb{R}$$

Then, from the definition of contisiuity and the properties of limits, it follows that:

1) $\forall p \in \mathbb{R}[x]$: p continuous on $\mathbb{R}$
2) $\forall p, q \in \mathbb{R}[x]$: p/q continuous on $\mathbb{R} - \{x \in \mathbb{R} \mid q(x) = 0\}$
3) sin continuous on $\mathbb{R}$
4) cos continuous on $\mathbb{R}$
5) tan continuous on $\mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$
6) cot continuous on $\mathbb{R} - \{k\pi \mid k \in \mathbb{Z}\}$

Consequences of the composition theorems

From the function composition theorem, it follows that:

(1) If $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: G \to \mathbb{R}$, then:

$$\left\{\begin{array}{l} g \text{ continuous on } x_0 \\ f \text{ continuous on } g(x_0) \end{array}\right. \Rightarrow f \circ g \text{ continuous on } x_0$$

(2) If $f: A \to \mathbb{R}$ with $(a,b) \subseteq A$ and (a_n) a sequence, then we have

$$\left\{\begin{array}{l} f \text{ continuous on } (a,b) \\ \forall n \in \mathbb{N}^*: a_n \in (a,b) \\ \lim\limits_{n \in \mathbb{N}^*} a_n = x \end{array}\right. \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = x$$

Both results are immediate consequences of the composition theorem.

EXAMPLE

Consider the function

$$f(x)=\begin{cases} x g(x) , & \text{with } x\in\mathbb{R}-\{0\} \\ a , & \text{with } x=0 \end{cases}$$

with g continuous on $\mathbb{R}$. Show that

f continuous on $\mathbb{R} \Leftrightarrow a=0$

Solution

Assume that g continuous on $\mathbb{R}$. Let $x_0\in\mathbb{R}-\{0\}$ be given. Then, we have:

g continuous on $\mathbb{R} \Rightarrow g$ continuous on x_0

$$\Rightarrow \lim_{x\to x_0} g(x) = g(x_0)$$

$$\Rightarrow \lim_{x\to x_0} f(x) = \lim_{x\to x_0} [x g(x)] = [\lim_{x\to x_0} x][\lim_{x\to x_0} g(x)]$$

$$= x_0 g(x_0) = f(x_0) \Rightarrow$$

$\Rightarrow$ f continuous on x_0

We have thus shown that:

$\forall x_0\in\mathbb{R}-\{0\}: f$ continuous on x_0

$\Rightarrow f$ continuous on $\mathbb{R}-\{0\}$. (1)

We also note that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} [x g(x)] = [\lim_{x\to 0} x][\lim_{x\to 0} g(x)] = 0g(0) = 0 \quad (2)$$

It follows that:

f continuous on $\mathbb{R} \Leftrightarrow f$ continuous on $x_0=0$ [via Eq.(1)]

$\Leftrightarrow \lim_{x\to 0} f(x) = f(0)$ [definition]

$\Leftrightarrow a=0$. [via Eq.(2)]

THEORY QUESTIONS

① Let $f: A \to \mathbb{R}$ with $x_0 \in \mathbb{R}$ and $S \subseteq \mathbb{R}$. Write the definition for the following statements

a) f continuous on x_0

b) f continuous on S

EXERCISES

② Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) + b & , \text{ if } x \in \mathbb{R} - \{0\} \\ a & , \text{ if } x = 0 \end{cases}$$

Show that:

f continuous on $\mathbb{R} \Leftrightarrow a = b$

③ Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ and define $h: \mathbb{R} \to \mathbb{R}$ such that

$\forall x \in \mathbb{R}: h(x) = \max\{f(x), g(x)\}$

Show that:

$$\begin{cases} f \text{ continuous on } \mathbb{R} \\ g \text{ continuous on } \mathbb{R} \end{cases} \Rightarrow h \text{ continuous on } \mathbb{R}.$$

↳ Hint: First, show that:

$\forall x \in \mathbb{R}: h(x) = (1/2)(f(x) + g(x)) + (1/2)|f(x) - g(x)|$

(4) Let $f:[a,c]\to\mathbb{R}$ and $g:[c,b]\to\mathbb{R}$ such that

$$\begin{cases} f \text{ continuous on } [a,c] \\ f \text{ continuous on } [c,b] \end{cases}$$

Define $h:[a,b]\to\mathbb{R}$ such that

$$h(x)=\begin{cases} f(x), & \text{if } x\in[a,c] \\ g(x), & \text{if } x\in[c,b] \end{cases}$$

Show that:

h continuous on $[a,b] \Leftrightarrow f(c)=g(c)$.

▼ Continuity and dense sets

Let $Seq(S)$ be the set of all sequences $a_n : \mathbb{N}^* \to S$ with $S \subseteq \mathbb{R}$ such that: $\forall n \in \mathbb{N}^* : a_n \in S$

Def: Let $S \subseteq \mathbb{R}$. We say that

$$S \text{ dense in } \mathbb{R} \iff \forall x \in \mathbb{R} : \exists a \in Seq(S) : \lim_{n \in \mathbb{N}^*} a_n = x$$

Our main result is the following theorem:

Thm: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and let $S \subseteq \mathbb{R}$.
Then, we have:

$$\left\{ \begin{array}{l} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in S : f(x) = g(x) \\ S \text{ dense on } \mathbb{R} \end{array} \right. \Rightarrow \forall x \in \mathbb{R} : f(x) = g(x)$$

Proof

Let $x \in \mathbb{R}$ be given. Since S dense on $\mathbb{R}$, choose $a \in Seq(S)$ such that $\lim_{n \in \mathbb{N}^*} a_n = x$. Then, we have:

$$\begin{aligned} f(x) &= f(\lim_{n \in \mathbb{N}^*} a_n) && [\text{definition of } (a_n)] \\ &= \lim_{n \in \mathbb{N}^*} f(a_n) && [f \text{ continuous on } \mathbb{R}] \\ &= \lim_{n \in \mathbb{N}^*} g(a_n) && [\text{via } a_n \in S \Rightarrow f(a_n) = g(a_n)] \end{aligned}$$

$$= g(\lim_{n\in\mathbb{N}^+} a_n) \qquad [g \text{ continuous on } \mathbb{R}]$$

$$= g(x) \qquad [\text{definition of } (a_n)]$$

We have thus shown that
$\forall x\in\mathbb{R} : f(x) = g(x)$ $\square$

In order to put this theorem to use, we will now show that:

(1) $\boxed{\mathbb{Q} \text{ dense in } \mathbb{R}}$

Proof
Let $x\in\mathbb{R}$ be given. Choose $a, b\in \mathrm{Seq}(\mathbb{Q})$ such that the interval sequence $([a_n, b_n])$ is nested with
$$\bigcap_{n\in\mathbb{N}^+} [a_n, b_n] = \{x\}$$
It follows that $\lim_{n\in\mathbb{N}^+} a_n = x$. We have thus shown that

$(\forall x\in\mathbb{R} : \exists a \in \mathrm{Seq}(\mathbb{Q}) : \lim_{n\in\mathbb{N}^+} a_n = x) \Rightarrow \mathbb{Q}$ dense in $\mathbb{R}$. $\square$

(2) $\boxed{\mathbb{R}-\mathbb{Q} \text{ dense in } \mathbb{R}}$

Proof
Let $x\in\mathbb{R}$ be given. Choose $a, b\in \mathrm{Seq}(\mathbb{Q})$ such that the interval sequence $([a_n, b_n])$ is nested with

$$\bigcap_{n\in\mathbb{N}^*} [a_n, b_n] = \{x\}$$

It follows that: $\lim_{n\in\mathbb{N}^*} a_n = \lim_{n\in\mathbb{N}^*} b_n = x$.

We <u>define (c_n)</u> such that
$\forall n\in\mathbb{N}^*: c_n = a_n + \sqrt{2}\,(b_n - a_n)$
and note that
$([a_n, b_n])$ nested $\Rightarrow \lim_{n\in\mathbb{N}^*} (b_n - a_n) = 0 \Rightarrow$

$$\Rightarrow \underline{\lim_{n\in\mathbb{N}^*}} c_n = \lim_{n\in\mathbb{N}^*} [a_n + \sqrt{2}(b_n - a_n)] =$$

$$= \lim_{n\in\mathbb{N}^*} a_n + \sqrt{2} \lim_{n\in\mathbb{N}^*} (b_n - a_n)$$

$$= x + \sqrt{2}\cdot 0 = \underline{x}$$

▸ We will show that $c \in \mathrm{Seq}(\mathbb{R} - \mathbb{Q})$.
Let $n\in\mathbb{N}^*$ be given. To show a contradiction, assume that $c_n \in \mathbb{Q}$. Then, choose $p, q \in \mathbb{Z}$ such that $c_n = p/q$.
It follows that

$$c_n = p/q \Rightarrow a_n + \sqrt{2}(b_n - a_n) = p/q \Rightarrow \sqrt{2}\,(b_n - a_n) = p/q - a_n$$

$$\Rightarrow \sqrt{2} = \frac{(p/q) - a_n}{b_n - a_n} \Rightarrow \sqrt{2} \in \mathbb{Q}$$

which is a contradiction. We have thus shown that
$(\forall n\in\mathbb{N}^*: c_n \notin \mathbb{Q}) \Rightarrow \underline{c \in \mathrm{Seq}(\mathbb{R} - \mathbb{Q})}$.
We conclude that:
$(\forall x\in\mathbb{R} : \exists c \in \mathrm{Seq}(\mathbb{R} - \mathbb{Q}) : \lim_{n\in\mathbb{N}^*} c_n = x) \Rightarrow \mathbb{R} - \mathbb{Q}$ dense on $\mathbb{R}$.

An immediate consequence of these results are the following statements:

① $$\boxed{\left\{\begin{array}{l} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in \mathbb{Q} : f(x) = g(x) \end{array}\right. \Rightarrow \forall x \in \mathbb{R} : f(x) = g(x)}$$

② $$\boxed{\left\{\begin{array}{l} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in \mathbb{R} - \mathbb{Q} : f(x) = g(x) \end{array}\right. \Rightarrow \forall x \in \mathbb{R} : f(x) = g(x)}$$

EXAMPLES

▶ The nowhere-continuous function

a) Consider the function

$$f(x) = \begin{cases} 0 & , \text{ if } x \in \mathbb{Q} \\ 1 & , \text{ if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that: $\forall x_0 \in \mathbb{R} : f$ not continuous on x_0

Solution

Let $x_0 \in \mathbb{R}$ be given. To show a contradiction, assume that f continuous on x_0. Since $\mathbb{Q}$ dense on $\mathbb{R}$, choose $(a_n) : \mathbb{N}^* \to \mathbb{Q}$ such that $\lim_{n \in \mathbb{N}^*} a_n = x_0$. Likewise, since $\mathbb{R} - \mathbb{Q}$ dense on $\mathbb{R}$, choose $(b_n) : \mathbb{N} \to \mathbb{R} - \mathbb{Q}$ such that $\lim_{n \in \mathbb{N}^*} b_n = x_0$. It follows that:

$$\begin{aligned}
f(x_0) &= \lim_{x \to x_0} f(x) && [f \text{ continuous on } x_0] \\
&= \lim_{n \in \mathbb{N}^*} f(a_n) && [\text{via } \lim_{n \in \mathbb{N}^*} a_n = x_0] \\
&= \lim_{n \in \mathbb{N}^*} 0 && [a_n \in \mathbb{Q} \Rightarrow f(a_n) = 0] \\
&= 0 && (1)
\end{aligned}$$

and

$$\begin{aligned}
f(x_0) &= \lim_{x \to x_0} f(x) && [f \text{ continuous on } x_0] \\
&= \lim_{n \in \mathbb{N}^*} f(b_n) && [\text{via } \lim_{n \in \mathbb{N}^*} b_n = x_0]
\end{aligned}$$

$$= \lim_{n \in \mathbb{N}^*} 1 \qquad [\text{via } b_n \in \mathbb{R} - \mathbb{Q} \Rightarrow f(b_n) = 1]$$

$$= 1 \qquad\qquad (2)$$

Eq.(1) and Eq.(2), therefore <u>f not continuous on x_0</u>

We have thus shown that

$\forall x_0 \in \mathbb{R}$: f not continuous on x_0.

6) Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that

$$\begin{cases} f \text{ continuous on } \mathbb{R} \\ \forall x,y \in \mathbb{R} : f(x+y) = f(x) + f(y) \end{cases}$$

Show that: $\exists a \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = ax$

Solution

▸ We will show that: $\forall x \in \mathbb{R} : \forall n \in \mathbb{Z} : f(nx) = nf(x)$. using proof by induction.

Let $x \in \mathbb{R}$ be given. For $n=0$, we have:

$f(0x) = f(0x+0x) = f(0x) + f(0x) \Rightarrow f(0x) = 0 = 0f(x)$

For $n=k$, we assume that: $f(kx) = kf(x)$.

For $n=k+1$, we will show that: $f((k+1)x) = (k+1)f(x)$.

We have:

$$f((k+1)x) = f(kx+x) = f(kx) + f(x) = kf(x) + f(x) = (k+1)f(x)$$

For $n=k-1$, we will show that: $f((k-1)x) = (k-1)f(x)$

We have:

$$kf(x) = f(kx) = f((k-1)x + x) = f((k-1)x) + f(x) \Rightarrow$$

$$\Rightarrow f((k-1)x) = kf(x) - f(x) = (k-1)f(x)$$

We have thus shown, by induction, that $\forall n \in \mathbb{Z} : f(nx) = nf(x)$ and conclude that:

$\forall x \in \mathbb{R} : \forall n \in \mathbb{Z} : f(nx) = nf(x)$

▸ Let $x \in \mathbb{Q}$ be given. Choose $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$ such that $x = p/q$. It follows that

$$\begin{cases} f(p) = f(q(p/q)) = f(qx) = qf(x) \\ f(p) = f(p \cdot 1) = pf(1) \end{cases} \Rightarrow qf(x) = pf(1) \Rightarrow$$

$$\Rightarrow f(x) = (p/q)f(1) = xf(1).$$

We have thus shown that

$\forall x \in \mathbb{Q} : f(x) = x f(1)$

Define: $\forall x \in \mathbb{R} : g(x) = x f(1)$. Then, we have:

$$\left\{ \begin{array}{l} f, g \text{ continuous on } \mathbb{R} \\ \forall x \in \mathbb{Q} : f(x) = g(x) \end{array} \right. \Rightarrow$$

$$\Rightarrow \forall x \in \mathbb{R} : f(x) = g(x) = x f(1)$$

$$\Rightarrow \exists a \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = ax \qquad (\text{for } a = f(1)) \qquad \square$$

Our methodology here is to first establish the claim on $\mathbb{Z}$ using proof by induction. Then, we generalize by proving the claim on $\mathbb{Q}$. Continuity is then used to rapidly extend the claim on $\mathbb{R}$.

THEORY QUESTIONS

(5) State the definition of the statement:
S dense on $\mathbb{R}$.

(6) Prove that:
a) $\mathbb{Q}$ dense on $\mathbb{R}$
b) $\mathbb{R}-\mathbb{Q}$ dense on $\mathbb{R}$

(7) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two functions. Prove that:

$$\begin{cases} S \text{ dense on } \mathbb{R} \\ f,g \text{ continuous on } \mathbb{R} \\ \forall x\in S: f(x)=g(x) \end{cases} \Rightarrow \forall x\in\mathbb{R}: f(x)=g(x)$$

EXERCISES

(8) Show that the set $S=\{a\sqrt{2} \mid a\in\mathbb{Q}\}$ is dense in $\mathbb{R}$.

(9) Let S be a set dense in $\mathbb{R}$ and let $a\in\mathbb{R}$ be some number. Show that the set

$$T=\{x+a \mid x\in S\}$$

is also dense in $\mathbb{R}$.

(10) Consider the function

$$f(x)=\begin{cases} x & \text{, if } x\in\mathbb{Q} \\ x+1 & \text{, if } x\in\mathbb{R}-\mathbb{Q} \end{cases}$$

Show that: $\forall x_0 \in \mathbb{R}$: f not continuous on x_0

(11) Consider the function

$$f(x) = \begin{cases} x & , \text{if } x \in \mathbb{Q} \\ 0 & , \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that:

$\forall x_0 \in \mathbb{R}$: (f continuous on $x_0 \Leftrightarrow x_0 = 0$)

<u>Hint:</u> The "$\Rightarrow$" proof uses rational and irrational sequences. However, the "$\Leftarrow$" requires proof by limit definition or properties of limits.

(12) Consider the functions

$$\begin{cases} f(x) = \begin{cases} x & , \text{if } x \in \mathbb{Q} \\ 2 - x & , \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases} \\ \forall x \in \mathbb{R} : g(x) = f(x) f(2-x) \end{cases}$$

Show that:

a) g continuous on $\mathbb{R}$

b) $\forall x_0 \in \mathbb{R}$: (f continuous on $x_0 \Leftrightarrow x_0 = 1$)

(13) Let $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ be two functions such that g_1, g_2 continuous on $\mathbb{R}$. Let $S \subseteq \mathbb{R}$ be the set

$S = \{x \in \mathbb{R} \mid g_1(x) = g_2(x)\}$

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by:

$$f(x) = \begin{cases} g_1(x) & , \text{if } x \in \mathbb{Q} \\ g_2(x) & , \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Show that:

$\forall x_0 \in \mathbb{R} : (f \text{ continuous on } x_0 \Leftrightarrow x_0 \in S)$

(14) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$\forall x, y \in \mathbb{R} : f(x+y) = f(x) + f(y)$

Show that:

f continuous at $x_0 = 0 \Rightarrow f$ continuous on $\mathbb{R}$.

(15) Let $f : (0, +\infty) \to \mathbb{R}$ such that

$\forall x, y \in (0, +\infty) : f(xy) = f(x) + f(y)$

Assume that f continuous on $x_0 = 1$

Show that:

a) $f(1) = 0$

b) $\forall x, y \in (0, +\infty) : f(x/y) = f(x) - f(y)$

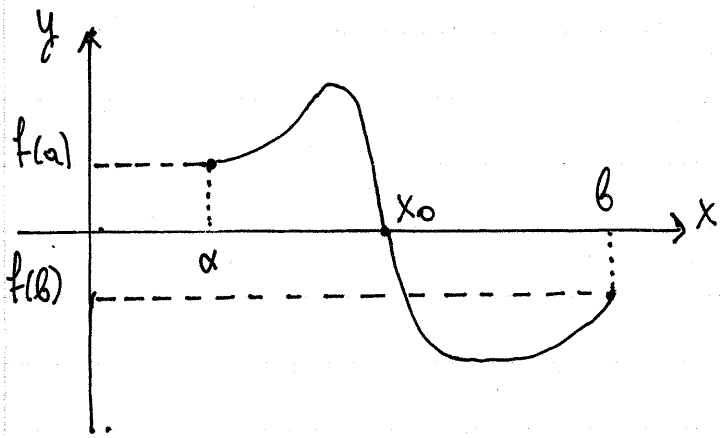
c) f continuous on $(0, +\infty)$

d) $\forall a \in \mathbb{R} : \forall x \in (0, +\infty) : f(x^a) = af(x)$.

Bolzano theorem

Thm: Let $f: A \to \mathbb{R}$ and let $[a,b] \subseteq A$. Then, we have:
$$\begin{cases} f \text{ continuous on } [a,b] \\ f(a)f(b) < 0 \end{cases} \Rightarrow \exists x_0 \in (a,b) : f(x_0) = 0$$

interpretation: When a function f is continuous on $[a,b]$, then if the values of f at $x=a$ and $x=b$ have opposite signs, then it has to pass through zero for some $x_0 \in (a,b)$. This result connects the formal definition of continuity with our intuitive understanding of the geometrical meaning of continuity.



Proof

Assume that f continuous on $[a,b] \wedge f(a)f(b) < 0$.
Since $f(a), f(b)$ are heterosigned, assume with no loss of generality that $f(a) < 0$ and $f(b) > 0$.

▸ We construct an interval sequence $([a_n,b_n])$ such that
$$\begin{cases} ([a_n,b_n]) \text{ nested} \\ \forall n \in \mathbb{N}^*: (f(a_n) \leq 0 \wedge f(b_n) \geq 0) \end{cases}$$

Define $[a_1,b_1] = [a,b]$ and note that trivially, we have:

$f(a_1) \leq 0 \wedge f(b_1) \geq 0$

Assume that $[a_n, b_n]$ has been defined such that

$f(a_n) \leq 0 \wedge f(b_n) \geq 0$

Let $c_n = (a_n + b_n)/2$ and define

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, c_n] &, \text{if } f(c_n) \geq 0 \\ [c_n, b_n] &, \text{if } f(c_n) < 0 \end{cases}$$

By construction, it follows that $([a_n, b_n])$ nested. We may therefore choose $\underline{x_0 \in [a, b]}$ such that

$$\bigcap_{n \in \mathbb{N}^*} [a_n, b_n] = \{x_0\}$$

Then, we have:

$$\begin{cases} f \text{ continuous on } [a, b] \\ \lim\limits_{n \in \mathbb{N}^*} a_n = \lim\limits_{n \in \mathbb{N}^*} b_n = x_0 \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} f(a_n) = \lim_{n \in \mathbb{N}^*} f(b_n) = f(x_0)$$

and therefore

$$\forall n \in \mathbb{N}^* : \begin{cases} f(a_n) \leq 0 \\ f(b_n) \geq 0 \end{cases} \Rightarrow$$

$$\Rightarrow f(x_0) = \lim_{n \in \mathbb{N}^*} f(a_n) \leq 0 \wedge f(x_0) = \lim_{n \in \mathbb{N}^*} f(b_n) \geq 0$$

$$\Rightarrow \underline{f(x_0) = 0}$$

We also note that

$$\begin{cases} f(a) < 0 \\ f(b) > 0 \end{cases} \Rightarrow \begin{cases} x_0 \neq a \\ x_0 \neq b \end{cases} \Rightarrow \underline{x_0 \in (a, b)}$$

We have thus shown that

$\exists x_0 \in (a, b) : f(x_0) = 0.$ □

EXAMPLES

a) Show that the equation

$$\sin(\cos 3x) = 0$$

has at least one solution on $(0, \pi)$.

Solution

Define $f(x) = \sin(\cos(3x))$, $\forall x \in \mathbb{R}$.

We note that f continuous on $[0, \pi]$ (1).

and also:

$f(0) = \sin(\cos(3 \cdot 0)) = \sin(\cos 0) = \sin 1$ (2)

$f(\pi) = \sin(\cos(3\pi)) = \sin(\cos \pi) = \sin(-1) = -\sin 1$ (3)

From Eq.(2) and Eq.(3):

$f(0)f(\pi) = (\sin 1)(-\sin 1) = -\sin^2 1 < 0$ (3)

From Eq.(1) and Eq.(3):

$(\exists x_0 \in (0, \pi) : f(x_0) = 0) \Rightarrow x_0$ solves $\sin(\cos(3x)) = 0.$

b) If $a, b \in \mathbb{R}$ with $0 < a < b < \pi/2$, show that the equation

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0$$

has at least one solution $x_0 \in (a, b)$.

Solution

We note that for $x \in (a,b)$, we have $(x-a)(x-b) \neq 0$, and therefore:

$$\frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0 \Leftrightarrow (x-b)\sin x + (x-a)\cos x = 0$$

Define $f(x) = (x-b)\sin x + (x-a)\cos x$, $\forall x \in \mathbb{R}$

Then: f continuous on $[a,b]$ (1)

$f(a) = (a-b)\sin a + (a-a)\cos a = (a-b)\sin a$ (2)

$f(b) = (b-b)\sin b + (b-a)\cos b = (b-a)\cos b$ (3)

From Eq.(2) and Eq.(3):

$$f(a)f(b) = [(a-b)\sin a][(b-a)\cos b] = (a-b)(b-a)\sin a \cos b$$
$$= -(a-b)^2 \sin a \cos b.$$

We note that $a \neq b \Rightarrow (a-b)^2 > 0$

and $0 < a < \pi/2 \Rightarrow \sin a > 0$

and $0 < b < \pi/2 \Rightarrow \cos b > 0$.

It follows that

$f(a)f(b) = -(a-b)^2 \sin a \cos b < 0$ (4)

From Eq.(1) and Eq.(4), via Bolzano theorem,

$$(\exists x_0 \in (a,b) : f(x_0) = 0) \Rightarrow x_0 \text{ solves } \frac{\sin x}{x-a} + \frac{\cos x}{x-b} = 0$$

THEORY QUESTIONS

(16) Let $f: A \to \mathbb{R}$ and $[a,b] \subseteq A$. Prove the Bolzano theorem:

$$\begin{cases} f \text{ continuous on } [a,b] \\ f(a)f(b) < 0 \end{cases} \Rightarrow \exists x_0 \in (a,b) : f(x_0) = 0$$

EXERCISES

(17) Let $a \in (0,+\infty)$. Show that the equation $x^n - a = 0$ has at least one solution on $(0,+\infty)$ using the Bolzano theorem.

↳ Note that if one also establishes uniqueness, then we have an alternate proof of the existence and uniqueness of $\sqrt[n]{a}$

(18) Let $a,b,c \in \mathbb{R}$ with $a<b<c$. Show that the equation
$(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$
has at least one solution in (a,b) and at least one additional solution in (b,c).

(19) Show that the equation $9x^3 - 6x^2 - 11x + 4 = 0$ has at least two solutions in $(0,2)$

(20) Show that the equation $x = \sin(x)$ has at least one solution on $(-\pi/2, \pi/2)$.

(21) Let $a, b \in \mathbb{R}$ with $a<b$. Show that the equation

$$\frac{x^2+1}{x-a} + \frac{x^4+1}{x-b} = 0$$

has at least one solution on (a,b).

(22) Let $a, b, c \in \mathbb{R}$ with $a<b<c$. Show that the equation

$$\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c} = 0$$

has at least one solution in (a,b) and at least one additional solution in (b,c).

(23) Let $f: [0,1] \to \mathbb{R}$ be a function such that

$$\begin{cases} f \text{ continuous on } [0,1] \\ \forall x \in [0,1]: 0 < f(x) < 1 \end{cases}$$

Show that the equation $f(x) = x$ has at least one solution on $(0,1)$

(24) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f, g \text{ continuous on } [a,b] \\ f(a) = g(b) \land f(b) = g(a) \end{cases}$$

Show that $\exists c \in [a,b]: f(c) = g(c)$

(25) Let $f: A \to \mathbb{R}$ with $[0, 2\pi] \subseteq A$ such that $f(0) = f(2\pi)$.

Show that:

f continuous on $[0, 2\pi] \implies \exists x_0 \in [0,\pi]: f(x_0+\pi) = f(x_0)$.

(26) Show that the equation $ax^3+x^2+x=1$ with $a\neq -1$ has at least one solution in the interval $(-1,1)$.
What happens when $a=-1$?

(27) Let $a,b\in\mathbb{R}$ with $a<b$. Show that the equation
$$\frac{x^2+1}{x-a}+\frac{x^6+1}{x-b}=0$$
has a solution in (a,b).

▼ Continuity and function bounds

Similarly to bounded nets (a_n) on a directed set $(D, <)$ we define bounded functions as follows:

Def: Let $f: A \to \mathbb{R}$ and let $S \subseteq A$. We say that

$$f \text{ upper bounded on } S \Leftrightarrow \exists b \in \mathbb{R} : \forall x \in S : f(x) \leq b$$

$$f \text{ lower bounded on } S \Leftrightarrow \exists b \in \mathbb{R} : \forall x \in S : f(x) \geq b$$

$$f \text{ bounded on } S \Leftrightarrow \begin{cases} f \text{ upper bounded on } S \\ f \text{ lower bounded on } S \end{cases}$$

and also show the following proposition:

Prop: Let $f: A \to \mathbb{R}$ and let $S \subseteq A$. Then, we have:

$$f \text{ bounded on } S \Leftrightarrow \exists p \in (0, +\infty) : \forall x \in S : |f(x)| \leq p$$

The following statements are also immediate consequences of the definition:

$$f \text{ bounded on } S \Leftrightarrow f(S) \text{ bounded}$$

$$\begin{cases} f \text{ bounded on } S_1 \\ f \text{ bounded on } S_2 \end{cases} \Rightarrow f \text{ bounded on } S_1 \cup S_2$$

The contrapositive of the last statement reads:

f not bounded on $S_1 \cup S_2 \Rightarrow$
$\Rightarrow (f$ not bounded on $S_1 \vee f$ not bounded on $S_2)$

Our main results are needed later for differential calculus and are the following theorems:

① → Bounded property of continuous functions on a closed interval

$\boxed{\begin{array}{l}\underline{\text{Thm}}: \text{Let } f: A \to \mathbb{R} \text{ with } [a,b] \subseteq A. \text{ Then, we have:} \\ f \text{ continuous on } [a,b] \Rightarrow f \text{ bounded on } [a,b]\end{array}}$

<u>Proof</u>
Assume that f continuous on $[a,b]$. To show a contradiction, assume that f not bounded on $[a,b]$. We will construct an interval sequence $([a_n,b_n])$ such that

$$\begin{cases} ([a_n,b_n]) \text{ nested} \\ \forall n \in \mathbb{N}^*: f \text{ not bounded on } [a_n,b_n] \end{cases}$$

as follows:
Choose $[a_1,b_1] = [a,b]$. By hypothesis, f not bounded on $[a_1,b_1]$.
Assume that $[a_k,b_k]$ has been constructed such that f not bounded on $[a_k,b_k]$.
Define $c_k = (a_k+b_k)/2$. Then, we have:
f not bounded on $[a_k,b_k] \Rightarrow$
$\Rightarrow f$ not bounded on $[a_k,c_k] \vee f$ not bounded on $[c_k,b_k]$

and we choose

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, c_k], & \text{if } f \text{ not bounded on } [a_k, c_k] \\ [c_k, b_k], & \text{otherwise} \end{cases}$$

By construction, it follows that f not bounded on $[a_{k+1}, b_{k+1}]$. The resulting interval sequence $([a_n, b_n])$ is nested and satisfies

$\forall n \in \mathbb{N}^*: f$ not bounded on $[a_n, b_n]$

Consequently, for each $n \in \mathbb{N}^*$ we can choose $q_n \in [a_n, b_n]$ such that $f(q_n) \geq n$. Then, we have:

$$\begin{cases} \forall n \in \mathbb{N}^*: f(q_n) \geq n \\ \lim_{n \in \mathbb{N}^*} n = +\infty \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} f(q_n) = +\infty$$

$$\Rightarrow f(q_n) \text{ not convergent} \quad (1)$$

Since $([a_n, b_n])$ nested, choose $x_0 \in \mathbb{R}$ such that $\forall n \in \mathbb{N}^*: x_0 \in [a_n, b_n]$

Then, we have:

$$\begin{cases} \forall n \in \mathbb{N}^*: a_n \leq q_n \leq b_n \\ \lim_{n \in \mathbb{N}^*} a_n = \lim_{n \in \mathbb{N}^*} b_n = x_0 \end{cases} \Rightarrow \lim_{n \in \mathbb{N}^*} q_n = x_0 \quad \text{[via squeeze thm]}$$

$$\Rightarrow \lim_{n \in \mathbb{N}^*} f(q_n) = f(x_0) \quad \text{[via } f \text{ continuous on } [a,b]\text{]}$$

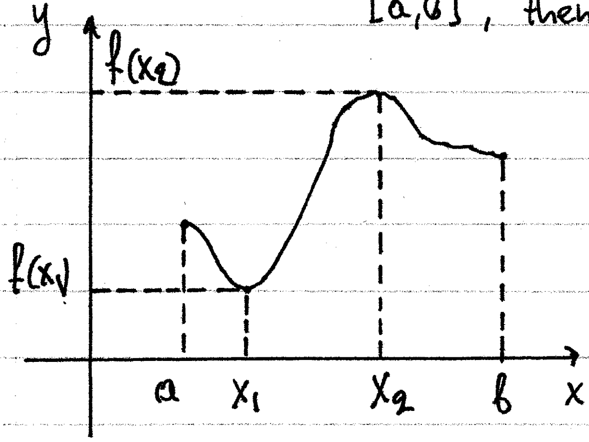
$$\Rightarrow f(q_n) \text{ convergent} \quad (2)$$

Eq. (1) contradicts Eq. (2). We have thus shown that: f bounded on $[a, b]$.

(9) → Extremum Value Theorem

Thm: Let $f: A \to \mathbb{R}$ with $[a,b] \subseteq A$. Then, we have:
f continuous on $[a,b] \Rightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

▸ interpretation: If f is continuous on a closed interval $[a,b]$, then f has a global minimum at $x = x_1$ and a global maximum at $x = x_2$ when the domain of f is restricted to the closed interval $[a,b]$.



Proof

Assume that f continuous on $[a,b]$. Then, we have:

f continuous on $[a,b] \Rightarrow f$ bounded on $[a,b]$
$\Rightarrow f([a,b])$ bounded
$\Rightarrow f([a,b])$ upper bounded

thus we can define $M = \sup(f([a,b]))$.

To show that $M \in f([a,b])$, we assume that $M \notin f([a,b])$ in order to show a contradiction. Then, we have:

$M \notin f([a,b]) \Rightarrow \forall x \in [a,b] : (f(x) \neq M \land f(x) \leq M)$
$\Rightarrow \forall x \in [a,b] : f(x) < M$
$\Rightarrow \forall x \in [a,b] : M - f(x) > 0$

Define $\forall x \in [a,b] : g(x) = 1/(M - f(x))$. Then, we have:
f continuous on $[a,b] \Rightarrow g$ continuous on $[a,b]$
$\Rightarrow g$ bounded on $[a,b]$
$\Rightarrow \exists p \in (0,+\infty) : \forall x \in [a,b] : |g(x)| \leq p$
Choose $p \in (0,+\infty)$ such that $\forall x \in [a,b] : |g(x)| \leq p$.
Let $\underline{x \in [a,b]}$ be given. Then, we have:

$$|g(x)| \leq p \Rightarrow \left| \frac{1}{M - f(x)} \right| \leq p \Rightarrow \frac{1}{|M - f(x)|} \leq p$$

$$\Rightarrow \frac{1}{M - f(x)} \leq p \qquad [\text{via } M - f(x) > 0]$$

$$\Rightarrow 1 \leq p(M - f(x)) \qquad [\text{via } M - f(x) > 0]$$

$$\Rightarrow M - f(x) \geq 1/p \qquad [\text{via } p > 0]$$

$$\Rightarrow -f(x) \geq 1/p - M \Rightarrow \underline{f(x) \leq M - 1/p}$$

We have thus shown that:

$$(\forall x \in [a,b] : f(x) \leq M - 1/p) \Rightarrow M - 1/p \text{ upper bound of } f([a,b])$$

$$\Rightarrow M - 1/p \geq \sup(f([a,b])) = M$$

$$\Rightarrow -1/p \geq 0 \Rightarrow p \leq 0$$

which is a contradiction, since $p > 0$.
We have thus shown that

$$M \in f([a,b]) \Rightarrow \exists x_2 \in [a,b] : f(x_2) = M = \sup(f([a,b]))$$

$$\Rightarrow \exists x_2 \in [a,b] : \forall x \in [a,b] : f(x) \leq f(x_2)$$

With a similar argument, we can show that

$$\exists x_1 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x)$$

Combining the two statements, we conclude that

$$\exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2). \qquad \square$$

THEORY QUESTIONS

(28) Let $f: A \to \mathbb{R}$ and let $S \subseteq A$. Write the definitions for the following statements.

a) f upper bounded on S

b) f lower bounded on S

c) f bounded on S

(29) Let $f: A \to \mathbb{R}$ with $[a,b] \subseteq A$. Prove that:

a) f continuous on $[a,b] \Rightarrow f$ bounded on $[a,b]$.

b) f continuous on $[a,b] \Rightarrow$

$$\Rightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$$

EXERCISES

(30) Let $f: A \to \mathbb{R}$ and let S, S_1, S_2 be subsets of A. Show the following statements

a) f bounded on $S \Leftrightarrow \exists p \in (0, +\infty) : \forall x \in S : |f(x)| \leq p$

b) f bounded on $S \Leftrightarrow f(S)$ bounded

c) $\begin{cases} f \text{ bounded on } S_1 \\ f \text{ bounded on } S_2 \end{cases} \Rightarrow f$ bounded on $S_1 \cup S_2$.

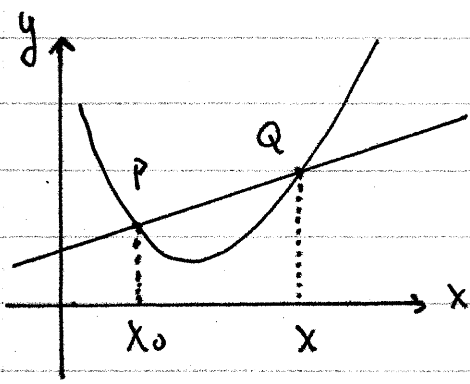
(31) Produce a counterexample to show that it is not possible to prove for all functions f the statement f continuous on $(a,b) \Rightarrow f$ bounded on (a,b).

RA 1.5: Derivatives

DIFFERENTIAL CALCULUS

▼ Definition of differentiability

The derivative of a function is defined in the usual way as follows. Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $(c): y = f(x)$ be the graph of the function f. Choose $x_0, x \in A$ and consider the points $P(x_0, f(x_0))$ and $Q(x, f(x))$. We denote the slope of the segment PQ as $\lambda(f|x, x_0)$ and it is given by



$$\boxed{\forall x, x_0 \in A : \lambda(f|x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}}$$

The slope of the tangent line at x_0 is given by the limit $\lim_{x \to x_0} \lambda(f|x, x_0)$ and that motivates the following definitions:

> **Def**: Let $f: A \to \mathbb{R}$ and $x_0 \in A$ and $S \subseteq A$. We say that
> f differentiable on $x_0 \iff \exists l \in \mathbb{R} : \lim_{x \to x_0} \lambda(f|x, x_0) = l$
> f differentiable on $S \iff \forall x_0 \in S : f$ differentiable on x_0
> $\iff \forall x_0 \in S : \exists l \in \mathbb{R} : \lim_{x \to x_0} \lambda(f|x, x_0) = l$

Differentiability implies continuity

Prop: Let $f: A \to \mathbb{R}$ and $x_0 \in A$. Then, we have:
f differentiable at $x_0 \Rightarrow f$ continuous at x_0

Proof

Assume that f differentiable at x_0. Then,
f differentiable at $x_0 \Rightarrow \exists \ell \in \mathbb{R}: \lim_{x \to x_0} \lambda(f|x, x_0) = \ell$
Choose $\ell \in \mathbb{R}$ such that $\lim_{x \to x_0} \lambda(f|x, x_0) = \ell$. Then, we have:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} [f(x_0) + (f(x) - f(x_0))] =$$

$$= \lim_{x \to x_0} \left[f(x_0) + \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right]$$

$$= \lim_{x \to x_0} [f(x_0) + \lambda(f|x, x_0)(x - x_0)]$$

$$= f(x_0) + \left[\lim_{x \to x_0} \lambda(f|x, x_0)\right]\left[\lim_{x \to x_0} (x - x_0)\right]$$

$$= f(x_0) + \ell(x_0 - x_0) = f(x_0) \Rightarrow f \text{ continuous at } x_0 \ \square$$

The contrapositive statement reads:
f not continuous at $x_0 \Rightarrow f$ not differentiable at x_0.

Def: (Corner points). Let $f: A \to \mathbb{R}$ and let $x_0 \in A$. We say that
x_0 corner point of $f \Leftrightarrow \begin{cases} f \text{ continuous at } x_0 \\ f \text{ NOT differentiable at } x_0. \end{cases}$

Corner points can emerge from

a) Sudden change in the direction of the function (example: $f(x) = |x|$ at $x=0$)

b) When the graph of the function becomes momentarily vertical at a particular point (example: $f(x) = \sqrt{x}$ at $x=0$)

These two examples of corner points are elaborated upon in the following examples.

EXAMPLE

a) For $f(x)=|x|, \forall x \in \mathbb{R}$ show that $x_0=0$ is a corner point.

Solution

Since

$$f(0)=|0|=0 \qquad (1)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \qquad (2)$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0 \qquad (3)$$

it follows that

$$\lim_{x\to 0} f(x) = 0 \qquad [\text{from Eq. (2) and Eq. (3)}]$$

$$= f(0) \qquad [\text{from Eq. (1)}]$$

$\Rightarrow f$ continuous at $x_0=0$. (4)

Furthermore:

$$\lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{|x|-|0|}{x-0} = \frac{|x|}{x} =$$

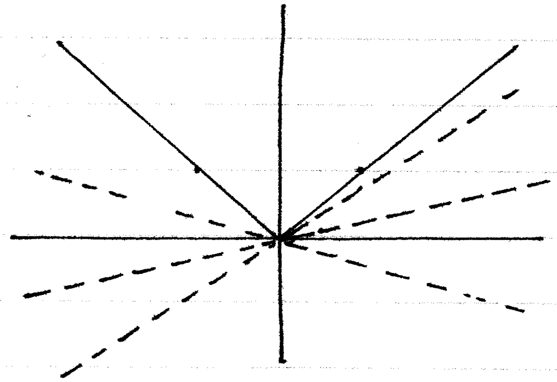
$$= \begin{cases} x/x, & \text{if } x>0 \\ -x/x, & \text{if } x<0 \end{cases} = \begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x<0 \end{cases} \Rightarrow$$

$$\Rightarrow \lim_{x\to 0^+} \lambda(f|x,0) = 1 \wedge \lim_{x\to 0^-} \lambda(f|x,0) = -1$$

$$\Rightarrow \lim_{x\to 0} \lambda(f|x,0) \text{ does not exist} \Rightarrow \forall \ell \in \mathbb{R}: \lim \frac{f(x)-f(0)}{x-0} \neq \ell$$

$\Rightarrow f$ not differentiable at $x_0=0$. (5)

From Eq.(4) and Eq.(5): $x_0=0$ corner point of f.



From the graph of $f(x)=|x|$, $\forall x \in \mathbb{R}$ we see that the corner point $x_0=0$, the function suddenly changes direction. As a result, we cannot draw a unique tangent line at $x_0=0$.

b) Show that $f(x)=\sqrt{x}$, $\forall x \in [0,+\infty)$ has a corner point at $x_0=0$.

Solution

$$f(0)=\sqrt{0}=0 \qquad (1)$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\sqrt{x} = \lim_{x\to 0^+}\sqrt{x} = \sqrt{0}=0 \qquad (2)$$

From Eq.(1) and Eq.(2):

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow f \text{ continuous at } x_0=0 \qquad (3)$$

Furthermore:

$$\lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{\sqrt{x}-\sqrt{0}}{x-0} = \frac{\sqrt{x}}{x} =$$

$$= \frac{\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}} = \frac{1}{\sqrt{x}}, \ \forall x \in (0,+\infty)$$

Since:

$$\begin{cases}\sqrt{x}>0, \ \forall x\in(0,+\infty) \\ \lim\limits_{x\to 0^+}\sqrt{x}=\sqrt{0}=0\end{cases} \Rightarrow \lim_{x\to 0^+}\frac{1}{\sqrt{x}}=+\infty \Rightarrow$$

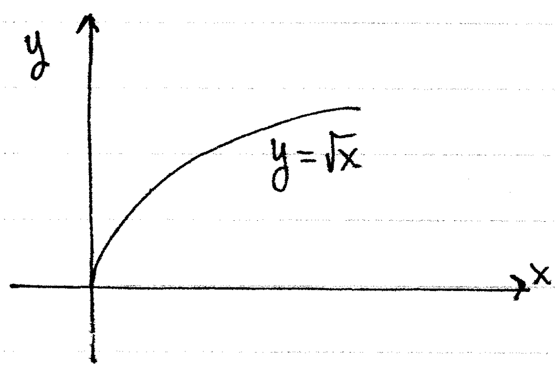
$$\Rightarrow \lim_{x\to 0^+}\lambda(f|x,0)=+\infty$$

$$\Rightarrow \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}\neq \ell, \ \forall \ell\in\mathbb{R}$$

$$\Rightarrow f \text{ not differentiable at } x_0=0. \quad (4)$$

From Eq. (3) and Eq. (4): $x_0=0$ corner point of f.

From the graph of $f(x)=\sqrt{x}$, $\forall x\in[0,+\infty)$ we see that the graph becomes vertical at $x_0=0$. This is the second way one may lose differentiability without losing continuity.



c) Consider the function

$$f(x) = \begin{cases} x^2[\sin(\pi/x) + \cos(\pi/x)], & \text{if } x \in \mathbb{R} - \{0\} \\ 0 & \text{, if } x = 0 \end{cases}$$

Show that f differentiable at $x_0 = 0$

<u>Solution</u>

Let $x \in \mathbb{R} - \{0\}$ be given. Then, we have:

$$\lambda(f|x,0) = \frac{f(x) - f(0)}{x - 0} = \frac{x^2[\sin(\pi/x) + \cos(\pi/x)] - 0}{x}$$

$$= x[\sin(\pi/x) + \cos(\pi/x)] \Rightarrow$$

$$\Rightarrow |\lambda(f|x,0)| = |x[\sin(\pi/x) + \cos(\pi/x)]| =$$

$$= |x| \cdot |\sin(\pi/x) + \cos(\pi/x)| \leq |x|[|\sin(\pi/x)| + |\cos(\pi/x)|]$$

$$\leq |x| \cdot (1 + 1) = 2|x| = |2x|$$

We have thus shown that

$$\begin{cases} \forall x \in \mathbb{R} - \{0\} : |\lambda(f|x,0)| \leq |2x| \\ \lim\limits_{x \to 0} (2x) = 0 \end{cases} \Rightarrow \lim_{x \to 0} \lambda(f|x,0) = 0$$

$$\Rightarrow f \text{ differentiable at } x_0 = 0$$

c) Consider the function

$$f(x) = \begin{cases} x^2+2x & , x\in[0,+\infty) \\ ax+b & , x\in(-\infty,0) \end{cases}$$

Find all $a,b\in\mathbb{R}$ for which f differentiable at $x_0=0$.

<u>Solution</u>

We note that

$$\forall x\in(0,+\infty): \lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{(x^2+2x)-(0^2+2\cdot 0)}{x}$$

$$= \frac{x^2+2x}{x} = \frac{x(x+2)}{x} = x+2$$

$$\forall x\in(-\infty,0): \lambda(f|x,0) = \frac{f(x)-f(0)}{x-0} = \frac{ax+b-0}{x} = \frac{ax+b}{x}$$

$$\lim_{x\to 0^+}\lambda(f|x,0) = \lim_{x\to 0^+}(x+2) = 0+2 = 2$$

The limit $\lim_{x\to 0^-}\lambda(f|x,0)$ may or may not exist depending on whether $b=0$ or $b\neq 0$, so we leverage continuity but must do, as a result, a split argument:

($\Rightarrow$): Assume that f differentiable at $x_0=0$. Since:

$f(0) = 0^2+2\cdot 0 = 0$

$\lim_{x\to 0^+} f(x) = 0^2+2\cdot 0 = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(ax+b) = a\cdot 0+b = b$

it follows that:

f differentiable at $x_0=0 \Rightarrow f$ continuous at $x_0=0$

$\Rightarrow \lim_{x\to 0} f(x) = f(0)$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \Rightarrow \underline{b=0}$$

For $b=0$:

$$\lim_{x\to 0^-} \lambda(f|x,0) = \lim_{x\to 0^-} \frac{ax+0}{x} = \lim_{x\to 0^-} a = a$$

and therefore:

f differentiable at $x_0=0 \Rightarrow \exists\, \ell\in\mathbb{R} : \lim_{x\to 0} \lambda(f|x,0) = \ell$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f|x,0) = \lim_{x\to 0^+} \lambda(f|x,0)$$

$$\Rightarrow a = 2.$$

We have thus shown that

f differentiable at $x_0=0 \Rightarrow (a=2 \wedge b=0)$

($\Leftarrow$): Assume that $a=2\wedge b=0$. Then:

$$a=2\wedge b=0 \Rightarrow \forall x\in(-\infty,0) : \lambda(f|x,0) = \frac{2x+0}{x} = \frac{2x}{x} = 2$$

$$\Rightarrow \lim_{x\to 0^-} \lambda(f|x,0) = 2 = \lim_{x\to 0^+} \lambda(f|x,0)$$

$$\Rightarrow \lim_{x\to 0} \lambda(f|x,0) = 2 \Rightarrow \exists\, \ell\in\mathbb{R} : \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow f \text{ differentiable at } x_0=0.$$

We have thus shown that:

f differentiable at $x_0=0 \Leftrightarrow a=2\wedge b=0$. $\square$

↳ Note that a direct argument of the form

$$f \text{ differentiable at } x_0=0 \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftrightarrow$$
$$\Leftrightarrow a=2\wedge b=0$$

is not possible if we wish to use continuity.

Consequently the forward ($\Rightarrow$) and backward ($\Leftarrow$) arguments need to be done separately.

THEORY QUESTIONS

① Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$ and $S \subseteq A$. State the definitions for
a) f differentiable at x_0
b) f differentiable on S
c) x_0 corner point of f

② Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$. Prove that:
f differentiable at $x_0 \Rightarrow f$ continuous at x_0.

EXERCISES

③ Show that the function
$$f(x) = \begin{cases} x^2+4x, & \text{if } x \in [0,+\infty) \\ x^2-4x, & \text{if } x \in (-\infty,0) \end{cases}$$
is continuous on $\mathbb{R}$ but not differentiable at $x_0 = 0$

④ Show that the function
$$f(x) = (x+|x|)^2, \quad \forall x \in \mathbb{R}$$
is continuous and differentiable at $x_0 = 0$

⑤ Define the function
$$f(x) = \begin{cases} x\sin(2x)\cos(\pi/x)[1+\sin(\pi/x)], & \text{if } x \in \mathbb{R}-\{0\} \\ 0, & \text{if } x=0 \end{cases}$$
Show that f is differentiable at $x_0 = 0$

⑥ Let $f: A \to \mathbb{R}$ be a function, and define $g: A \to \mathbb{R}$ such that

$\forall x \in A: \ g(x) = x f(x)$

Show that:

f continuous at $x_0 = 0 \Rightarrow g$ differentiable at $x_0 = 0$

⑦ Find all $a, b \in \mathbb{R}$ such that the following functions are differentiable at x_0:

a) $f(x) = \begin{cases} ax + b, & \text{if } x \in (-\infty, 3) \\ x^2, & \text{if } x \in [3, +\infty) \end{cases}$ at $x_0 = 3$

b) $f(x) = \begin{cases} ax^2 + 2bx, & \text{if } x \in [1, +\infty) \\ bx - a, & \text{if } x \in (-\infty, 1) \end{cases}$ at $x_0 = 1$

⑧ Let $f: A \to \mathbb{R}$ be a function and define $g: A \to \mathbb{R}$ such that

$\forall x \in A: \ g(x) = |f(x)|$

Show that:

$\begin{cases} f \text{ differentiable at } x_0 \in A \\ f(x_0) \neq 0 \end{cases} \Rightarrow g$ differentiable at x_0

$\hookrightarrow$ Hint: We write:

$$\lambda(g|x, x_0) = \frac{(|f(x)| - |f(x_0)|)(|f(x)| + |f(x_0)|)}{(x - x_0)(|f(x)| + |f(x_0)|)}$$

and continue from there.

▼ Derivative function

- Let $f: A \to \mathbb{R}$ be a function and let $S \subseteq A$. We say that
  f differentiable at $S \Leftrightarrow \forall x_0 \in S$: f differentiable at x_0
- If $f: A \to \mathbb{R}$ is differentiable at S, then we define the derivative function $f': S \to \mathbb{R}$ as:
  $$\forall x_0 \in S : f'(x_0) = \lim_{x \to x_0} \lambda(f | x, x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$
- The notation $f'(x)$ is attributed to Newton. The Leibnitz notation of the derivative is:
  $$\frac{df}{dx} = f' \quad \text{and} \quad \left.\frac{df}{dx}\right|_{x = x_0} = f'(x_0)$$
- If f' is also differentiable at S, then the derivative of f' is denoted as f'' and is called the <u>2nd derivative</u> of f. Likewise we define
  $$f'' = \frac{df'}{dx} = \frac{d^2 f}{dx^2}$$
  $$f''' = \frac{df''}{dx} = \frac{d^3 f}{dx^3}$$

  Beyond the 3rd derivative, we use the notation $f^{(4)}, f^{(5)}, \ldots, f^{(n)}$ and write:
  $$f^{(n)} = \frac{d f^{(n-1)}}{dx} = \frac{d^n f}{dx^n}$$

- If we can define $f^{(n)}$ at x_0 we say that f is n-times differentiable at x_0. Likewise, for $S \subseteq A$, we say that f n-times differentiable at $S \Leftrightarrow \forall x_0 \in S : f$ n-times differentiable at x_0.

Derivatives of basic functions

① $\boxed{f(x) = ax + b,\ \forall x \in \mathbb{R} \Rightarrow f'(x) = a,\ \forall x \in \mathbb{R}}$

Proof

Since

$$\forall x, x_0 \in \mathbb{R} : \lambda(f \mid x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{(ax+b) - (ax_0+b)}{x - x_0}$$

$$= \frac{ax - ax_0}{x - x_0} = \frac{a(x - x_0)}{x - x_0} = a \Rightarrow$$

$$\Rightarrow \forall x_0 \in \mathbb{R} : f'(x_0) = \lim_{x \to x_0} \lambda(f \mid x, x_0) = a. \qquad \square$$

For the next result we use the identity

$$\boxed{\forall a, b \in \mathbb{R} : \forall n \in \mathbb{N} - \{0\} : a^n - b^n = (a-b) \sum_{k=0}^{n-1} (a^{n-k-1} b^k)}$$

Note that:

$n = 2$: $\quad a^2 - b^2 = (a-b)(a+b)$

$n = 3$: $\quad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$n = 4$: $\quad a^4 - b^4 = (a-b)(a^3 + a^2 b + ab^2 + b^3)$

Proof

$$(a-b)\sum_{k=0}^{n-1} a^{n-k-1} b^k = \sum_{k=0}^{n-1} (a-b) a^{n-k-1} b^k =$$

$$= \sum_{k=0}^{n-1} (a^{n-k} b^k - a^{n-k-1} b^{k+1}) =$$

$$= \sum_{k=0}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-1} a^{n-k-1} b^{k+1} =$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=0}^{n-2} a^{n-k-1} b^{k+1} - a^{n-(n-1)-1} b^{(n-1)+1}$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k} b^k - \sum_{k=1}^{n-1} a^{n-k} b^k - b^n =$$

$$= a^n - b^n$$ □

② $\boxed{f(x) = ax^n,\ \forall x \in \mathbb{R} \Rightarrow f'(x) = nax^{n-1},\ \forall x \in \mathbb{R}}$

Proof

Since:

$$\lambda(f \mid x, x_0) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{ax^n - ax_0^n}{x - x_0} = \frac{a(x^n - x_0^n)}{x - x_0} =$$

$$= \frac{a(x - x_0)\sum_{k=0}^{n-1} x^{n-k-1} x_0^k}{x - x_0} =$$

$$= a\sum_{k=0}^{n-1} x^{n-k-1} x_0^k \Rightarrow$$

$$\Rightarrow f'(x_0) = \lim_{x\to x_0} \lambda(f|x,x_0) = \lim_{x\to x_0}\left[a\sum_{k=0}^{n-1} x^{n-k-1}x_0^k\right] =$$

$$= a\lim_{x\to x_0}\sum_{k=0}^{n-1} x^{n-k-1}x_0^k = a\sum_{k=0}^{n-1}\lim_{x\to x_0}(x^{n-k-1}x_0^k)$$

$$= a\sum_{k=0}^{n-1} x_0^{n-k-1}x_0^k = a\sum_{k=0}^{n-1} x_0^{n-1} = anx_0^{n-1} =$$

$$= nax_0^{n-1},\ \forall x_0\in\mathbb{R}. \qquad \square$$

③ $\boxed{f(x)=\sqrt{x},\ \forall x\in[0,+\infty) \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}},\ \forall x\in(0,+\infty)}$

Proof

$$\forall x,x_0\in[0,+\infty): \lambda(f|x,x_0) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} =$$

$$= \frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x})^2-(\sqrt{x_0})^2} = \frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})} = \frac{1}{\sqrt{x}+\sqrt{x_0}} \Rightarrow$$

$$\Rightarrow \forall x_0\in(0,+\infty): f'(x_0) = \lim_{x\to x_0}\lambda(f|x,x_0) = \lim_{x\to x_0}\frac{1}{\sqrt{x}+\sqrt{x_0}} =$$

$$= \frac{1}{\sqrt{x_0}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}. \qquad \square$$

↳ Note that, as was shown previously, although the function $f(x)=\sqrt{x}$ is defined at $x=0$, it is <u>not</u> differentiable at $x=0$.

EXAMPLES

a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function that is differentiable at $x_0\in\mathbb{R}$ and let $g:\mathbb{R}\to\mathbb{R}$ such that
$\forall x\in\mathbb{R}: g(x)=[f(x)]^2$
Show that g differentiable at x_0 with $g'(x_0)=2f(x_0)f'(x_0)$, without using the chain rule

Solution

We have:

$$\forall x\in\mathbb{R}: \lambda(g|x,x_0)=\frac{g(x)-g(x_0)}{x-x_0}=\frac{[f(x)]^2-[f(x_0)]^2}{x-x_0}=$$

$$=\frac{[f(x)-f(x_0)][f(x)+f(x_0)]}{x-x_0}=$$

$$=\lambda(f|x,x_0)[f(x)+f(x_0)]$$

and

$$f \text{ differentiable at } x_0 \Rightarrow \lim_{x\to x_0}\lambda(f|x,x_0)=f'(x_0)$$

and

f differentiable at $x_0 \Rightarrow f$ continuous at $x_0 \Rightarrow$

$$\Rightarrow \lim_{x\to x_0} f(x)=f(x_0)$$

$$\Rightarrow \lim_{x\to x_0}[f(x)+f(x_0)]=\lim_{x\to x_0}[f(x)]+f(x_0)=$$

$$=f(x_0)+f(x_0)=2f(x_0)$$

therefore,

$$g'(x_0) = \lim_{x \to x_0} \partial(g|x,x_0) = \lim_{x \to x_0} \{\partial(f|x,x_0)[f(x)+f(x_0)]\} =$$

$$= \left[\lim_{x \to x_0} \partial(f|x,x_0)\right]\left[\lim_{x \to x_0} (f(x)+f(x_0))\right]$$

$$= f'(x_0)[2f(x_0)] = 2f(x_0)f'(x_0) \qquad \square$$

b) Let $f:(0,+\infty)\to\mathbb{R}$ such that

$$\begin{cases} f \text{ differentiable at } x_0=1 \\ \forall a,b\in(0,+\infty): f(ab)=f(a)+f(b) \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } (0,+\infty) \\ \forall x\in(0,+\infty): f'(x)=\dfrac{f'(1)}{x} \end{cases}$$

Solution

Choose some $b\in(0,+\infty)$. Then, we have:

$f(b)=f(1b)=f(1)+f(b)\Rightarrow f(1)=0$

and therefore

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1} \quad [\text{via } f \text{ differentiable at } x_0=1]$$

$$=\lim_{x\to 1}\frac{f(x)}{x-1} \quad [\text{via } f(1)=0] \qquad (1)$$

Let $x_0\in(0,+\infty)$ be given. Then, we have:

$$\lim_{x\to x_0}\lambda(f\,|\,x,x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

$$=\lim_{t\to 1}\frac{f(tx_0)-f(x_0)}{tx_0-x_0} \quad [\text{via composition thm}]$$

$$=\lim_{t\to 1}\frac{f(t)+f(x_0)-f(x_0)}{x_0(t-1)} \quad [\text{via hypothesis}]$$

$$=\lim_{t\to 1}\frac{f(t)}{x_0(t-1)}=\frac{1}{x_0}\lim_{t\to 1}\frac{f(t)}{t-1}$$

$$=\frac{f'(1)}{x_0} \quad [\text{via Eq. (1)}]$$

We have thus shown that

$$\begin{cases} f \text{ differentiable on } (0,+\infty) \\ \forall x \in (0,+\infty) : f'(x) = f'(1)/x. \end{cases}$$

EXERCISES

(9) Let $f: A \to \mathbb{R}$ with $x_0 \in A$ such that f differentiable at x_0. Show that:

$$\lim_{x \to x_0} \frac{x f(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0)$$

(10) Let $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{cases} f \text{ differentiable at } x_0 = 0 \\ \forall a, b \in \mathbb{R}: f(a+b) = f(a) + f(b) \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R}: f'(x) = f'(0) \end{cases}$$

(11) Let $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{cases} f \text{ differentiable at } x_0 = 0 \\ \forall x \in \mathbb{R}: f(x) \neq 0 \\ \forall a, b \in \mathbb{R}: f(a+b) = f(a) f(b) \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R}: f'(x) = f'(0) f(x) \end{cases}$$

(12) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \forall a, b \in \mathbb{R}: f(a+b) = f(a) f(b) \\ \forall x \in \mathbb{R}: f(x) = 1 + x g(x) \\ \lim_{x \to 0} g(x) = 1 \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R}: f'(x) = f(x). \end{cases}$$

(13) Let $f:\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} \forall a,b\in\mathbb{R}: f(a+b)+a+b=(f(a)+a)(f(b)+b) \\ \forall x\in\mathbb{R}: f(x)\neq 0 \end{cases}$$

Show that:

a) $f(0)=1$ b) $\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall x\in\mathbb{R}: f'(x)=(f(x)+x)(f'(0)+1)-1 \end{cases}$

(14) Let $f:\mathbb{R}\to\mathbb{R}$ such that

$$\begin{cases} \forall a,b\in\mathbb{R}: f(a+b)\leqslant f(a)+f(b) \\ f \text{ differentiable on } \mathbb{R} \\ f'(0)=f(0)=1 \end{cases}$$

Show that: $\forall x\in\mathbb{R}: f'(x)=f(x)$

(15) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ and let $a\in\mathbb{R}$ such that

$$\begin{cases} f,g \text{ differentiable on } \mathbb{R} \\ f(a)=g(a) \\ \forall x\in\mathbb{R}: f(x)+x\leqslant g(x)+a \end{cases}$$

Show that: $f'(a)+1=g'(a)$

(16) Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients and one variable. Show that:

$\forall f\in\mathbb{R}[x]: [(f')^2=f \iff \exists b\in\mathbb{R}: \forall x\in\mathbb{R}: f(x)=(1/4)x^2+bx+b^2]$

Basic differentiation rules

Let f, g be functions differentiable at a set $A \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$. Then:

$$\boxed{\begin{array}{l} h(x) = f(x) + g(x), \forall x \in A \Rightarrow h'(x) = f'(x) + g'(x), \forall x \in A \\ h(x) = af(x), \forall x \in A \Rightarrow h'(x) = af'(x), \forall x \in A \\ h(x) = f(x)g(x), \forall x \in A \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x), \forall x \in A \end{array}}$$

Proof

a) Assume that $h(x) = f(x) + g(x), \forall x \in A$. Then

$$\forall x, x_0 \in A : \lambda(h \mid x, x_0) = \frac{h(x) - h(x_0)}{x - x_0} =$$

$$= \frac{[f(x) + g(x)] - [f(x_0) + g(x_0)]}{x - x_0} =$$

$$= \frac{[f(x) - f(x_0)] + [g(x) - g(x_0)]}{x - x_0} =$$

$$= \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} = \lambda(f \mid x, x_0) + \lambda(g \mid x, x_0)$$

$$\Rightarrow \forall x_0 \in A : h'(x_0) = \lim_{x \to x_0} \lambda(h \mid x, x_0) =$$

$$= \lim_{x \to x_0} [\lambda(f \mid x, x_0) + \lambda(g \mid x, x_0)]$$

$$= \lim_{x \to x_0} \lambda(f \mid x, x_0) + \lim_{x \to x_0} \lambda(g \mid x, x_0)$$

$$= f'(x_0) + g'(x_0).$$

b) Assume that $h(x) = af(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \lambda(h \mid x, x_0) = \frac{h(x)-h(x_0)}{x-x_0} = \frac{af(x)-af(x_0)}{x-x_0} =$$

$$= \frac{a[f(x)-f(x_0)]}{x-x_0} = a\lambda(f \mid x, x_0) \Rightarrow$$

$$\Rightarrow \forall x_0 \in A: h'(x_0) = \lim_{x \to x_0} \lambda(h \mid x, x_0) = \lim_{x \to x_0} [a\lambda(f \mid x, x_0)]$$

$$= a \lim_{x \to x_0} \lambda(f \mid x, x_0) = af'(x_0)$$

c) Assume that $h(x) = f(x)g(x)$, $\forall x \in A$. Then

$$\forall x, x_0 \in A: \lambda(h \mid x, x_0) = \frac{h(x)-h(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \frac{g(x)[f(x)-f(x_0)]+f(x_0)[g(x)-g(x_0)]}{x-x_0} =$$

$$= f(x_0)\frac{g(x)-g(x_0)}{x-x_0} + g(x)\frac{f(x)-f(x_0)}{x-x_0} =$$

$$= f(x_0)\lambda(g \mid x, x_0) + \lambda(f \mid x, x_0)g(x)$$

We note that:

g differentiable at $x_0 \Rightarrow g$ continuous at x_0

$$\Rightarrow \lim_{x \to x_0} g(x) = g(x_0)$$

and therefore:

$$\forall x_0 \in A: h'(x_0) = \lim_{x \to x_0} \lambda(h|x, x_0) =$$

$$= \lim_{x \to x_0} \left[f(x_0)\, \lambda(g|x, x_0) + \lambda(f|x, x_0)\, g(x) \right]$$

$$= f(x_0) \lim_{x \to x_0} \lambda(g|x, x_0) + \lim_{x \to x_0} \lambda(f|x, x_0) \lim_{x \to x_0} g(x)$$

$$= f(x_0)\, g'(x_0) + f'(x_0)\, g(x_0)$$

$$= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0). \qquad \square$$

THEORY QUESTIONS

(17) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be functions differentiable on A. Prove that:

$\forall x \in A : [f(x)+g(x)]' = f'(x) + g'(x)$

$\forall a \in \mathbb{R} : \forall x \in A : [a f(x)]' = a f'(x)$

$\forall x \in A : [f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

EXERCISES

(18) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ such that:

$$\begin{cases} f, g \text{ 3-times differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : f'(x)g'(x) = a \\ \forall x \in \mathbb{R} : f(x)g(x) \neq 0 \end{cases}$$

and define $h: \mathbb{R} \to \mathbb{R}$ such that

$\forall x \in \mathbb{R} : h(x) = f(x)g(x)$

Show that:

a) $\forall x \in \mathbb{R} : \dfrac{h''(x)}{h(x)} = \dfrac{f''(x)}{f(x)} + \dfrac{2a}{f(x)g(x)} + \dfrac{g''(x)}{g(x)}$

b) $\forall x \in \mathbb{R} : \dfrac{h'''(x)}{h(x)} = \dfrac{f'''(x)}{f(x)} + \dfrac{g'''(x)}{g(x)}$

(19) Let $f \in \mathbb{R}[x]$ be a polynomial $f: \mathbb{R} \to \mathbb{R}$ with degree $\deg(f) = n \geq 2$. We say that:

p double zero of $f \Leftrightarrow \exists q \in \mathbb{R}[x] : \forall x \in \mathbb{R} : f(x) = (x-p)^2 q(x)$

Show that:
p double zero of $f \Leftrightarrow f(p)=0 \wedge f'(p)=0$

(20) Let $f \in \mathbb{R}[x]$ be a polynomial with degree 3 and three distinct roots $p_1, p_2, p_3 \in \mathbb{R}$. Show that

$$\frac{p_1}{f'(p_1)} + \frac{p_2}{f'(p_2)} + \frac{p_3}{f'(p_3)} = 0$$

(21) Let $f \in \mathbb{R}[x]$ be a polynomial with degree $n \in \mathbb{N}^*$. Show that:

$$\forall x \in \mathbb{R}: f(x) = \sum_{a=0}^{n} \frac{f^{(a)}(0)\, x^a}{a!}$$

using proof by induction.

(22) Let $f_1 : A \to \mathbb{R}$, $f_2 : A \to \mathbb{R}$, $g_1 : A \to \mathbb{R}$, $g_2 : A \to \mathbb{R}$ be functions that are differentiable on $\mathbb{R}$ and let

$$\forall x \in A : h(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$$

Show that:

$$\forall x \in A: h'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$$

▼ Chain rule

- The chain rule is a superrule that is used to generate differentiation rules that are then used in problems. We seldomly use the chain rule directly.
- Recall the definition of function composition:
  For $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$, we define $f \circ g: C \to \mathbb{R}$ with

$$\begin{cases} \text{dom}(f \circ g) = \{x \in \text{dom}(g) \mid g(x) \in \text{dom}(f)\} \\ \qquad\qquad\quad = \{x \in B \mid g(x) \in A\} = C \\ \forall x \in C : (f \circ g)(x) = f(g(x)) \end{cases}$$

  Note that by definition, the belonging condition for $\text{dom}(f \circ g)$ is:

$$x \in \text{dom}(f \circ g) \Leftrightarrow \begin{cases} x \in \text{dom}(g) \\ g(x) \in \text{dom}(f) \end{cases}$$

- The chain rule claims that:

$$\boxed{\begin{cases} g \text{ differentiable at } x_0 \\ f \text{ differentiable at } g(x_0) \end{cases} \Rightarrow \begin{cases} f \circ g \text{ differentiable at } x_0 \\ (f \circ g)'(x_0) = f'(g(x_0))\, g'(x_0). \end{cases}}$$

We postpone the proof. Every choice of f generates a new generalized differentiation rule. For example:

1) For $f(x) = x^n$ with $n \in \mathbb{N}^*$, using

$$(x^n)' = nx^{n-1}$$

we obtain:

$$\boxed{([g(x)]^n)' = n\,[g(x)]^{n-1}\,g'(x)}$$

2) For $f(x)=\sqrt{x}$, using $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$, we obtain:

$$\boxed{\left(\sqrt{g(x)}\right)' = \frac{g'(x)}{2\sqrt{g(x)}}}$$

↳ Note that for each generalization, starting from the initial differentiation rule:
(a) All x are replaced with $g(x)$
(b) The entire result is then multiplied with $g'(x)$.
Step (a) corresponds to the $f'(g(x_0))$ factor
Step (b) corresponds to the $g'(x_0)$ factor.
We see therefore that every basic differentiation rule can give a more powerful generalized differentiation rule via the chain rule.

Proof of chain rule

Assume that g differentiable at x_0 and f differentiable at $g(x_0)$. It follows that $f\circ g$ can be defined on a neighborhood $N(x_0,\delta)$ for some $\delta>0$.

We define $y_0=g(x_0)$ and

$$F(y)=\begin{cases}\lambda(f\,|\,y,y_0), & \text{if } y\neq y_0\\ f'(y_0), & \text{if } y=y_0\end{cases}$$

We claim that $\lambda(f\circ g\,|\,x,x_0)=F(g(x))\,\lambda(g\,|\,x,x_0)$, $\forall x\in N(x_0,\delta)$ (1)

To show the claim, let $x\in N(x_0,\delta)$ be given. We distinguish between the following cases:

Case 1: If $g(x)\neq g(x_0)$ then:

$$\lambda(f\circ g\,|\,x,x_0)=\frac{(f\circ g)(x)-(f\circ g)(x_0)}{x-x_0}=\frac{f(g(x))-f(g(x_0))}{x-x_0}=$$

$$=\frac{f(g(x))-f(g(x_0))}{g(x)-g(x_0)}\,\frac{g(x)-g(x_0)}{x-x_0}=$$

$$=\lambda(f\,|\,g(x),y_0)\,\lambda(g\,|\,x,x_0)=F(g(x))\,\lambda(g\,|\,x,x_0)$$

Case 2: If $g(x)=g(x_0)$, then:

$$\lambda(f\circ g\,|\,x,x_0)=\frac{(f\circ g)(x)-(f\circ g)(x_0)}{x-x_0}=\frac{f(g(x))-f(g(x_0))}{x-x_0}$$

$$=\frac{f(g(x_0))-f(g(x_0))}{x-x_0}=0$$

and

$$\lambda(g|x,x_0) = \frac{g(x)-g(x_0)}{x-x_0} = \frac{g(x_0)-g(x_0)}{x-x_0} = 0$$

and therefore $\lambda(f\circ g|x,x_0) = F(g(x))\lambda(g|x,x_0)$ holds trivially since both sides are zero.

This proves the claim.

Now, we note that

g differentiable at $x_0 \Rightarrow g$ continuous at $x_0 \Rightarrow$

$$\Rightarrow \lim_{x\to x_0} g(x) = g(x_0) = y_0 \qquad (2)$$

and

$$\begin{aligned}\lim_{y\to y_0} F(y) = \lim_{y\to y_0} \lambda(f|y,y_0) &= \qquad [\text{def of } F(y)]\\ &= f'(y_0) = \qquad [f \text{ differentiable at } y_0]\\ &= F(y_0) \Rightarrow \qquad [\text{def of } F(y)]\end{aligned}$$

$\Rightarrow F$ continuous at y_0. $\qquad (3)$

Via the composition theorem, from Eq. (2) and Eq. (3):

$$\begin{aligned}\lim_{x\to x_0} F(g(x)) &= F(\lim_{x\to x_0} g(x)) \qquad [\text{via composition thm}]\\ &= F(y_0) \qquad [\text{via eq. (2)}]\\ &= f'(y_0) \qquad [\text{def of } F(y)]\\ &= f'(g(x_0)) \quad (4) \quad [\text{def of } y_0]\end{aligned}$$

and it follows that

$$[f(g(x_0))]' = \lim_{x\to x_0} \lambda(f\circ g|x,x_0) = \lim_{x\to x_0} [F(g(x))\lambda(g|x,x_0)] =$$

$$= \lim_{x\to x_0} F(g(x)) \cdot \lim_{x\to x_0} \lambda(g|x,x_0) = f'(g(x_0))\, g'(x_0). \quad \square$$

THEORY QUESTIONS

(23) Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ and $f \circ g: G \to \mathbb{R}$ and let $x_0 \in G$

a) Write the definition of G in terms of A and B

b) Prove that

$$\begin{cases} g \text{ differentiable at } x_0 \\ f \text{ differentiable at } g(x_0) \end{cases} \Rightarrow \begin{cases} f \circ g \text{ differentiable at } x_0 \\ (f \circ g)'(x_0) = f'(g(x_0))\, g'(x_0) \end{cases}$$

EXERCISES

(24) Let $f: \mathbb{R} \to \mathbb{R}$ such that f differentiable on $\mathbb{R}$.
Use the chain rule to show that.

a) f odd $\Rightarrow f'$ even

b) f periodic $\Rightarrow f'$ periodic

$\hookrightarrow$ Recall the following definitions

f even $\Leftrightarrow \forall x \in \mathbb{R}: f(-x) = f(x)$

f odd $\Leftrightarrow \forall x \in \mathbb{R}: f(-x) = -f(x)$

f periodic $\Leftrightarrow \exists a \in \mathbb{R}: \forall x \in \mathbb{R}: f(x+a) = f(x)$

(25) Let $f: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f \text{ 2-times differentiable on } \mathbb{R} \\ f \text{ odd} \\ \forall x \in \mathbb{R}: f(x) f'(x) \neq 0 \end{cases}$$

and let $g : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R} : g(x) = f(x) f'(x)$
Show that:

a) $f(0) = f''(0) = 0$

b) g' even

c) $\forall x \in \mathbb{R} : \dfrac{g'(x)}{g(x)} = \dfrac{f'(x)}{f(x)} + \dfrac{f''(x)}{f'(x)}$

26 Let $f \in \mathbb{R}[x]$ be a polynomial $f : \mathbb{R} \to \mathbb{R}$ and let $p \in \mathbb{R}$ and $n \in \mathbb{N}^*$. We say that

p zero of f with multiplicity $n \Leftrightarrow$

$\Leftrightarrow \exists q \in \mathbb{R}[x] : \forall x \in \mathbb{R} : f(x) = (x-p)^n q(x)$

Use proof by induction to show that

p zero of f with multiplicity $n \Leftrightarrow$

$\Leftrightarrow \forall k \in \{0\} \cup [n-1] : f^{(k)}(p) = 0$

27 Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that

$$\begin{cases} \forall x \in \mathbb{R} : f(x) = x + a \\ g \text{ differentiable on } \mathbb{R} \\ f \circ g = g \circ f \end{cases} \Rightarrow g' \text{ periodic}$$

▼ The quotient rule

The quotient rule is derived from the chain rule as follows.

•1 First we show that

$$\boxed{\left(\forall x \in \mathbb{R}^*: f(x) = \frac{1}{x}\right) \Rightarrow \forall x \in \mathbb{R}^*: f'(x) = \frac{-1}{x^2}}$$

Proof

Since

$$\forall x, x_0 \in \mathbb{R}-\{0\}: \lambda(f \mid x, x_0) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{\frac{1}{x}-\frac{1}{x_0}}{x-x_0} =$$

$$= \frac{\left(\frac{x_0-x}{xx_0}\right)}{x-x_0} = \frac{-(x-x_0)}{xx_0(x-x_0)} = \frac{-1}{xx_0} \Rightarrow$$

$$\Rightarrow \forall x_0 \in \mathbb{R}-\{0\}: f'(x_0) = \lim_{x \to x_0} \lambda(f \mid x, x_0) = \lim_{x \to x_0} \left(\frac{-1}{xx_0}\right) =$$

$$= \frac{-1}{x_0 x_0} = \frac{-1}{x_0^2} \qquad \square$$

•2 Via the chain rule, this result immediately generalizes to the <u>reduced quotient rule</u>:

$$\boxed{h(x) = \frac{1}{g(x)}, \ \forall x \in A \Rightarrow h'(x) = \frac{-g'(x)}{[g(x)]^2}, \ \forall x \in A}$$

•3 Combined with the product rule, the reduced quotient rule gives the <u>quotient rule</u>:

$$\boxed{h(x) = \frac{f(x)}{g(x)}, \forall x \in A \Rightarrow h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

<u>Proof</u>

$$h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \left[f(x) \cdot \frac{1}{g(x)}\right]' =$$

$$= f'(x)\frac{1}{g(x)} + f(x) \cdot \left[\frac{1}{g(x)}\right]' = \quad \text{[product rule]}$$

$$= \frac{f'(x)}{g(x)} + f(x)\frac{-g'(x)}{[g(x)]^2} = \quad \text{[reduced quotient rule]}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \quad \square$$

EXERCISES

(28) Let $f : \mathbb{R}^* \to \mathbb{R}$ with $\forall x \in \mathbb{R}^* : f(x) = 1/x$.
Use proof by induction to show that:
$$\forall n \in \mathbb{N}^* : \forall x \in \mathbb{R}^* : f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

(29) Let $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R} : f(x) = \sqrt{x + \sqrt{1+x^2}}$
Show that.
a) $\forall x \in \mathbb{R} : f(x) = 2\sqrt{1+x^2}\, f'(x)$
b) $\forall x \in \mathbb{R} : 4(1+x^2) f''(x) + 4x f'(x) = f(x)$

(30) Let $f \in \mathbb{R}[x]$ be a polynomial with degree $n \in \mathbb{N}^*$ with distinct zeroes $\rho_1, \rho_2, \ldots, \rho_n \in \mathbb{R}$. Show that:
a) $\forall x \in \mathbb{R} - \{\rho_k \mid k \in [n]\} : \dfrac{f'(x)}{f(x)} = \displaystyle\sum_{k=1}^{n} \frac{1}{x - \rho_k}$
b) The function $g : \mathbb{R} \to \mathbb{R}$ given by
$$\forall x \in \mathbb{R} : g(x) = f(x) f''(x) - [f'(x)]^2$$
satisfies
$$\forall x \in \mathbb{R} : g(x) \neq 0$$

(31) Define $p : \mathbb{R} \to \mathbb{R}$ such that
$$\forall x \in \mathbb{R} : p(x) = ax^2 + bx + c = a(x - \rho_1)(x - \rho_2)$$
with ρ_1, ρ_2 the distinct zeroes of p. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ \forall k \in \{1,2\} : \forall x \in \mathbb{R} : x f'(x) - f(x) \leq \rho_k f(x) \end{cases}$$

Show that:

a) $\forall x \in \mathbb{R} - \{\rho_1, \rho_2\} : \dfrac{p'(x)}{p(x)} = \dfrac{1}{x-\rho_1} + \dfrac{1}{x-\rho_2}$

b) $\forall x \in \mathbb{R} : f(x) p(x) p''(x) \leq p'(x) \begin{vmatrix} f(x) & f'(x) \\ p(x) & p'(x) \end{vmatrix}$

For part (b) introduce the function $w(x) = \dfrac{p'(x)}{p(x)} f(x)$

Calculate $w'(x)$ and show, using (a), that $w'(x) \geq 0$.

▼ Trigonometric derivatives

- The derivative of $\sin x$ can be derived via the result
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$
and the trigonometric identity for factoring the sum/difference of sine functions:
$$\sin a \pm \sin b = 2\sin\left(\frac{a\pm b}{2}\right)\cos\left(\frac{a\mp b}{2}\right)$$
The main result is:

① $\boxed{\forall x\in\mathbb{R} : (\sin x)' = \cos x}$

Proof

Let $x, x_0 \in \mathbb{R}$ be given with $x \neq x_0$.
$$\lambda(\sin \mid x, x_0) = \frac{\sin x - \sin x_0}{x - x_0} = \frac{2\sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)}{x-x_0} =$$
$$= \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cos\left(\frac{x+x_0}{2}\right), \forall x, x_0 \in \mathbb{R} \quad (1)$$

Since
$$\left.\begin{array}{l} \lim\limits_{x\to x_0} \frac{x+x_0}{2} = \frac{x_0+x_0}{2} = x_0 \\ \cos \text{ continuous on } \mathbb{R} \end{array}\right\} \Rightarrow \lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right) = \cos x_0 \quad (2)$$

and

$$\left.\begin{array}{l} \lim\limits_{x \to x_0} \dfrac{x-x_0}{2} = 0 \\ \dfrac{x-x_0}{2} \neq 0, \forall x \in N(x_0, \delta) \\ \lim\limits_{y \to 0} \dfrac{\sin y}{y} = 1 \end{array}\right\} \Rightarrow \lim_{x \to 0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = 1 \quad (3)$$

From Eq. (1), Eq. (2), Eq. (3):

$$(\sin x_0)' = \lim_{x \to x_0} \lambda(\sin | x, x_0) =$$

$$= \lim_{x \to x_0} \left[\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \cdot \cos\left(\frac{x+x_0}{2}\right) \right]$$

$$= \lim_{x \to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} \lim_{x \to x_0} \cos\left(\frac{x+x_0}{2}\right) =$$

$$= 1 \cdot \cos\left(\frac{x_0+x_0}{2}\right) = \cos x_0 \qquad \square$$

↳ Note that the proof of this result depends on the continuity of cos and the limit $\lim\limits_{x \to 0} (\sin x)/x$. Consequently continuity has to be established first before establishing differentiability.

- For the derivative of cos we use the chain rule generalization of the above result

$$[\sin(g(x))]' = g'(x) \cos(g(x))$$

and the cofactor identities:

$$\forall x \in \mathbb{R}: \sin(\pi/2 - x) = \cos x$$
$$\forall x \in \mathbb{R}: \cos(\pi/2 - x) = \sin x$$

as follows:

(2) $\boxed{(\cos x)' = -\sin x \ , \forall x \in \mathbb{R}}$

Proof

$$(\cos x)' = [\sin(\pi/2 - x)]' = (\pi/2 - x)' \cos(\pi/2 - x)$$
$$= -\cos(\pi/2 - x) = -\sin x \ , \ \forall x \in \mathbb{R}.$$

(3) $\boxed{(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x \ , \forall x \in \mathbb{R} - \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}}$

Proof

$$(\tan x)' = \left[\frac{\sin x}{\cos x}\right]' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} =$$
$$= \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \qquad (1)$$

From Eq. (1):

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\tan x)' = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} =$$
$$= 1 + \left(\frac{\sin x}{\cos x}\right)^2 = 1 + \tan^2 x \qquad \square$$

- Via the chain rule, we obtain the following generalized differentiation rules:

$(\sin x)' = \cos x$	$[\sin(g(x))]' = g'(x)\cos(g(x))$
$(\cos x)' = -\sin x$	$[\cos(g(x))]' = -g'(x)\sin(g(x))$
$(\tan x)' = \dfrac{1}{\cos^2 x}$	$[\tan(g(x))]' = \dfrac{g'(x)}{\cos^2(g(x))}$
$(\tan x)' = 1+\tan^2 x$	$[\tan(g(x))]' = [1+\tan^2(g(x))]\,g'(x)$

EXAMPLE

Consider the function

$$f(x) = \begin{cases} \sin(\pi x^2)/x & , \text{ if } x \in \mathbb{R}^* \\ 0 & , \text{ if } x = 0 \end{cases}$$

Show that:

$$\begin{cases} f \text{ differentiable on } \mathbb{R} \\ f' \text{ continuous on } \mathbb{R}. \end{cases}$$

Solution

Since,

$$\forall x \in \mathbb{R}^* : f'(x) = \left[\frac{\sin(\pi x^2)}{x}\right]' = \frac{[\sin(\pi x^2)]' x - \sin(\pi x^2)(x)'}{x^2}$$

$$= \frac{(\pi x^2)' \cos(\pi x^2) x - \sin(\pi x^2)}{x^2} = \frac{2\pi x \cos(\pi x^2) x - \sin(\pi x^2)}{x^2}$$

$$= \frac{2\pi x^2 \cos(\pi x^2) - \sin(\pi x^2)}{x^2} = 2\pi \cos(\pi x^2) - \sin(\pi x^2)/x^2.$$

and

$$\lambda(f|x,0) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \frac{\sin(\pi x^2)/x}{x} = \frac{\sin(\pi x^2)}{x^2}$$

$$\Rightarrow \lim_{x \to 0} \lambda(f|x,0) = \lim_{x \to 0} \frac{\sin(\pi x^2)}{x^2} = \pi \lim_{x \to 0} \frac{\sin(\pi x^2)}{\pi x^2}$$

$$= \pi \lim_{x \to 0} \frac{\sin x}{x} = \pi$$

it follows that f differentiable on $\mathbb{R}$ with

$$f'(x) = \begin{cases} 2\pi\cos(\pi x^2) - \sin(\pi x^2)/x^2, & \text{if } x \in \mathbb{R}^* \\ \pi &, \text{if } x = 0 \end{cases}$$

Since,

$\forall x \in \mathbb{R}^* : f'(x) = 2\pi\cos(\pi x^2) - \sin(\pi x^2)/x^2$

$\Rightarrow f'$ continuous on $\mathbb{R}^*$

and

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left[2\pi\cos(\pi x^2) - \sin(\pi x^2)/x^2\right] =$$

$$= 2\pi\cos 0 - \pi\lim_{x\to 0}\frac{\sin(\pi x^2)}{\pi x^2} =$$

$$= 2\pi - \pi\lim_{x\to 0}\frac{\sin x}{x} = 2\pi - \pi\cdot 1 = \pi = f'(0)$$

$\Rightarrow f'$ continuous at $x = 0$.

we conclude that f' continuous on $\mathbb{R}$.

EXERCISES

(32) Consider the function

$$f(x) = \begin{cases} \sin^2(\pi x)/(x-1) & , \text{ if } x \in \mathbb{R}-\{1\} \\ 0 & , \text{ if } x = 0 \end{cases}$$

Show that f differentiable on $\mathbb{R}$ and f' continuous on $\mathbb{R}$

(33) Let $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ and consider the function

$\forall x \in \mathbb{R}: f(x) = \sin(ax+b)$

Use proof by induction to show that

$\forall n \in \mathbb{N}^*: \forall x \in \mathbb{R}: f^{(n)}(x) = a^n \sin(ax+b+n\pi/2)$

(34) Let $a \in \mathbb{R}$ and consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x) + ax & , \text{ if } x \in \mathbb{R}^* \\ 0 & , \text{ if } x = 0 \end{cases}$$

Show that f differentiable on $\mathbb{R}$.

(35) Consider the function

$$\forall x \in \mathbb{R}: f(x) = \frac{\cos^2 t}{1+\sin^2 t}$$

Show that : $f(\pi/4) - 3f'(\pi/4) = 3$.

36 Let $f:\mathbb{R}\to\mathbb{R}$ with $\forall x\in\mathbb{R}: f(x) = x\sin(ax)$. Show that

a) $\forall n\in\mathbb{N}^*: f^{(2n)}(x) = (-1)^n\left[a^{2n}x\sin(ax) - 2na^{2n-1}\cos(ax)\right]$

b) $|a|<1 \Rightarrow \lim\limits_{n\in\mathbb{N}^*} f^{(2n)}(x) = 0$

37 Let $f:(0,1)\to\mathbb{R}$ be a function such that

$\forall x\in(0,\pi/2): f(\sin x) = \sin^2 x - \cos x$

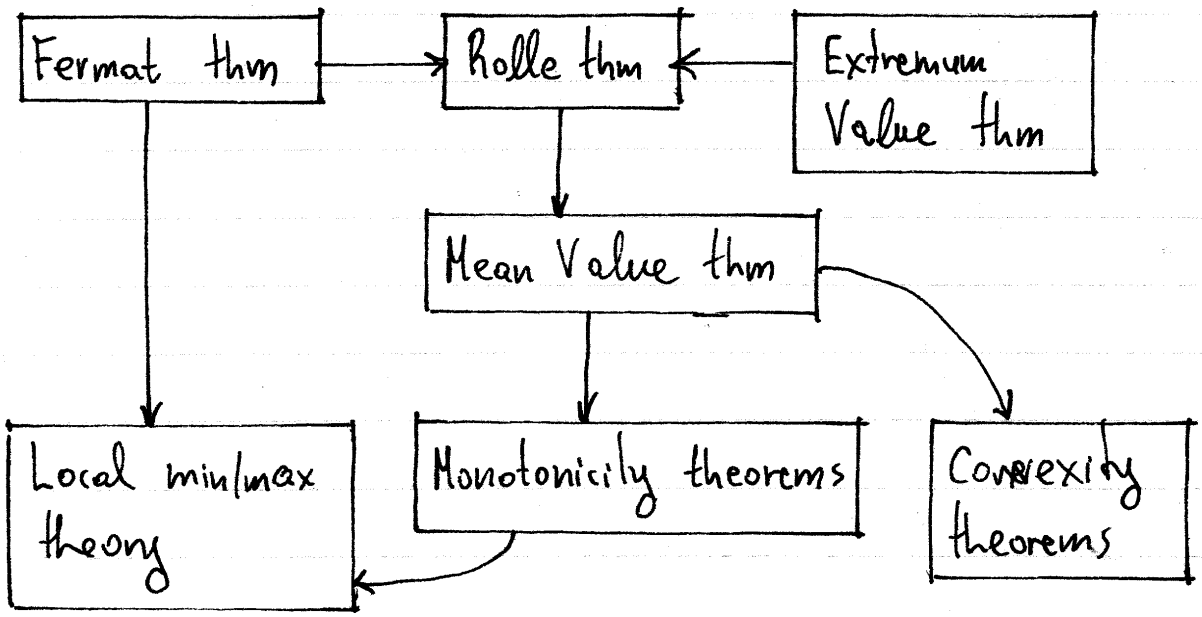
Show that: $3f''(1/2) - 2f'(1/2) = 4+2\sqrt{3}$.

RA 1.6: Differential Calculus

DIFFERENTIAL CALCULUS

Foundation of Differential Calculus

The applications of derivatives are based on a collection of theorems that have the following interdependence amongst themselves



(1) → Fermat theorem

Def: (Interior points)
Let A be a set $A \subseteq \mathbb{R}$. We say that
x_0 interior point of $A \iff \exists \delta \in (0, +\infty) : (x_0 - \delta, x_0 + \delta) \subseteq A$

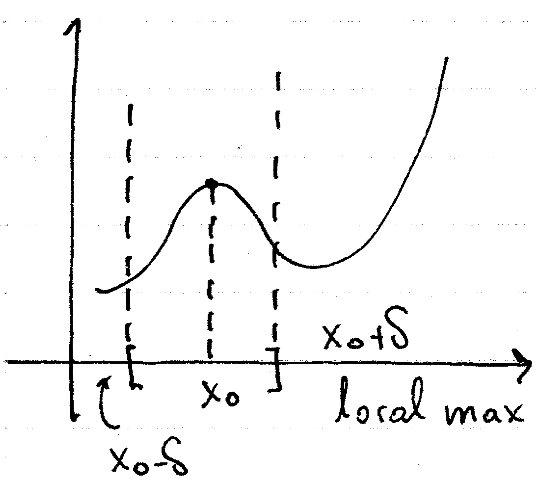
<u>notation</u>: The set of all interior points of a set A is denoted as

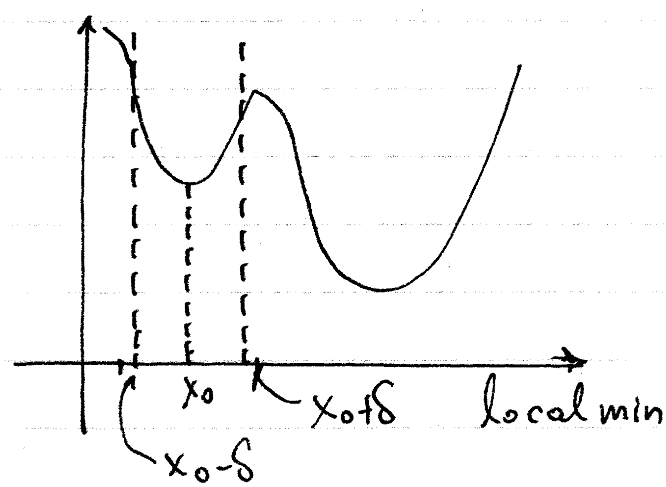
$$\mathrm{int}(A) = \{x_0 \in A \mid x_0 \text{ interior to } A\}$$
$$= \{x_0 \in A \mid \exists \delta \in (0,+\infty) : (x_0-\delta, x_0+\delta) \subseteq A\}$$

▸ In general, given a set defined as a union of intervals, $\mathrm{int}(A)$ can be obtained by changing all closed intervals to open intervals

<u>example</u>: For $A = (1,3] \cup [5,+\infty)$, we have
$$\mathrm{int}(A) = (1,3) \cup (5,+\infty).$$
Consequently, 2 is interior to A but for $x_0 \in \{1,3,5\}$, x_0 is <u>not</u> interior to A.

<u>Def</u>: (<u>Local min/max</u>)
Let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$.
We say that
a) x_0 <u>local max</u> of $f \Leftrightarrow$
$\Leftrightarrow \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \leq f(x_0)$
b) x_0 <u>local min</u> of $f \Leftrightarrow$
$\Leftrightarrow \exists \delta \in (0,+\infty) : \forall x \in (x_0-\delta, x_0+\delta) \cap A : f(x) \geq f(x_0)$



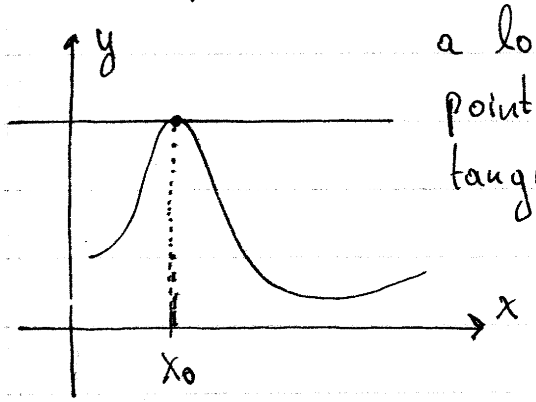


interpretation: A point $x_0 \in A$ is local min of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the minimum value of f in a small enough interval around the point x_0. Likewise, a point $x_0 \in A$ is local max of $f: A \to \mathbb{R}$ if and only if $f(x_0)$ is the maximum value of f in a small enough interval around the point x_0.

Thm: (Fermat theorem)
Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $x_0 \in A$.
We have:

$$\begin{cases} x_0 \in \text{int}(A) \\ x_0 \text{ local min or max of } f \\ f \text{ differentiable on } x_0 \end{cases} \Rightarrow f'(x_0) = 0$$

▸ interpretation: If a function is differentiable and has a local max or min at an interior point x_0 of its domain, then the tangent line (ℓ) to the graph of f at the point x_0 is horizontal.



Proof
With no loss of generality, assume that

$$\begin{cases} x_0 \in \text{int}(A) \wedge x_0 \text{ local max of } f \\ f \text{ differentiable on } x_0 \end{cases}$$

It follows that

$x_0 \in \operatorname{int}(A) \Rightarrow \exists \delta_1 \in (0,+\infty): (x_0-\delta_1, x_0+\delta_1) \subseteq A$

x_0 local max of $f \Rightarrow$

$\Rightarrow \exists \delta_2 \in (0,+\infty): \forall x \in (x_0-\delta_2, x_0+\delta_2) \cap A: f(x) \leq f(x_0)$

Choose $\delta_1, \delta_2 \in (0,+\infty)$ such that

$$\begin{cases} (x_0-\delta_1, x_0+\delta_1) \subseteq A \\ \forall x \in (x_0-\delta_2, x_0+\delta_2) \cap A: f(x) \leq f(x_0) \end{cases}$$

Define $\delta = \min\{\delta_1, \delta_2\}$ and define

$$\forall x, x_0 \in A: \lambda(x, x_0) = \frac{f(x)-f(x_0)}{x-x_0}$$

Since

$(x_0-\delta, x_0+\delta) \subseteq (x_0-\delta_1, x_0+\delta_1) \subseteq A \Rightarrow$

$\Rightarrow (x_0-\delta, x_0+\delta) \subseteq A \Rightarrow (x_0-\delta, x_0+\delta) \cap A = (x_0-\delta, x_0+\delta)$

$\Rightarrow \forall x \in (x_0-\delta, x_0+\delta): f(x) \leq f(x_0)$

$\Rightarrow \forall x \in (x_0-\delta, x_0+\delta): f(x)-f(x_0) \leq 0$

$$\Rightarrow \begin{cases} \forall x \in (x_0-\delta, x_0): \lambda(x, x_0) = \dfrac{f(x)-f(x_0)}{x-x_0} \geq 0 & (1) \\[2ex] \forall x \in (x_0, x_0+\delta): \lambda(x, x_0) = \dfrac{f(x)-f(x_0)}{x-x_0} \leq 0 & (2) \end{cases}$$

Since f differentiable at x_0

$$f'(x_0) = \lim_{x \to x_0^-} \lambda(x, x_0) \geq 0 \quad \text{, from Eq. (1)}$$

$$f'(x_0) = \lim_{x \to x_0^+} \lambda(x, x_0) \leq 0 \quad \text{, from Eq. (2)}$$

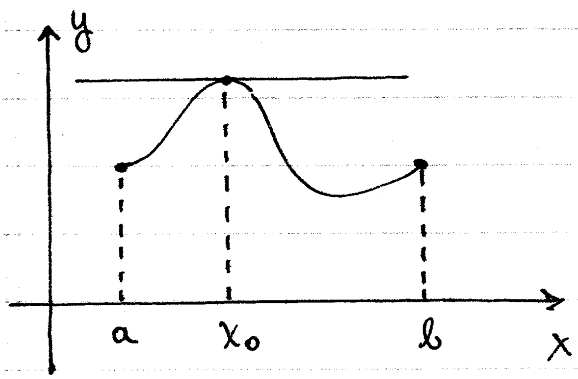
and it follows that $f'(x_0) = 0$. $\square$

(2) → Rolle theorem

Thm : Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $\alpha, b \in A$ with $[a,b] \subseteq A$. Then,

$$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$$

interpretation :



If a function f is continuous on $[a,b]$ and differentiable on (a,b) and if $f(a) = f(b)$, then there is a point $x_0 \in (a,b)$ where the tangent line to the graph of the function becomes horizontal.

Proof

Assume that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{cases}$$

Using the Extremum Value Theorem,

f continuous on $[a,b] \Rightarrow$

$\Rightarrow \exists x_1, x_2 \in [a,b] : \forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$.

Choose $x_1, x_2 \in [a,b]$ such that

$\forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

We distinguish between the following cases.

Case 1: Assume that $x_1 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \geq f(x_1)) \Rightarrow x_1$ local min of f (1)

We also know that

$\begin{cases} x_1 \text{ interior to } (a,b) & (2) \\ f \text{ differentiable on } (a,b) \end{cases}$

From Eq. (1) and Eq. (2), via the Fermat theorem:

$f'(x_1) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_1$)

Case 2: Assume that $x_2 \in (a,b)$. Then

$(\forall x \in [a,b] : f(x) \leq f(x_2)) \Rightarrow x_2$ local max of f (3)

We also know that

$\begin{cases} x_2 \text{ interior to } (a,b) & (4) \\ f \text{ differentiable on } (a,b) \end{cases}$

From Eq. (3) and Eq. (4), via the Fermat theorem:

$f'(x_2) = 0 \Rightarrow \exists x_0 \in (a,b) : f'(x_0) = 0$ (for $x_0 = x_2$).

Case 3: Assume that $x_1 = a \wedge x_2 = b$.

We define $c = f(a) = f(b)$. Then:

$\forall x \in [a,b] : f(x_1) \leq f(x) \leq f(x_2)$

$\Rightarrow \forall x \in [a,b] : f(a) \leq f(x) \leq f(b)$

$\Rightarrow \forall x \in [a,b] : c \leq f(x) \leq c$

$\Rightarrow \forall x \in [a,b] : f(x) = c$

$\Rightarrow \forall x \in [a,b] : f'(x) = c$

$\Rightarrow \exists x_0 \in [a,b] : f'(x_0) = c.$

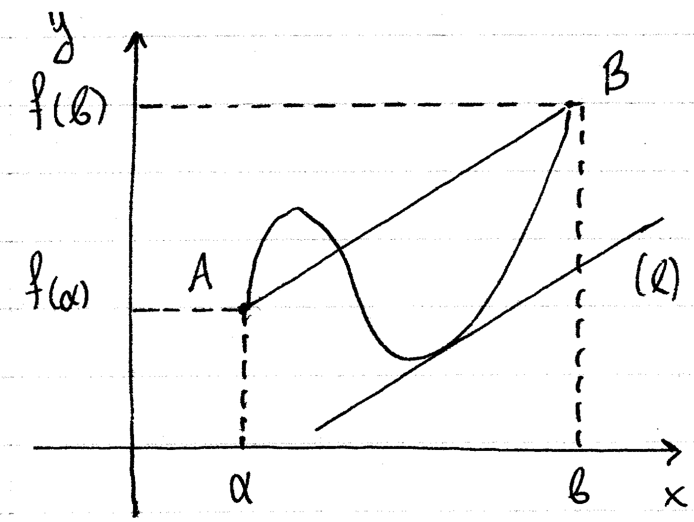
In all cases we conclude that $\exists x_0 \in [a,b] : f'(x_0) = c$.

(3) → Mean Value Theorem

Thm: (Lagrange's Mean Value Theorem)
Let $f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ be a function and let $a, b \in A$ such that $[a,b] \subseteq A$. Then

$$\left.\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases}\right. \Rightarrow \exists\, x_0 \in (a,b) : f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

Interpretation:



If the function f is continuous on $[a,b]$ and differentiable on (a,b), then given the points $A(a, f(a))$ and $B(b, f(b))$ on the graph of f, there is at least one $x_0 \in (a,b)$ such that the tangent line (l) at $x = x_0$ to the graph of f satisfies $(l) \parallel (AB)$.

Proof

Assume that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases}$$

Define

$\forall x \in [a,b]: F(x) = (a-b)f(x) + [f(b)-f(a)]x + [bf(a) - af(b)]$

and note that

f continuous on $[a,b] \Rightarrow F$ continuous on $[a,b]$ (1)

and

f differentiable on $(a,b) \Rightarrow F$ differentiable on (a,b) (2)

with $\forall x \in (a,b): F'(x) = (a-b)f'(x) - [f(a)-f(b)]$ (3)

We also have

$$\begin{aligned} F(a) &= (a-b)f(a) + [f(b)-f(a)]a + [bf(a) - af(b)] = \\ &= (a-b)f(a) + af(b) - af(a) + bf(a) - af(b) = \\ &= (a-b-a+b)f(a) + (a-a)f(b) = \\ &= 0f(a) + 0f(b) = 0 \qquad (4) \end{aligned}$$

and

$$\begin{aligned} F(b) &= (a-b)f(b) + [f(b)-f(a)]b + [bf(a) - af(b)] = \\ &= (a-b)f(b) + bf(b) - bf(a) + bf(a) - af(b) = \\ &= (-b+b)f(a) + (a-b+b-a)f(b) = \\ &= 0f(a) + 0f(b) = 0 \qquad (5) \end{aligned}$$

From Eq.(4) and Eq.(5): $F(a) = F(b) = 0$ (6).

From Eq.(1) and Eq.(2) and Eq.(6), via the Rolle theorem:

$$\left\{ \begin{array}{l} F \text{ continuous on } [a,b] \\ F \text{ differentiable on } (a,b) \\ F(a) = F(b) \end{array} \right. \Rightarrow \exists x_0 \in (a,b): F'(x_0) = 0$$

$$\begin{aligned} &\Rightarrow \exists x_0 \in (a,b): (a-b)f'(x_0) - [f(a)-f(b)] = 0 \\ &\Rightarrow \exists x_0 \in (a,b): (b-a)f'(x_0) = f(b)-f(a) \\ &\Rightarrow \exists x_0 \in (a,b): f'(x_0) = \frac{f(b)-f(a)}{b-a} \qquad \square \end{aligned}$$

Remark: During the early development of Calculus, many arguments were based on the concept of the linear approximation

$f(x+\Delta x) \approx f(x) + \Delta x f'(x)$

where Δx is very small relative to x (i.e. $\Delta x \ll x$). The linear approximation assumes that the graph of the function f in the interval $[x, x+\Delta x]$ is approximately a straight line as long as Δx is small enough, and can be therefore represented by a linear function with respect to Δx. The linear approximation can be used to argue, e.g. that if a function has $f'(x) > 0$, then it is increasing from x to $x+\Delta x$. The problem is that such arguments are not rigorous because they are based on a statement that is true only approximately.

According to the Mean Value Theorem, if f satisfies

$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases}$ with $a = x$ and $b = x+\Delta x$

then we conclude that

$\exists x_0 \in (x, x+\Delta x) : f(x+\Delta x) = f(x) + \Delta x\, f'(x_0)$

It follows that the linear approximation statement becomes exact if we replace $f'(x)$ with $f'(x_0)$ for some choice of $x_0 \in (x, x+\Delta x)$. This in turn makes it possible to formulate rigorous arguments based on the overall linear approximation concept.

Immediate corollaries of the Mean Value Theorem

The following theorems are immediate consequences of the Mean Value Theorem. We use the assumption that a set $I \subseteq \mathbb{R}$ is an interval, as opposed to a union of disjoint intervals (e.g. $I=[a,b]$ or $I=(a,b]$ or $I=[a,b)$ etc....). A practical definition that encompasses all possibilities is the following:

Def: Let $I \subseteq \mathbb{R}$. We say that
$$I \text{ interval} \iff \forall x_1, x_2 \in I : (x_1 < x_2 \Rightarrow [x_1, x_2] \subseteq I)$$

We also define the concept of a constant function:

Def: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. We say that
$$f \text{ constant on } I \iff \forall x_1, x_2 \in I : f(x_1) = f(x_2)$$

We will now show that

Thm: Let $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:
$$\begin{cases} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{cases} \Rightarrow f \text{ constant on } I.$$

Proof

Assume that

$$\begin{cases} I \text{ interval} \\ f \text{ differentiable on } I \\ \forall x \in I : f'(x) = 0 \end{cases}$$

▸ We will show that $\forall x_1, x_2 \in I : f(x_1) = f(x_2)$.

Let $x_1, x_2 \in I$ be given and assume with no loss of generality that $x_1 < x_2$. Then

$$\begin{cases} I \text{ interval} \\ x_1, x_2 \in I \land x_1 < x_2 \end{cases} \Rightarrow [x_1, x_2] \subseteq I$$

and therefore:

$$f \text{ differentiable on } I \Rightarrow f \text{ differentiable on } [x_1, x_2] \Rightarrow$$

$$\Rightarrow \begin{cases} f \text{ continuous on } [x_1, x_2] \\ f \text{ differentiable on } (x_1, x_2) \end{cases}$$

$$\Rightarrow \exists x_0 \in (x_1, x_2) : f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Choose $x_0 \in (x_1, x_2)$ such that $f'(x_0) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

It follows that

$$\begin{aligned} f(x_2) - f(x_1) &= f'(x_0)(x_2 - x_1) \\ &= 0(x_2 - x_1) \qquad [\text{via } \forall x \in I : f'(x) = 0] \\ &= 0 \Rightarrow f(x_1) = f(x_2) \end{aligned}$$

and therefore:

$$(\forall x_1, x_2 \in I : f(x_1) = f(x_2)) \Rightarrow$$

$$\Rightarrow f \text{ constant on } I.$$

<u>Thm</u>: Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$ and let $I \subseteq A$. Then:

$$\left.\begin{cases} I \text{ interval} \\ f, g \text{ differentiable on } I \\ \forall x \in I: f'(x) = g'(x) \end{cases}\right. \Rightarrow \exists c \in \mathbb{R}: \forall x \in I: f(x) = g(x) + c$$

<u>Proof</u>

Assume that

$$\begin{cases} I \text{ interval} & (1) \\ f, g \text{ differentiable on } I \\ \forall x \in I: f'(x) = g'(x) \end{cases}$$

Define $\forall x \in I: h(x) = f(x) - g(x)$. Then

f, g differentiable on $I \Rightarrow h$ differentiable on I (2)

with

$$\begin{aligned} \forall x \in I: h'(x) &= [f(x) - g(x)]' = f'(x) - g'(x) \\ &= f'(x) - f'(x) = 0 \qquad (3) \end{aligned}$$

From Eq.(1), Eq.(2), Eq.(3):

h constant on $I \Rightarrow \exists c \in \mathbb{R}: \forall x \in I: h(x) = c$

$\Rightarrow \exists c \in \mathbb{R}: \forall x \in I: f(x) - g(x) = c$

$\Rightarrow \exists c \in \mathbb{R}: \forall x \in I: f(x) = g(x) + c$

Method - Examples

(1) To show that an equation has a unique solution (i.e. $f(x)=0$) in (a,b).

- •1 Use the Bolzano theorem to establish EXISTENCE of a solution $x_0 \in (a,b)$.
- •2 Show that $f'(x) \neq 0$, $\forall x \in (a,b)$
- •3 Assume there are two solutions $x_0, x_1 \in (a,b)$ with $x_0 \neq x_1$ and use the Rolle theorem to reach a contradiction.

EXAMPLES

a) Show that $x^3-3x+1=0$ has a unique solution at $(-1,1)$

Solution

• Existence: Let $f(x) = x^3-3x+1$. Then

$$\left.\begin{array}{l} f(-1) = (-1)^3-3(-1)+1 = -1+3+1=3 \\ f(1) = 1^3-3\cdot 1+1 = -1 \end{array}\right\} \Rightarrow$$

$\Rightarrow f(-1)f(1) = 3\cdot(-1) < 0$ (1)

f continuous at $[-1,1]$ (2)

From (1) and (2):

$\exists\, x_0 \in (-1,1) : f(x_0)=0$

- <u>Uniqueness</u> : Assume that the equation is satisfied by $x_0, x_1 \in (-1,1)$ with $x_0 < x_1$

We note that

$f'(x) = (x^3 - 3x + 1)' = 3x^2 - 3 = 3(x^2 - 1) < 0,\ \forall x \in (-1,1) \Rightarrow$

$\Rightarrow f'(x) \neq 0,\ \forall x \in (-1,1).$ (3)

$$\text{Since } \left.\begin{array}{l} f(x_0) = f(x_1) = 0 \\ f \text{ continuous at } [x_0, x_1] \\ f \text{ differentiable at } (x_0, x_1) \end{array}\right\} \Rightarrow$$

$\Rightarrow \exists\, x_2 \in (x_0, x_1) : \underline{f'(x_2) = 0.}$

From (3): $f'(x_2) \neq 0$, thus we have a contradiction.

It follows that the solution x_0 is unique.

B) Show that $x^5 + 2x^3 + 7x + 12 = 0$ has a unique solution in $\mathbb{R}$.

<u>Solution</u>

- <u>Existence</u> : Let $f(x) = x^5 + 2x^3 + 7x + 12$.

We note that:

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} (x^5 + 2x^3 + 7x + 12) = \lim\limits_{x \to +\infty} x^5 = +\infty \Rightarrow$

$\Rightarrow \exists\, b \in (0, +\infty) : f(b) > 0$ (1)

and

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} (x^5 + 2x^3 + 7x + 12) = \lim\limits_{x \to -\infty} x^5 = -\infty \Rightarrow$

$\Rightarrow \exists\, a \in (-\infty, 0) : f(a) < 0$ (2)

From (1) and (2):

$$\left.\begin{array}{l} f(a)f(b)<0 \\ f \text{ continuous at } [a,b] \end{array}\right\} \Rightarrow \exists x_0 \in (a,b): f(x_0)=0 \Rightarrow$$

$\Rightarrow$ x_0 solves the equation.

- Uniqueness : Assume that $x_0, x_1 \in \mathbb{R}$ solve the equation with $x_0 < x_1$. We note that

$$f'(x) = (x^5+2x^3+7x+12)' = 5x^4+6x^2+7 >$$
$$> 5x^4+6x^2 \geq 0 \,,\ \forall x \in \mathbb{R} \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R}: f'(x) > 0 \quad (3)$$

Furthermore:

$$\left.\begin{array}{l} f(x_0)=f(x_1)=0 \\ f \text{ continuous at } [x_0,x_1] \\ f \text{ differentiable at } (x_0,x_1) \end{array}\right\} \Rightarrow \exists x_2 \in (x_0,x_1): \underline{f'(x_2)=0}.$$

From (3): $\underline{f'(x_2)>0}$, so we have a contradiction.
It follows that the equation cannot have more than one solution in $\mathbb{R}$.

$\hookrightarrow$ In the above solution we have used the statements:

$$\lim_{x\to+\infty} f(x) = +\infty \Rightarrow \exists a \in (0,+\infty): f(a) > 0$$

$$\lim_{x\to-\infty} f(x) = -\infty \Rightarrow \exists a \in (-\infty,0): f(a) < 0$$

which are immediate consequences of the limit definition. More generally:

$$\lim_{x\to\sigma} f(x) = L \Rightarrow \exists a \in N(\sigma,\delta): f(a) \in I(L,\varepsilon)$$

② Inequalities: In general, using the Mean Value Theorem, an inequality satisfied by $f'(x)$ implies an inequality satisfied by $f(x)$.

EXAMPLES

a) Let f be a function differentiable in $\mathbb{R}$. Show that if $\forall x \in \mathbb{R}: 3 \leq f'(x) \leq 5$, then $18 \leq f(8) - f(2) \leq 30$.

Solution

f differentiable in $\mathbb{R} \Rightarrow$ MVT applies on $[2,8] \Rightarrow$

$\Rightarrow \exists x_0 \in (2,8): f(8) - f(2) = f'(x_0)(8-2) = 6f'(x_0) \quad (1)$

It follows that

$3 \leq f'(x) \leq 5, \forall x \in \mathbb{R} \Rightarrow 3 \leq f'(x_0) \leq 5 \Rightarrow$

$\Rightarrow 18 \leq 6f'(x_0) \leq 30 \Rightarrow 18 \leq f(8) - f(2) \leq 30.$

↳ Inequalities involving $\boxed{\text{two}}$ variables can be proved via the Mean Value Theorem if it is possible, with or without, some manipulation, to produce an expression of the form $f(b) - f(a)$. Then we can use:

$$f(b) - f(a) = f'(x_0)(b-a)$$

for some $x_0 \in (a,b)$.

b) Show that:

$$0 < a < b < \pi/2 \Rightarrow \frac{a}{b} < \frac{\sin a}{\sin b}$$

Solution

Since $0 < a < b < \pi/2 \Rightarrow b\sin b > 0$ and $ab > 0$.
It follows that

$$\frac{a}{b} < \frac{\sin a}{\sin b} \overset{*}{\Leftrightarrow} \frac{a}{b}(b\sin b) < \frac{\sin a}{\sin b}(b\sin b) \Leftrightarrow$$

$$\Leftrightarrow a\sin b < b\sin a \Leftrightarrow a\sin b - b\sin a < 0 \overset{*}{\Leftrightarrow}$$

$$\Leftrightarrow \frac{a\sin b - b\sin a}{ab} < 0 \Leftrightarrow \frac{\sin b}{b} - \frac{\sin a}{a} < 0 \quad (1).$$

Define $f(x) = \frac{\sin x}{x}$. It follows that:

$$f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{(\sin x)' x - \sin x\,(x)'}{x^2} =$$

$$= \frac{x\cos x - \sin x}{x^2}$$

Since:

$\left.\begin{array}{l} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } [a,b] \end{array}\right\} \Rightarrow$ The Mean-Value-Theorem applies on $[a,b] \Rightarrow$

$$\Rightarrow \exists x_0 \in (a,b) : f(b) - f(a) = f'(x_0)(b-a) \Rightarrow$$

$$\Rightarrow \frac{\sin b}{b} - \frac{\sin a}{a} = f(b) - f(a) = f'(x_0)(b-a) =$$

$$= \frac{x_0\cos x_0 - \sin x_0}{x_0^2} \cdot (b-a) =$$

$$= \frac{(x_0 \cos x_0 - \sin x_0)(b-a)}{x_0^2} \qquad (2)$$

Note that

$a < b \Rightarrow b - a > 0 \qquad (3)$

and $\quad x_0^2 > 0 \qquad (4)$

and

$$\left.\begin{array}{l} |\tan x_0| > |x_0| \\ x_0 \in (0, \pi/2) \end{array}\right\} \Rightarrow \tan x_0 > x_0 \Rightarrow \frac{\sin x_0}{\cos x_0} > x_0 \Rightarrow$$

$$\Rightarrow \sin x_0 > x_0 \cos x_0 \Rightarrow x_0 \cos x_0 - \sin x_0 < 0 \qquad (5)$$

From (2), (3), (4), (5):

$$\frac{\sin b}{b} - \frac{\sin a}{a} < 0 \underset{(1)}{\Rightarrow} \frac{a}{b} < \frac{\sin a}{\sin b} \qquad \square$$

↳ Note the 3-step process:

- •1 Reduce the inequality to be shown to an equivalent simpler inequality that exposes the $f(b) - f(a)$ expression
- •2 Define $f(x)$ and calculate $f'(x)$.
- •3 Apply the MVT and establish a relation between f and f'.
- •4 Determine if $f'(x_0)$ is positive or negative and backtrack your way back to the original inequality.

↳ Also recall the inequalities:

$|\tan x| > |x|$, $\forall x \in (-\pi/2, 0) \cup (0, \pi/2)$

$|\sin x| < |x|$, $\forall x \in \mathbb{R} - \{0\}$.

EXERCISES

Problems on the Rolle theorem

(1) Use the Bolzano and Rolle theorems to show that the following equations have a unique solution in the corresponding sets

a) $\dfrac{\cos x}{2} + \dfrac{1}{(1+x)^2} = 0$ on $A = (2\pi, 3\pi)$

b) $x^5 + x^3 + x = a^2(b-x) + b^2(c-x) + c^2(a-x)$
on $\mathbb{R}$ with $a, b, c \in \mathbb{R}$.

c) $\cos x = x$ on $A = (0, \pi)$

(2) Show that the equation $x^2 = x\sin x + \cos x$ has only 2 distinct solutions on $A = (-\pi, \pi)$

(3) Let $f : \mathbb{R} \to \mathbb{R}$ such that
$$\begin{cases} f \text{ twice differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : f''(x) \neq 0 \end{cases}$$
Show that the equation $f(x) = 0$ cannot have more than two distinct solutions on $\mathbb{R}$.

(4) Show that the equation $x^n + ax + b = 0$ with $n \in \mathbb{N}^*$ has
a) at most 2 real solutions when n even and $n \geqslant 2$.
b) no more than 3 real solutions when n odd with $n \geqslant 3$.

(5) Show that the equation $x^n + nx + 1 = 0$ with $n \in \mathbb{N}^*$ has

a) only one real solution when n odd

b) at most 2 real solutions when n even

(6) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a, b] \subseteq A$ such that

$$\begin{cases} f, g \text{ differentiable on } (a, b) \\ f, g \text{ continuous on } [a, b] \\ \dfrac{f(a)}{g(a)} = \dfrac{f(b)}{g(b)} \\ \forall x \in [a, b] : g(x) \neq 0 \\ \forall x \in (a, b) : g'(x) \neq 0 \end{cases}$$

Show that:

$$\exists x_0 \in (a, b) : \frac{f'(x_0)}{g'(x_0)} = \frac{f(x_0)}{g(x_0)}$$

(7) Let $f: A \to \mathbb{R}$ and $a \in (0, +\infty)$ with $[-a, a] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [-a, a] \\ f \text{ twice-differentiable on } (-a, a) \\ f(-a) = a \wedge f(a) = -a \wedge f(0) = 0 \end{cases}$$

Show that

$$\exists x_0 \in (-a, a) : f''(x_0) = 0$$

(8) Let $f: A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \\ f(a) = f(b) \end{cases}$$

Let $c \in \mathbb{R} - [a,b]$ and define $g: [a,b] \to \mathbb{R}$ such that

$$\forall x \in [a,b]: g(x) = \frac{f(x)}{x-c}$$

Show that: $\exists x_0 \in (a,b): g'(x_0) = 0$

(9) Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ and $0 \notin [a,b]$ such that

$$\begin{cases} f, g \text{ differentiable on } [a,b] \\ f, g \text{ continuous on } [a,b] \\ f(a) = g(b) = 0 \\ \forall x \in [a,b]: f(x)g(x) \neq 0 \end{cases}$$

Show that:

$$\exists x_0 \in (a,b): \frac{f'(x_0)}{f(x_0)} + \frac{g'(x_0)}{g(x_0)} = \frac{1}{x_0}$$

$\hookrightarrow$ Hint: Apply the Rolle theorem on the function $h(x) = f(x)g(x)/x$

(10) Let $f: A \to \mathbb{R}$ and $a, b \in (0, +\infty)$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ twice-differentiable on } [a,b] \\ f(a) = f(b) = 0 \\ \forall x \in (a,b): f'(x) \neq 0 \end{cases}$$

Show that the equation $xf'(x) - f(x) = 0$ has a unique solution on the interval (a,b).

↳ Use the Rolle theorem on the function $g(x) = f(x)/x$.

(11) Let $f : A \to \mathbb{R}$ with $[0,1] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [0,1] \\ f \text{ differentiable on } (0,1) \\ f(1) = f(0) + 1/2 \end{cases}$$

Show that the equation $f'(x) = x$ has at least one solution on the interval $(0,1)$

↳ Use Rolle theorem on the appropriate function $g(x)$ to establish the existence of at least one solution.

(12) Let $f : A \to \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ twice-differentiable on } [a,b] \\ \forall x \in [a,b] : f(x) f'(x) \neq 0 \\ \dfrac{f(a)}{f'(a)} = \dfrac{f'(b)}{f'(b)} \end{cases}$$

Show that

$$\exists c_1, c_2 \in (a,b) : f'(c_1) f''(c_1) + f(c_2) f''(c_2) > 0$$

↳ Use the Rolle theorem on the functions $g(x) = \dfrac{f(x)}{f'(x)}$ and $h(x) = \dfrac{f'(x)}{f(x)}$

(13) Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ and $h : A \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f, g, h \text{ continuous on } [a,b] \\ f, g, h \text{ differentiable on } (a,b) \end{cases}$$

Show that the equation

$$\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

has at least one solution on (a,b).

(14) Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f, g \text{ continuous on } [a,b] \\ f, g \text{ differentiable on } (a,b) \end{cases}$$

Show that:

$$\exists x_0 \in (a,b) : \frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Problems on the Mean Value Theorem

(15) Use the mean-value theorem to prove the following inequalities

a) $a, b \in (-\pi/2, \pi/2) \Rightarrow |\sin a - \sin b| \leq |a - b|$

b) $\forall n \in \mathbb{N}^* : (0 < a < b \Rightarrow n(b-a)a^{n-1} \leq b^n - a^n \leq n(b-a)b^{n-1})$

c) $0 < a \leq b < \pi/2 \Rightarrow \dfrac{a-b}{\cos^2 b} \leq \tan a - \tan b \leq \dfrac{a-b}{\cos^2 a}$

d) $0 < a < a+b < \pi/2 \Rightarrow \sin(a+b) < \sin a + b\cos a$

e) $0 < a < b < \pi/2 \Rightarrow \dfrac{\tan a}{\tan b} < \dfrac{b}{a}$

$\hookrightarrow$ Use the mean-value theorem on $g(x) = x\tan x$

(16) Let $f : A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a,b] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [a,b] \\ f \text{ differentiable on } (a,b) \end{cases}$$

Show that: $\exists c_1, c_2 \in (a,b) : f'(c_1) + f'(c_2) = 0$

(17) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f, g \text{ differentiable on } \mathbb{R} \\ f(0) = 0 \land g(0) = 1 \\ \forall x \in \mathbb{R} : (f'(x) - g(x) = 0 \land f(x) + g'(x) = 0) \end{cases}$$

Show that: $\forall x \in \mathbb{R} : f^2(x) + g^2(x) = 1$.

(18) Let $f: \mathbb{R} \to \mathbb{R}$ such that
$\forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq |x-y|^2$
Show that f is constant on $\mathbb{R}$.

(19) Let $f: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f \text{ twice-differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : f''(x) + f(x) = 0 \\ f(0) = f'(0) = 0 \end{cases}$$

Show that
$\exists c \in \mathbb{R} : \forall x \in \mathbb{R} : [f(x)]^2 + [f'(x)]^2 = c$

(20) Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} f, g \text{ differentiable on } \mathbb{R} \\ \forall x \in \mathbb{R} : (f'(x) = g(x) \land g'(x) = f(x)) \\ f(0) = 1 \land g(0) = 1 \end{cases}$$

Show that:
$\forall x \in \mathbb{R} : [f(x)]^2 = [g(x)]^2 + 1$

(21) Let $f: A \to \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $[a, b] \subseteq A$ such that

$$\begin{cases} f \text{ continuous on } [a, b] \\ f \text{ differentiable on } (a, b) \end{cases}$$

Show that:

$$\exists x_1, x_2, x_3 \in (a, b) : \begin{cases} x_1 \neq x_2 \neq x_3 \neq x_1 \\ (b-a)(f'(x_1) + f'(x_2) + f'(x_3)) = f(b) - f(a) \end{cases}$$