Lecture Notes on Ordinary Differential Equations

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ODE 1: Introduction to ODEs

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

Definitions

- An ordinary differential equation (ODE) is an equation that contains one or more derivatives of the unknown function. A function that satisfies the equation is called a solution of the ODE.
- The most general form of an ODE is:

$$\boxed{F(x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)) = 0} \qquad (1)$$

with $F: \mathbb{R} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$.
If we define $Y(x) = (y(x), y'(x), y''(x), \dots, y^{(n)}(x))$, then the equation above can be rewritten as:

$$\boxed{F(x, Y(x)) = 0} \qquad (2)$$

- The natural number n is the order of the ODE.

Linear vs. nonlinear ODEs

Let $\mathbb{V}$ be the set of all continuous functions $Y: \mathbb{R} \to \mathbb{R}^n$.
We say that the ODE $F(x, Y(x)) = 0$ is linear if and only

if F satisfies

$$\forall x, \lambda, \mu \in \mathbb{R} : \forall Y, Z \in V : F(x, \lambda Y + \mu Z) = \lambda F(x, Y) + \mu F(x, Z)$$

otherwise we say that the ODE is <u>nonlinear.</u>

- It can be shown that the most general form of a linear ODE is:

$$p_n(x) y^{(n)}(x) + \dots + p_2(x) y''(x) + p_1(x) y'(x) + p_0(x) y(x) = q(x)$$

- <u>Types of ODE problems</u>

We distinguish between the following types of ODE problems:

1 → <u>Initial Value Problem</u>

These are problems of the form:

$$\boxed{\begin{cases} F(x, y(x), y'(x), \dots, y^{(n-1)}(x), y^{(n)}(x)) = 0 \\ y(x_0) = a_0 \wedge y'(x_0) = a_1 \wedge \dots \wedge y^{(n-1)}(x_0) = a_{n-1} \end{cases}}$$

where $y, y', y'', \dots, y^{(n-1)}$ are all fixed at the same point $x_0 \in \mathbb{R}$. These additional equations are called <u>initial conditions.</u>

(2) → Boundary Value Problem

These are problems of the form

$$\boxed{\begin{array}{l} F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \\ y^{(k_1)}(x_1) = a_1 \wedge y^{(k_2)}(x_2) = a_2 \wedge \dots \wedge y^{(k_n)}(x_n) = a_n \end{array}}$$

where $y^{(k_1)}, y^{(k_2)}, \dots, y^{(k_n)}$ are specified on more than just a unique point. These additional equations are called <u>Boundary conditions.</u>

● Techniques for solving ODEs

Solution techniques are classified under the following categories.

a) <u>Exact analytic methods:</u> We obtain an exact solution in closed form.

b) <u>Approximate methods:</u> We obtain an approximate solution in closed form.

  i) <u>Local methods:</u> We obtain an approximate solution which is good in a neighborhood of some special point.

  ii) <u>Global methods:</u> Obtain an approximate solution which is good on the entire domain of the ODE.

c) <u>Numerical methods:</u> We obtain an approximate discretized solution with the use of a computer.

d) <u>Existence/Uniqueness:</u> We prove rigorously that a given ODE problem has a unique solution, without actually being able to find the solution exactly or approximately.

Systems of ODEs

- A system of m ODEs is any problem of the form

$$\boxed{\begin{cases} F_1(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \\ F_2(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \\ \vdots \\ F_m(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0 \end{cases}}$$

where we require the logical conjunction of all equations.

- Every n^{th}-order ODE of the form

$$y^{(n+1)} = F(x, y(x), y'(x), \dots, y^{(n)}(x))$$

can be rewritten as a system of 1st-order equations.

$$\begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = F(x, y_0, y_1, y_2, \dots, y_n) \end{cases}$$

ODE 2: First-order ODEs

FIRST-ORDER ODEs

• A 1st-order ordinary differential equation (ODE) is an equation of the form $y' = f(x,y)$ satisfied by a function $y(x)$ of x. A corresponding 1st-order initial value problem is a problem of the form

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

with $x_0, y_0 \in \mathbb{R}$ given.

• An implicit solution to the initial value problem above is a solution of the form $F(x,y)=0$ where we have shown that

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \iff F(x,y) = 0$$

• An explicit solution to the initial value problem above is a solution of the form $y = g(x)$ such that

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \iff y = g(x).$$

• There is no general solution method that can give an implicit or explicit solution to a 1st-order ODE. However, solution methods exist for some special cases, including the following:

(1) → Separable ODEs

These are problems of the form $\boxed{\begin{cases} y' = g(x)h(y) \\ y(x_0) = y_0 \end{cases}}$ (1)

Note that we say that

y_0 is a fixed point of Eq. (1) $\Leftrightarrow$ $h(y_0) = 0$

If we initialize the system at a fixed point, then $y' = 0$, and we expect $y(x)$ to remain at the fixed point for all $x \in \mathbb{R}$. Furthermore, if we initialize at y_0 with $h(y_0) \neq 0$ then the solution cannot cross over any fixed point. We can therefore expect that $h(y(x)) \neq 0$ for all $x \in \mathbb{R}$ for which $y(x)$ can be obtained.

<u>Methodology</u>: Based on the above remarks we begin by assuming that $h(y) \neq 0$, and therefore:

$$y' = g(x)h(y) \Leftrightarrow \frac{y'}{h(y)} = g(x) \Leftrightarrow \int \frac{dy}{h(y)} = \int g(x)dx \Leftrightarrow$$

$$\Leftrightarrow H(y) = G(x) + c_1$$

To determine c we use the initial condition $y(x_0) = y_0$:

$H(y_0) = G(x_0) + c_1 \Leftrightarrow c_1 = H(y_0) - G(x_0)$.

Note that in the above argument we assume that the system has not been initialized at a fixed point. If the goal is to find a general solution, then it is necessary to explore whether the general solution continuous to hold when y_0 is a fixed point.

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y'(x) = (1+y^2(x))\cos x \\ y(0) = 1 \end{cases}$$

Solution

Since $1+y^2 > 0$, then the system has no fixed points.

We note that

$$y' = (1+y^2)\cos x \Leftrightarrow \frac{y'}{1+y^2} = \cos x \Leftrightarrow \int \frac{dy}{1+y^2} = \int \cos x \, dx \quad (1)$$

with $\int \cos x \, dx = \sin x + c_1$ and $\int \frac{dy}{1+y^2} = \text{Arctan}(y) + c_2$

thus

$$(1) \Leftrightarrow \text{Arctan}(y) = \sin x + c \Leftrightarrow y = \tan(\sin x + c)$$

From the initial condition:

$$y(0) = 1 \Leftrightarrow \text{Arctan}(1) = \sin 0 + c \Leftrightarrow$$
$$\Leftrightarrow c = \text{Arctan}(1) = \pi/4$$

and therefore: $y(x) = \tan(\sin x + \pi/4)$.

We note that with increasing x, this solution becomes singular

when:

$$\sin x + \pi/4 = \pi/2 \Leftrightarrow \sin x = \pi/4 - \pi/2 \Leftrightarrow \sin x = \pi/4 \in [-1,1]$$
$$\Leftrightarrow x = \text{Arcsin}(\pi/4).$$

▶ We say that the solution has a finite-time singularity at $x = \text{Arcsin}(\pi/4)$.

6) Solve the initial value problem

$$\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$$

Solution

We note that $y=0$ is a fixed point. We assume that initially $y_0 \neq 0$. Then $y \neq 0$, and it follows that

$$y' = y^2 \Leftrightarrow \frac{y'}{y^2} = 1 \Leftrightarrow \int \frac{dy}{y^2} = \int dx \Leftrightarrow \frac{y^{-1}}{-1} = x + c_1$$

$$\Leftrightarrow y^{-1} = -x - c \Leftrightarrow y = \frac{1}{-x-c} = \frac{-1}{x+c_1}$$

Since $y(0) = y_0 \Leftrightarrow y_0^{-1} = -0 - c \Leftrightarrow c = -y_0^{-1} = \frac{-1}{y_0}$ it follows that

$$y = \frac{-1}{x+c_1} = \frac{-1}{x - y_0^{-1}} = \frac{-y_0}{y_0(x - y_0^{-1})} = \frac{-y_0}{y_0 x - 1}, \text{ with } y_0 \neq 0$$

For the fixed point initialization $y_0 = 0$, the above equation correctly gives $y = \frac{-0}{0x - 1} = 0$, therefore it is valid for all $y_0 \in \mathbb{R}$.

The solution has a finite time singularity when

$y_0 x - 1 = 0 \Leftrightarrow y_0 x = 1 \Leftrightarrow x = 1/y_0$.

c) Solve the initial value problem

$$\begin{cases} y' = 2x(y-1) \\ y(1) = y_0 \end{cases}$$

Solution

We note that $y-1=0 \Leftrightarrow y=1$, so $y=1$ is the fixed point. We assume initialization $y_0 \neq 1$, thus $y \neq 1$. Then,

$$y' = 2x(y-1) \Leftrightarrow \frac{y'}{y-1} = 2x \Leftrightarrow \int \frac{dy}{y-1} = \int 2x\,dx$$

$$\Leftrightarrow \ln|y-1| = x^2 + c \qquad (1)$$

From the initial condition

$$y(1) = y_0 \Leftrightarrow \ln|y_0-1| = 1^2 + c \Leftrightarrow c = \ln|y_0-1| - 1$$

and therefore

$$\ln|y-1| = x^2 + \ln|y_0-1| - 1 \Leftrightarrow$$

$$\Leftrightarrow |y-1| = \exp(x^2 + \ln|y_0-1| - 1) = \exp(x^2-1)\exp(\ln|y_0-1|)$$

$$= |y_0-1|\exp(x^2-1) \Leftrightarrow$$

$$\Leftrightarrow y-1 = \pm|y_0-1|\exp(x^2-1) \qquad (2)$$

Since $y=1$ is a fixed point, for $y_0-1>0$ we will have $y-1>0$ and for $y_0-1<0$ we will have $y-1<0$. It follows that

$$(2) \Leftrightarrow y-1 = (y_0-1)\exp(x^2-1) \Leftrightarrow$$

$$\Leftrightarrow y = 1 + (y_0-1)\exp(x^2-1) \quad \text{for } y_0 \neq 1.$$

For $y_0=1$, the above solution gives $y=1$, so the general solution also works for $y_0=1$.

EXERCISES

(1) Solve the following initial value problems

a) $\begin{cases} y' = x^3/y \\ y(1) = y_0 \end{cases}$
b) $\begin{cases} (1+x^2)\, y' = y \\ y(0) = y_0 \end{cases}$

c) $\begin{cases} y' + y^2 \cos x = 0 \\ y(0) = y_0 \end{cases}$
d) $\begin{cases} (y+1)\, y' = x^2 - 4 \\ y(0) = y_0 \end{cases}$

e) $\begin{cases} y' = \sqrt{1-y^2} \\ y(0) = y_0 \end{cases}$
f) $\begin{cases} e^{-x} y y' + x^2 = 0 \\ y(1) = y_0 \end{cases}$

g) $\begin{cases} y' = x y^3 (1+x^2)^{-1/2} \\ y(0) = y_0 \end{cases}$
h) $\begin{cases} y' = x^2 y \ln|x| \\ y(0) = y_0 \end{cases}$

i) $\begin{cases} y' = y^2 \operatorname{Arctan}(x) \\ y(0) = y_0 \end{cases}$
j) $\begin{cases} \cos(2x)\, y' + \sin y = 0 \\ y(\pi/2) = \pi/3 \end{cases}$

k) $\begin{cases} y' = \sqrt{y^2 + 3y + 2} \\ y(0) = y_0 \end{cases}$
l) $\begin{cases} e^{-x} y' = y^{-1} \cos(2x) \\ y(1) = y_0 \end{cases}$

m) $\begin{cases} dy/dt = y^2 - 4 \\ y(0) = y_0 \end{cases}$

↳ For the solution of the above ODEs it may be necessary to review techniques of integration from Calculus 2.

(2) Logistic Population Model

The logistic population model is intended to model population growth under finite resources. If $y(t)$ is the population at time t, λ is the population growth rate, and N is the carrying capacity, then according to the logistic model, $y(t)$ is governed by

$$dy/dt = \lambda y(N-y)$$

Using initial condition $y(0) = y_0$, show that

$$y(t) = \frac{Ny_0}{y_0 + (N-y_0)\exp(-\lambda N t)}$$

2 → Homogeneous ODEs

<u>Def</u>: A homogeneous ODE is an equation of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

<u>Solution method</u>: Let $y(x) = xu(x)$. It follows that:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \Leftrightarrow x\frac{du}{dx} + u = f(u) \Leftrightarrow x\frac{du}{dx} = f(u) - u \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{f(u)-u}\frac{du}{dx} = \frac{1}{x} \Leftrightarrow \int \frac{d\tilde{u}}{f(\tilde{u}) - \tilde{u}} = \int \frac{dx}{x} \Leftrightarrow \text{etc...}$$

EXAMPLE

Solve $\dfrac{dy}{dx} = \dfrac{2xy + y^2}{x^2}$ with $y_0 = -1/2$ for $x_0 = 1$.

<u>Solution</u>

We note that

$$\frac{dy}{dx} = \frac{2xy+y^2}{x^2} = \frac{2xy}{x^2} + \frac{y^2}{x^2} = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \quad (1)$$

Let $y = xu \Rightarrow u = y/x$. It follows that

$$(1) \Leftrightarrow x\frac{du}{dx} + u = 2u + u^2 \Leftrightarrow x\frac{du}{dx} = u^2 + 2u - u \Leftrightarrow$$

$$\Leftrightarrow x\frac{du}{dx} = u(u+1) \Leftrightarrow \frac{1}{u(u+1)}\frac{du}{dx} = \frac{1}{x} \Leftrightarrow$$

$$\Leftrightarrow \int \frac{du}{u(u+1)} = \int \frac{dx}{x} \quad (2)$$

Since
$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$
with
$$A = \left.\frac{1}{u+1}\right|_{u=0} = \frac{1}{0+1} = 1 \quad , \text{ and}$$
$$B = \left.\frac{1}{u}\right|_{u=-1} = \frac{1}{-1} = -1$$
it follows that
$$\int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = \ln|u| - \ln|u+1| + C_1$$
$$= \ln\left|\frac{u}{u+1}\right| + C_1$$

and
$$\int \frac{dx}{x} = \ln|x| + c_2$$

and therefore
$$(2) \Leftrightarrow \ln\left|\frac{u}{u+1}\right| = \ln|x| + C \qquad (3)$$

Apply the initial condition:
$$y(1) = -1/2 \Leftrightarrow u(1) = y(1)/1 = -1/2 \Leftrightarrow$$
$$\Leftrightarrow \ln\left|\frac{-1/2}{-1/2+1}\right| = \ln|1| + C \Leftrightarrow$$
$$\Leftrightarrow c = \ln\left|\frac{-1/2}{-1/2+1}\right| = \ln\left|\frac{-1}{-1+2}\right| = \ln|-1| = 0$$

and therefore:

$$(3) \Leftrightarrow \ln\left|\frac{u}{u+1}\right| = \ln|x| \Leftrightarrow \left|\frac{u}{u+1}\right| = |x| \Leftrightarrow$$

$$\Leftrightarrow \frac{u}{u+1} = x \;\vee\; \frac{u}{u+1} = -x \qquad (4)$$

From the initial condition $u(1) = -1/2$ we note that $\frac{u}{u+1} < 0$ and $x > 0$, and therefore we reject the first equation on (4) and have:

$$(4) \Leftrightarrow \frac{u}{u+1} = -x \Leftrightarrow u = -x(u+1) \Leftrightarrow u = -xu - x \Leftrightarrow$$

$$\Leftrightarrow (1+x)u = -x \Leftrightarrow u = \frac{-x}{1+x} \Leftrightarrow \frac{y}{x} = \frac{-x}{x+1}$$

$$\Leftrightarrow y = \frac{-x^2}{x+1}$$

EXERCISES

(3) Solve the following homogeneous ODEs using initial condition $y(1) = y_0$.

a) $3xy' + y = x$ b) $(x-2y)y' = x+y$

c) $(x+3y)y' = 3x+y$ d) $x^2 y' = y(x+y)$

e) $xy^2 y' = y^3 - x^3$ f) $(x^2+y^2)y' = xy$

g) $xy' + y\sqrt{x^2-y^2} = 0$ h) $y'\sqrt{x} = -\sqrt{x+y}$

(4) Consider an ordinary differential equation of the form $M(x,y) + N(x,y)\,y' = 0$ such that

$$\forall \lambda \in (0,+\infty): \begin{cases} M(\lambda x, \lambda y) = \lambda^a M(x,y) \\ N(\lambda x, \lambda y) = \lambda^a N(x,y) \end{cases}$$

with $a \in \mathbb{R}$.

a) Show that this ODE is homogeneous by reducing it to the form

$$\frac{dy}{dx} = \frac{-M(1, y/x)}{N(1, y/x)}$$

b) Show that the substitution $u = y/x$ reduces this ODE to the separable form:

$$\frac{1}{x} + \frac{N(1,u)}{M(1,u) + uN(1,u)} \frac{du}{dx} = 0$$

(3) → Integrating Factors Method

This method can be applied to ODEs of the form:

$$\boxed{y' + f(x)y = g(x)}$$

with f, g continuous on $\mathbb{R}$.

Solution method

Define $h(x) = \exp\left(\int f(x)\,dx\right)$ and note that $h'(x) = f(x)h(x)$.

Then we multiply both sides of the ODE with $h(x)$:

$$y' + f(x)y = g(x) \Leftrightarrow y'h(x) + h(x)f(x)y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow y'h(x) + h'(x)\,y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dx}\left[y h(x)\right] = h(x)g(x) \Leftrightarrow$$

$$\Leftrightarrow h(x)y = \int h(x)g(x)\,dx + C_1$$

$$\Leftrightarrow y = \frac{1}{h(x)}\int h(x)g(x)\,dx + \frac{C_1}{h(x)} \qquad (1)$$

→ Note that for $g(x) = 0$, the above solution simplifies to

$$y = \frac{C_1}{h(x)} = C_1 \exp\left(-\int f(x)\,dx\right)$$

This is called the homogeneous term to Eq. (1).
The integral term is called the particular term.

EXAMPLE

a) Solve the ODE $y' + xy = x^2$ with $y(0) = y_0$.

Solution

Use the integrating factor

$$h(x) = \exp\left(\int x\,dx\right) = \exp(x^2/2) \Rightarrow h'(x) = x\,h(x)$$

and therefore:

$$y' + xy = x^2 \Leftrightarrow y'h(x) + xh(x)y = x^2 h(x) \Leftrightarrow y'h(x) + h'(x)y = x^2 h(x) \Leftrightarrow$$

$$\Leftrightarrow [y\,h(x)]' = x^2 h(x) \Leftrightarrow yh(x) = c + \int_0^x t^2 h(t)\,dt \qquad (1)$$

For $x = 0$: $y_0 h(0) = c + 0 \Leftrightarrow c = y_0 h(0) = y_0 \exp(0) = y_0$

and therefore,

$$(1) \Leftrightarrow yh(x) = y_0 + \int_0^x t^2 h(t)\,dt \Leftrightarrow$$

$$\Leftrightarrow y = \frac{y_0}{h(x)} + \frac{1}{h(x)} \int_0^x t^2 h(t)\,dt =$$

$$= \frac{y_0}{\exp(x^2/2)} + \frac{1}{\exp(x^2/2)} \int_0^x t^2 \exp(t^2/2)\,dt =$$

$$= y_0 \exp(-x^2/2) + \exp(-x^2/2) \int_0^x t^2 \exp(t^2/2)\,dt$$

→ The integrating factor method can be applied to the more general problem of the form

$$\boxed{f(x)y' + g(x)y = h(x)}$$

However, if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then x_0 is a singular point of the ODE and the ODE will only yield a unique solution if x is restricted to an interval between neighboring singular points.

EXAMPLE

Solve the ODE $(x^2-1)y' + xy = 0$ with $y(x_0) = y_0$.

Solution

We have

$$(x^2-1)y' + xy = 0 \Leftrightarrow y' + \frac{x}{x^2-1}y = 0 \qquad (1)$$

We will use the integrating factor

$$h(x) = \exp\left(\int \frac{x}{x^2-1}\,dx\right) = \exp\left(\frac{1}{2}\int \frac{(x^2-1)'}{x^2-1}\,dx\right) =$$

$$= \exp\left(\frac{1}{2}\ln|x^2-1|\right) = \exp\left(\ln\sqrt{|x^2-1|}\right) =$$

$$= \sqrt{|x^2-1|}$$

$$\Rightarrow h'(x) = h(x)\frac{x}{x^2-1}.$$ It follows that

$$(1) \Leftrightarrow y'h(x) + \frac{x}{x^2-1}h(x)y = 0 \Leftrightarrow y'h(x) + yh'(x) = 0$$

$$\Leftrightarrow (d/dx)[yh(x)] = 0 \Leftrightarrow (d/dx)\left[y\sqrt{|x^2-1|}\right] = 0$$

$$\Leftrightarrow y\sqrt{|x^2-1|} = c_1 \Leftrightarrow y = \frac{c_1}{\sqrt{|x^2-1|}}$$

We note that the ODE has singular points on $x=1$ and $x=-1$. From the initial condition:

$$y(x_0) = y_0 \Leftrightarrow \frac{c_1}{\sqrt{|x_0^2-1|}} = y_0 \Leftrightarrow c = y_0\sqrt{|x_0^2-1|}$$

and therefore:

$$y = \frac{y_0\sqrt{|x_0^2-1|}}{\sqrt{|x^2-1|}}$$

We distinguish between the following cases:

Case 1 : If $x_0 \in (-\infty, -1)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (-\infty, -1)$$

Case 2 : If $x_0 \in (-1, 1)$, then $|x_0^2-1| = 1-x_0^2$ and

$$y = \frac{y_0\sqrt{1-x_0^2}}{\sqrt{1-x^2}}, \quad \forall x \in (-1, 1)$$

Case 3 : If $x_0 \in (1, +\infty)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (1, +\infty).$$

EXERCISES

(5) Solve the following initial value problems.

a) $\begin{cases} y' - 2y = x e^{-2x} \\ y(0) = y_0 \end{cases}$

b) $\begin{cases} xy' - 2y = x^4 \\ y(1) = y_0 \end{cases}$

c) $\begin{cases} y' + y \tan x = \sin(2x) \\ y(0) = y_0 \end{cases}$

d) $\begin{cases} y' - (\cot x)\, y = 3x \sin x \\ y(\pi/4) = y_0 \end{cases}$

e) $\begin{cases} xy' + y = 3x^3 - 1 \\ y(1) = y_0 \end{cases}$

f) $\begin{cases} y' + e^x y = 2e^x \\ y(0) = y_0 \end{cases}$

g) $\begin{cases} y' + 2xy = x \exp(-x^2) \\ y(0) = y_0 \end{cases}$

(6) Consider the initial value problem

$$\begin{cases} y' - 2xy = 1 \\ y(0) = y_0 \end{cases}$$

Show that its unique solution is:

$$y(x) = \exp(x^2) \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + y_0 \right]$$

with erf(t) the error function defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$$

7 Bernoulli equations

A Bernoulli ordinary differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n$$

with $n \in \mathbb{R}$.

a) Show that the substitution $u = y^{1-n}$ reduces the Bernoulli equation to a linear ordinary differential equation of the form

$$u' + (1-n)p(x)u = (1-n)q(x)$$

b) Use this substitution to solve the following Bernoulli initial value problem.

$$\begin{cases} y' + xy = xy^2 \\ y(0) = y_0 \end{cases}$$

ODE 3: Review of Linear Algebra

LINEAR ALGEBRA REVIEW

General linear differential equations are analogous to linear systems of equations. It is therefore useful to briefly review basic concepts of linear algebra

Vectors in $\mathbb{R}^n$

Consider two n-dimensional vectors $x, y \in \mathbb{R}^n$ with

$$x = (x_1, x_2, \dots, x_n)$$
$$y = (y_1, y_2, \dots, y_n)$$

We define the following vector operations:

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \leftarrow \text{vector addition}$$
$$\forall \lambda \in \mathbb{R}: \lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_n) \leftarrow \text{scalar multiplication}$$

We also define the zero vector

$$\mathbf{0} = (0, 0, 0, \dots, 0)$$

Linearly independent vectors

Def: Let $u_1, u_2, \dots, u_m \in \mathbb{R}^n$ be m n-dimensional vectors.

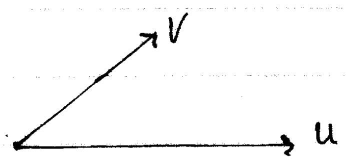
We say that

$$\boxed{\begin{aligned} &u_1, u_2, \dots, u_m \text{ linearly independent} \iff \\ &\iff \forall \lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}: (\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_m u_m = \mathbf{0} \implies \\ &\qquad \implies \lambda_1 = \lambda_2 = \dots = \lambda_m = 0) \end{aligned}}$$

Interpretation

The equation $\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_m u_m = \mathbf{0}$ implies that each of the m vectors can be written as a linear combination of the other vectors. If the vectors are linearly independent, it is impossible for the equation to be satisfied with non-zero coefficients, therefore none of the vectors can be written as a linear combination of the other vectors.

▶ In two dimensions:

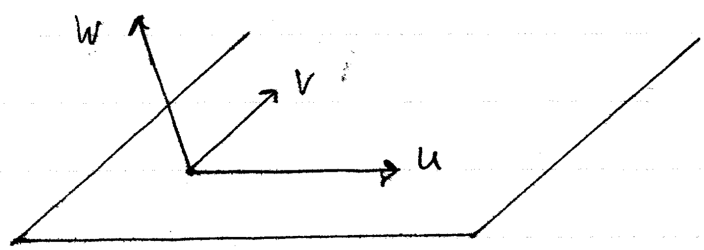


$u = (u_1, u_2)$

$v = (v_1, v_2)$

u, v are linearly independent if and only if they point in different directions.

▶ In three dimensions:



$u = (u_1, u_2, u_3)$

$v = (v_1, v_2, v_3)$

$w = (w_1, w_2, w_3)$

u, v, w are linearly independent if and only if u and v are not on the same line and w does not lie on the plane defined by u, v.

● Matrices

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be an arbitrary vector. A matrix $A \in M_n(\mathbb{R})$ represents a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined as:

$$\begin{cases} y_1 = (Ax)_1 = A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ y_2 = (Ax)_2 = A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ y_n = (Ax)_n = A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n \end{cases}$$

For $y = (y_1, y_2, \ldots, y_n)$ we write: $y = Ax$

- The numbers A_{ab} are the components of the matrix A and we write

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Alternatively, if $A_1, A_2, \ldots, A_n \in \mathbb{R}^n$ are vectors representing the rows of A such that

$$A_1 = (A_{11}, A_{12}, \ldots, A_{1n})$$
$$A_2 = (A_{21}, A_{22}, \ldots, A_{2n})$$
$$\vdots$$
$$A_n = (A_{n1}, A_{n2}, \ldots, A_{nn})$$

we write $A = (A_1, A_2, \ldots, A_n)$.

- We note that

$$\boxed{\forall \lambda_1, \lambda_2 \in \mathbb{R} : \forall u, v \in \mathbb{R}^n : A(\lambda_1 u + \lambda_2 v) = \lambda_1 (Au) + \lambda_2 (Av).}$$

Matrix operations

Let $A, B \in M_n(\mathbb{R})$ be two matrices and let $\lambda \in \mathbb{R}$ be a number. We define $A+B$, AB, and λA as follows:

$\forall x \in \mathbb{R}^n : (A+B)x = Ax + By$

$\forall x \in \mathbb{R}^n : (AB)x = A(Bx)$

$\forall x \in \mathbb{R}^n : (\lambda A)x = \lambda(Ax)$

It follows that the components of these new matrices are given by:

$\forall a, b \in [n] : (A+B)_{ab} = A_{ab} + B_{ab}$

$$\forall a, b \in [n] : (AB)_{ab} = \sum_{c \in [n]} A_{ac} B_{cb}$$

$\forall a, b \in [n] : (\lambda A)_{ab} = \lambda A_{ab}$

Identity Matrix

Given the unit vectors $e_1, e_2, \ldots, e_n$ defined as:

$e_1 = (1, 0, \ldots, 0)$

$e_2 = (0, 1, \ldots, 0)$

$\vdots$

$e_n = (0, 0, \ldots, 1)$

we define the $n \times n$ identity matrix as

$I = (e_1, e_2, \ldots, e_n)$

or equivalently as

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

We note that

$$\forall A \in M_n(\mathbb{R}): IA = AI = A$$

Matrix Inverse

Let $A \in M_n(\mathbb{R})$ be a matrix. We say that

$$B = A^{-1} \iff AB = BA = I.$$

- interpretation: The inverse matrix A^{-1} undoes the effect of the operation A on any vector x, since

$$A^{-1}(Ax) = (A^{-1}A)x = Ix = x$$

Not all matrices have an inverse. If a matrix A has an inverse, we say that A is non-singular

- inverse of a 2×2 matrix

Let A be a matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A non-singular if and only if $ad - bc \neq 0$ and A^{-1} is given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Determinant of a matrix

The existence of an inverse can be queried via the determinant $\det(A)$ of the matrix A. We define the determinant as follows:

• 1 <u>Permutations</u> : A permutation σ is a mapping $\sigma : [n] \to [n]$ that rearranges the order of the elements of $[n]$.
e.g.: $\sigma = (3,1,2)$ is the permutation with $\sigma(1) = 3$, $\sigma(2) = 1$, and $\sigma(3) = 2$.
The set of all permutations $\sigma : [n] \to [n]$ is denoted as S_n.

• 2 <u>Parity of a permutation</u> :
Let $\sigma \in S_n$ be a permutation. We define the <u>parity</u> $s(\sigma)$ of σ as:

$$\boxed{\begin{aligned} s(\sigma) &= \operatorname{sign}\left[\prod_{b=1}^{n-1} \prod_{a=b+1}^{n} (\sigma(a) - \sigma(b))\right] \\ \operatorname{sign}(x) &= \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases} \end{aligned}}$$

• 3 <u>Determinant</u>
Let $A \in M_n(\mathbb{R})$ be a matrix. We define the determinant $\det(A)$ of A as:

$$\boxed{\det A = \sum_{\sigma \in S_n} \left[s(\sigma) \prod_{a \in [n]} A_{a,\sigma(a)} \right]}$$

▸ <u>Determinant of a 2x2 matrix</u>
The determinant of
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
is given by: $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

▶ Properties of determinants

- 1 Determinant and matrix non-singularity.
  $\forall A \in M_n(\mathbb{R})$: A non-singular $\Leftrightarrow \det A \neq 0$
- 2 Determinant of a matrix product
  $\forall A \in M_n(\mathbb{R})$: $\det(AB) = \det(A)\det(B)$
- 3 Determinant of a matrix with two identical rows:
  Let $a_1, a_2, \dots, a_n \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ be vectors. Then
  $\det(a_1, a_2, \dots, b, \dots, b, \dots, a_n) = 0$
- 4 Determinant linearity:
  Let $a_1, a_2, \dots, a_n \in \mathbb{R}^n$ and $b, c \in \mathbb{R}^n$ be vectors. Then
  $$\forall \lambda, \mu \in \mathbb{R}: \det(a_1, \dots, \lambda b + \mu c, \dots, a_n) = \lambda \det(a_1, \dots, b, \dots, a_n) + \mu \det(a_1, \dots, c, \dots, a_n)$$

▶ Evaluation of determinants

The efficient evaluation of derivatives can be done using the following results:

(1) A 2×2 determinant can be evaluated as:
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

(2) From properties 3 and 4 above it follows that we can a multiple of one row to another row without changing the value of the derivative. For example,
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{array}{l} \cdot\lambda \\ \leftarrow \\ \end{array} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+\lambda a_1 & b_2+\lambda a_2 & b_3+\lambda a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The same property also holds for columns. For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3+\lambda a_1 \\ b_1 & b_2 & b_3+\lambda b_1 \\ c_1 & c_2 & c_3+\lambda c_1 \end{vmatrix}$$

We can use this to zero-out a row or column of the matrix.

(3) Determinants with a row or column of the form $(0,0,\ldots,0,a,0,\ldots,0)$ can be reduced into an equal determinant of smaller size by deleting both the row and column that pass through a. We then multiply with a ± 1 factor, depending on the location of a, according to a "chessboard pattern" of the form

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

in which the upper-left corner is always "+". For example,

$$\begin{vmatrix} a_1 & 0 & b_1 \\ a_2 & 0 & b_2 \\ a_3 & \lambda & b_3 \end{vmatrix} = (-1)\lambda \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\lambda(a_1 b_2 - a_2 b_1).$$

$$\begin{vmatrix} 0 & 0 & \lambda \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (+1)\lambda \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \lambda(a_1 b_2 - a_2 b_1).$$

Linear system of equations

Consider the linear system $Ax=b$ with $A\in M_n(\mathbb{R})$ and $x,b\in\mathbb{R}^n$ which can be expanded as:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n = b_n \end{cases}$$

Cramer rule

If $\det A \neq 0$, then the system $Ax=b$ has a unique solution $x=(x_1,x_2,\dots,x_n)$ with

$\forall k\in[n]:\ x_k = D_k/D$

where $D=\det A$ and

$$D_1 = \begin{vmatrix} b_1 & A_{12} & \cdots & A_{1n} \\ b_2 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & A_{n2} & \cdots & A_{nn} \end{vmatrix}, \quad D_2 = \begin{vmatrix} A_{11} & b_1 & \cdots & A_{1n} \\ A_{21} & b_2 & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & b_n & \cdots & A_{nn} \end{vmatrix}, \dots,$$

$$D_n = \begin{vmatrix} A_{11} & A_{12} & \cdots & b_1 \\ A_{21} & A_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & b_n \end{vmatrix}$$

In other words D_k is the definition of the matrix obtained by replacing the column k of A with the components of b.

- null space

We now consider the case $\det A = 0$. We define the null-space of the matrix A as:

$$\text{null}(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\}$$

with corresponding belonging condition given by

$$x \in \text{null}(A) \Leftrightarrow Ax = \mathbf{0}.$$

- Given a particular solution $p \in \mathbb{R}^n$ of $Ax = b$, the entire solution set of the system is given by:

  $$S = \{x \in \mathbb{R}^n \mid Ax = b\} = \{p + x \mid x \in \text{null}(A)\}.$$

  We will see that an analogous result holds for linear differential equations with respect to homogeneous and particular solutions.

- We can also show that

  $$\text{null}(A) = \{\mathbf{0}\} \Leftrightarrow \det A \neq 0$$

  therefore null(A) has non-trivial content only if $\det A = 0$.

  Specifically we can show that:

a) $\text{null}(A) \cap (\mathbb{R}^n - \{\mathbf{0}\}) \neq \emptyset \Leftrightarrow \det A = 0$

   or equivalently:

   $(\exists x \in \mathbb{R}^n - \{\mathbf{0}\} : Ax = \mathbf{0}) \Leftrightarrow \det A = 0$

b) If $\det A = 0$, then:

   $$\exists u_1, \dots, u_k \in \mathbb{R}^n : \begin{cases} u_1, u_2, \dots, u_k \text{ linearly independent} \\ \text{null}(A) = \text{span}\{u_1, u_2, \dots, u_k\} \end{cases}$$

   where we define:

   $$\text{span}\{u_1, u_2, \dots, u_n\} = \{\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k \mid \lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}\}.$$

ODE 4: Linear Differential Equations

LINEAR DIFFERENTIAL EQUATIONS

▼ Basic Definitions - Terminology

- A linear differential equation is any equation of the form
$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = f(x). \quad (1)$$
- The functions $p_0, p_1, \ldots, p_n$ are called the coefficients of the linear differential equation and it is usually assumed that they are continuous functions.
- $n \in \mathbb{N}^*$ is the order of the linear differential equation.
- Given the linear differential equation of Eq.(1), we say that for a point $x_0 \in \mathbb{R}$:
$$x_0 \text{ is regular} \Leftrightarrow p_n(x_0) \neq 0$$
$$x_0 \text{ is singular} \Leftrightarrow p_n(x_0) = 0$$
- A linear differential equation of the form of Eq.(1) is homogeneous on a set $A \subseteq \mathbb{R}$ if and only if
$$\forall x \in A : f(x) = 0.$$
otherwise, we say that it is inhomogeneous.
- If an linear differential equation is regular for every point in some interval $A \subseteq \mathbb{R}$ (i.e. if $\forall x \in A : p_n(x) \neq 0$) then we can rewrite it as:
$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x) \quad (2)$$
with
$$\forall k \in [n-1] \cup \{0\} : a_k(x) = \frac{p_k(x)}{p_n(x)} \quad \text{and} \quad g(x) = \frac{f(x)}{p_n(x)}.$$

Function operators and linear operators

- Let $A \subseteq \mathbb{R}$ be an interval. We define the following function spaces via belonging conditions as follows:

  a) Space of continuous functions $C^0(A)$:
  $$y \in C^0(A) \Leftrightarrow \begin{cases} y: A \to \mathbb{R} \\ y \text{ continuous on } A. \end{cases}$$

  b) Space of n-times continuously differentiable functions $C^n(A)$.
  $$y \in C^n(A) \Leftrightarrow \begin{cases} y: A \to \mathbb{R} \\ y \text{ n-times differentiable on } A \\ y^{(n)} \text{ continuous on } A \end{cases}$$

  c) Space of infinitely differentiable functions $C^\infty(A)$.
  $$y \in C^\infty(A) \Leftrightarrow \forall n \in \mathbb{N} : y \in C^n(A)$$

- Given the linear differential equation from Eq.(2) we define the mapping $L: C^n(A) \to C^0(A)$ such that
$$\forall y \in C^n(A): L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y \quad (3)$$
Then, the linear differential equation
$$y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y'(x) + a_0(x) y(x) = g(x)$$
can be rewritten as:
$$L(y) = g. \quad \text{or also: } Ly = g.$$
Note that by analogy the <u>operator L</u> is to a function $y \in C^n(A)$ what a matrix A is to some vector $x \in \mathbb{R}^n$.

- The operator L defined by Eq.(3) satisfies the following definition of a <u>linear operator</u>

<u>Def</u>: Consider an operator $L: C^n(A) \to C^0(A)$. We say that L is a linear operator if and only if it satisfies the following conditions:

a) $\forall y_1, y_2 \in C^n(A): L(y_1+y_2) = Ly_1 + Ly_2$

b) $\forall \lambda \in \mathbb{R}: \forall y \in C^n(A): L(\lambda y) = \lambda L(y)$.

<u>Prop</u>: Let $L: C^n(A) \to C^0(A)$ be a linear operator. Then:

$\forall \lambda, \mu \in \mathbb{R}: \forall y_1, y_2 \in C^n(A): L(\lambda y_1 + \mu y_2) = \lambda L(y_1) + \mu L(y_2)$.

<u>Proof</u>

Let $\lambda, \mu \in \mathbb{R}$ and $y_1, y_2 \in C^n(A)$ be given. Then:

$$\begin{aligned} L(\lambda y_1 + \mu y_2) &= L(\lambda y_1) + L(\mu y_2) \\ &= \lambda L(y_1) + \mu L(y_2). \end{aligned}$$

It follows that

$\forall \lambda, \mu \in \mathbb{R}: \forall y_1, y_2 \in C^n(A): L(\lambda y_1 + \mu y_2) = \lambda L(y_1) + \mu L(y_2)$.

$\to$ Note that the definition

$$Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

is given in terms of function algebra, i.e function addition and function multiplication. In terms of regular algebra, we write:

$$\forall x \in A: (Ly)(x) = y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x).$$

▼ Homogeneous linear differential equations

We begin by presenting the theory needed for solving homogeneous linear differential equations of the form $Ly = \mathbb{0}$ given a linear operator $L : C^n(A) \to C^0(A)$.

● Solution set of the homogeneous ODE

We begin by stating some needed definitions. Then we state the main result without proof.

Def: Let $y_1, y_2, \dots, y_n \in C^0(A)$ be functions. We say that $y_1, y_2, \dots, y_n$ linearly independent $\iff$

$$\iff \forall \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} : (\lambda_1 y_1 + \dots + \lambda_n y_n = \mathbb{0} \implies \lambda_1 = \lambda_2 = \dots = \lambda_n = 0)$$

↳ We note that this definition is analogous to the linear independence of vectors on $\mathbb{R}^n$. However, the statement

$\lambda_1 y_1 + \dots + \lambda_n y_n = \mathbb{0}$

is equivalent to the algebraic statement

$\forall x \in A : \lambda_1 y_1(x) + \lambda_2 y_2(x) + \dots + \lambda_n y_n(x) = 0.$

Def: Let $y_1, y_2, \dots, y_n \in C^0(A)$. We define the space spanned by the functions $y_1, \dots, y_n$ as

$$\operatorname{span}\{y_1, y_2, \dots, y_n\} = \{\lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_n y_n \mid \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}\}$$

↳ The corresponding belonging condition reads:

$y \in \text{span}\{y_1, y_2, \ldots, y_n\} \Leftrightarrow$

$\Leftrightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: y = \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n.$

Def: Let $L: C^n(A) \to C^0(A)$ be an operator. We define the null space of L as:

$\text{null}(L) = \{y \in C^n(A) \mid Ly = 0\}.$

↳ Thus, the problem of solving the homogeneous linear differential equation $Ly = 0$ is equivalent to the problem of finding the null space $\text{null}(L)$ or the operator L.

Thm: Let $a_0, a_1, \ldots, a_{n-1} \in C^0(A)$ for some interval $A \subseteq \mathbb{R}$ and define the operator $L: C^n(A) \to C^0(A)$ such that

$\forall y \in C^n(A): Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$

Then there exist $y_1, y_2, \ldots, y_n \in C^n(A)$ such that they satisfy the following conditions:

(a) $y_1, y_2, \ldots, y_n$ are linearly independent

(b) $\text{null}(L) = \text{span}\{y_1, y_2, \ldots, y_n\}$

↳ It follows from this theorem that the general solution to the linear differential equation $Ly = 0$ takes the form

$\forall x \in A: y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ are constant coefficients and $y_1, y_2, \ldots, y_n$ are linearly independent functions.

EXAMPLES

a) Consider the functions

$\forall x \in \mathbb{R}: (f(x) = x \wedge g(x) = x^2 \wedge h(x) = x^3)$

Show that f, g, h are linearly independent.

Solution

▸ We give two different methods for solving this problem. Only one method is needed for a complete solution.

1st method: By definition.

It is sufficient to show that

$\forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}: (\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbb{0} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0)$

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be given and assume that $\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbb{0}$.

It follows that:

$\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbb{0} \Rightarrow \forall x \in \mathbb{R}: \lambda_1 f(x) + \lambda_2 g(x) + \lambda_3 h(x) = 0$

$\Rightarrow \forall x \in \mathbb{R}: \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0 \qquad (1)$

For $x = 1$: $\lambda_1 + \lambda_2 + \lambda_3 = 0$

For $x = 2$: $2\lambda_1 + 4\lambda_2 + 8\lambda_3 = 0$

For $x = -1$: $-\lambda_1 + \lambda_2 - \lambda_3 = 0$

Consider the system of equations:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ 2\lambda_1 + 4\lambda_2 + 8\lambda_3 = 0 \\ -\lambda_1 + \lambda_2 - \lambda_3 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

We note that:

$$\begin{vmatrix}1&1&1\\2&4&8\\-1&1&-1\end{vmatrix}\begin{matrix}(-2)&1\\ & \\ & \end{matrix} = \begin{vmatrix}1&1&1\\0&2&6\\0&2&0\end{vmatrix} = \begin{vmatrix}2&6\\2&0\end{vmatrix} = 2\cdot 0 - 6\cdot 2 = -12 \neq 0$$

$\Rightarrow$ Eq.(1) has a unique solution $(\lambda_1,\lambda_2,\lambda_3)=(0,0,0) \Rightarrow$
$\Rightarrow \lambda_1=\lambda_2=\lambda_3=0$

We have thus shown that

$\forall \lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}: (\lambda_1 f + \lambda_2 g + \lambda_3 h = \mathbf{0} \Rightarrow \lambda_1=\lambda_2=\lambda_3=0)$

$\Rightarrow f,g,h$ linearly independent.

b) Consider the functions $\forall x \in \mathbb{R}: (f(x) = e^{ax} \wedge g(x) = e^{bx})$

Show that: $a \neq b \Rightarrow f,g$ linearly independent.

<u>Solution</u>

Assume that $a \neq b$. It is sufficient to show that:

$\forall \lambda_1,\lambda_2 \in \mathbb{R}: (\lambda_1 f + \lambda_2 g = \mathbf{0} \Rightarrow \lambda_1=\lambda_2=0)$

Let $\lambda_1,\lambda_2 \in \mathbb{R}$ be given. Assume that $\lambda_1 f + \lambda_2 g = \mathbf{0}$. Then:

$\lambda_1 f + \lambda_2 g = \mathbf{0} \Rightarrow \forall x \in \mathbb{R}: \lambda_1 f(x) + \lambda_2 g(x) = 0$

$\Rightarrow \forall x \in \mathbb{R}: \lambda_1 e^{ax} + \lambda_2 e^{bx} = 0 \qquad (1)$

For $x=0$: $\lambda_1 e^0 + \lambda_2 e^0 = 0 \Leftrightarrow \lambda_1 + \lambda_2 = 0$

For $x=1$: $\lambda_1 e^a + \lambda_2 e^b = 0$

It follows that:

$$\begin{cases}\lambda_1+\lambda_2=0\\ e^a\lambda_1 + e^b\lambda_2 = 0\end{cases} \Rightarrow \begin{bmatrix}1&1\\e^a&e^b\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad (2)$$

and note that:

$$\begin{vmatrix}1&1\\e^a&e^b\end{vmatrix} = e^b - e^a$$

Since: $a \neq b \Rightarrow e^a \neq e^b \Rightarrow e^b - e^a \neq 0 \Rightarrow \begin{vmatrix} 1 & 1 \\ e^a & e^b \end{vmatrix} \neq 0$

$\Rightarrow$ Eq. (2) has a unique solution $(\lambda_1, \lambda_2) = (0, 0)$

$\Rightarrow \lambda_1 = \lambda_2 = 0$.

We have thus shown that

$\forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 f + \lambda_2 g = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = 0)$

$\Rightarrow f, g$ linearly independent.

EXERCISES

(1) Show that the functions f, g, h, defined below, are linearly independent, using the definition.

a) $\begin{cases} \forall x \in \mathbb{R}: f(x) = 3x \\ \forall x \in \mathbb{R}: g(x) = x+2 \\ \forall x \in \mathbb{R}: h(x) = (x-1)^2 \end{cases}$

b) $\begin{cases} \forall x \in \mathbb{R}: f(x) = \sin x \\ \forall x \in \mathbb{R}: g(x) = \cos x \\ \forall x \in \mathbb{R}: h(x) = x \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R}: f(x) = 1-x \\ \forall x \in \mathbb{R}: g(x) = 1+x \\ \forall x \in \mathbb{R}: h(x) = 1-x^2 \end{cases}$

d) $\begin{cases} \forall x \in \mathbb{R}: f(x) = 1 \\ \forall x \in \mathbb{R}: g(x) = e^x \\ \forall x \in \mathbb{R}: h(x) = e^{2x} \end{cases}$

e) $\begin{cases} \forall x \in \mathbb{R}: f(x) = e^{3x} \\ \forall x \in \mathbb{R}: g(x) = xe^{3x} \\ \forall x \in \mathbb{R}: h(x) = x^2 e^{3x} \end{cases}$

The initial value problem

In an initial value problem we consider the homogeneous linear differential equation $Ly = 0$ where we introduce the restrictions

$$y(x_0) = a_0 \wedge y'(x_0) = a_1 \wedge y''(x_0) = a_2 \wedge \cdots \wedge y^{(n-1)}(x_0) = a_{n-1}$$

Given the general solution

$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$$

the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ can be uniquely solved by the following system of equations:

$$\begin{cases} \lambda_1 y_1(x_0) + \lambda_2 y_2(x_0) + \cdots + \lambda_n y_n(x_0) = a_0 \\ \lambda_1 y_1'(x_0) + \lambda_2 y_2'(x_0) + \cdots + \lambda_n y_n'(x_0) = a_1 \\ \vdots \\ \lambda_1 y_1^{(n-1)}(x_0) + \lambda_2 y_2^{(n-1)}(x_0) + \cdots + \lambda_n y_n^{(n-1)}(x_0) = a_{n-1} \end{cases}$$

which can be rewritten in terms of matrices as follows:

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

The determinant of the matrix is called the <u>Wronskian</u> and we will prove later that it is non-zero. It follows that solving with respect to the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ will give a unique solution.

The Wronskian and its properties

Def: Let $y_1, y_2, \dots, y_n \in C^{n-1}(A)$, for some interval $A \subseteq \mathbb{R}$. We define:

a) The matrix $W[y_1, \dots, y_n](x)$ as:

$$\forall x \in A: W[y_1, \dots, y_n](x) = \begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & & & \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix}$$

b) The Wronskian $w[y_1, \dots, y_n](x)$ as:

$$\forall x \in A: w[y_1, \dots, y_n](x) = \det W[y_1, \dots, y_n](x)$$

We now show that the Wronskian satisfies the following properties:

1 → Nonzero Wronskian implies linear independence

Thm: Let $y_1, y_2, \dots, y_n \in C^{n-1}(A)$ with $A \subseteq \mathbb{R}$ an interval. Then:

$$(\exists x \in A: w[y_1, \dots, y_n](x) \neq 0) \Rightarrow$$
$$\Rightarrow y_1, y_2, \dots, y_n \text{ linearly independent}$$

Proof

Assume that $\exists x \in A: w[y_1, \dots, y_n](x) \neq 0$. Choose an $x_0 \in A$ such that $w[y_1, \dots, y_n](x_0) \neq 0$. It is sufficient to show that $\forall \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}: (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0)$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ be given and assume that
$\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = \mathbf{0} \Rightarrow$
$\Rightarrow \forall x \in A: \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0$
Differentiating with respect to x gives the equations:
$\forall x \in A: \lambda_1 y_1'(x) + \lambda_2 y_2'(x) + \cdots + \lambda_n y_n'(x) = 0$
$\forall x \in A: \lambda_1 y_1''(x) + \lambda_2 y_2''(x) + \cdots + \lambda_n y_n''(x) = 0$
$\vdots$
$\forall x \in A: \lambda_1 y_1^{(n-1)}(x) + \lambda_2 y_2^{(n-1)}(x) + \cdots + \lambda_n y_n^{(n-1)}(x) = 0$
These equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ y_1''(x) & y_2''(x) & \cdots & y_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \forall x \in A$$

We define $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the matrix equation is
$W[y_1, y_2, \ldots, y_n](x)\,\Lambda = \mathbf{0}, \quad \forall x \in A.$
For $x = x_0$, we have:

$$\left\{ \begin{array}{l} W[y_1, \ldots, y_n](x_0)\,\Lambda = \mathbf{0} \\ \det W[y_1, \ldots, y_n](x_0) = w[y_1, \ldots, y_n](x_0) \neq 0 \end{array} \right. \Rightarrow$$

$\Rightarrow \Lambda = \mathbf{0} \Rightarrow (\lambda_1, \lambda_2, \ldots, \lambda_n) = (0, 0, \ldots, 0)$
$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$
We have thus shown that
$\forall \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0)$
$\Rightarrow y_1, y_2, \ldots, y_n$ linearly independent $\square$

(2) → Linearly independent solutions of a linear differential equation give a non-zero Wronskian

The previous property can be used to prove that a set of functions are linearly independent, if the corresponding Wronskian is nonzero for at least one point. The converse statement is not always true. However we will now show that if some functions $y_1, \dots, y_n$ solve the SAME linear differential equation and are linearly independent, then they will give a nonzero Wronskian for all points.

Thm: Define the operator $L : C^n(A) \to C^0(A)$, for some interval $A \subseteq \mathbb{R}$, such that:

$$\forall x \in A : Ly(x) = y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \dots + a_1(x) y'(x) + a_0(x) y(x)$$

We assume that

a) $y_1, y_2, \dots, y_n \in C^n(A)$ are linearly independent

b) $\forall k \in [n] : Ly_k = 0$

Then, it follows that

a) $\forall x \in A : w'[y_1, \dots, y_n](x) + a_{n-1}(x) w[y_1, \dots, y_n](x) = 0$

b) For some $c \in A$:

$$\forall c, x \in A : w[y_1, \dots, y_n](x) = w[y_1, \dots, y_n](c) \exp\left(-\int_c^x a_{n-1}(t)\,dt\right)$$

c) $\forall x \in A : w[y_1, \dots, y_n](x) \neq 0$

Proof

a) Define the vector-valued function $y : A \to \mathbb{R}^n$ with $y = (y_1, y_2, \dots, y_n)$. Since

$$(\forall k \in [n] : L y_k = 0) \Rightarrow \left(\forall k \in [n] : y_k^{(n)} = -\sum_{p=0}^{n-1} a_p y_k^{(p)}\right) \Rightarrow$$

$$\Rightarrow y^{(n)} = -\sum_{p=0}^{n-1} a_p y^{(p)} \qquad (1)$$

Note that $y^{(p)}$ is a vector-valued function whereas a_p is a scalar function. It follows that

$$\begin{aligned}
(d/dx)\, w[y_1, \dots, y_n](x) &= (d/dx) \det(y, y', y'', \dots, y^{(n-1)}) = \\
&= \det(y, y', \dots, y^{(n-2)}, y^{(n)}) = \\
&= \det\left(y, y', \dots, y^{(n-2)}, -\sum_{p=0}^{n-1} a_p y^{(p)}\right) = \\
&= \sum_{p=0}^{n-1} \det(y, y', \dots, y^{(n-2)}, -a_p y^{(p)}) = \\
&= -\sum_{p=0}^{n-1} a_p \det(y, y', \dots, y^{(n-2)}, y^{(p)}) = \\
&= -\sum_{p=0}^{n-2} a_p \det(y, \dots, y^{(p)}, \dots, y^{(n-2)}, y^{(p)}) + \\
&\quad a_{n-1} \det(y, \dots, y^{(n-2)}, y^{(n-1)}) \\
&= 0 - a_{n-1} w[y] = -a_{n-1} w[y] \Rightarrow
\end{aligned}$$

$$\Rightarrow \forall x \in A : \ w'[y](x) + a_{n-1}(x)\, w[y](x) = 0$$

b) Define the integrating factor

$$\forall x \in A : h(x) = \exp\left(\int_c^x a_{n-1}(t)\,dt\right)$$

and note that

$$\forall x \in A : h'(x) = (d/dx)\exp\left(\int_c^x a_{n-1}(t)\,dt\right) =$$

$$= \exp\left(\int_c^x a_{n-1}(t)\,dt\right)\frac{d}{dx}\int_c^x a_{n-1}(t)\,dt =$$

$$= h(x)\,a_{n-1}(x).$$

We may now solve the differential equation satisfied by the Wronskian as follows:

$$w'[y](x) + a_{n-1}(x)\,w[y](x) = 0 \Leftrightarrow$$

$$\Leftrightarrow w'[y](x)\,h(x) + h(x)\,a_{n-1}(x)\,w[y](x) = 0 \Leftrightarrow$$

$$\Leftrightarrow w'[y](x)\,h(x) + w[y](x)\,h'(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow (d/dx)\,[w[y](x)\,h(x)] = 0 \Leftrightarrow w[y](x)\,h(x) = c_0$$

$$\Leftrightarrow w[y](x) = \frac{c_0}{h(x)} = c_0 \exp\left(-\int_c^x a_{n-1}(t)\,dt\right)$$

For $x = c$: $w[y](c) = c_0 \cdot 1 = c_0$, and therefore

$$\forall x \in A : w[y](x) = w[y](c)\exp\left(-\int_c^x a_{n-1}(t)\,dt\right)$$

c) From (b) we see that it is sufficient to show that $\exists c \in A : w[y](c) \neq 0$. To show a contradiction, we assume the opposite statement: $\forall c \in A : w[y](c) = 0$. Choose some $c \in A$ and consider the linear system of equations $W[y](c)\Lambda = \mathbf{0}$ with $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. It follows that

$w[y](c)=0 \Rightarrow \det W[y](c)=0 \Rightarrow \exists \Lambda \in \mathbb{R}^n-\{0\}: W[y]\Lambda = \mathbf{0}$

Choose some $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n - \{\mathbf{0}\}$ such that $W[y](c)\Lambda = \mathbf{0}$ and define the function $f: A \to \mathbb{R}$ with

$\forall x \in A: f(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$

It follows that

$$Lf = L\left(\sum_{k=1}^n \lambda_k y_k\right) = \sum_{k=1}^n L(\lambda_k y_k) = \sum_{k=1}^n \lambda_k L y_k =$$

$$= \sum_{k=1}^n \lambda_k \cdot \mathbf{0} = \mathbf{0} \Rightarrow f \in \text{null}(L).$$

We also know that

$$\forall p \in \{0\} \cup [n-1]: f^{(p)}(c) = \sum_{k=1}^n \lambda_k y_k^{(p)}(c) = \sum_{k=1}^n [W[y](c)]_{pk} \lambda_k =$$

$$= [W[y](c)\Lambda]_p = 0$$

We will now claim that given the initial condition

$f(c) = f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$

the function f will satisfy $\forall x \in A: f(x) = 0$. To show this, we rewrite the equation as a system of first-order ODEs by defining

$\forall k \in [n]: \forall x \in A: g_k(x) = f^{(k-1)}(x)$.

The ODE $Lf = \mathbf{0}$ can be rewritten as

$$\begin{cases} g_1'(x) = g_2(x) \\ g_2'(x) = g_3(x) \\ \vdots \\ g_{n-1}'(x) = g_n(x) \\ g_n'(x) = -\sum_{k=1}^n a_{k-1}(x) g_k(x) \end{cases}$$

and the corresponding initial condition is

$$g_1(x) = g_2(x) = \cdots = g_n(x) = 0$$

It is easy to see that all derivatives $g_1'(x), g_2'(x), \ldots, g_n'(x)$ are then zero, and therefore all functions $g_1, \ldots, g_n$ will remain constant and be equal to zero for all $x \in A$. This proves the claim. From the claim we have:

$$(\forall x \in A : f(x) = 0) \Rightarrow f = \mathbf{0} \Rightarrow \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = \mathbf{0} \quad (1)$$

By hypothesis, we also know that

$y_1, y_2, \ldots, y_n$ linearly independent (2)

From Eq.(1) and Eq.(2):

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \Rightarrow \lambda = \mathbf{0}$$

This is a contradiction, since by construction λ satisfies $\lambda \in \mathbb{R}^n - \{\mathbf{0}\}$. It follows that

$$\exists c \in A : w[y](c) \neq 0$$

Fix a $c \in A$ such that $w[y](c) \neq 0$. Then, from (6), it follows that

$$\forall x \in A : w[y](x) = w[y](c) \exp\left(-\int_c^x a_{n-1}(t)\,dt\right) \neq 0$$

because $\forall x \in \mathbb{R} : \exp(x) > 0$. This concludes the proof. $\square$

EXAMPLES

a) Consider the functions

$\forall x \in \mathbb{R} : \left(f(x) = x \wedge g(x) = x^2 \wedge h(x) = x^3 \right)$

Use the Wronskian to show that f, g, h are linearly independent

<u>Solution</u>

Since,

$$w[f,g,h](x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} (-x) =$$

$$= \begin{vmatrix} 0 & x^2 - 2x^2 & x^3 - 3x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = \begin{vmatrix} 0 & -x^2 & -2x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} =$$

$$= (-1) \begin{vmatrix} -x^2 & -2x^3 \\ 2 & 6x \end{vmatrix} = -\left[(-x^2) 6x - 2(-2x^3) \right] =$$

$$= -(-6x^3 + 4x^3) = -(-2x^3) = 2x^3, \quad \forall x \in \mathbb{R} \Rightarrow$$

$\Rightarrow w[f,g,h](1) = 2 \neq 0 \Rightarrow \exists x \in \mathbb{R} : w[f,g,h](x) \neq 0$

$\Rightarrow f, g, h$ linearly independent.

b) Show that for $\forall x \in \mathbb{R} : (f(x) = e^{2x} \wedge g(x) = xe^{2x})$ f, g are linearly independent.

Solution

(We use the Wronskian)

Since,

$\forall x \in \mathbb{R} : f'(x) = (e^{2x})' = e^{2x}(2x)' = 2e^{2x}$

$\forall x \in \mathbb{R} : g'(x) = (xe^{2x})' = (x)'e^{2x} + x(e^{2x})' = e^{2x} + xe^{2x}(2x)' =$

$= e^{2x} + xe^{2x} \cdot 2 = (1+2x)e^{2x}$

it follows that

$$\forall x \in \mathbb{R} : w[f, g](x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix} =$$

$$= e^{2x} \begin{vmatrix} e^{2x} & xe^{2x} \\ 2 & 1+2x \end{vmatrix} = e^{2x} e^{2x} \begin{vmatrix} 1 & x \\ 2 & 1+2x \end{vmatrix} =$$

$$= e^{4x}[1(1+2x) - 2x] = e^{4x}(1+2x-2x) = e^{4x} > 0$$

$\Rightarrow \forall x \in \mathbb{R} : w[f, g](x) \neq 0$

$\Rightarrow \exists x \in \mathbb{R} : w[f, g](x) \neq 0$

$\Rightarrow f, g$ linearly independent

EXERCISES

(2) Use the Wronskian to show that the functions f, g, h, defined below, are linearly independent.

a) $\begin{cases} \forall x \in \mathbb{R}: f(x) = e^{ax} \\ \forall x \in \mathbb{R}: g(x) = x e^{ax} \\ \forall x \in \mathbb{R}: h(x) = x^2 e^{ax} \end{cases}$ with $a \in \mathbb{R}$

b) $\begin{cases} \forall x \in \mathbb{R}: f(x) = \sin(ax) \\ \forall x \in \mathbb{R}: g(x) = \cos(ax) \\ \forall x \in \mathbb{R}: h(x) = x^2 \end{cases}$ with $a \in \mathbb{R}$

c) $\begin{cases} \forall x \in \mathbb{R}: f(x) = e^{ax} \cos x \\ \forall x \in \mathbb{R}: g(x) = e^{ax} \sin x \\ \forall x \in \mathbb{R}: h(x) = e^{ax} \end{cases}$ with $a \in \mathbb{R}$

d) $\begin{cases} \forall x \in \mathbb{R}: f(x) = x^2 \\ \forall x \in \mathbb{R}: g(x) = x^2 \ln x \\ \forall x \in \mathbb{R}: h(x) = x^2 [\ln x]^2 \end{cases}$

e) $\begin{cases} \forall x \in \mathbb{R}: f(x) = x^3 \cos(2 \ln x) \\ \forall x \in \mathbb{R}: g(x) = x^3 \sin(2 \ln x) \\ \forall x \in \mathbb{R}: h(x) = x^2 \end{cases}$

(3) Consider a general linear differential equation of the form
$\forall x \in A: y''(x) + a_1(x) y'(x) + a_0(x) y(x) = 0$
for some interval $A \subseteq \mathbb{R}$ with $a_0, a_1 \in C^0(A)$. Assume that $y_1 \in C^2(A)$ is a solution, and define $y_2 \in C^2(A)$ as:

$$\forall x \in A: \quad y_2(x) = y_1(x) \int_c^x \frac{Q(t)}{[y_1(t)]^2}\, dt$$

with $c \in A$ and with $Q(t)$ given by

$$\forall t \in A: \quad Q(t) = \exp\left(- \int a_1(t)\, dt \right)$$

a) Show that $y_2(x)$ is also a solution.
(Hint: start with $y_2(x) = y_1(x)u(x)$ and substitute to the ODE to derive a sufficient condition for $u(x)$)

b) Show that y_1, y_2 are linearly independent.
(Hint: Use the Wronskian)

↳ Note that an immediate consequence of (a) and (b) is that if we define an operator $L: C^2(A) \to C^0(A)$ with $Ly = y'' + a_1 y' + a_0 y$, then it follows that its null space is
$\text{null}(L) = \text{span}\{y_1, y_2\}$
The corresponding general solution of the equation $Ly = 0$ is given by
$\forall x \in A: \; y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$
This exercise shows that if we can guess one solution of the 2nd-order linear ODE $Ly = 0$, we have an equation that can be used to find a second linearly independent solution. Then given the aforementioned theorems, we have the null space and the general solution.

(4) Find a solution of the form $\forall x \in \mathbb{R}: y_1(x) = e^{bx}$ for the linear ODE:
$\forall x \in \mathbb{R}: \; y''(x) + 2ay'(x) + a^2 y(x) = 0$
with $a \in \mathbb{R}$. Use exercise 2 to find the second solution and write the general solution.

(5) Find a solution of the form $\forall x \in (0,+\infty): y_1(x) = x^b$ for the linear ODE
$\forall x \in (0,+\infty): x^2 y'' + (2\lambda+1) x y' + \lambda^2 y = 0$
with $\lambda \in \mathbb{R}$. Use exercise **3** to find the second solution and write the general solution.

Solving homogeneous linear differential equations

To solve a homogeneous linear differential equation
$y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = 0$
we need to find the linearly independent solutions
$y_1(x), y_2(x), \ldots, y_n(x)$ that form the general solution
$$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x)$$
There is no general method for finding the functions $y_1(x), \ldots, y_n(x)$. However, an exact solution is possible for the following cases.

(1) → Constant coefficient case

Consider the linear ODE
$y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0$
with $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ given constants. Let L be the corresponding operator.

Solution method

- 1 Find the characteristic polynomial $P(b)$:
$$\begin{aligned} L(e^{bx}) &= (b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0) e^{bx} \\ &= P(b) e^{bx} \end{aligned}$$
- 2 Let $\rho_1, \rho_2, \ldots, \rho_n \in \mathbb{C}$ be the zeroes of the characteristic polynomial P. Then:
  a) Each single zero ρ_k contributes a solution
  $$y_k(x) = \exp(\rho_k x)$$

b) Each zero ρ_k with multiplicity m (i.e. $P(b)$ has a factor $(x-\rho_k)^m$) contributes the following linearly independent solutions:

$$y_k(x) = \exp(\rho_k x)$$
$$y_{k+1}(x) = x\exp(\rho_k x)$$
$$y_{k+2}(x) = x^2\exp(\rho_k x)$$
$$\vdots$$
$$y_{k+m-1}(x) = x^{m-1}\exp(\rho_k x).$$

• 3 We write the general solution and apply the initial conditions if given.

↳ Remark: Complex zeroes appear as complex conjugate pairs $\rho_k = \gamma + i\omega$ and $\rho_{k+1} = \gamma - i\omega$, because the coefficients of the characteristic polynomial are real numbers. We use the De Moivre identity:

$$\forall \vartheta \in \mathbb{R}: \ e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$$

and note that the corresponding solutions satisfy:

$$y_k(x) = \exp(\rho_k x) = \exp((\gamma+i\omega)x) = \exp(\gamma x + i\omega x) =$$
$$= \exp(\gamma x)\exp(i\omega x) = e^{\gamma x}(\cos(\omega x) + i\sin(\omega x))$$
$$y_{k+1}(x) = \exp(\rho_{k+1}x) = \exp((\gamma - i\omega)x) = \exp(\gamma x - i\omega x) =$$
$$= \exp(\gamma x)\exp(-i\omega x) = e^{\gamma x}(\cos(-\omega x) + i\sin(-\omega x)) =$$
$$= e^{\gamma x}(\cos(\omega x) - i\sin(\omega x))$$

It follows that any linear combination of $y_k(x)$ and $y_{k+1}(x)$ can be rewritten as:

$$\lambda_k y_k(x) + \lambda_{k+1} y_{k+1}(x) =$$
$$= \lambda_k e^{\gamma x}(\cos(\omega x) + i\sin(\omega x)) + \lambda_{k+1} e^{\gamma x}(\cos(\omega x) - i\sin(\omega x)) =$$
$$= e^{\gamma x}\left[(\lambda_k + \lambda_{k+1})\cos(\omega x) + i(\lambda_k - \lambda_{k+1})\sin(\omega x)\right] =$$
$$= (\lambda_k + \lambda_{k+1})\left[e^{\gamma x}\cos(\omega x)\right] + i(\lambda_k - \lambda_{k+1})\left[e^{\gamma x}\sin(\omega x)\right]$$
$$= \mu_k e^{\gamma x}\cos(\omega x) + \mu_{k+1} e^{\gamma x}\sin(\omega x)$$

with

$$\begin{cases} \mu_k = \lambda_k + \lambda_{k+1} \\ \mu_{k+1} = i(\lambda_k - \lambda_{k+1}) \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} \lambda_k \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_k \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}\begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} =$$
$$= \frac{1}{-i-i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}\begin{bmatrix} \mu_k \\ \mu_{k+1} \end{bmatrix} \Leftrightarrow$$
$$\Leftrightarrow \lambda_k = \frac{-i\mu_k - \mu_{k+1}}{-2i} = \frac{\mu_{k+1} + i\mu_k}{2i} \;\wedge$$
$$\wedge\; \lambda_{k+1} = \frac{-i\mu_k + \mu_{k+1}}{-2i} = \frac{-\mu_{k+1} + i\mu_k}{2i}$$

It follows that an equivalent set of solutions are

$$z_k(x) = e^{\gamma x}\cos(\omega x)$$
$$z_{k+1}(x) = e^{\gamma x}\sin(\omega x)$$

In general: given complex conjugate zeroes $\gamma + i\omega$ and $\gamma - i\omega$ with multiplicity m, it is best practice to use the following set of linearly independent solutions:

$$y_k(x) = e^{\gamma x}\cos(\omega x), \quad y_{k+2} = x e^{\gamma x}\cos(\omega x), \quad \ldots,$$
$$y_{k+1}(x) = e^{\gamma x}\sin(\omega x) \quad y_{k+3} = x e^{\gamma x}\sin(\omega x)$$

$$y_{k+2m-2}(x) = x^{m-1} e^{\gamma x} \cos(\omega x)$$
$$y_{k+2m-1}(x) = x^{m-1} e^{\gamma x} \sin(\omega x).$$

EXAMPLE

a) Write the general solution to $y'''(x) - 2y'(x) = 0$.

Solution

Define $Ly(x) = y'''(x) - 2y'(x)$ and note that

$$L(e^{bx}) = (e^{bx})''' - 2(e^{bx})' = b^3 e^{bx} - 2b e^{bx} =$$
$$= (b^3 - 2b) e^{bx} = b(b^2 - 2) e^{bx} = b(b-\sqrt{2})(b+\sqrt{2}) e^{bx}$$

The characteristic polynomial $P(b) = b(b-\sqrt{2})(b+\sqrt{2})$ has zeroes: $0, \sqrt{2}, -\sqrt{2}$ and therefore

$$y(x) = \lambda_1 e^{0x} + \lambda_2 e^{\sqrt{2}x} + \lambda_3 e^{-\sqrt{2}x} =$$
$$= \lambda_1 + \lambda_2 e^{x\sqrt{2}} + \lambda_3 e^{-x\sqrt{2}}$$

b) Solve the initial value problem

$$\begin{cases} y''(x) - 8y'(x) + 16y(x) = 0 \\ y(0) = 1 \wedge y'(0) = 3 \end{cases}$$

Solution

Define $Ly(x) = y''(x) - 8y'(x) + 16y(x)$ and note that

$$L(e^{bx}) = (e^{bx})'' - 8(e^{bx})' + 16e^{bx} =$$
$$= b^2 e^{bx} - 8b e^{bx} + 16 e^{bx} = (b^2 - 8b + 16) e^{bx}$$
$$= (b-4)^2 e^{bx}$$

The characteristic polynomial $P(b) = (b-4)^2$ has zeroes: $4, 4$ and therefore:

$$y(x) = \lambda_1 e^{4x} + \lambda_2 x e^{4x}$$

To apply the initial condition, we note that

$$y'(x) = \lambda_1 (e^{4x})' + \lambda_2 (xe^{4x})' = 4\lambda_1 e^{4x} + \lambda_2 (e^{4x} + 4xe^{4x})$$
$$= (4\lambda_1 + \lambda_2) e^{4x} + 4\lambda_2 x e^{4x}$$

and therefore

$$\begin{cases} y(0)=1 \\ y'(0)=3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 e^0 + \lambda_2 e^0 \cdot 0 = 1 \\ (4\lambda_1 + \lambda_2) e^0 + 4\lambda_2 \cdot 0 e^0 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + 0\lambda_2 = 1 \\ 4\lambda_1 + \lambda_2 = 3 \end{cases}$$
$$\Leftrightarrow \begin{cases} \lambda_1 = 1 \\ 4 \cdot 1 + \lambda_2 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3-4 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

It follows that the solution is

$$y(x) = e^{4x} - xe^{4x} = (1-x)e^{4x}.$$

c) Linear Oscillator problem:

Solve the initial value problem

$$\begin{cases} y''(x) + \omega^2 y(x) = 0 \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

Solution

Define $Ly(x) = y''(x) + \omega^2 y(x)$ and note that

$$L(e^{bx}) = (e^{bx})'' + \omega^2 e^{bx} = b^2 e^{bx} + \omega^2 e^{bx} = (b^2 + \omega^2) e^{bx}$$
$$= (b + i\omega)(b - i\omega) e^{bx}$$

The characteristic polynomial $P(b) = (b+i\omega)(b-i\omega)$ has zeroes $+i\omega, -i\omega$. It follows that

$$y(x) = \lambda_1 e^{0x} \cos(\omega x) + \lambda_2 e^{0x} \sin(\omega x) =$$
$$= \lambda_1 \cos(\omega x) + \lambda_2 \sin(\omega x)$$

To apply the initial conditions, we calculate:

$$y'(x) = \lambda_1(-w\sin(wx)) + \lambda_2(w\cos(wx)) =$$
$$= -w\lambda_1 \sin(wx) + w\lambda_2 \cos(wx)$$

and therefore

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \iff \begin{cases} \lambda_1 \cos 0 + \lambda_2 \sin 0 = y_0 \\ -w\lambda_1 \sin 0 + w\lambda_2 \cos 0 = y_1 \end{cases} \iff$$

$$\iff \begin{cases} \lambda_1 + 0\lambda_2 = y_0 \\ -0\lambda_1 + w\lambda_2 = y_1 \end{cases} \iff \begin{cases} \lambda_1 = y_0 \\ w\lambda_2 = y_1 \end{cases} \iff$$

$$\iff \begin{cases} \lambda_1 = y_0 \\ \lambda_2 = y_1/w \end{cases}$$

It follows that the solution is:

$$y(x) = y_0 \cos(wx) + (y_1/w)\sin(wx).$$

EXERCISES

(6) Find the general solution for the following linear differential equations

a) $y'''(x) - 5y''(x) + 6y'(x) = 0$

b) $y'''(x) - y'(x) = 0$

c) $y'''(x) - y(x) = 0$

d) $y''(x) + y'(x) + y(x) = 0$

e) $y^{(4)}(x) - 16y(x) = 0$

f) $y'''(x) - 4y'(x) + 3y(x) = 0$

g) $y^{(4)}(x) + 2y''(x) + y(x) = 0$

(7) Show that the initial value problem

$$\begin{cases} y''(x) - 2(p+a)y'(x) + p^2 y(x) = 0 \\ y(0) = 0 \wedge y'(0) = 1 \end{cases}$$

with $a, p \in (0, +\infty)$ has solution

$$y(x|a,p) = \frac{\exp(A(p,a)x) - \exp(B(p,a)x)}{2\sqrt{a(2p+a)}}$$

with $A(p,a) = p + a + \sqrt{a(2p+a)}$

$B(p,a) = p + a - \sqrt{a(2p+a)}$

<u>without</u> substituting the solution to the ODE. Then show that $\lim\limits_{a \to 0} y(x|a,p) = x e^{px}$

(8) Solve the following initial value problems; with $\mu \in (0, +\infty)$

a) $\begin{cases} y'''(x) - \mu y''(x) + \mu^2 y'(x) - \mu^3 y(x) = 0 \\ y(0) = 0 \wedge y'(0) = 0 \wedge y''(0) = 1 \end{cases}$

b) $\begin{cases} y'''(x) - \mu^2 y'(x) = 0 \\ y(0) = y'(0) = 0 \wedge y''(0) = 1 \end{cases}$

c) $\begin{cases} y^{(4)}(x) - \mu y(x) = 0 \\ y(0) = y'(0) = y''(0) = 0 \wedge y'''(0) = 1 \end{cases}$

2 → Equidimensional case (Euler-Cauchy equation)

Consider the linear ODE:

$$x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y(x) = 0$$

with $a_0, a_1, a_2, \ldots, a_{n-1} \in \mathbb{R}$ given constants. Let L be the corresponding operator.

Solution method

- 1 We evaluate the characteristic polynomial P from:

$$L(x^b) = P(b) x^b$$

- 2 Let $p_1, p_2, \ldots, p_n \in \mathbb{C}$ be the zeroes of $P(b)$. Then
  (a) If p_k is a single zero, it contributes a solution

$$y_k(x) = x^{p_k}.$$

  (b) If p_k is a zero with multiplicity m, it contributes the following linearly independent solutions.

$$y_k(x) = x^{p_k}$$
$$y_{k+1}(x) = x^{p_k} \ln x$$
$$y_{k+2}(x) = x^{p_k} [\ln x]^2$$
$$\vdots$$
$$y_{k+m-1}(x) = x^{p_k} [\ln x]^{m-1}$$

  (c) Given a complex conjugate pair $p_k = \gamma + i\omega$ and $p_{k+1} = \gamma - i\omega$, from (a) we obtain (see remark below) the following linearly independent solutions:

$$y_k(x) = x^\gamma \cos(\omega \ln x)$$
$$y_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$

(d) Given a complex conjugate pair $\rho_k = \gamma + i\omega$ and $\rho_k = \gamma - i\omega$ of multiplicity m, from (6), we obtain the following linearly independent solutions:

$$y_k(x) = x^\gamma \cos(\omega \ln x)$$
$$y_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$
$$y_{k+2}(x) = x^\gamma \cos(\omega \ln x) \ln x$$
$$y_{k+3}(x) = x^\gamma \cos(\omega \ln x) \ln x$$
$$\vdots$$
$$y_{k+2m-2}(x) = x^\gamma \cos(\omega \ln x)[\ln x]^{m-1}$$
$$y_{k+2m-1}(x) = x^\gamma \sin(\omega \ln x)[\ln x]^{m-1}$$

•3 We write the general solution and apply the initial conditions, if given.

↳ Remark: For the case of a single pair of complex conjugate zeroes $\rho_k = \gamma + i\omega$ and $\rho_{k+1} = \gamma - i\omega$, we have the following contributed solutions:

$$y_k(x) = x^{\rho_k} = x^{\gamma + i\omega} = \exp((\gamma + i\omega)\ln x) = \exp(\gamma \ln x)\exp(i\omega \ln x) =$$
$$= x^\gamma[\cos(\omega \ln x) + i\sin(\omega \ln x)]$$

and similarly:

$$y_{k+1}(x) = x^{\rho_{k+1}} = x^{\gamma - i\omega} = x^\gamma[\cos(\omega \ln x) - i\sin(\omega \ln x)]$$

Via an argument similar to that of case 1, we obtain the following alternate linearly independent solutions:

$$z_k(x) = x^\gamma \cos(\omega \ln x)$$
$$z_{k+1}(x) = x^\gamma \sin(\omega \ln x)$$

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} x^2 y''(x) + x y'(x) + 4y(x) = 0 \\ y(2) = p \wedge y'(2) = q \end{cases}$$

Solution

Define $Ly(x) = x^2 y''(x) + x y'(x) + 4y(x)$. It follows that

$$\begin{aligned} L(x^b) &= x^2 (x^b)'' + x (x^b)' + 4x^b = \\ &= x^2 b(b-1) x^{b-2} + x\, b x^{b-1} + 4x^b = \\ &= b(b-1) x^b + b x^b + 4x^b = [b(b-1) + b + 4] x^b = \\ &= (b^2 - b + b + 4) x^b = (b^2 + 4) x^b = (b+2i)(b-2i) x^b \end{aligned}$$

which gives the characteristic polynomial

$$P(b) = (b+2i)(b-2i)$$

with zeroes $\rho_1 = 2i$ and $\rho_2 = -2i$. It follows that the general solution reads

$$y(x) = \lambda_1 \cos(2\ln x) + \lambda_2 \sin(2\ln x)$$

To apply the initial condition we note that

$$y(2) = \lambda_1 \cos(2\ln 2) + \lambda_2 \sin(2\ln 2)$$

and

$$\begin{aligned} y'(x) &= \lambda_1 [\cos(2\ln x)]' + \lambda_2 [\sin(2\ln x)]' = \\ &= \lambda_1 [-\sin(2\ln x)] (2\ln x)' + \lambda_2 [\cos(2\ln x)] (2\ln x)' = \\ &= (2/x)[-\lambda_1 \sin(2\ln x) + \lambda_2 \cos(2\ln x)] \Rightarrow \end{aligned}$$

$$\begin{aligned} \Rightarrow y'(2) &= (2/2)[-\lambda_1 \sin(2\ln 2) + \lambda_2 \cos(2\ln 2)] = \\ &= -\lambda_1 \sin(2\ln 2) + \lambda_2 \cos(2\ln 2) \end{aligned}$$

and it follows that:

$$\begin{cases} y(2) = p \\ y'(2) = q \end{cases} \Leftrightarrow \begin{cases} \lambda_1 \cos(2\ln 2) + \lambda_2 \sin(2\ln 2) = p \\ -\lambda_1 \sin(2\ln 2) + \lambda_2 \cos(2\ln 2) = q \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \cos(2\ln 2) & \sin(2\ln 2) \\ -\sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \cos(2\ln 2) & \sin(2\ln 2) \\ -\sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} =$$

$$= \frac{1}{\cos^2(2\ln 2) + \sin^2(2\ln 2)} \begin{bmatrix} \cos(2\ln 2) & -\sin(2\ln 2) \\ \sin(2\ln 2) & \cos(2\ln 2) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$= \frac{1}{1} \begin{bmatrix} p\cos(2\ln 2) - q\sin(2\ln 2) \\ p\sin(2\ln 2) + q\cos(2\ln 2) \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda_1 = p\cos(2\ln 2) - q\sin(2\ln 2) \\ \lambda_2 = p\sin(2\ln 2) + q\cos(2\ln 2) \end{cases}$$

Thus, the solution reads

$$y(x) = [p\cos(2\ln 2) - q\sin(2\ln 2)]\cos(2\ln x) + [p\sin(2\ln 2) + q\cos(2\ln 2)]\sin(2\ln x)$$

$$= p[\cos(2\ln 2)\cos(2\ln x) + \sin(2\ln 2)\sin(2\ln x)] +$$

$$+ q[-\sin(2\ln 2)\cos(2\ln x) + \sin(2\ln x)\cos(2\ln 2)] =$$

$$= p\cos(2\ln x - 2\ln 2) + q\sin(2\ln x - 2\ln 2)$$

b) Solve the initial value problem

$$\begin{cases} 4x^2y''(x) + 8xy'(x) + y(x) = 0 \\ y(3) = p \wedge y'(3) = q \end{cases}$$

Solution

Define $Ly(x) = 4x^2y''(x) + 8xy'(x) + y(x)$. It follows that

$$\begin{aligned} L(x^b) &= 4x^2(x^b)'' + 8x(x^b)' + x^b = \\ &= 4x^2 b(b-1)x^{b-2} + 8x\, b x^{b-1} + x^b = \\ &= 4b(b-1)x^b + 8bx^b + x^b = [4b(b-1) + 8b + 1]x^b = \\ &= (4b^2 - 4b + 8b + 1)x^b = (4b^2 + 4b + 1)x^b = (2b+1)^2 x^b \end{aligned}$$

which gives the characteristic polynomial $P(b) = (2b+1)^2$ with a double zero $\rho = -1/2$. Thus, the general solution reads:

$$y(x) = \lambda_1 x^{-1/2} + \lambda_2 x^{-1/2}\ln x = \frac{\lambda_1 + \lambda_2 \ln x}{\sqrt{x}}$$

To apply the initial condition, we note that

$$y(3) = \frac{\lambda_1 + \lambda_2 \ln 3}{\sqrt{3}}$$

and

$$\begin{aligned} y'(x) &= \frac{(\lambda_1 + \lambda_2 \ln x)'\sqrt{x} - (\lambda_1 + \lambda_2 \ln x)(\sqrt{x})'}{(\sqrt{x})^2} = \\ &= \frac{1}{x}\left[\lambda_2 \frac{1}{x}\sqrt{x} - \frac{\lambda_1 + \lambda_2 \ln x}{2\sqrt{x}}\right] = \\ &= \frac{1}{x}\left[\lambda_2 \frac{1}{\sqrt{x}} - \frac{\lambda_1 + \lambda_2 \ln x}{2\sqrt{x}}\right] = \\ &= \frac{1}{2x\sqrt{x}}\left[2\lambda_2 - (\lambda_1 + \lambda_2 \ln x)\right] = \frac{(2\lambda_2 - \lambda_1) - \lambda_2 \ln x}{2x\sqrt{x}} \end{aligned}$$

$$\Rightarrow y'(3) = \frac{(2\lambda_2 - \lambda_1) - \lambda_2 \ln 3}{2\cdot 3\sqrt{3}} = \frac{-\lambda_1 + (2-\ln 3)\lambda_2}{6\sqrt{3}}$$

and therefore

$$\begin{cases} y(3) = p \\ y'(3) = q \end{cases} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 \ln 3 = p\sqrt{3} \\ -\lambda_1 + (2-\ln 3)\lambda_2 = 6q\sqrt{3} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & \ln 3 \\ -1 & 2-\ln 3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & \ln 3 \\ -1 & 2-\ln 3 \end{bmatrix}^{-1} \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{(2-\ln 3)+\ln 3} \begin{bmatrix} 2-\ln 3 & -\ln 3 \\ +1 & 1 \end{bmatrix} \begin{bmatrix} p\sqrt{3} \\ 6q\sqrt{3} \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} (2-\ln 3)p\sqrt{3} - 6q\sqrt{3}\ln 3 \\ p\sqrt{3} + 6q\sqrt{3} \end{bmatrix}$$

It follows that the solution to the initial value problem is:

$$y(x) = \frac{\lambda_1 + \lambda_2 \ln x}{\sqrt{x}} =$$

$$= \frac{1}{2\sqrt{x}} \left[(2-\ln 3)p\sqrt{3} - 6q\sqrt{3}\ln 3 + (p\sqrt{3} + 6q\sqrt{3})\ln x \right] =$$

$$= \frac{1}{2\sqrt{x}} \left[p\sqrt{3}(2 - \ln 3 + \ln x) + 6q\sqrt{3}(\ln x - \ln 3) \right]$$

c) Solve the initial value problem

$$\begin{cases} x^3y'''(x) - xy'(x) - 3y(x) = 0 \\ y(1) = 0 \wedge y'(1) = 0 \wedge y''(1) = p \end{cases}$$

Solution

Define $Ly(x) = x^3y'''(x) - xy'(x) - 3y(x)$. It follows that

$$\begin{aligned} L(x^b) &= x^3(x^b)''' - x(x^b)' - 3x^b = \\ &= x^3 b(b-1)(b-2)x^{b-3} - x(bx^{b-1}) - 3x^b = \\ &= b(b-1)(b-2)x^b - bx^b - 3x^b = [b(b-1)(b-2) - b - 3]x^b = \\ &= [b(b^2-3b+2) - b - 3]x^b = (b^3-3b^2+2b-b-3)x^b \\ &= (b^3-3b^2+b-3)x^b \end{aligned}$$

and therefore the characteristic polynomial is:

$$P(b) = b^3-3b^2+b-3 = b^2(b-3)+(b-3) = (b-3)(b^2+1)$$

with zeroes $\rho_1 = 3$; $\rho_2 = i$, and $\rho_3 = -i$. Thus, the general solution is given by:

$$y(x) = \lambda_1 x^3 + \lambda_2 \cos(\ln x) + \lambda_3 \sin(\ln x).$$

To apply the initial condition, we note that

$$\begin{aligned} y(1) &= \lambda_1 \cdot 1^3 + \lambda_2 \cos(\ln 1) + \lambda_3 \sin(\ln 1) = \\ &= \lambda_1 + \lambda_2 \cos 0 + \lambda_3 \sin 0 = \lambda_1 + \lambda_2 \end{aligned}$$

and

$$\begin{aligned} y'(x) &= 3\lambda_1 x^2 + \lambda_2 [\cos(\ln x)]' + \lambda_3 [\sin(\ln x)]' = \\ &= 3\lambda_1 x^2 + \lambda_2[-\sin(\ln x)](\ln x)' + \lambda_3[\cos(\ln x)](\ln x)' \\ &= 3\lambda_1 x^2 + \frac{-\lambda_2 \sin(\ln x) + \lambda_3 \cos(\ln x)}{x} \implies \end{aligned}$$

$$\implies y'(1) = 3\lambda_1 \cdot 1^2 + \frac{-\lambda_2 \sin(\ln 1) + \lambda_3 \cos(\ln 1)}{1} =$$

$$= 3\lambda_1 - \lambda_2 \sin 0 + \lambda_3 \cos 0 = 3\lambda_1 - 0\lambda_2 + \lambda_3.$$

and

$$y''(x) = 6\lambda_1 x + \frac{d}{dx}\left[\frac{-\sin(\ln x)}{x}\right]\lambda_2 + \frac{d}{dx}\left[\frac{\cos(\ln x)}{x}\right]\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{-\left[(\sin(\ln x))' x - \sin(\ln x)(x)'\right]}{x^2}\lambda_2$$

$$+ \frac{[\cos(\ln x)]' x - \cos(\ln x)(x)'}{x^2}\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{-\left[\cos(\ln x)(\ln x)' x - \sin(\ln x)\right]}{x^2}\lambda_2$$

$$+ \frac{-\sin(\ln x)(\ln x)' x - \cos(\ln x)}{x^2}\lambda_3 =$$

$$= (6x)\lambda_1 + \frac{\sin(\ln x) - \cos(\ln x)}{x^2}\lambda_2 + \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}\lambda_3$$

$$\Rightarrow y''(1) = 6\lambda_1 + [\sin(\ln 1) - \cos(\ln 1)]\lambda_2 - [\sin(\ln 1) + \cos(\ln 1)]\lambda_3 =$$

$$= 6\lambda_1 + (\sin 0 - \cos 0)\lambda_2 - (\sin 0 + \cos 0)\lambda_3 =$$

$$= 6\lambda_1 - \lambda_2 - \lambda_3$$

and therefore:

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ 3\lambda_1 + \lambda_3 = 0 \\ 6\lambda_1 - \lambda_2 - \lambda_3 = p \end{cases} \iff \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

Apply Cramer rule:

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 6 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 9 & -1 & 0 \\ 6 & -1 & -1 \end{vmatrix} = +(-1)\begin{vmatrix} 1 & 1 \\ 9 & -1 \end{vmatrix} = -(-1-9) = 10$$

$$D_1 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & -1 & -1 \end{vmatrix} = p \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = p$$

$$D_2 = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 6 & p & -1 \end{vmatrix} = (-1)p \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = -p(1\cdot 1 - 0\cdot 3) = -p$$

$$D_3 = \begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 6 & -1 & p \end{vmatrix} = (+1)p \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = p(1\cdot 0 - 1\cdot 3) = -3p$$

thus

$$\lambda_1 = \frac{D_1}{D} = \frac{p}{10}$$

$$\lambda_2 = \frac{D_2}{D} = \frac{-p}{10}$$

$$\lambda_3 = \frac{D_3}{D} = \frac{-3p}{10}$$

and the solution reads

$$y(x) = \frac{px^3}{10} - \frac{p\cos(\ln x)}{10} - \frac{3p\sin(\ln x)}{10}$$

$$= (p/10)\left[x^3 - \cos(\ln x) - 3\sin(\ln x)\right]$$

EXERCISES

(9) Solve the following linear differential equations on the interval $(0,+\infty)$ using initial conditions $y(1)=y_0 \wedge y'(1)=y_1$.

a) $x^2y''(x)-2xy'(x)+2y(x)=0$

b) $x^2y''(x)-2y(x)=0$

c) $x^2y''(x)-xy'(x)+y(x)=0$

(10) Similarly, solve the following linear differential equations on the interval $(0,+\infty)$ using initial conditions $y(1)=y_0 \wedge y'(1)=y_1 \wedge y''(1)=y_2$:

a) $x^3y'''(x)-6y(x)=0$

b) $xy'''(x)+y''(x)=0$

d) $x^3y'''(x)+3x^2y''(x)+xy'(x)-8y(x)=0$

e) $x^3y'''(x)+3x^2y''(x)-6y(x)=0$

(11) Consider the linear differential equation
$ax^2y''(x)+bxy'(x)+cy(x)=0$
with $a,b,c\in\mathbb{R}$, and let Δ be the discriminant of the equation's characteristic polynomial $p(x)=Ax^2+Bx+C$.
Show that $\Delta=B^2-4AC=a^2+b^2-2a(b+2c)$.

(12) Show that the linear differential equation
$ax^3y'''(x)+(b+3a)x^2y''(x)+(a+b+c)xy'(x)+dy(x)=0$
with $a,b,c,d\in\mathbb{R}$ has characteristic polynomial
$p(x)=ax^3+bx^2+cx+d$.

Solving inhomogeneous linear differential equations

We will now consider the general problem of the linear inhomogeneous linear differential equation of the form
$\forall x \in A: y^{(n)}(x) + a_{n-1}(x) y^{(n-1)}(x) + \cdots + a_1(x) y'(x) + a_0(x) y(x) = f(x)$ (1)
with $a_0, a_1, a_2, \ldots, a_{n-1}, f \in C^0(A)$. The general method is as follows:

1) Given the solutions $y_1, \ldots, y_n$ of the homogeneous equation and at least one solution y_p of the inhomogeneous equation we show that the general solution of Eq. (1) is:
$y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) + y_p(x)$
2) Given $y_1, y_2, \ldots, y_n$ there is a general result that gives the solution y_p.

<u>Terminology</u>: The terms $\lambda_1 y_1 + \cdots + \lambda_n y_n$ are the homogeneous solution and y_p are the particular solution to the problem.

We now give the details of the theory:

<u>Thm</u>: Consider the linear operator $L: C^n(A) \to C^0(A)$ for some interval $A \subseteq \mathbb{R}$ such that $Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$ with $a_0, a_1, \ldots, a_n \in C^0(A)$. Let $f \in C^0(A)$, and assume that
(a) $\operatorname{null}(L) = \operatorname{span}\{y_1, y_2, \ldots, y_n\}$ with $y_1, y_2, \ldots, y_n \in C^n(A)$.
(b) $Ly_p = f$
Then: $Ly = f \Leftrightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$

Proof

($\Rightarrow$): Assume that $Ly=f$. Then it follows that

$L(y-y_p) = Ly - Ly_p = f - f = 0 \Rightarrow (y-y_p) \in \text{null}(L) \Rightarrow$

$\Rightarrow y - y_p \in \text{span}\{y_1, y_2, \ldots, y_n\} \Rightarrow$

$\Rightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: y - y_p = \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$

$\Rightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$

($\Leftarrow$): Assume that: $\exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}: y = y_p + \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n$

Then, it follows that:

$$Ly = L\left(y_p + \sum_{k=1}^{n} \lambda_k y_k\right) = Ly_p + L\left(\sum_{k=1}^{n} \lambda_k y_k\right) = f + \sum_{k=1}^{n} L(\lambda_k y_k) =$$

$$= f + \sum_{k=1}^{n} \lambda_k (Ly_k) = f + \sum_{k=1}^{n} \lambda_k 0 = f + 0 = f. \qquad \square$$

$\underline{\text{Thm}}$: Let $L: C^n(A) \to C^0(A)$, with $A \subseteq \mathbb{R}$ an interval, be a linear operator defined as:

$\forall y \in C^n(A): Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$

with $a_0, a_1, \ldots, a_{n-1} \in C^0(A)$, and let $f \in C^0(A)$. Assume that $\text{null}(L) = \text{span}\{y_1, y_2, \ldots, y_n\}$. Then, the inhomogeneous ODE $Ly = f$ has a particular solution $y_p \in C^n(A)$ such that $Ly_p = f$ given by:

$$\forall x \in A: \quad y_p(x) = \int_A G(x,t) f(t)\, dt$$

with $\forall x, t \in A: G(x,t) = \begin{cases} \sum_{k=1}^{n} B_k(t) y_k(x) & , \text{ if } x \geq t \\ 0 & , \text{ if } x < t \end{cases}$

where $B_1(t), B_2(t), \ldots, B_n(t)$ is the unique solution of the system

$W[y_1, y_2, \ldots, y_n](t) (B_1(t), B_2(t), \ldots, B_n(t)) = (0, 0, \ldots, 0, 1)$

Remarks :

a) The proof of this theorem is based on generalized functions and will be given later.

b) An alternative proof is to substitute the solution $y_p \in C^n(\mathbb{R})$ to the equation $Ly_p = f$ and confirm that the solution satisfies the equation. This method is known as "variation of parameters".

c) The function $G(x,t)$ is called the Green's function. It captures the effect of the value of the forcing function f at t to the solution y_p at x. The Green's function is not unique, but can be made unique if we introduce the assumption that $G(x,t) = 0$ for $x<t$. This is known as the causality assumption that "the future value $f(t)$ should not have an effect on the past solution $y_p(x)$".

↳ Special case: 2nd-order linear ODE on $A=[c,d]$

Consider the 2nd-order linear ODE of the form

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x). \quad \text{with } a_0, a_1, f \in C^0(A)$$

Given two linearly independent solutions $y_1, y_2 \in C^2(A)$ such that

$$\begin{cases} y_1''(x) + \alpha_1(x)y_1'(x) + a_0(x)y_1(x) = 0 \\ y_2''(x) + \alpha_1(x)y_2'(x) + a_0(x)y_2(x) = 0 \end{cases}$$

a corresponding particular solution $y_p \in C^2(A)$ is given by

$$y_p(x) = -y_1(x)\int_c^x \frac{f(t)y_2(t)}{W(t)}\,dt + y_2(x)\int_c^x \frac{f(t)y_1(t)}{W(t)}\,dt$$

with $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

Proof

The Green's function is given by

$$G(x,t) = \begin{cases} B_1(t)y_1(x) + B_2(t)y_2(x) & , \text{ if } x \geq t \\ 0 & , \text{ if } x < t \end{cases}$$

with $B_1(t), B_2(t)$ given by:

$$W[y_1, y_2](t)\,(B_1(t), B_2(t)) = (0,1) \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{W(t)} \begin{bmatrix} -y_2(t) \\ y_1(t) \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow B_1(t) = \frac{-y_2(t)}{W(t)} \wedge B_2(t) = \frac{y_1(t)}{W(t)}.$$

and therefore, a particular solution is:

$$y_p(x) = \int_c^d G(x,t)f(t)\,dt = \int_c^x [B_1(t)y_1(x) + B_2(t)y_2(x)]\,f(t)\,dt =$$

$$= y_1(x)\int_c^x B_1(t)f(t)\,dt + y_2(x)\int_c^x B_2(t)f(t)\,dt =$$

$$= -y_1(x)\int_c^x \frac{f(t)y_2(t)}{W(t)}\,dt + y_2(x)\int_c^x \frac{f(t)y_1(t)}{W(t)}$$

Note that the lower limit $-\infty$ can be replaced with any constant c. Then the $(-\infty, c)$ integrals gives a contribution that can be moved to the homogeneous solution.

EXAMPLES

a) Solve the initial-value problem:

$$\begin{cases} y''(x) - 2y'(x) + y(x) = (3x+2)e^x \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

Solution

Define $\forall y \in C^2(\mathbb{R}) : Ly = y'' - 2y' + y$, and note that

$$Le^{bx} = (e^{bx})'' - 2(e^{bx})' + e^{bx} = b^2 e^{bx} - 2be^{bx} + e^{bx} =$$
$$= (b^2 - 2b + 1)e^{bx} = (b-1)^2 e^{bx}.$$

The characteristic polynomial $P(b) = (b-1)^2$ has a double zero $b = 1$, therefore $null(L) = span\{y_1, y_2\}$ with $\forall x \in \mathbb{R} : (y_1(x) = e^x \wedge y_2(x) = xe^x)$.

The corresponding Wronskian is:

$$w(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t) =$$
$$= e^x(xe^x)' - (e^x)'(xe^x) = e^x(e^x + xe^x) - e^x xe^x =$$
$$= e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

and a particular solution is:

$$y_p(x) = -y_1(x)\int_0^x \frac{f(t)y_2(t)}{w(t)}\,dt + y_2(x)\int_0^x \frac{f(t)y_1(t)}{w(t)}\,dt$$

with $f(t) = (3t+2)e^t$. It follows that

$$y_p(x) = -e^x \int_0^x \frac{(3t+2)e^t \cdot te^t}{e^{2t}}\,dt + xe^x \int_0^x \frac{(3t+2)e^t \cdot e^t}{e^{2t}}\,dt$$
$$= -e^x \int_0^x (3t^2+2t)\,dt + xe^x \int_0^x (3t+2)\,dt =$$

$$= -e^x \left[\frac{3t^3}{3} + \frac{2t^2}{2}\right]_0^x + xe^x\left[\frac{3t^2}{2} + 2t\right]_0^x =$$

$$= -e^x\left(x^3+x^2\right) + xe^x\left(\frac{3x^2}{2} + 2x\right) =$$

$$= -x^2e^x(x+1) + x^2e^x(3x/2+2) =$$

$$= \frac{+x^2e^x}{2}\left(-2(x+1)+3x+4\right) = (1/2)x^2e^x(-2x-2+3x+4)$$

$$= (1/2)x^2e^x(x+2).$$

and therefore the general solution is

$$y(x) = \lambda_1 e^x + \lambda_2 xe^x + (1/2)x^2e^x(x+2)$$

To apply the initial condition, we note that

$$y(0) = \lambda_1 e^0 + \lambda_2 \cdot 0 e^0 + (1/2)0^2 e^0(0+2) = \lambda_1$$

and

$$\begin{aligned}
y'(x) &= \lambda_1(e^x)' + \lambda_2(xe^x)' + (1/2)\left[x^2e^x(x+2)\right]' = \\
&= \lambda_1 e^x + \lambda_2(e^x + xe^x) + (1/2)\left[e^x(x^3+2x^2)\right]' = \\
&= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)\left[(e^x)'(x^3+2x^2) + e^x(x^3+2x^2)'\right] = \\
&= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)\left[e^x(x^3+2x^2+3x^2+4x)\right] \\
&= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)e^x(x^3+5x^2+4x) \\
&= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)e^x x(x^2+5x+4) \\
&= (\lambda_1+\lambda_2)e^x + \lambda_2 xe^x + (1/2)xe^x(x+4)(x+1) \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\Rightarrow y'(0) &= (\lambda_1+\lambda_2)e^0 + \lambda_2 \cdot 0 \cdot e^0 + (1/2)\cdot 0 e^0(0+4)(0+1) = \\
&= \lambda_1+\lambda_2
\end{aligned}$$

and therefore:

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_1+\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ y_0+\lambda_2 = y_1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = y_0 \\ \lambda_2 = y_1 - y_0 \end{cases}$$

thus $y(x) = y_0 e^x + (y_1 - y_0)xe^x + (1/2)x^2e^x(x+2)$.

b) Solve the ODE value problem

$$x^3y'''(x) + x^2y''(x) - 2xy'(x) + 2y(x) = f(x), \quad \forall x \in [1, +\infty)$$

Solution

Define $Ly(x) = x^3y'''(x) + x^2y''(x) - 2xy'(x) + 2y(x)$. Then, since

$$Lx^b = x^3(x^b)''' + x^2(x^b)'' - 2x(x^b)' + 2x^b =$$

$$= x^3 b(b-1)(b-2)x^{b-3} + x^2 b(b-1)x^{b-2} - 2x\,b\,x^{b-1} + 2x^b =$$

$$= [b(b-1)(b-2) + b(b-1) - 2b + 2]x^b$$

the characteristic polynomial is given by

$$P(b) = b(b-1)(b-2) + b(b-1) - 2b + 2 = b(b^2 - 3b + 2) + b^2 - b - 2b + 2$$

$$= b^3 - 3b^2 + 2b + b^2 - b - 2b + 2 =$$

$$= b^3 + (-3+1)b^2 + (2-1-2)b + 2$$

$$= b^3 - 2b^2 - b + 2 = b^2(b-2) - (b-2) = (b^2-1)(b-2)$$

$$= (b-1)(b+1)(b-2)$$

and has single zeroes $b_1 = -1 \wedge b_2 = 1 \wedge b_3 = 2$.

Thus the general solution is:

$$y(x) = \lambda_1 x^{-1} + \lambda_2 x + \lambda_3 x^2 + y_p(x)$$

Define: $y_1(x) = x^{-1} \wedge y_2(x) = x \wedge y_3(x) = x^2$, $\forall x \in [1, +\infty)$

The particular solution is given by

$$y_p(x) = \int_1^{+\infty} G(x,t) f(t)\, dt, \quad \forall x \in [1, +\infty)$$

with
$$G(x,t) = \begin{cases} B_1(t)x^{-1} + B_2(t)x + B_3(t)x^2, & \text{if } x \geq t \\ 0, & \text{if } x < t \end{cases}$$

with $B_1(t), B_2(t), B_3(t)$ the solution of

$$W[y_1, y_2, y_3](B_1(t), B_2(t), B_3(t)) = (0, 0, 1).$$

and therefore

$$y_p(x) = \int_1^{+\infty} G(x,t) f(t)\,dt = \int_1^x \left[B_1(t)x^{-1} + B_2(t)x + B_3(t)x^2\right] f(t)\,dt$$

$$= x^{-1}\int_1^x B_1(t) f(t)\,dt + x\int_1^x B_2(t) f(t)\,dt + x^2\int_1^x B_3(t) f(t)\,dt$$

Since

$$W[y_1, y_2, y_3](t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{bmatrix} = \begin{bmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{bmatrix} \begin{bmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We apply Cramer's rule:

$$D = \begin{vmatrix} t^{-1} & t & t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} (-t) = \begin{vmatrix} t^{-1}+t^{-1} & t-t & t^2-2t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} =$$

$$= \begin{vmatrix} 2t^{-1} & 0 & -t^2 \\ -t^{-2} & 1 & 2t \\ 2t^{-3} & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2t^{-1} & -t^2 \\ 2t^{-3} & 2 \end{vmatrix} = (2t^{-1})2 - (-t^2)(2t^{-3})$$

$$= 4t^{-1} + 2t^{-1} = 6t^{-1}$$

and

$$D_1 = \begin{vmatrix} 0 & t & t^2 \\ 0 & 1 & 2t \\ 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t(2t) - t^2 = 2t^2 - t^2 = t^2.$$

$$D_2 = \begin{vmatrix} t^{-1} & 0 & t^2 \\ -t^{-2} & 0 & 2t \\ 2t^{-3} & 1 & 2 \end{vmatrix} = - \begin{vmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{vmatrix} = -\left[t^{-1}(2t) - t^2(-t^{-2})\right] =$$

$$= -(2+1) = -3$$

and

$$D_3 = \begin{vmatrix} t^{-1} & t & 0 \\ -t^{-2} & 1 & 0 \\ 2t^{-3} & 0 & 1 \end{vmatrix} = \begin{vmatrix} t^{-1} & t \\ -t^{-2} & 1 \end{vmatrix} = t^{-1} - t(-t^{-2}) = t^{-1} + t^{-1} = 2t^{-1}$$

and therefore:

$$B_1(t) = \frac{D_1(t)}{D(t)} = \frac{t^2}{6t^{-1}} = \frac{t^3}{6}$$

$$B_2(t) = \frac{D_2(t)}{D(t)} = \frac{-3}{6t^{-1}} = \frac{-t}{2}$$

$$B_3(t) = \frac{D_3(t)}{D(t)} = \frac{2t^{-1}}{6t^{-1}} = \frac{1}{3}$$

The particular solution is:

$$y_p(x) = x^{-1} \int_1^x \frac{t^3}{6} f(t)\,dt + x \int_1^x \frac{-t}{2} f(t)\,dt + x^2 \int_1^x \frac{1}{3} f(t)\,dt =$$

$$= \frac{1}{6x} \int_1^x t^3 f(t)\,dt - \frac{x}{2} \int_1^x t f(t)\,dt + \frac{x^2}{3} \int_1^x f(t)\,dt.$$

It follows that the general solution is given by

$$y(x) = \left[\lambda_1 + \int_1^x \frac{t^3 f(t)}{6}\,dt\right] x^{-1} + \left[\lambda_2 - \int_1^x \frac{t f(t)}{2}\,dt\right] x + \left[\lambda_3 + \int_1^x \frac{f(t)}{3}\,dt\right] x^2$$

↳ Note that the integrals can start from numbers other than 1. This will result in a constant shift (i.e. independent of x) in the value of the integrals that can be absorbed by $\lambda_1, \lambda_2, \lambda_3$. In general, it is convenient for the integrals to begin at the location where the initial condition is given.

EXERCISES

(13) Derive the general solution for the following inhomogeneous linear differential equations

a) $y''(x)+y(x)=\sin(ax)$, with $a\in(0,+\infty)$

b) $y''(x)+y'(x)+y(x)=\sin(ax)$, with $a\in(0,+\infty)$

c) $y''(x)-2y'(x)+y(x)=e^x/x$ on $x\in(0,+\infty)$

d) $x^2y''(x)-2xy'(x)+2y(x)=x\ln x$ on $x\in[1,+\infty)$

e) $x^2y''(x)-xy'(x)=x^3e^x$ on $x\in[1,+\infty)$

f) $y'''(x)-y'(x)=x^2-3x$ on $x\in[1,+\infty)$

g) $x^3y'''(x)+3x^2y''(x)=1$ on $x\in[1,+\infty)$

(14) Solve the following initial value problem:

$$\begin{cases} x^2y''-2xy'+2y=f(x) \\ y(1)=y_0 \wedge y'(1)=y_1 \end{cases}$$

(15) Solve the general damped oscillator problem, which is defined as the following initial value problem.

$$\begin{cases} y''(x)+by'(x)+w^2y(x)=f(x) \\ y(0)=y_0 \wedge y'(0)=y_1 \end{cases}$$

with $b,w\in(0,+\infty)$, $y_0,y_1\in\mathbb{R}$. Distinguish 3 different cases:

Case 1: $b<2w$ (underdamped oscillator)

Case 2: $b=2w$ (critically damped oscillator)

Case 3: $b>2w$ (overdamped oscillator)

ODE 5: Series Solution of Linear Differential Equations

SERIES SOLUTION OF ODES

We begin by reviewing, and in some cases, extending, results from Calculus II needed for solving linear ODEs via convergent series methods.

▼ The Gamma function

We recall from my Calculus 2 lecture notes the definition of the factorial and the double factorial:

▶ Factorial:

$$\boxed{\begin{aligned} &0! = 1 \\ &\forall n \in \mathbb{N}^*: \; n! = \prod_{k=1}^{n} k = 1\cdot 2\cdot 3\cdot \ldots \cdot n \end{aligned}}$$

▶ Double Factorial:

$$\boxed{\begin{aligned} &0!! = 1 \text{ and } 1!! = 1 \\ &\forall n \in \mathbb{N}^*: \; (2n)!! = \prod_{k=1}^{n} (2k) = 2^n n! \\ &\forall n \in \mathbb{N}^*: \; (2n+1)!! = \prod_{k=1}^{n} (2k+1) = 1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1) \end{aligned}}$$

The Gamma function $\Gamma(n)$ generalizes the factorial and is defined, first on $(0,+\infty)$ and then on a wider set as follows.

Def: (Gamma function on $(0,+\infty)$)

$$\forall n \in (0,+\infty): \; \Gamma(n) = \int_{0^+}^{+\infty} x^{n-1} e^{-x}\, dx$$

Then, we show that:

$$\boxed{\begin{array}{l}\underline{\text{Prop}}: \\ \text{a) } \forall n \in (0,+\infty): \text{ The } \Gamma(n) \text{ integral converges} \\ \text{b) } \Gamma(1) = 1 \\ \text{c) } \forall n \in (0,+\infty): \Gamma(n+1) = n\Gamma(n)\end{array}}$$

It immediately follows that

$$\boxed{\forall n \in \mathbb{N}^* : \Gamma(n) = (n-1)!}$$

But n is a continuous variable and $\Gamma(n)$ has been defined on $n \in (0,+\infty)$. So, $\Gamma(n)$ generalizes the factorial on a continuous set. We can now use the equation $\Gamma(n) = \Gamma(n+1)/(n+1)$ to extend the definition of the Gamma function for negative n as follows:

$$\forall n \in (-1,0): \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\forall n \in (-2,-1): \Gamma(n) = \frac{\Gamma(n+1)}{n} = \frac{\Gamma(n+2)}{n(n+1)}$$

$$\forall n \in (-3,-2): \Gamma(n) = \frac{\Gamma(n+2)}{n(n+1)} = \frac{\Gamma(n+3)}{n(n+1)(n+2)}$$

and so on. The general definition of the Gamma function for negative numbers is:

$$\boxed{\begin{array}{l}\underline{\text{Def}}: (\underline{\text{Gamma function for negative numbers}})\\[1em] \forall k\in\mathbb{N}^*: \forall n\in(-k,-k+1): \Gamma(n)=\dfrac{\Gamma(n+k)}{\prod_{a=0}^{k-1}(n-a)}=\dfrac{\Gamma(n+k)}{n(n+1)\cdots(n+k-1)}\end{array}}$$

● Proof of proposition

The proof requires the following lemma.

Lemma: $\boxed{\forall a\in\mathbb{R}: \lim_{x\to+\infty} x^a e^{-x}=0}$

Proof

Let $a\in\mathbb{R}$ be given. We distinguish between the following cases.

Case 1 : For $a\in(-\infty,0)$, we have

$$\left(\lim_{x\to+\infty} x^a=0 \wedge \lim_{x\to+\infty} e^{-x}=0\right) \Rightarrow \lim_{x\to+\infty} x^a e^{-x}=0.$$

Case 2 : For $a=0$, we have

$$\lim_{x\to+\infty} x^a e^{-x}=\lim_{x\to+\infty} x^0 e^{-x}=\lim_{x\to+\infty} e^{-x}=0$$

Case 3 : For $a\in(0,+\infty)$, we define $n=\max\{k\in\mathbb{N} \mid a-k>0\}$.

We evaluate the limit by applying De L'Hospital $n+1$ times:

$$\lim_{x\to+\infty} x^a e^{-x}=\lim_{x\to+\infty}\frac{x^a}{e^x}=\lim_{x\to+\infty}\frac{a(a-1)\cdots(a-n)x^{a-(n+1)}}{e^x}=$$

$$=a(a-1)\cdots(a-n)\lim_{x\to+\infty} x^{a-(n+1)}e^{-x}=0$$

because, by definition of n, $a-(n+1)<0$. □

For the convergence proof we use the following theorems from Calculus II:

1) Comparison test

$$\left.\begin{array}{l}\forall x\in S: 0\leq f(x)\leq g(x)\\ \int_S g(x)\,dx \text{ converges}\end{array}\right\}\Rightarrow \int_S f(x)\,dx \text{ converges.}$$

2) Ratio test

$$\left.\begin{array}{l}\forall x\in S: (f(x)\geq 0 \wedge g(x)\geq 0)\\ \lim_{x\to\sigma}\dfrac{f(x)}{g(x)}=0\end{array}\right\}\Rightarrow\left(\int_S g(x)\,dx \text{ converges}\Rightarrow\int_S f(x)\,dx \text{ converges}\right)$$

The proofs are as follows:

<u>Proof of (a)</u>: Let $n\in(0,+\infty)$ be given.

We write

$$\Gamma(n)=\int_{0^+}^{+\infty}x^{n-1}e^{-x}\,dx=\int_{0^+}^{1}x^{n-1}e^{-x}\,dx+\int_{1}^{+\infty}x^{n-1}e^{-x}\,dx$$

For the $(1,+\infty)$ integral, we define

$$\begin{cases}\forall x\in(1,+\infty): f(x)=x^{n-1}e^{-x}>0\\ \forall x\in(1,+\infty): g(x)=1/x^2>0\end{cases}$$

and therefore:

$$\forall x\in(1,+\infty): \frac{f(x)}{g(x)}=\frac{x^{n-1}e^{-x}}{1/x^2}=x^2x^{n-1}e^{-x}=x^{n+1}e^{-x}$$

$$\Rightarrow \lim_{x\to+\infty}\frac{f(x)}{g(x)}=\lim_{x\to+\infty}(x^{n+1}e^{-x})=0 \qquad (1)$$

From Eq.(1) and the ratio test it follows that:

$$\int_1^{+\infty} \frac{dx}{x^2} \text{ converges} \Rightarrow \int_1^{+\infty} x^{n-1} e^{-x}\, dx \text{ converges.} \quad (2)$$

For the $(0,1)$ integral, let $x \in (0,1)$ be given. Then:
$x \in (0,1) \Rightarrow 0 < x < 1 \Rightarrow -1 < -x < 0 \Rightarrow 0 < e^{-x} < e^0 \Rightarrow$
$\Rightarrow 0 < e^{-x} < 1 \Rightarrow 0 < x^{n-1} e^{-x} < x^{n-1}$ (since $x^{n-1} > 0$).
It follows that
$\forall x \in (0,1) : 0 < x^{n-1} e^{-x} < x^{n-1}$ (3)
From Eq. (3) and via the comparison test, we argue that
$n > 0 \Rightarrow n-1 > -1 \Rightarrow \int_{0^+}^1 x^{n-1} dx$ converges $\Rightarrow \int_{0^+}^1 x^{n-1} e^{-x}\, dx$ converges. (4)

From Eq. (2) and Eq. (4): $\Gamma(n) = \int_{0^+}^{+\infty} x^{n-1} e^{-x}\, dx$ converges. $\square$

<u>Proof of (b)</u> : Claim $\Gamma(1) = 1$

$$\Gamma(1) = \int_{0^+}^{+\infty} x^{1-1} e^{-x}\, dx = \int_{0^+}^{+\infty} x^0 e^{-x}\, dx = \int_{0^+}^{+\infty} e^{-x}\, dx = \left[-e^{-x} \right]_{0^+}^{+\infty} =$$
$$= \lim_{x \to +\infty} (-e^{-x}) - \lim_{x \to 0^+} (-e^{-x}) = (-0) - (-e^0) = 1$$

<u>Proof of (c)</u> : Claim $\forall n \in (0, +\infty) : \Gamma(n+1) = n\Gamma(n)$
Let $n \in (0, +\infty)$ be given. Then:
$$\Gamma(n+1) = \int_{0^+}^{+\infty} x^{(n+1)-1} e^{-x}\, dx = \int_{0^+}^{+\infty} x^n e^{-x}\, dx = \int_{0^+}^{+\infty} x^n (-e^{-x})'\, dx$$
$$= \left[-x^n e^{-x} \right]_{0^+}^{+\infty} - \int_{0^+}^{+\infty} (x^n)' (-e^{-x})\, dx =$$
$$= \lim_{x \to +\infty} (-x^n e^{-x}) - (-0^n e^0) - \int_{0^+}^{+\infty} n x^{n-1} (-e^{-x})\, dx =$$

$$= 0 - 0 + n \int_{0^+}^{+\infty} x^{n-1} e^{-x} dx = n\Gamma(n)$$

and therefore $\forall n \in (0, +\infty): \Gamma(n+1) = n\Gamma(n)$. $\square$

● Value of $\Gamma(1/2)$: $\boxed{\Gamma(1/2) = \sqrt{\pi}}$

To show that $\Gamma(1/2) = \sqrt{\pi}$ we use the following result from Calculus 3:

$$\int_0^{+\infty} dx \int_0^{+\infty} dy\, f(x,y) = \int_0^{+\infty} r dr \int_0^{\pi/2} d\theta\, f(r\cos\theta, r\sin\theta)$$

Proof

We define $u = \sqrt{x} \Rightarrow du = \dfrac{dx}{2\sqrt{x}} \Rightarrow x^{-1/2} dx = 2du$

and note that $x = 0 \Leftrightarrow u = 0$ and $x \to +\infty \Leftrightarrow u \to +\infty$.

It follows that

$$\Gamma(1/2) = \int_{0^+}^{+\infty} x^{1/2-1} e^{-x} dx = \int_{0^+}^{+\infty} x^{-1/2} e^{-x} dx = \int_{0^+}^{+\infty} e^{-u^2} 2du$$

$$= 2\int_0^{+\infty} \exp(-u^2) du \Rightarrow$$

$$\Rightarrow [\Gamma(1/2)]^2 = \left[2\int_0^{+\infty} \exp(-u^2) du\right]\left[2\int_0^{+\infty} \exp(-v^2) dv\right] =$$

$$= 4\int_0^{+\infty} du \int_0^{+\infty} dv\, \exp(-u^2 - v^2)$$

$$= 4\int_0^{+\infty} r dr \int_0^{\pi/2} d\theta\, \exp(-r^2\cos^2\theta - r^2\sin^2\theta)$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\vartheta \, \exp(-r^2(\sin^2\vartheta+\cos^2\vartheta))$$

$$= 4\int_0^{+\infty} r\,dr \int_0^{\pi/2} d\vartheta \, \exp(-r^2) = 4\int_0^{+\infty} r\exp(-r^2)\left[\int_0^{\pi/2} d\vartheta\right] dr$$

$$= 4\int_0^{+\infty} r\exp(-r^2)\,(\pi/2)\,dr = \pi\int_0^{+\infty} 2r\exp(-r^2)\,dr =$$

$$= \pi\int_0^{+\infty} [-\exp(-r^2)]'\,dr = \pi\Big[-\exp(-r^2)\Big]_0^{+\infty} =$$

$$= \pi\left[\lim_{x\to+\infty}(-\exp(-r^2)) - (-\exp(-0))\right] = \pi[0-(-1)] = \pi$$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi} \vee \Gamma(1/2) = -\sqrt{\pi}. \qquad (1)$$

Since $(\forall u \in (0, +\infty) : \exp(-u^2) \geq 0) \Rightarrow$

$$\Rightarrow \Gamma(1/2) = 2\int_0^{+\infty} \exp(-u^2)\,du \geq 0 \qquad (2)$$

From Eq. (1) and Eq. (2) it follows that $\Gamma(1/2) = \sqrt{\pi}$.

EXAMPLE

Use proof by induction to show that given an $a \in \mathbb{R} - (-1)\mathbb{N}^*$ with $(-1)\mathbb{N}^* = \{-x \mid x \in \mathbb{N}^*\} = \{-1, -2, -3, \dots\}$, we have:

$$\boxed{\forall n \in \mathbb{N}^*: \prod_{k=1}^{n} (k+a) = \frac{\Gamma(n+1+a)}{\Gamma(a+1)}}$$

→ This result is VERY useful for rewritting products in terms of Gamma functions.

Solution

For $n=1$, we have:

$$\prod_{k=1}^{n} (k+a) = (1+a) = \frac{(a+1)\Gamma(a+1)}{\Gamma(a+1)} = \frac{\Gamma(a+2)}{\Gamma(a+1)} = \frac{\Gamma(1+1+a)}{\Gamma(a+1)} =$$

$$= \frac{\Gamma(n+1+a)}{\Gamma(a+1)}$$

For $n=m$, we assume that $\prod_{k=1}^{m} (k+a) = \frac{\Gamma(m+1+a)}{\Gamma(a+1)}$

For $n=m+1$, we will show that $\prod_{k=1}^{m+1} (k+a) = \frac{\Gamma((m+1)+1+a)}{\Gamma(a+1)}$ as follows:

$$\prod_{k=1}^{m+1} (k+a) = (m+1+a)\prod_{k=1}^{m} (k+a) = (m+1+a) \cdot \frac{\Gamma(m+1+a)}{\Gamma(a+1)} =$$

$$= \frac{(m+1+a)\Gamma(m+1+a)}{\Gamma(a+1)} = \frac{\Gamma(m+1+a+1)}{\Gamma(a+1)} = \frac{\Gamma((m+1)+1+a)}{\Gamma(a+1)}$$

EXERCISES

① Learn the proofs of the following statements:

a) The integrals $\Gamma(n) = \int_0^{+\infty} x^{n-1} e^{-x}\,dx$ converges for $n > 0$.

b) $\Gamma(1) = 1$

c) $\forall n \in \mathbb{N}^* : \Gamma(n+1) = n\Gamma(n)$

d) $\Gamma(1/2) = \sqrt{\pi}$

② Show that $\forall n \in \mathbb{N}^* : (2n+1)!! = \dfrac{2^{n+1}}{\sqrt{\pi}} \Gamma\left(\dfrac{2n+3}{2}\right)$.

③ Recall from Calculus 2 that the binomial series is given by

$$\forall x \in (-1,1) : (1+x)^a = \sum_{n=0}^{+\infty} \binom{a}{n} x^n$$

with $\binom{a}{0} = 1$ and $\binom{a}{n} = \prod_{k=1}^{n} \dfrac{a+1-k}{k}$, $\forall n \in \mathbb{N}^*$

Show that:

a) $\forall n \in \mathbb{N}^* : \binom{-1/2}{n} = \dfrac{(-1)^n (2n-1)!!}{(2n)!!} = \dfrac{(-1)^n}{\sqrt{\pi}\,\Gamma(n+1)} \Gamma\left(\dfrac{2n+1}{2}\right)$

b) $\forall a \in (1,+\infty) : \forall n \in \mathbb{N}^* : \binom{1/a}{n} = \dfrac{(-1)^n \Gamma(n-1/a)}{n\Gamma(n)\Gamma(-1/a)}$

(4) Show that :

a) $$\int_0^{+\infty} x(2x+3)^2 e^{-x}\,dx = 57$$

b) $$\int_0^{+\infty} \frac{(x+1)(x-1)e^{-x}}{\sqrt{x}}\,dx = \frac{-\sqrt{\pi}}{4}$$

c) $$\int_0^{+\infty} (\sqrt{x}+2)^2 e^{-x}\,dx = 2\sqrt{\pi}+5$$

d) $$\int_0^{+\infty} \sqrt{x}\,(\sqrt{x}-1)^3 e^{-x}\,dx = 5 - \frac{11\sqrt{\pi}}{4}$$

(5) Use the method of substitution and the Gamma function integral to show that

a) $$\int_0^{+\infty} \sqrt{x}\,\exp(-x^3)\,dx = \frac{\sqrt{\pi}}{3}$$

b) $$\int_0^{+\infty} 2^{-x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{\ln 2}}$$

c) $$\int_{0^+}^{1^-} \frac{dx}{\sqrt{|\ln x|}} = \sqrt{\pi}$$

d) $$\int_{0^+}^{1} (\ln x)^3\,dx = -6$$

e) $$\int_{0^+}^{1} \sqrt{\ln(1/x)}\,dx = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

f) $$\int_{0^+}^{1} (x\ln x)^2\,dx = \frac{2}{27}$$

▼ Review of power series

We review basic results from Calculus II concerning power series expansion of functions.

● Definitions

- A power series is a series of the form

$$\forall x \in A : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

with $a \in \text{Seq}(\mathbb{R})$ and $x_0 \in \mathbb{R}$.

- The domain A is chosen to be the widest possible subset of $\mathbb{R}$ for which the series converges. If $A = (x_0 - \mu, x_0 + \mu)$ then we say that $\mu > 0$ is the radius of convergence.

Def: Let $f : A \to \mathbb{R}$ be a function with $x_0 \in A$. We say that f analytic at $x = x_0 \Leftrightarrow$

$$\Leftrightarrow \exists a \in \text{Seq}(\mathbb{R}) : \exists \mu \in (0, +\infty) : \forall x \in (x_0 - \mu, x_0 + \mu) : f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

f analytic on $S \subseteq A \Leftrightarrow \forall x_0 \in S : f$ analytic on $x = x_0$

- The space of all functions analytic on S is denoted as $C^\omega(S)$. Note that $C^\omega(S) \subseteq C^\infty(S)$ which means that in general

$$f \in C^{\omega}(S) \Rightarrow f \in C^{\infty}(S).$$

However, the converse statement is not always true.

General properties of power series

Let f, g be two functions that are analytic at $x = x_0$ such that

$$\forall x \in (x_0 - \mu, x_0 + \mu) : \left(f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x - x_0)^n \right)$$

Then, we can show that:

a) $(\forall x \in (x_0 - \mu, x_0 + \mu) : f(x) = g(x)) \Leftrightarrow (\forall n \in \mathbb{N} : a_n = b_n)$

b) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f(x) + g(x) = \sum_{n=0}^{+\infty} (a_n + b_n)(x - x_0)^n$

c) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f(x) g(x) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] (x - x_0)^n$

d) $\forall x \in (x_0 - \mu, x_0 + \mu) : \quad f'(x) = \sum_{n=1}^{+\infty} n a_n (x - x_0)^{n-1}$

e) $\forall k \in \mathbb{N}^* : \forall x \in (x_0 - \mu, x_0 + \mu) : f^{(k)}(x) = \sum_{n=k}^{+\infty} \left[\prod_{l=0}^{k-1} (n - l) \right] a_n (x - x_0)^{n-k}$

$$= \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n (x - x_0)^{n-k}$$

e) $\forall x_1, x_2 \in (x_0 - \mu, x_0 + \mu) : \int_{x_1}^{x_2} f(t)\, dt = \sum_{n=0}^{+\infty} \left[a_n \int_{x_1}^{x_2} (t - x_0)^n \, dt \right]$

$$= \sum_{n=0}^{+\infty} \left[\frac{a_n \left[(x_2 - x_0)^{n+1} - (x_1 - x_0)^{n+1} \right]}{n+1} \right]$$

Some important power series

$$\forall x \in (-1,1): \quad \frac{1}{1-x} = \sum_{k=0}^{+\infty} x^k = 1 + x + x^2 + \cdots$$

$$\forall x \in (-1,1): \quad (1+x)^p = \sum_{n=0}^{+\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2 \cdot 1} x^2 + \cdots$$

$$\forall x \in \mathbb{R}: \quad e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\forall x \in \mathbb{R}: \quad \sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\forall x \in \mathbb{R}: \quad \cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\forall x \in (-1,1]: \quad \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\forall x \in [-1,1]: \quad \operatorname{Arctan} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

The detailed theory on the above series is given in my Calculus 2 notes.

Convergence tests

The proofs of the relevant theorems for series solution of linear ODEs depend on the comparison test and the absolute ratio test. Applied on power series these tests reduce to the following statements:

1 → Comparison test

Given $a, b \in \mathrm{Seq}(\mathbb{R})$ and $x_0 \in \mathbb{R}$, then

$$\forall x \in \mathbb{R}: \left(\begin{cases} \forall n \in \mathbb{N}: |a_n| \leq b_n \\ \sum_{n=0}^{+\infty} b_n (x-x_0)^n \text{ converges} \end{cases} \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges} \right)$$

2 → Absolute Ratio test

Given $a \in \mathrm{Seq}(\mathbb{R})$ and $x_0 \in \mathbb{R}$, then:

$$\left(\lim_{n\in\mathbb{N}} \left| \frac{a_{n+1}(x-x_0)}{a_n} \right| < 1 \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ converges} \right), \forall x \in \mathbb{R}$$

$$\left(\lim_{n\in\mathbb{N}} \left| \frac{a_{n+1}(x-x_0)}{a_n} \right| > 1 \Rightarrow \sum_{n=0}^{+\infty} a_n (x-x_0)^n \text{ diverges} \right), \forall x \in \mathbb{R}$$

In practice we get convergence for free via the relevant theorems as we solve the linear ODE. Therefore the above convergence tests are only required in the proofs of the necessary theorems.

• Merten' theorem

Thm: Let (a_n) and (b_n) be two sequences with $n \in \mathbb{N}$. Then, we have:

$$\left\{\begin{array}{l} \sum_{n=0}^{+\infty} |a_n| \text{ converges} \\ \sum_{n=0}^{+\infty} b_n \text{ converges} \end{array}\right. \Rightarrow \left[\sum_{n=0}^{+\infty} a_n\right]\left[\sum_{n=0}^{+\infty} b_n\right] = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right]$$

Merten's theorem can be used safely to multiply power series because when they converge, they converge absolutely. A useful shortcut is to note that if

$$\boxed{\forall x \in A : \left(f(x) = \sum_{n=0}^{+\infty} a_n (x-x_0)^n \wedge g(x) = \sum_{n=0}^{+\infty} b_n (x-x_0)^n \right)}$$

then, it follows that

$$\boxed{\forall x \in A : f(x) g(x) = \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] (x-x_0)^n}$$

For more details, see my Calculus 2 lecture notes.

EXAMPLES

a) Write the series expansion around $x_0 = 0$ of the function

$$f(x) = \frac{e^x}{2x+1}$$

and find the radius of convergence.

<u>Solution</u>

We have:

$$f(x) = \frac{e^x}{2x+1} = e^x \cdot \frac{1}{1-(-2x)} = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-2x)^n\right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n 2^n x^n\right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^k 2^k}{(n-k)!}\right] x^n$$

The series expansion of e^x converges on $\mathbb{R}$. The series expansion of $1/(1-(-2x))$ requires $|-2x| < 1$. Since:

$$|-2x| < 1 \Leftrightarrow |2x| < 1 \Leftrightarrow 2|x| < 1 \Leftrightarrow |x| < 1/2 \Leftrightarrow$$

$$\Leftrightarrow -1/2 < x < 1/2 \Leftrightarrow x \in (-1/2, 1/2).$$

b) Write the series expansion of the function $f(x) = e^x \cos x$ and find the radius of convergence

<u>Solution</u>

Since:

$$f(x) = e^x \cos x = \left[\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \right] \left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right]$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \right] \left[\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k}}{(2k)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!} \right] +$$

$$+ \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} (-1)^{n-k} \frac{x^{2n-2k}}{(2n-2k)!} \right] =$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k)!(2n-2k)!} \right] x^{2n} +$$

$$+ \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(2k+1)!(2n-2k)!} \right] x^{2n+1}$$

c) Write a series expansion of $f(x) = \sin(2x)$ around $x = \pi/8$ and find the radius of convergence.

Solution

$$f(x) = \sin(2x) = \sin(2x - \pi/4 + \pi/4) = \sin(2(x-\pi/8) + \pi/4) =$$

$$= \sin(2(x-\pi/8)) \cos(\pi/4) + \cos(2(x-\pi/8)) \sin(\pi/4) =$$

$$= (\sqrt{2}/2) [\cos(2(x-\pi/8)) + \sin(2(x-\pi/8))]$$

$$= \frac{\sqrt{2}}{2} \left[\sum_{n=0}^{+\infty} (-1)^n \frac{[2(x-\pi/8)]^{2n}}{(2n)!} + \sum_{n=0}^{+\infty} (-1)^n \frac{[2(x-\pi/8)]^{2n+1}}{(2n+1)!} \right]$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{\sqrt{2}\, 2^{2n}}{2(2n)!} (x-\pi/8)^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{\sqrt{2}\, 2^{2n+1}}{2(2n+1)!} (x-\pi/8)^{2n+1}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n-1}\sqrt{2}}{(2n)!} (x-\pi/8)^{2n} + \sum_{n=0}^{+\infty} (-1)^n \frac{2^{2n}\sqrt{2}}{(2n+1)!} (x-\pi/8)^{2n+1}$$

The convergence set for all series expansions here is $\mathbb{R}$

EXERCISES

(6) Show that

a) $\forall x \in \mathbb{R}: \sin^2 x = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2^{2n-1}}{(2n)!} x^{2n}$

b) $\forall x \in (-2,2): \frac{x}{2-x} = \sum_{n=1}^{+\infty} \left(\frac{x}{2}\right)^n$

c) $\forall x \in \mathbb{R}: \sin^3 x = \frac{3}{4} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}(3^{2n}-1)}{(2n+1)!} x^{2n+1}$

d) $\forall x \in (-1,1): \ln\left(\sqrt{\frac{1+x}{1-x}}\right) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1}$

e) $\forall x \in (-1,1): \frac{1}{x^2+x+1} = \frac{2}{\sqrt{3}} \sum_{n=0}^{+\infty} \sin\left(\frac{2\pi(n+1)}{3}\right) x^n$

(7) Derive the series expansions for the following functions around the indicated points, and find the convergence radius.

a) $f(x) = e^x \sin x$ (around $x_0 = 0$)

b) $f(x) = \sin(2x)$ (around $x_0 = \pi/6$)

c) $f(x) = e^x \ln(1+x)$ (around $x_0 = 0$)

d) $f(x) = \frac{\cos x}{1-x^2}$ (around $x_0 = 0$)

(8) Consider the function f defined by the power series

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}$$

a) Show that the power-series converges on $\mathbb{R}$.

b) Show that $\forall x \in \mathbb{R}: \; x f''(x) + f'(x) = -x f(x)$

(9) Consider the function f defined by the power series

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1} \, n! \, (n+1)!}$$

a) Show that the series converges on $\mathbb{R}$.

b) Show that $\forall x \in \mathbb{R}: \; x^2 f''(x) + x f'(x) = (1 - x^2) f(x)$

Series solution of 2nd-order linear ODES

We consider a 2nd-order linear ordinary differential equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

and we seek the general solution approximated as a power series around the point $x = x_0$.

We distinguish between the following 3 cases:

1) $x = x_0$ is a <u>regular point</u> $\Longleftrightarrow$ $\begin{cases} p(x) \text{ analytic at } x = x_0 \\ q(x) \text{ analytic at } x = x_0 \end{cases}$

2) $x = x_0$ is a <u>regular singular point</u> $\Longleftrightarrow$ $\begin{cases} x = x_0 \text{ is NOT a regular point} \\ (x-x_0)p(x) \text{ analytic at } x = x_0 \\ (x-x_0)^2 q(x) \text{ analytic at } x = x_0 \end{cases}$

3) $x = x_0$ is an <u>irregular singular point</u> $\Longleftrightarrow$ $\big((x-x_0)p(x) \text{ NOT analytic at } x = x_0\big) \vee \big((x-x_0)^2 q(x) \text{ NOT analytic at } x = x_0\big)$.

- The first two cases can be solved with convergent power series methods. The third case can be only investigated with asymptotic techniques or may be current research.

1 → Regular linear ODEs

Thm: Consider an initial value problem of the form

$$\begin{cases} y''(x) + p(x)y'(x) + q(x)y(x) = 0 \\ y(x_0) = a_0 \wedge y'(x_0) = a_1 \end{cases}$$

with $p, q \in C^\omega((x_0 - \mu, x_0 + \mu))$ (i.e. p, q analytic at $x = x_0$) such that

$$\forall x \in (x_0 - \mu, x_0 + \mu) : \left(p(x) = \sum_{n=0}^{+\infty} p_n (x - x_0)^n \wedge q(x) = \sum_{n=0}^{+\infty} q_n (x - x_0)^n \right)$$

The unique solution to this initial value problem is given by

$$\forall x \in (x_0 - \mu, x_0 + \mu) : y(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

with $a \in \mathrm{Seq}(\mathbb{R})$ a sequence defined recursively by

$$\forall n \in \mathbb{N} : a_{n+2} = \frac{-1}{(n+1)(n+2)} \sum_{k=0}^{n} \left[(k+1) a_{k+1} p_{n-k} + a_k q_{n-k} \right]$$

with $a_0, a_1 \in \mathbb{R}$ given via the above initial conditions.

Remarks

1) The unique sequence defined by the above recursion combined with initial values $a_0, a_1 \in \mathbb{R}$ will be denoted for convenience as: $a_n = A_n(a_0, a_1 | p, q)$.

2) The convergence of the power series for $y(x)$ is provided for by the theorem and has the same radius of convergence as the functions p, q. It is therefore not necessary to establish convergence when solving problems.

3) To find the two linearly independent solutions y_1, y_2 we solve, by convention, the following initial value problems:

$$\begin{cases} y(x_0)=1 \\ y'(x_0)=0 \end{cases} \longleftrightarrow y_1(x)=\sum_{n=0}^{+\infty} b_n (x-x_0)^n$$

$$\begin{cases} y(x_0)=0 \\ y'(x_0)=1 \end{cases} \longleftrightarrow y_2(x)=\sum_{n=0}^{+\infty} c_n (x-x_0)^n$$

with $\forall n \in \mathbb{N} : \begin{cases} b_n = A_n(1,0 \mid p,q) \\ c_n = A_n(0,1 \mid p,q) \end{cases}$

To show that y_1, y_2 are indeed linearly independent we note that

$$\begin{cases} y_1(x_0)=1 \\ y_1'(x_0)=0 \end{cases} \wedge \begin{cases} y_2(x_0)=0 \\ y_2'(x_0)=1 \end{cases}$$

and therefore:

$$W[y_1,y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 0 = 1$$

$\Longrightarrow y_1, y_2$ linearly independent.

4) In practice it is customary to derive the recursion formula for the power series on a case by case basis. However, given the theorem, it is not necessary to prove convergence.

EXAMPLES

a) Find the general solution to the $\boxed{\text{Airy equation}}$ initial value problem

$$\begin{cases} y''(x) - xy(x) = 0 \\ y(0) = a_0 \wedge y'(0) = a_1 \end{cases}$$

Solution

Consider a solution of the form

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

and note that

$$y'(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$$

and

$$y''(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{+\infty} n(n+1) a_{n+1} x^{n-1} =$$

$$= \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n .$$

Then, we have:

$$y''(x) - xy(x) = 0 \Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - x \sum_{n=0}^{+\infty} a_n x^n = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{+\infty} a_n x^{n+1} = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{+\infty} a_{n-1} x^n = 0$$

$$\Leftrightarrow (0+1)(0+2)a_2 + \sum_{n=1}^{+\infty} [(n+1)(n+2)a_{n+2} - a_{n-1}] x^n = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N}^* : (n+1)(n+2)a_{n+2} - a_{n-1} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N}^* : a_{n+2} = \dfrac{a_{n-1}}{(n+1)(n+2)} \end{cases}$$

$$\Leftrightarrow \begin{cases} a_2 = 0 \\ \forall n \in \mathbb{N} - \{0,1,2\} : a_n = \dfrac{a_{n-3}}{n(n-1)} \end{cases}$$

▶ Now we can derive direct results for the sequence a_n. Starting from a_0, we get $a_3, a_6, \dots, a_{3k}, \dots$
we have:

$$\forall n \in \mathbb{N}^* : a_{3n} = a_0 \prod_{\lambda=1}^{n} \frac{1}{3\lambda(3\lambda-1)} = a_0 \prod_{\lambda=1}^{n} \frac{3\lambda-2}{3\lambda(3\lambda-1)(3\lambda-2)}$$

$$= a_0 \frac{1}{(3n)!} \prod_{\lambda=1}^{n} (3\lambda-2)$$

$$= a_0 \frac{3^n}{(3n)!} \prod_{\lambda=1}^{n} (\lambda - 2/3)$$

$$= a_0 \frac{3^n \Gamma(n-2/3+1)}{(3n)! \Gamma(-2/3+1)}$$

$$= a_0 \frac{3^n \Gamma(n+1/3)}{(3n)! \Gamma(1/3)}$$

and we note that this equation is also satisfied for $n=0$.

Starting from a_1, we get $a_4, a_7, \dots, a_{3n+1}, \dots$ and therefore

$$\forall n \in \mathbb{N}^* : a_{3n+1} = a_1 \prod_{\lambda=1}^{n} \frac{1}{(3\lambda+1)((3\lambda+1)-1)} =$$

$$= a_1 \prod_{\lambda=1}^{n} \frac{1}{3\lambda(3\lambda+1)} = a_1 \prod_{\lambda=1}^{n} \frac{3\lambda-1}{(3\lambda-1)(3\lambda)(3\lambda+1)}$$

$$= a_1 \frac{1}{(3n+1)!} \prod_{\lambda=1}^{n} (3\lambda-1) = a_1 \frac{3^n}{(3n+1)!} \prod_{\lambda=1}^{n} (\lambda - 1/3)$$

$$= a_1 \frac{3^n \Gamma(n-1/3+1)}{(3n+1)!\,\Gamma(-1/3+1)} = a_1 \frac{3^n \Gamma(n+2/3)}{(3n+1)!\,\Gamma(2/3)}$$

and we note that this equation is also satisfied for $n=0$.

Since $a_2 = 0$, it follows that $\forall n \in \mathbb{N} : a_{3n+2} = 0$

It follows that the general solution is:

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_{3n} x^{3n} + \sum_{n=0}^{+\infty} a_{3n+1} x^{3n+1}$$

$$= \sum_{n=0}^{+\infty} a_0 \frac{3^n \Gamma(n+1/3)}{(3n)!\,\Gamma(1/3)} x^{3n} + \sum_{n=0}^{+\infty} a_1 \frac{3^n \Gamma(n+2/3)}{(3n+1)!\,\Gamma(2/3)} x^{3n+1}$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

with

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{3^n \Gamma(n+1/3)}{(3n)!\,\Gamma(1/3)} x^{3n} \wedge y_2(x) = \sum_{n=0}^{+\infty} \frac{3^n \Gamma(n+2/3)}{(3n+1)!\,\Gamma(2/3)} x^{3n+1}$$

These series will converge uniformly on $\mathbb{R}$ and define the two linearly independent homogeneous solutions that span the null-space of the Airy equation.

↳ In the above argument, we have used the following identity

$$\boxed{\prod_{k=1}^{n} (k+a) = \frac{\Gamma(n+a+1)}{\Gamma(a+1)}}$$

to eliminate the products in the formula for $y_1(x)$ and $y_2(x)$ and extend their validity to from $n \in \mathbb{N}^*$ to $n \in \mathbb{N}$.

EXAMPLE

Solve the linear ODE: $y''(x) + \cos(x) y(x) = 0$.
with a series around $x = 0$.

Solution

We consider a solution of the form

$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$

and note that

$$y'(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n \Rightarrow$$

$$\Rightarrow y''(x) = \frac{d}{dx} \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{+\infty} n(n+1) a_{n+1} x^{n-1} =$$

$$= \sum_{n=0}^{+\infty} (n+1)(n+2) a_{n+2} x^n$$

and

$$(\cos x) y(x) = \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_n x^n \right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_{2n} x^{2n} + \sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1} \right] =$$

$$= \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_{2n} x^{2n} \right] + \left[\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \right] \left[\sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1} \right]$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{(2n-2k)}}{(2n-2k)!} a_{2k} x^{2k} \right] + \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\frac{x^{2n-2k}}{(2n-2k)!} a_{2k+1} x^{2k+1} \right]$$

$$= \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!} \right] x^{2n} + \sum_{n=0}^{+\infty} \left[\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!} \right] x^{2n+1}$$

It follows that

$y''(x) - \cos(x)y(x) = 0 \Leftrightarrow$

$$\Leftrightarrow \sum_{n=0}^{+\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} \frac{a_{2k}}{(2n-2k)!}\right]x^{2n} - $$

$$- \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} \frac{a_{2k+1}}{(2n-2k)!}\right]x^{2n+1} = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{n=0}^{+\infty}\left[(2n+1)(2n+2)a_{2n+2} - \sum_{k=0}^{n}\frac{a_{2k}}{(2n-2k)!}\right]x^{2n} +$$

$$+ \sum_{n=0}^{+\infty}\left[((2n+1)+1)((2n+1)+2)a_{(2n+1)+2} - \sum_{k=0}^{n}\frac{a_{2k+1}}{(2n-2k)!}\right]x^{2n+1} = 0$$

$$\Leftrightarrow \sum_{n=0}^{+\infty}\left[(2n+1)(2n+2)a_{2n+2} - \sum_{k=0}^{n}\frac{a_{2k}}{(2n-2k)!}\right]x^{2n} +$$

$$+ \sum_{n=0}^{+\infty}\left[(2n+2)(2n+3)a_{2n+3} - \sum_{k=0}^{n}\frac{a_{2k+1}}{(2n-2k)!}\right]x^{2n+1} = 0$$

$$\Leftrightarrow \forall n \in \mathbb{N}: \begin{cases} (2n+1)(2n+2)a_{2n+2} - \displaystyle\sum_{k=0}^{n}\frac{a_{2k}}{(2n-2k)!} = 0 \\ (2n+2)(2n+3)a_{2n+3} - \displaystyle\sum_{k=0}^{n}\frac{a_{2k+1}}{(2n-2k)!} = 0 \end{cases}$$

$$\Leftrightarrow \boxed{\forall n \in \mathbb{N}: \begin{cases} a_{2n+2} = \dfrac{1}{(2n+1)(2n+2)} \displaystyle\sum_{k=0}^{n}\frac{a_{2k}}{(2n-2k)!} \\ a_{2n+3} = \dfrac{1}{(2n+2)(2n+3)} \displaystyle\sum_{k=0}^{n}\frac{a_{2k+1}}{(2n-2k)!} \end{cases} \qquad (1)}$$

Initializing the power series requires a_0 and a_1.

Note that it is not possible to express the series in closed form. We can only use Eq. (1) to generate as many series terms as needed. To obtain two linearly independent solutions $y_1(x)$ and $y_2(x)$ we initialize Eq. (1) using $(a_0, a_1) = (1, 0)$ and $(a_0, a_1) = (0, 1)$ respectively, ~~to~~ This will yield the power series for $y_1(x)$ and $y_2(x)$.

In order to multiply power series expansions to calculate $\cos(x) y(x)$ we used Merten's theorem from my Calculus 2 lecture notes:

$$\boxed{\left\{\begin{array}{l}\displaystyle\sum_{n=0}^{+\infty} |a_n| \text{ converges}\\ \displaystyle\sum_{n=0}^{+\infty} b_n \text{ converges}\end{array}\right. \Longrightarrow \left[\sum_{n=0}^{+\infty} a_n\right]\left[\sum_{n=0}^{+\infty} b_n\right] = \sum_{n=0}^{+\infty}\left[\sum_{k=0}^{n} a_k b_{n-k}\right]}$$

The required assumptions are always satisfied by power series within their convergence interval.

EXERCISES

(10) Show that Hermite's equation $y''(x)-2xy'(x)+2ay(x)=0$ with $a\in(0,+\infty)$ has the following linearly independent solutions:

$$y_1(x)=\Gamma(1+a/2)\sum_{n=0}^{+\infty}\frac{(-1)^n(2x)^n}{(2n)!\,\Gamma(a/2-n+1)}$$

$$y_2(x)=\Gamma(1/2+a/2)\sum_{n=0}^{+\infty}\frac{(-1)^n(2x)^{n+1}}{(2n+1)!\,\Gamma(a/2-n+1/2)}$$

(11) Show that Chebyshev's equation

$$(1-x^2)y''(x)-xy'(x)+a^2y(x)=0$$

with $a\in(0,+\infty)$ has the following linearly independent solutions

$$y_1(x)=1+\sum_{n=1}^{+\infty}\frac{1}{(2n)!}\left[\prod_{k=0}^{n-1}(4k^2-a^2)\right]x^{2n}$$

$$y_2(x)=x+\sum_{n=1}^{+\infty}\frac{1}{(2n+1)!}\left[\prod_{k=0}^{n-1}(4k^2+4k+1-a^2)\right]x^{2n+1}$$

(12) Show that the equation $y''(x)+xy'(x)+y(x)=0$ has the following linearly independent solutions:

$$y_1(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2^n n!}x^{2n}$$

$$y_2(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n 2^n n!}{(2n+1)!}x^{2n+1}$$

(13) Show that the equation $y''(x)+x^2y'(x)+xy(x)=0$ has the following linearly independent solutions:

$$y_1(x) = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{(3n)!}\left[\prod_{k=1}^{n}(3k-2)^2\right]x^{3n}$$

$$y_2(x) = x + \sum_{n=1}^{+\infty} \frac{(-1)^n}{(3n+1)!}\left[\prod_{k=1}^{n}(3k-1)^2\right]x^{3n+1}$$

(14) Show that the equation $y''(x)+x^2y(x)=0$ has the following linearly independent solutions:

$$y_1(x) = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{4^n n!}\left[\prod_{k=1}^{n}\frac{1}{4k-1}\right]x^{4n}$$

$$y_2(x) = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{4^n n!}\left[\prod_{k=0}^{n}\frac{1}{4k+1}\right]x^{4n+1}$$

(15) Consider the equation $y''(x)+a^2y(x)=0$ with $a\in\mathbb{R}$. Use the power-series method to "rediscover" the well-known general solution $y(x)=\lambda_1\cos(ax)+\lambda_2\sin(ax)$.

② → Regular singular linear ODEs (Frobenius method)

We consider a linear ODE of the form

$$y''(x) + \frac{p(x)}{x - x_0} y'(x) + \frac{q(x)}{(x-x_0)^2} y(x) = 0 \qquad (1)$$

or equivalently

$$(x-x_0)^2 y''(x) + (x-x_0) p(x) y'(x) + q(x) y(x) = 0$$

with p, q analytic at $x = x_0$ with power-series expansions

$$\forall x \in (x_0 - \mu, x_0 + \mu) : \left(p(x) = \sum_{n=0}^{+\infty} p_n (x-x_0)^n \wedge q(x) = \sum_{n=0}^{+\infty} q_n (x-x_0)^n \right)$$

Since $x = x_0$ is not a regular point, the ODE does not admit linearly independent solutions $y_1(x), y_2(x)$ that can be expressed as a power series. Nonetheless, a general solution method for Eq.(1), where $x = x_0$ is a regular singular point, has been developed by Frobenius as follows.

Prop: Consider a function y defined as

$$y(x) = |x - x_0|^\lambda \sum_{n=0}^{+\infty} a_n (x-x_0)^n$$

If $y(x)$ solves Eq.(1), then:

(a) $F(\lambda | p_0, q_0) \equiv \lambda(\lambda - 1) + p_0 \lambda + q_0 = 0$

(b) $\forall n \in \mathbb{N}^* : F(\lambda + n | p_0, q_0) a_n = - \sum_{k=0}^{n-1} [(k+\lambda) p_{n-k} + q_{n-k}] a_k$

Remarks

(a) The polynomial $F(\lambda|p_0,q_0)$ is the indicial polynomial and the equation

$$\lambda(\lambda-1)+p_0\lambda+q_0=0$$

is the indicial equation associated with the linear ODE Eq.(1).

(b) Using the recurrence for the sequence a_n given by the above proposition with a given initial value a_0, we can show that $a_1, a_2, \dots, a_n, \dots$ are proportional to a_0 and the resulting sequence will be denoted as

$$\forall n \in \mathbb{N}^*: \ a_n = a_0 \Phi_n(\lambda|p,q)$$

with $p,q \in \mathrm{Seq}(\mathbb{R})$ representing the sequences $p_1, p_2, \dots, p_n, \dots$ and $q_1, q_2, \dots, q_n, \dots$ Note that Φ_n is independent of x_0.

(c) We may now define the general function

$$y(x,\lambda|p,q) = |x-x_0|^\lambda \sum_{n=0}^{+\infty} \Phi_n(\lambda|p,q)(x-x_0)^n$$

For most values of λ this function does not solve Eq.(1). From the following propositions we see that $y(x,\lambda|p,q)$ solves Eq.(1) when λ is one of the zeroes of the indicial equation.

Prop: If p,q converge on $(x_0-\mu, x_0+\mu)$ then the series expansion for $y(x,\lambda|p,q)$ also converges both uniformly and absolutely on $(x_0-\mu, x_0+\mu)$.

<u>Prop</u>: Let $A = (x_0 - \mu, x_0 + \mu)$ and let $L: C^2(A) \to C^0(A)$ be the linear operator associated with the linear ODE Eq.(1) such that

$$\forall y \in C^2(A): (Ly)(x) = y''(x) + \frac{p(x)}{x - x_0} y'(x) + \frac{q(x)}{(x-x_0)^2} y(x).$$

It follows that

$$\begin{aligned} Ly(x,\lambda|p,q) &= |x - x_0|^{\lambda-2} F(\lambda|p_0,q_0) \\ &= |x-x_0|^{\lambda-2}[\lambda(\lambda-1) + p_0\lambda + q_0] \end{aligned}$$

Using the above results and notations, and some additional considerations needed for the proofs, we establish the main result:

<u>Thm</u>: Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the zeroes of the indicial polynomial $F(\lambda|p_0,q_0)$ and with no loss of generality we assume that $Re(\lambda_1) \geq Re(\lambda_2)$. We then distinguish between the following cases:

<u>Case 1</u>: If $\lambda_1 \neq \lambda_2 \wedge \lambda_1 - \lambda_2 \notin \mathbb{N}^*$ then Eq.(1) has two linearly independent solutions given by:

$$\forall x \in (x_0 - \mu, x_0 + \mu): \begin{cases} y_1(x) = y(x,\lambda_1|p,q) = |x-x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1|p,q)(x-x_0)^n \\ y_2(x) = y(x,\lambda_2|p,q) = |x-x_0|^{\lambda_2} \sum_{n=0}^{+\infty} \phi_n(\lambda_2|p,q)(x-x_0)^n \end{cases}$$

<u>Case 2</u>: If $\lambda_1 = \lambda_2$, then the two linearly independent solutions are

$$y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n$$

$$y_2(x) = \frac{\partial}{\partial \lambda} y(x, \lambda | p, q) \Big|_{\lambda = \lambda_1} =$$

$$= y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} b_n (x - x_0)^n$$

with $\forall n \in \mathbb{N} : b_n = \frac{\partial}{\partial \lambda} \phi_n(\lambda | p, q) \Big|_{\lambda = \lambda_1}$

Case 3: If $\lambda_1 - \lambda_2 = N \in \mathbb{N}^*$, then the two linearly independent solutions are

$$y_1(x) = y(x, \lambda_1 | p, q) = |x - x_0|^{\lambda_1} \sum_{n=0}^{+\infty} \phi_n(\lambda_1 | p, q)(x - x_0)^n$$

$$y_2(x) = \frac{\partial}{\partial \lambda} \left[(\lambda - \lambda_2) y(x, \lambda | p, q) \right] \Big|_{\lambda = \lambda_2} =$$

$$= C_1 y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_2} \sum_{n=0}^{+\infty} c_n (x - x_0)^n$$

with $C_1 = \lim_{\lambda \to \lambda_2} \left[(\lambda - \lambda_2) \phi_N(\lambda | p, q) \right]$

$$\forall n \in \mathbb{N} : \quad c_n = \frac{\partial}{\partial \lambda} \left[(\lambda - \lambda_2) \phi_n(\lambda | p, q) \right] \Big|_{\lambda = \lambda_2}$$

↳ Given the solutions $y_1(x)$ and $y_2(x)$, the general solution is:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

with $c_1, c_2 \in \mathbb{R}$.

Methodology / Remarks

(a) It is recommended that you use the above theorems and propositions to determine the indicial polynomial and the recurrence relationship defining the sequence $a_n = a_0 \Phi_n(\lambda | p, q)$. Although both can be obtained from substituting the solution forms to the original ODE, that tends to be cumbersome.

(b) An explicit expression for a_n as a function of λ is needed for cases 2, 3 in order to differentiate them with respect to λ. For case 1 it is not needed, and it is sufficient to have explicit equations for a_n only for $\lambda = \lambda_1$ and $\lambda = \lambda_2$

(c) For the calculation of $y_2(x)$ in cases 2, 3 it is often necessary to calculate the derivatives (with respect to λ) of a function defined as a product or ratio of a large number of factors. A technique known as logarithmic differentiation can be used to evaluate such products as follows:

$$\boxed{\frac{d}{dx}\prod_{a=1}^{n}[f_a(x)]^{c_a} = \prod_{a=1}^{n}[f_a(x)]^{c_a}\left[\sum_{a=1}^{n} c_a \frac{f_a'(x)}{f_a(x)}\right]}$$

as long as $\forall a \in [n]: f_a(x) \neq 0$.

(d) Gamma functions are used to simplify linear products:

$$\boxed{\prod_{k=1}^{n}(ak+b) = a^n \prod_{k=1}^{n}(k+b/a) = \frac{a^n \Gamma(n+1+b/a)}{\Gamma(1+b/a)}}$$

EXAMPLES

a) Solve the linear ODE

$$x^2 y''(x) + x(x-1/2)y'(x) + (1/2)y(x) = 0$$

with a series around $x=0$.

Solution

We rewrite the ODE as:

$$y''(x) + \frac{1}{x}(x-1/2)y'(x) + \frac{1}{x^2}\frac{1}{2}y(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow y''(x) + \frac{p(x)}{x}y'(x) + \frac{q(x)}{x^2}y(x) = 0$$

with $p(x) = x - 1/2 \rightarrow p_0 = -1/2 \wedge p_1 = 1 \wedge p_2 = p_3 = \cdots = 0$

and $q(x) = 1/2 \Rightarrow q_0 = 1/2 \wedge q_1 = q_2 = \cdots = 0$

Consider a solution

$$y(x) = |x|^\lambda \sum_{n=0}^{+\infty} a_n x^n$$

Substituting to the ODE gives the indicial polynomial

$$F(\lambda) = \lambda(\lambda-1) + p_0\lambda + q_0 = \lambda(\lambda-1) - (1/2)\lambda + 1/2 =$$

$$= \lambda(\lambda-1) - (1/2)(\lambda-1) = (\lambda-1/2)(\lambda-1)$$

and the recurrence

$$\forall n \in \mathbb{N}^*: F(\lambda+n)a_n = -\sum_{k=0}^{n-1}[(k+\lambda)p_{n-k} + q_{n-k}]a_k =$$

$$= -[(n-1+\lambda)p_1 + q_1]a_{n-1} - \sum_{k=0}^{n-2}[(k+\lambda)p_{n-k} + q_{n-k}]a_k$$

$$= -[(n+\lambda-1)p_1 + 0]a_{n-1} = -(n+\lambda-1)a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda+n-1/2)(\lambda+n-1)a_n = -(\lambda+n-1)a_{n-1} \Leftrightarrow$$
$$\Leftrightarrow (\lambda+n-1/2)a_n = -a_{n-1} \Leftrightarrow a_n = \frac{-1}{\lambda+n-1/2}a_{n-1}.$$

It follows that
$$\forall n\in\mathbb{N}^*: a_n = a_0\prod_{k=1}^{n}\left(\frac{-1}{\lambda+k-1/2}\right) = a_0(-1)^n\prod_{k=1}^{n}\frac{1}{\lambda+k-1/2}.$$

Solving the indicial equation:
$$F(\lambda)=0 \Leftrightarrow (\lambda-1/2)(\lambda-1)=0 \Leftrightarrow \lambda-1/2=0 \vee \lambda-1=0 \Leftrightarrow$$
$$\Leftrightarrow \lambda=1/2 \vee \lambda=1.$$

For $\lambda=1/2$:
$$\forall n\in\mathbb{N}^*: a_n = a_0\prod_{k=1}^{n}\frac{-1}{1/2+k-1/2} = a_0(-1)^n\prod_{k=1}^{n}\frac{1}{k} =$$
$$= a_0\frac{(-1)^n}{n!}$$

and therefore the first homogeneous solution is:
$$y_1(x) = |x|^{1/2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}x^n = |x|^{1/2}\sum_{n=0}^{+\infty}\frac{(-x)^n}{n!} = |x|^{1/2}e^{-x}$$

For $\lambda=1$:
$$\forall n\in\mathbb{N}^*: a_n = a_0\prod_{k=1}^{n}\frac{-1}{1+k-1/2} = a_0(-1)^n\prod_{k=1}^{n}\frac{1}{k+1/2} =$$
$$= a_0(-1)^n\left[\prod_{k=1}^{n}(k+1/2)\right]^{-1} = a_0(-1)^n\left[\frac{\Gamma(n+1+1/2)}{\Gamma(1+1/2)}\right]^{-1}$$
$$= a_0\frac{(-1)^n\Gamma(3/2)}{\Gamma(n+3/2)}$$

and therefore the second homogeneous solution is:

$$y_2(x) = |x| \sum_{n=0}^{+\infty} \frac{(-1)^n \Gamma(3/2)}{\Gamma(n+3/2)} x^n$$

The general solution is:

$$y(x) = \lambda_1 |x|^{1/2} e^{-x} + \lambda_2 |x| \sum_{n=0}^{+\infty} \frac{(-1)^n \Gamma(3/2)}{\Gamma(n+3/2)} x^n.$$

Since p, q converge on $\mathbb{R}$, the general solution $y(x)$ converges on $\mathbb{R}$.

b) Solve the linear ODE

$$x(1-x)y''(x) + (1-x)y'(x) - y(x) = 0$$

around $x=0$.

<u>Solution</u>

We note that

$$x(1-x)y''(x) + (1-x)y'(x) - y(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow y''(x) + \frac{1-x}{x(1-x)}y'(x) - \frac{1}{x(x-1)}y(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow y''(x) + \frac{1}{x}y'(x) + \frac{1}{x^2}\frac{-x}{x-1}y(x) = 0$$

$$\Leftrightarrow y''(x) + \frac{1}{x}p(x)y'(x) + \frac{1}{x^2}q(x)y(x) = 0$$

with $p(x) = 1 = \sum_{n=0}^{+\infty} p_0 x^n \Rightarrow p_0 = 1 \wedge p_1 = p_2 = \dots = 0$

and $q(x) = \frac{-x}{1-x} = (-x)\frac{1}{1-x} = (-x)\sum_{n=0}^{+\infty} x^n =$

$$= \sum_{n=0}^{+\infty}(-x^{n+1}) = \sum_{n=1}^{+\infty}(-1)x^n = \sum_{n=0}^{+\infty} q_n x^n \Rightarrow$$

$$\Rightarrow q_0 = 0 \wedge q_1 = q_2 = \dots = -1$$

Note that the convergence interval for $q(x)$ is $(-1,1)$.

Using a candidate solution

$$y(x) = |x|^\lambda \sum_{n=0}^{+\infty} a_n x^n$$

we find that the indicial polynomial is:

$$F(\lambda) = \lambda(\lambda-1) + p_0\lambda + q_0 = \lambda(\lambda-1) + \lambda + 0 = \lambda(\lambda-1+1) = \lambda^2$$

and the sequence a_n must satisfy

$$\forall n \in \mathbb{N}^*: F(\lambda+n)a_n = -\sum_{k=0}^{n-1} [(k+\lambda)p_{n-k} + q_{n-k}]a_k =$$

$$= -\sum_{k=0}^{n-1} (k+\lambda)p_{n-k}a_k - \sum_{k=0}^{n-1} q_{n-k}a_k =$$

$$= -0 - \sum_{k=0}^{n-1} (-1)a_k = a_0 + a_1 + \cdots + a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow (\lambda+n)^2 a_n = a_0 + a_1 + \cdots + a_{n-1}, \quad \forall n \in \mathbb{N}^* \qquad (1)$$

For $n=1$: $(\lambda+1)^2 a_1 = a_0 \Leftrightarrow a_1 = \dfrac{a_0}{(\lambda+1)^2}$ (2)

For $n \geq 2$ we note that $(\lambda+n-1)^2 a_{n-1} = a_0 + a_1 + \cdots + a_{n-2}$ and therefore from Eq. (1)

$$(1) \Leftrightarrow (\lambda+n)^2 a_n = (a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1} =$$

$$= (\lambda+n-1)^2 a_{n-1} + a_{n-1} =$$

$$= [(\lambda+n-1)^2 + 1] a_{n-1} \Leftrightarrow$$

$$\Leftrightarrow a_n = \frac{(\lambda+n-1)^2+1}{(\lambda+n)^2} a_{n-1} \qquad (3)$$

Note that for $n=1$, $(\lambda+n-1)^2+1 = (\lambda+1-1)^2+1 = \lambda^2+1 \neq 0$ so equation (3) does not reduce to equation (2) for $n=1$. To mitigate that, we choose $a_0 = \lambda^2+1$. Then:

$$a_1 = \frac{\lambda^2+1}{(\lambda+1)^2} = \frac{(\lambda+1-1)^2+1}{(\lambda+1)^2}$$

and it follows that

$$\forall n \in \mathbb{N}^*: a_n = \prod_{k=1}^{n} \frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}$$

Solving the indicial equation gives:

$F(\lambda)=0 \Leftrightarrow \lambda^2=0 \Leftrightarrow \lambda=0 \longleftarrow$ double zero.

For $\lambda=0$:

$$\forall n\in\mathbb{N}^*: \quad a_n=\prod_{k=1}^{n}\frac{(0+k-1)^2+1}{(0+k)^2}=\frac{1}{(n!)^2}\prod_{k=1}^{n}[(k-1)^2+1]$$
$$=\frac{1}{(n!)^2}\prod_{k=0}^{n-1}(k^2+1)$$

and $a_0=0^2+1=1$, therefore the first homogeneous solution is given by

$$y_1(x)=1+\sum_{n=1}^{+\infty}\left[\frac{1}{(n!)^2}\prod_{k=0}^{n-1}(k^2+1)\right]x^n.$$

and the second linearly independent solution is given by:

$$y_2(x)=y_1(x)\ln|x|+\sum_{n=0}^{+\infty}b_nx^n \quad \text{with } b_n=\left.\frac{\partial a_n}{\partial\lambda}\right|_{\lambda=0}$$

To calculate b_n, we note that

$$\frac{\partial a_0}{\partial\lambda}=\frac{\partial}{\partial\lambda}(\lambda^2+1)=2\lambda \Rightarrow b_0=\left.\frac{\partial a_0}{\partial\lambda}\right|_{\lambda=0}=\left.2\lambda\right|_{\lambda=0}=0$$

and

$$\forall n\in\mathbb{N}^*: \frac{\partial a_n}{\partial\lambda}=\frac{\partial}{\partial\lambda}\prod_{k=1}^{n}\frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}=$$
$$=\prod_{k=1}^{n}\left(\frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}\right)\left[\sum_{k=1}^{n}\frac{(\partial/\partial\lambda)[(\lambda+k-1)^2+1]}{(\lambda+k-1)^2+1}-\right.$$
$$\left.-2\sum_{k=1}^{n}\frac{(\partial/\partial\lambda)(\lambda+k)}{\lambda+k}\right]=$$

$$= \left[\prod_{k=1}^{n} \frac{(\lambda+k-1)^2+1}{(\lambda+k)^2}\right]\left[\sum_{k=1}^{n} \frac{2(\lambda+k-1)}{(\lambda+k-1)^2+1} - 2\sum_{k=1}^{n} \frac{1}{\lambda+k}\right] \Rightarrow$$

$$\Rightarrow b_n = \left.\frac{\partial a_n}{\partial \lambda}\right|_{\lambda=0} = a_n \sum_{k=1}^{n}\left[\frac{2(k-1)}{(k-1)^2+1} - \frac{2}{k}\right] =$$

$$= a_n \sum_{k=1}^{n}\left[\frac{2(k-1)k - 2[(k-1)^2+1]}{k[(k-1)^2+1]}\right] =$$

$$= a_n \sum_{k=1}^{n}\left[\frac{2k^2-2k-2(k^2-2k+1+1)}{k(k^2-2k+1+1)}\right] =$$

$$= a_n \sum_{k=1}^{n} \frac{2k^2-2k-2k^2+4k-4}{k(k^2-2k-2)} =$$

$$= a_n \sum_{k=1}^{n} \frac{2k-4}{k(k^2-2k-2)} =$$

$$= \frac{1}{(n!)^2}\prod_{k=0}^{n-1}(k^2+1)\left[\sum_{k=1}^{n}\frac{2(k-2)}{k(k^2-2k-2)}\right]$$

It follows that the second solution is given by

$$y_2(x) = y_1(x)\ln|x| + \sum_{n=1}^{+\infty}\left[\frac{1}{(n!)^2}\left(\prod_{k=0}^{n-1}(k^2+1)\right)\left(\sum_{k=1}^{n}\frac{2(k-2)}{k(k^2-2k-2)}\right)\right]x^n$$

and the general solution is $y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$.
The solution will converge on $(-1,1)$ since p converges on $\mathbb{R}$ and q converges on $(-1,1)$.

c) Solve the linear ODE $xy''(x)+2y'(x)-y(x)=0$ with a series around $x=0$

Solution

We note that

$$xy''(x)+2y'(x)-y(x)=0 \Leftrightarrow y''(x)+(2/x)y'(x)-(1/x)y(x)=0$$
$$\Leftrightarrow y''(x)+(1/x)2y'(x)+(1/x^2)(-x)y(x)=0$$
$$\Leftrightarrow y''(x)+(1/x)p(x)y'(x)+(1/x^2)q(x)y(x)=0$$

with

$$p(x)=2=\sum_{n=0}^{+\infty} p_n x^n \Rightarrow p_0=2 \wedge p_1=p_2=\cdots=0$$

and

$$q(x)=-x=\sum_{n=0}^{+\infty} q_n x^n \Rightarrow q_0=0 \wedge q_1=-1 \wedge q_2=q_3=\cdots=0.$$

Using a candidate solution $y(x)=|x|^\lambda \sum_{n=0}^{+\infty} a_n x^n$ the corresponding indicial polynomial is

$$F(\lambda)=\lambda(\lambda-1)+p_0\lambda+q_0=\lambda(\lambda-1)+2\lambda=\lambda(\lambda-1+2)=\lambda(\lambda+1)$$

and a_n satisfies:

$$\forall n\in\mathbb{N}^*: F(\lambda+n)a_n=-\sum_{k=0}^{n-1}[(k+\lambda)p_{n-k}+q_{n-k}]a_k=$$
$$=-\sum_{k=0}^{n-1}(k+\lambda)p_{n-k}a_k-\sum_{k=0}^{n-1}q_{n-k}a_k=$$
$$=-0-q_1a_{n-1}=-(-1)a_{n-1}=a_{n-1}\Leftrightarrow$$
$$\Leftrightarrow (\lambda+n)(\lambda+n+1)a_n=a_{n-1}\Leftrightarrow a_n=\frac{1}{(\lambda+n)(\lambda+n+1)}a_{n-1}$$

and therefore $\forall n\in\mathbb{N}^*: a_n=a_0\prod_{k=1}^{n}\frac{1}{(\lambda+k+1)(\lambda+k)}$

Solving the indicial equation gives:

$F(\lambda)=0 \Leftrightarrow \lambda(\lambda+1)=0 \Leftrightarrow \lambda=0 \vee \lambda+1=0 \Leftrightarrow \lambda=0 \vee \lambda=-1.$

For $\lambda=0$, we have

$$a_n = a_0 \prod_{k=1}^{n} \frac{1}{(0+k+1)(0+k)} = a_0 \prod_{k=1}^{n} \frac{1}{k(k+1)} =$$

$$= a_0 \left[\prod_{k=1}^{n} \frac{1}{k}\right]\left[\prod_{k=1}^{n} \frac{1}{k+1}\right] =$$

$$= \frac{a_0}{n!} \prod_{k=2}^{n+1} \frac{1}{k} = \frac{a_0}{n!} \prod_{k=1}^{n+1} \frac{1}{k} = \frac{a_0}{n!\,(n+1)!} =$$

$$= \frac{a_0}{(n!)^2(n+1)}, \quad \forall n \in \mathbb{N}^*$$

and the corresponding solution is:

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(n!)^2(n+1)}$$

Since $0-(-1)=1$, the second solution is

$$y_2(x) = C_1 y_1(x) \ln|x| + |x|^{-1} \sum_{n=0}^{+\infty} c_n x^n$$

Using $a_0(\lambda)=a_0$, we have:

$$C_1 = \lim_{\lambda\to-1} \left[(\lambda-(-1))\, a_1(\lambda)\right] = \lim_{\lambda\to-1} \left[(\lambda+1) \frac{a_0}{(\lambda+1)(\lambda+2)}\right] =$$

$$= \lim_{\lambda\to-1} \frac{a_0}{\lambda+2} = \frac{a_0}{-1+2} = a_0$$

and

$$c_n = \frac{\partial}{\partial\lambda}\left[(\lambda-(-1))a_n(\lambda)\right]\Big|_{\lambda=-1} = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_n(\lambda)\right]\Big|_{\lambda=-1}$$

$$= \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\prod_{k=1}^{n}\frac{1}{(\lambda+k+1)(\lambda+k)}\right]\Bigg|_{\lambda=-1}, \quad \forall n\in\mathbb{N}^*$$

We distinguish between the following cases.

For $n=0$:

$$c_0 = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\right]\Big|_{\lambda=-1} = a_0\Big|_{\lambda=-1} = a_0$$

For $n=1$:

$$c_1 = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\frac{1}{(\lambda+1+1)(\lambda+1)}\right]\Bigg|_{\lambda=-1} = \frac{\partial}{\partial\lambda}\left[\frac{a_0}{\lambda+2}\right]\Bigg|_{\lambda=-1}$$

$$= \left[\frac{-a_0(\partial/\partial\lambda)(\lambda+2)}{(\lambda+2)^2}\right]\Bigg|_{\lambda=-1} = \left[\frac{-a_0}{(\lambda+2)^2}\right]_{\lambda=-1} =$$

$$= \frac{-a_0}{(-1+2)^2} = \frac{-a_0}{1^2} = -a_0 .$$

For $n>1$:

$$c_n = \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_n(\lambda)\right]\Big|_{\lambda=-1} =$$

$$= \frac{\partial}{\partial\lambda}\left[(\lambda+1)a_0\prod_{k=1}^{n}\left(\frac{1}{(\lambda+k+1)(\lambda+k)}\right)\right]\Bigg|_{\lambda=-1} =$$

$$= \frac{\partial}{\partial\lambda}\left[a_0\prod_{k=1}^{n}\left(\frac{1}{\lambda+k+1}\right)\prod_{k=2}^{n}\left(\frac{1}{\lambda+k}\right)\right]\Bigg|_{\lambda=-1} =$$

$$= a_0\frac{\partial}{\partial\lambda}\left[\frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\right]\Bigg|_{\lambda=-1} =$$

$$= a_0 \frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\left[\frac{-(\partial/\partial\lambda)(\lambda+n+1)}{\lambda+n+1}+\sum_{k=2}^{n}\frac{-2(\partial/\partial\lambda)(\lambda+k)}{\lambda+k}\right]\Bigg|_{\lambda=-1}$$

$$= -a_0 \frac{1}{\lambda+n+1}\left(\prod_{k=2}^{n}\frac{1}{\lambda+k}\right)^2\left[\frac{1}{\lambda+n+1}+\sum_{k=2}^{n}\frac{2}{\lambda+k}\right]\Bigg|_{\lambda=-1}$$

$$= -a_0 \frac{1}{-1+n+1}\left(\prod_{k=2}^{n}\frac{1}{-1+k}\right)^2\left[\frac{1}{-1+n+1}+\sum_{k=2}^{n}\frac{2}{-1+k}\right]$$

$$= -a_0 \frac{1}{n}\left(\prod_{k=1}^{n-1}\frac{1}{k}\right)^2\left[\frac{1}{n}+\sum_{k=1}^{n-1}\frac{2}{k}\right] =$$

$$= -a_0 \frac{1}{n[(n-1)!]^2}\left[\frac{-1}{n}+\sum_{k=1}^{n}\frac{2}{k}\right]$$

$$= \frac{-a_0}{n!(n-1)!}\left[\frac{-1}{n}+2\sum_{k=1}^{n}\frac{1}{k}\right]$$

Note that this result, for $n=1$, agrees with our previous result for $n=1$. It follows that the second solution is

$$y_2(x) = y_1(x)\ln|x| + |x|^{-1}\left[1-\sum_{n=1}^{+\infty}\frac{1}{n!(n-1)!}\left[\frac{-1}{n}+2\sum_{k=1}^{n}\frac{1}{k}\right]x^n\right]$$

The general solution is $y(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$.

EXERCISES

(16) Show that the equation $4xy''(x) + 2y'(x) + y(x) = 0$ has the following linearly independent solutions

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^n$$

$$y_2(x) = |x|^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^n$$

(17) Show that the equation $9x^2 y''(x) + 9xy'(x) + (9x^2 - 1)y(x) = 0$ has the following linearly independent solutions

$$y_1(x) = |x|^{1/3} \left[1 + \sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{(2n)!!} \left[\prod_{k=1}^{n} \frac{1}{6k+2} \right] x^{2n} \right]$$

$$y_2(x) = |x|^{-1/3} \left[1 + \sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{(2n)!!} \left[\prod_{k=1}^{n} \frac{1}{6k-2} \right] x^{2n} \right]$$

(18) Show that the equation $x^2 y'' + (x^2 - 7/36)$

$x^2 y''(x) + (x^2 - 7/36) y(x) = 0$

has the following linearly independent solutions

$$y_1(x) = |x|^{7/6} \left[1 + \sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{2^{2n} n!} \left[\prod_{k=1}^{n} \frac{1}{3k+2} \right] x^{2n} \right]$$

$$y_2(x) = |x|^{-1/6} \left[1 + \sum_{n=1}^{+\infty} \frac{(-1)^n 3^n}{2^{2n} n!} \left[\prod_{k=1}^{n} \frac{1}{3k-2} \right] x^{2n} \right]$$

(19) Show that the equation $x^2 y''(x) + (x^2 - x) y'(x) + y(x) = 0$ has the following linearly independent solution

$$y_1(x) = |x| \exp(-x)$$
$$y_2(x) = y_1(x) \ln|x| + |x| \left[\sum_{n=1}^{+\infty} \frac{(-1)^{n+1} \varphi(n)}{n!} x^n \right]$$

with $\varphi(n) = \sum_{K=1}^{n} \frac{1}{K}$

(20) Show that the equation

$$x(1-x) y''(x) + (1-5x) y'(x) - 4y(x) = 0$$

has the following linearly independent solutions:

$$y_1(x) = \sum_{n=0}^{+\infty} (1+n)^2 x^n$$

$$y_2(x) = y_1(x) \ln|x| - 2 \sum_{n=1}^{+\infty} n(n+1) x^n$$

(21) Show that the equation

$$(x^2 + x^3) y''(x) - (x + x^2) y'(x) + y(x) = 0$$

has the following linearly independent solutions:

$$y_1(x) = x(1+x)$$
$$y_2(x) = y_1(x) \ln|x| + |x| \left[-2x - \sum_{n=2}^{+\infty} \frac{(-1)^n}{n(n-1)} x^n \right]$$

(22) Show that the equation

$$x^2 y''(x) + 2x y'(x) + x y(x) = 0$$

has the following linearly independent solutions:

$$y_1(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,(n+1)!} x^n$$

$$y_2(x) = -y_1(x)\ln|x| + |x|^{-1}\left[1 - \sum_{n=1}^{+\infty} \frac{(-1)^n(2\varphi(n-1)+1/n)}{n!\,(n-1)!} x^n\right]$$

with $\forall n \in \mathbb{N}^*: \varphi(n) = \sum_{k=1}^{n} (1/k)$.

23 Show that the equation $x(1-x)y''(x) - 3xy'(x) - y(x) = 0$ has the following linearly independent solutions

$y_1(x) = x(1-x)^{-2}$

$y_2(x) = y_1(x)\ln|x| + (1-x)^{-1}$

↳ Use the Frobenius method to solve the above differential equations.

▼ Theory of Bessel functions

● Summary of main results

The Bessel function $J_a(x)$ is defined via the following power series:

$$\boxed{\forall a \in \mathbb{R} : \forall x \in \mathbb{R}-\{0\} : J_a(x) = \left|\frac{x}{2}\right|^a \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+a)} \left(\frac{x}{2}\right)^{2n}}$$

For integer $a = m \in \mathbb{N}$, the above definition reduces to

$$\boxed{\forall x \in \mathbb{R}-\{0\} : \quad J_m(x) = \left|\frac{x}{2}\right|^m \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,(n+m)!} \left(\frac{x}{2}\right)^{2n}}$$

This function arises from using the Frobenius method to solve the Bessel equation, which is given by

$$\boxed{x^2 y''(x) + x y'(x) + (x^2 - a^2) y(x) = 0 , \quad \forall x \in \mathbb{R}-\{0\}}$$

With no loss of generality we will assume that $a \geq 0$ (since the transformation $a \to -a$ leaves the Bessel equation invariant). To write the general solution to the Bessel equation we distinguish between the following cases:

Case 1 : If $a \notin \mathbb{N}$ (with $a > 0$), then the general solution is
$\forall x \in \mathbb{R} - \{0\}: \; y(x) = \lambda_1 J_a(x) + \lambda_2 J_{-a}(x)$

Case 2 : If $a = 0$, then the general solution is
$\forall x \in \mathbb{R} - \{0\}: \; y(x) = \lambda_1 J_0(x) + \lambda_2 J^0(x)$
with $J^0(x)$ given by

$$\forall x \in \mathbb{R} - \{0\}: \; J^0(x) = J_0(x) \ln|x| - \sum_{n=1}^{+\infty} \frac{(-1)^n \varphi(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

with $\forall n \in \mathbb{N}^*: \; \varphi(n) = \sum_{k=1}^{n} \frac{1}{k}$ and $\varphi(0) = 0$.

Case 3 : If $a \in \mathbb{N}^*$, then the general solution is
$\forall x \in \mathbb{R} - \{0\}: \; y(x) = \lambda_1 J_a(x) + \lambda_2 J^a(x)$
with $J^a(x)$ given by

$$\forall x \in \mathbb{R} - \{0\}: \; J^a(x) = J_a(x) \ln|x| - \frac{1}{2}\left(\frac{x}{2}\right)^{-a} \sum_{n=0}^{a-1} \frac{(a-n-1)!}{n!} \left(\frac{x}{2}\right)^{2n}$$

$$- \frac{1}{2}\left(\frac{x}{2}\right)^{a} \sum_{n=0}^{+\infty} \frac{(-1)^n [\varphi(n) + \varphi(n+a)]}{n!\,(n+a)!} \left(\frac{x}{2}\right)^{2n}$$

The above results can be obtained by application of the Frobenius method and a lot of tedious calculations.

Properties of Bessel functions

We prove some interesting properties of Bessel functions and leave the rest as exercises.

① $\rightarrow$ $$\boxed{\forall x \in \mathbb{R}^*: \forall t \in \mathbb{R}^*;\ G(x,t) \equiv \exp\left(\frac{1}{2}x\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{+\infty} J_n(x)t^n}$$

Proof

Let $x \in \mathbb{R}^*$ and $t \in \mathbb{R}^*$ be given. It follows that

$$G(x,t) = \exp((1/2)x(t-1/t)) = \exp((1/2)xt)\exp(-(1/2)(x/t))$$

$$= \left[\sum_{p=0}^{+\infty} \frac{1}{p!}\left(\frac{xt}{2}\right)^p\right]\left[\sum_{q=0}^{+\infty} \frac{1}{q!}\left(\frac{-x}{2t}\right)^q\right] =$$

$$= \left[\sum_{p=0}^{+\infty} \frac{x^p t^p}{2^p p!}\right]\left[\sum_{q=0}^{+\infty} \frac{(-1)^q x^q}{2^q q! t^q}\right] =$$

$$= \sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\left[\frac{(-1)^q x^{p+q} t^{p-q}}{2^{p+q}\, p!\, q!}\right]$$

Let $n=p-q$. Then n ranges from $-\infty$ to $+\infty$ and we replace the sum over p with a sum over n. The sum over q is retained. We note that $p=n+q$ and $p+q=n+2q$. and therefore

$$G(x,t) = \sum_{n=-\infty}^{+\infty}\sum_{q=0}^{+\infty}\left[\frac{(-1)^q x^{n+2q} t^n}{2^{n+2q}(n+q)!\, q!}\right] =$$

$$= \sum_{n=-\infty}^{+\infty}\left[t^n \frac{x^n}{2^n}\sum_{q=0}^{+\infty}\left(\frac{(-1)^q x^{2q}}{2^{2q}\, q!\,(n+q)!}\right)\right] =$$

$$= \sum_{n=-\infty}^{+\infty}\left[t^n\left(\frac{x}{2}\right)^n \sum_{q=0}^{+\infty}\left((-1)^q \frac{1}{q!\,\Gamma(n+q+1)}\left(\frac{x}{2}\right)^{2q}\right)\right]$$

$$= \sum_{n=-\infty}^{+\infty} t^n J_n(x).$$

(2) $\boxed{\forall n \in \mathbb{N} : \forall x \in \mathbb{R}^* : J_n(x) = (-1)^n J_{-n}(x)}$

Proof

Let $n \in \mathbb{N}$ and $x \in \mathbb{R}^*$ be given. Using the previous result, we note that

$$G(x, -1/t) = \sum_{n=-\infty}^{+\infty} J_n(x)(-1/t)^n = \sum_{n=-\infty}^{+\infty} (-1)^{-n} J_n(x) t^{-n} =$$

$$= \sum_{n=-\infty}^{+\infty} (-1)^n J_{-n}(x) t^n \ , \ \forall t \in \mathbb{R}^*$$

and

$$G(x, -1/t) = \exp\left(\frac{1}{2} x \left((-1/t) - \frac{1}{-1/t}\right)\right) = \exp\left(\frac{x}{2}\left(-\frac{1}{t} - (-t)\right)\right)$$

$$= \exp\left(\frac{1}{2} x \left(t - \frac{1}{t}\right)\right) = G(x,t) = \sum_{n=-\infty}^{+\infty} J_n(x) t^n, \forall t \in \mathbb{R}^*$$

It follows that

$$\forall t \in \mathbb{R}^* : \sum_{n=-\infty}^{+\infty} (-1)^n J_{-n}(x) t^n = \sum_{n=-\infty}^{+\infty} J_n(x) t^n$$

$$\Rightarrow \forall n \in \mathbb{N} : J_n(x) = (-1)^n J_{-n}(x). \quad \square$$

It follows from the above result that for integer order, $J_n(x)$ and $J_{-n}(x)$ are not linearly independent. This is the reason why it becomes necessary to introduce the function $J^{\alpha}(x)$ when $\alpha \in \mathbb{N}$ for the second solution.

3 $\rightarrow$ $\boxed{\forall a \in \mathbb{R} : \forall x \in (0,+\infty) : \; x J_a'(x) = a J_a(x) - x J_{a+1}(x)}$

Proof

Let $a \in \mathbb{R}$ and $x \in (0,+\infty)$ be given. Then since

$$J_a(x) = \left|\frac{x}{2}\right|^a \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n} =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a} \Rightarrow$$

$$\Rightarrow x J_a'(x) = x \frac{d}{dx} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a}$$

$$= x \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \frac{d}{dx} \left(\frac{x}{2}\right)^{2n+a}$$

$$= x \sum_{n=0}^{+\infty} \frac{(-1)^n (2n+a)}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a-1} \cdot \left(\frac{1}{2}\right)$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n (2n+a)}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a} =$$

$$= a \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a} + \sum_{n=1}^{+\infty} \frac{(-1)^n (2n)}{n!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a}$$

$$= a J_a(x) + x \sum_{n=1}^{+\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(n+a+1)} \left(\frac{x}{2}\right)^{2n+a-1}$$

$$= a J_a(x) + x \sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{n!\,\Gamma((n+1)+a+1)} \left(\frac{x}{2}\right)^{2(n+1)+a-1}$$

$$= a J_a(x) - x \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+(a+1)+1)} \left(\frac{x}{2}\right)^{2n+(a+1)}$$

$$= a J_a(x) - x J_{a+1}(x).$$

$\square$

(4) $\boxed{\forall x \in (0,+\infty): \; J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$

Proof

Let $x \in (0,+\infty)$ be given. Then

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!\,\Gamma(n+1/2+1)} \left(\frac{x}{2}\right)^{2n} =$$

$$= \left(\frac{x}{2}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left[\prod_{k=0}^{n} \frac{1}{k+1/2}\right] \frac{1}{\Gamma(1/2)} \left(\frac{x}{2}\right)^{2n} =$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma(1/2)} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left[2^{n+1} \prod_{k=0}^{n} \frac{1}{2k+1}\right] \frac{x^{2n}}{2^{2n}} =$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{2(-1)^n}{2^n n!} \frac{1}{(2n+1)!!} x^{2n} =$$

$$= \left(\frac{x}{2\pi}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{2(-1)^n}{(2n)!!\,(2n+1)!!} x^{2n} =$$

$$= 2\left(\frac{x}{2\pi}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} =$$

$$= \left(\frac{2}{\pi x}\right)^{1/2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} =$$

$$= \left(\frac{2}{\pi x}\right)^{1/2} \sin x = \sqrt{\frac{2}{\pi x}} \sin x. \qquad \square$$

EXERCISES

(24) Given $n \in \mathbb{N}$ and $x \in (0, +\infty)$, show the following identities.

a) $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$

b) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

c) $2n J_n(x) = x \left[J_{n-1}(x) + J_{n+1}(x) \right]$

d) $(d/dx)(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$

e) $(d/dx)(x^{n} J_n(x)) = x^{n} J_{n-1}(x)$

f) $(d/dx)(x J_n(x) J_{n+1}(x)) = x (J_n^2(x) - J_{n+1}^2(x))$

↳ We have already showed that
$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$.
This result can be combined with (a) to prove the other results, without directly using power series expansions.

(25) Given $x \in (0, +\infty)$, show that

a) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$

b) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

c) $J_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left(\dfrac{\sin x}{x} - \cos x \right)$

d) $J_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}} \left(-\dfrac{\cos x}{x} - \sin x \right)$

(26) Mini-project

The goal of this mini-project is to establish the following integral representation:

$$\boxed{\forall n \in \mathbb{N} : \forall x \in \mathbb{R}^* : J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\vartheta - x\sin\vartheta)\, d\vartheta}$$

We recall that $\forall \vartheta \in \mathbb{R} : e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.

a) Show that $\exp(ix\sin\vartheta) = G(x, e^{i\vartheta})$

b) Use (a) to establish the following identities:

$$\cos(x\sin\vartheta) = \left[\sum_{n=0}^{+\infty} 2J_{2n}(x)\cos(2n\vartheta)\right] - J_0(x)$$

$$\sin(x\sin\vartheta) = \sum_{n=0}^{+\infty} 2J_{2n+1}(x)\sin((2n+1)\vartheta)$$

c) Show that

$$\forall a,b \in \mathbb{N} : \int_0^\pi \cos(a\vartheta)\cos(b\vartheta)\,d\vartheta = \begin{cases} 0 & , \text{ if } a \neq b \\ \pi/2 & , \text{ if } a = b \end{cases}$$

$$\forall a,b \in \mathbb{N} : \int_0^\pi \sin(a\vartheta)\sin(b\vartheta)\,d\vartheta = \begin{cases} 0 & , \text{ if } a \neq b \\ \pi/2 & , \text{ if } a = b \end{cases}$$

d) Combine the results (b) and (c) to show that

$$\int_0^\pi \cos(n\vartheta - x\sin\vartheta)\,d\vartheta = \pi J_n(x)$$

(Hint: it will be necessary to distinguish between two cases: n even vs. n odd).

ODE 6: Generalized Functions

GENERALIZED FUNCTIONS

▼ Introduction - Motivation

In 1920, Paul Dirac introduced the Dirac delta function $\delta(x)$ for which he postulated the following properties

(a) $\forall x \in \mathbb{R} - \{0\}: \ \delta(x) = 0,$

(b) $\displaystyle\int_{-\infty}^{+\infty} \delta(x)\,dx = 1,$ (c) $\displaystyle\int_{-\infty}^{+\infty} \delta(x-a) f(x)\,dx = f(a)$

No such function can be defined, but the idea was to introduce such exotic functions as a way of EXPANDING the space of available functions. Functions like $\delta(x)$ are called generalized functions or distributions. Rigorous theories formalizing the concept of distributions have been proposed by Schwarz, Mikusinski, Lighthill, and Sato. Below we will adopt and review the approach of Schwarz.

Generalized functions arise usually in the following contexts:

(1) Probability theory

Consider a random variable $x \in \mathbb{R}$ with probability density function $p(x)$ such that the probability $P(a \leq x \leq b)$ is given by

$$P(a \leq x \leq b) = \int_a^b p(x)\,dx$$

Obviously $p(x)$ will satisfy the normalization condition:

$$\int_{-\infty}^{+\infty} p(x)\,dx = P(x\in\mathbb{R}) = 1.$$

Note that x is continuously distributed on $\mathbb{R}$, therefore the probability that x is EXACTLY equal to some $a\in\mathbb{R}$ is zero:

$$P(x=a) = \int_a^a p(x)\,dx = 0$$

If we use the random variable x to evaluate $f(x)$, then the average value of $f(x)$ is given by:

$$\langle f(x)\rangle = \int_{-\infty}^{+\infty} f(x)p(x)\,dx$$

In this context, the Dirac delta function $\delta(x)$ can be thought of as the probability density function of a "random" variable such that $P(x=0)=1$. Then, indeed

$$\langle f(x)\rangle = \int_{-\infty}^{+\infty} f(x)\delta(x)\,dx = f(0).$$

In general, a discrete random variable with

$$\begin{cases} \forall k\in[n]: P(x=a_k) = p_k \\ \sum_{k\in[n]} p_k = 1 \end{cases}$$

can be represented with the probability density function

$$p(x) = \sum_{k\in[n]} p_k\,\delta(x-a_k)$$

such that the average of some evaluation $f(x)$ is:

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x) p(x) dx = \int_{-\infty}^{+\infty} f(x) \left[\sum_{k \in [n]} p_k \delta(x - a_k) \right] dx =$$

$$= \sum_{k \in [n]} \left[p_k \int_{-\infty}^{+\infty} f(x) \delta(x - a_k) dx \right] =$$

$$= \sum_{k \in [n]} p_k f(a_k).$$

② Theory of Green's functions

Given a linear differential operator $L: C^n(\mathbb{R}) \to C^0(\mathbb{R})$, then the corresponding Green's function $G(x,t)$ can be found by solving the problem

$$L y(x) = \delta(x - t) \qquad (1)$$

Generalized functions are used to establish the theory of for calculating a particular solution $y_p(x)$ for the more general problem $Ly(x) = f(x)$, given the homogeneous solutions for the homogeneous problem $Ly(x) = 0$. This is explained in detail at the end of the lecture notes. chapter. The above problem given by Eq.(1) is the stepping stone for solving the general problem.

(3) Distributional derivatives

Functions with discontinuities or corner points cannot be differentiated in the usual sense. With the theory of distributions we can define a more general definition of the distributional derivative. Then non-differentiable functions will have a distributional derivative but it will be a generalized function, not a regular function. For example, the function

$$f(x) = \begin{cases} 1, & \text{if } x \in (a, +\infty) \\ 0, & \text{if } x \in (-\infty, a] \end{cases}$$

is not differentiable at $x = a$, however it has a distributional derivative:

$$f'(x) = \delta(x-a)$$

▼ Schwarz definition of generalized functions

We define generalized functions via the following sequence of definitions.

<u>Def</u>: (<u>Compact support</u>)

Let $f: A \to \mathbb{R}$ be a function. We define the <u>support</u> $\mathrm{supp}(f)$ of A as:

$$\mathrm{supp}(f) = \{x \in A \mid f(x) \neq 0\}.$$

We say that

$$f \text{ has compact support} \Leftrightarrow \exists a, b \in A : \mathrm{supp}(f) \subseteq [a, b]$$

<u>Def</u>: (<u>Test functions</u>)

We define the space $\mathcal{D}(A)$ of <u>test functions</u> as:

$$\begin{aligned} \mathcal{D}(A) &= \{f \in C^\infty(A) \mid f \text{ has compact support}\} \\ &= \{f \in C^\infty(A) \mid \exists a, b \in \mathbb{R} : \mathrm{supp}(f) \subseteq [a, b]\} \end{aligned}$$

or equivalently, in terms of a belonging condition, as:

$$f \in \mathcal{D}(A) \Leftrightarrow \begin{cases} f \in C^\infty(A) \\ \exists a, b \in \mathbb{R} : \mathrm{supp}(f) \subseteq [a, b] \end{cases}$$

<u>Def</u>: (<u>Convergence in $\mathcal{D}(\mathbb{R})$</u>)

Consider a sequence $\varphi_1, \varphi_2, \ldots \in \mathcal{D}(\mathbb{R})$ of test functions and also a test function $\varphi \in \mathcal{D}(\mathbb{R})$. We say that

$$\varphi_n \xrightarrow{\mathcal{D}(\mathbb{R})} \varphi \Leftrightarrow \begin{cases} \exists a, b \in \mathbb{R} : (\mathrm{supp}(\varphi) \subseteq [a, b] \wedge \forall n \in \mathbb{N}^* : \mathrm{supp}(\varphi_n) \subseteq [a, b]) \\ \forall k \in \mathbb{N} : \varphi_n^{(k)} \text{ converges uniformly to } \varphi^{(k)} \text{ on } [a, b]. \end{cases}$$

We recall from my Calculus 2 lecture notes that the definition of uniform convergence is:

$\varphi_n^{(k)}$ converges uniformly to $\varphi^{(k)}$ on $[a,b] \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon > 0 : \exists n_0 \in \mathbb{N}^* : \forall x \in [a,b] : \forall n \in \mathbb{N}^* - [n_0] : |\varphi_n^{(k)}(x) - \varphi^{(k)}(x)| < \varepsilon$

For the next definitions we define:

a) $\mathrm{Seq}(A)$ as the set of all sequences $a : \mathbb{N}^* \to A$

b) $\mathcal{I}(\mathbb{R})$ as the set of all <u>locally integrable</u> functions as follows:

$f \in \mathcal{I}(\mathbb{R}) \Leftrightarrow \forall a,b \in \mathbb{R} : (a < b \Rightarrow f \text{ integrable on } [a,b])$

We can now give the formal definition for a generalized function (or distribution).

<u>Def</u>: (Generalized function or distribution)

A functional $F : \mathcal{X}(\mathbb{R}) \to \mathbb{C}$ is a <u>generalized function (or distribution)</u> if and only if it satisfies the following conditions:

(a) $\forall \lambda, \mu \in \mathbb{C} : \forall \varphi, \psi \in \mathcal{X}(\mathbb{R}) : F(\lambda\varphi + \mu\psi) = \lambda F(\varphi) + \mu F(\psi)$

(b) $\forall \varphi \in \mathrm{Seq}(\mathcal{X}(\mathbb{R})) : \forall \psi \in \mathcal{X}(\mathbb{R}) : (\varphi_n \xrightarrow{\mathcal{X}(\mathbb{R})} \psi \Rightarrow \lim_{n \in \mathbb{N}} F(\varphi_n) = F(\psi)$

<u>notation:</u>

(a) $\mathcal{X}'(\mathbb{R})$ is the set of all distributions $F : \mathcal{X}(\mathbb{R}) \to \mathbb{C}$

(b) By convention, we write $(F, \varphi) = F(\varphi)$.

<u>Remark:</u>

Given any integrable function $f \in \mathcal{I}(\mathbb{R})$ we define the distribution $F \in \mathcal{X}'(\mathbb{R})$ generated by f as:

$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : (F, \varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx$$

As a result, every locally integrable function f can be also thought of a distribution, and such trivial distributions are called <u>regular distributions</u>. One can prove that the distribution F defined above satisfies the formal definition of a distribution. (proof omitted). This motivates the following definition:

<u>Def</u>: Consider a distribution $F \in \mathcal{X}'(\mathbb{R})$. We say that
a) F is a regular distribution $\Leftrightarrow \exists f \in I(\mathbb{R}) : \forall \varphi \in \mathcal{X}(\mathbb{R}) : (F, \varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx$

b) F is a singular distribution $\Leftrightarrow$ F is NOT a regular distribution

Singular distributions can be defined as limits of regular distributions. For example, given a sequence $f \in \mathrm{Seq}(I(\mathbb{R}))$ of locally integrable functions, we can define a possibly singular distribution $F \in \mathcal{X}'(\mathbb{R})$ according to:

$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : (F, \varphi) = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,dx$$

as long as the limit exists. If F is indeed a singular distribution we may still introduce a fictitious function $F(x)$ and claim that

$$\forall \varphi \in \mathcal{X}(\mathbb{R}) : \int_{-\infty}^{+\infty} F(x)\varphi(x)\,dx \equiv (F, \varphi) = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,dx$$

$F(x)$ is not an actual function, in the usual sense, but it can be interpreted as a singular limit of the function sequence f_n.

We may then say that $f_n \xmapsto{\mathcal{D}(\mathbb{R})} F$, in the sense of distributions. The precise definition of the above statement is:

Def: Let $f \in \mathrm{Seq}(I(\mathbb{R}))$ be a sequence of locally integrable functions and let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution. We say that

$$f_n \xmapsto{\mathcal{D}(\mathbb{R})} F \Leftrightarrow \forall k \in \mathbb{N} : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (F, \varphi^{(k)}) = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} f_n(x) \varphi^{(k)}(x) dx$$

and we write:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_{-\infty}^{+\infty} F(x) \varphi(x) dx \equiv (F, \varphi)$$

Given F defined as $f_n \xmapsto{\mathcal{D}(\mathbb{R})} F$, we also define integrals over an $[a,b]$ interval with $a, b \in \mathbb{R}$ as follows:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_a^b F(x) \varphi(x) dx = \lim_{n \in \mathbb{N}^*} \int_a^b f_n(x) \varphi(x) dx$$

The Dirac delta function

The Dirac delta function is a singular distribution that is defined as a limit of regular distributions as follows:

- 1 We define a sequence of Gaussian distributions $\Delta_n(x)$ as:

$$\forall n \in \mathbb{N}^* : \forall x \in \mathbb{R} : \Delta_n(x) = \sqrt{\frac{n}{2\pi}} \exp(-nx^2)$$

- 2 The Dirac delta function is a singular distribution defined as

$$\Delta_n(x) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(x)$$

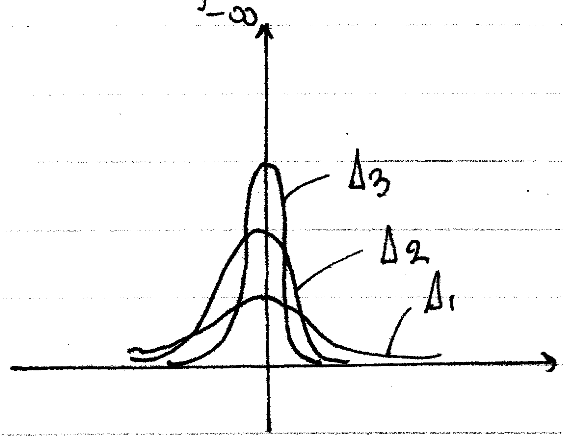
- 3 Now, we can show that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_{-\infty}^{+\infty} \delta(x)\varphi(x)\,dx = \lim_{n \in \mathbb{N}^*} \int_{-\infty}^{+\infty} \Delta_n(x)\varphi(x)\,dx = \varphi(0)$$

Geometric interpretation

The function $\Delta_n(x)$ is a bell-shaped function. With increasing n, the peak becomes taller and the graph becomes narrower such that the following constraint is always satisfied:

$$\forall n \in \mathbb{N}^* : \int_{-\infty}^{+\infty} \Delta_n(x)\,dx = 1$$



In probability theory these functions are known as Gaussian distributions. As $n \to +\infty$, we obtain the Dirac delta function that can be visualised as a spike located at $x=0$ with infinitesimal width and infinite height.

▸ Adjusted delta functions

We can likewise define the following singular distributions:

a) $\Delta_n(x-a) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(x-a)$ (shifting)

b) $\Delta_n(ax) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(ax)$ (dilation)

with $a \in \mathbb{R}-\{0\}$. Then, we can show that

$$\boxed{\begin{aligned} &\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(x-a)\varphi(x)\,dx = \varphi(a) \\ &\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(ax)\varphi(x)\,dx = \frac{\varphi(0)}{|a|} \end{aligned}}$$

Shifting and dilation can be combined to define $\delta(ax+b)$ with $a \in \mathbb{R}-\{0\}$ and $b \in \mathbb{R}$ via:

$$\Delta_n(ax+b) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(ax+b)$$

Then, it follows that

$$\boxed{\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(ax+b)\varphi(x)\,dx = \frac{\varphi(-b/a)}{|a|}}$$

EXAMPLES

a) Evaluate $I = \int_{-\infty}^{+\infty} \delta(6x-\pi)\cos^2(x+\pi/4)\,dx$

Solution

$$I = \int_{-\infty}^{+\infty} \delta(6x-\pi)\cos^2(x+\pi/4)\,dx = \frac{1}{|6|}\cos^2(\pi/6+\pi/4) =$$

$$= \frac{1}{6}\left[\cos(\pi/6)\cos(\pi/4)-\sin(\pi/6)\sin(\pi/4)\right]^2 =$$

$$= \frac{1}{6}\left[\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}-\frac{1}{2}\frac{\sqrt{2}}{2}\right]^2 = \frac{1}{6}\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\sqrt{3}-1}{2}\right)^2 =$$

$$= \frac{1}{6}\,\frac{1}{2}\,\frac{(\sqrt{3})^2-2\sqrt{3}+1}{4} = \frac{3-2\sqrt{3}+1}{48} = \frac{4-2\sqrt{3}}{48} =$$

$$= \frac{2-\sqrt{3}}{24}$$

b) Evaluate $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \cos x \cos y \, \delta(6x-\pi)\,\delta(4y-\pi)$

Solution

$$I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \cos x \cos y \, \delta(6x-\pi)\,\delta(4y-\pi) =$$

$$= \int_{-\infty}^{+\infty} dx \; \cos x \, \delta(6x-\pi)\left[\int_{-\infty}^{+\infty} dy \; \cos y \, \delta(4y-\pi)\right] =$$

$$= \left[\int_{-\infty}^{+\infty} dy \cos y \,\delta(4y-\pi) \right] \left[\int_{-\infty}^{+\infty} dx \cos x \,\delta(6x-\pi) \right]$$

$$= \left[\frac{\cos(\pi/4)}{|4|} \right] \left[\frac{\cos(\pi/6)}{|6|} \right] = \frac{1}{24} \cos(\pi/4) \cos(\pi/6) =$$

$$= \frac{1}{24} \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{96}$$

EXERCISES

① Evaluate the following integrals.

a) $I = \int_{-\infty}^{+\infty} \cos^2 x \, \delta(x - \pi/6)\, dx$

b) $I = \int_{-\infty}^{+\infty} \delta(3x) \, \mathrm{Arccos}(x)\, dx$

c) $I = \int_{-\infty}^{+\infty} \delta(3x - \pi) \sin(x + \pi/4)\, dx$

d) $I = \int_{-\infty}^{+\infty} \delta(2 - 5x)(x^2 + 3x)^2\, dx$

e) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; xy(x-y)\, \delta(2x+3)\, \delta(y-2)$

f) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; x^2(x+y)\, \delta(2x - a)\, \delta(3y + a - 2)$

Operations with distributions

Def: (Equality of distributions)
Let $F, G \in \mathcal{D}'(\mathbb{R})$ be two distributions. We say that
$F = G \Leftrightarrow \forall \varphi \in \mathcal{D}(\mathbb{R}) : (F, \varphi) = (G, \varphi)$

Given the definition of equality of distributions, we can now introduce algebra with distributions as follows:

Def: (Addition of distributions)
Let $F, G \in \mathcal{D}'(\mathbb{R})$ be two distributions. We say that
We define the distribution $(F+G) \in \mathcal{D}'(\mathbb{R})$ as:
$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (F+G, \varphi) = (F, \varphi) + (G, \varphi)$

Note that there are many technical difficulties with respect to defining multiplication of distributions. However, we may define multiplication of a smooth function with a distribution

Def: (Function-distribution multiplication)
Let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution and let $g \in C^\infty(\mathbb{R})$ be a smooth function. We define the distribution $(gF) \in \mathcal{D}'(\mathbb{R})$ as:
$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (gF, \varphi) = (F, g\varphi)$

Writing F, G as generalized functions $F(x), G(x)$, the above definitions can be rewritten equivalently as:

(a) For addition of distributions:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (F+G, \varphi) = \int_{-\infty}^{+\infty} [F(x) + G(x)]\varphi(x)\,dx =$$
$$= \int_{-\infty}^{+\infty} F(x)\varphi(x)\,dx + \int_{-\infty}^{+\infty} G(x)\varphi(x)\,dx$$
$$= (F, \varphi) + (G, \varphi)$$

(b) For function-distribution multiplication

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (gF, \varphi) = \int_{-\infty}^{+\infty} [g(x)F(x)]\varphi(x)\,dx =$$
$$= \int_{-\infty}^{+\infty} F(x)[g(x)\varphi(x)]\,dx = (F, g\varphi)$$

- Multiplying a number $\lambda \in \mathbb{R}$ with a distribution $F \in \mathcal{D}'(\mathbb{R})$ is a special case of the function-distribution product gF using the function $\forall x \in \mathbb{R} : g(x) = \lambda$.

<u>Prop:</u> $\boxed{\forall \lambda \in \mathbb{R} : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (\lambda F, \varphi) = \lambda (F, \varphi)}$

<u>Proof</u>

Let $\lambda \in \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R})$ be given. Then

$(\lambda F, \varphi) = (F, \lambda\varphi)$ [definition of function-distribution product]

$= \lambda (F, \varphi)$ [definition of distribution-linearity of F]

It follows that $\forall \lambda \in \mathbb{R} : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (\lambda F, \varphi) = \lambda (F, \varphi)$

EXAMPLES

a) Use the definition of $\delta(ax)$:

$$\Delta_n(ax) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta(ax)$$

to show that $\delta(ax) = \frac{1}{|a|}\delta(x)$, for $a \neq 0$.

Solution

▸ It is sufficient to show that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}): \int_{-\infty}^{+\infty} \delta(ax)\varphi(x)\,dx = \int_{-\infty}^{+\infty} \left[\frac{1}{|a|}\delta(x)\right]\varphi(x)\,dx$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$ and let $n \in \mathbb{N}^*$ be given. Then:

$$(\Delta_n(ax), \varphi) = \int_{-\infty}^{+\infty} \Delta_n(ax)\varphi(x)\,dx = \int_{-\infty}^{+\infty} \Delta_n(|a|x)\varphi(x)\,dx$$

because

$$\Delta_n(ax) = \sqrt{\frac{n}{2\pi}}\exp(-n(ax)^2) = \sqrt{\frac{n}{2\pi}}\exp(-na^2x^2)$$

$$= \sqrt{\frac{n}{2\pi}}\exp(-n|a|^2x^2) = \sqrt{\frac{n}{2\pi}}\exp(-n(|a|x)^2)$$

$$= \Delta_n(|a|x), \quad \forall x \in \mathbb{R}.$$

Define $y = |a|x$. Then $dy = |a|dx \Leftrightarrow dx = (1/|a|)dy$ and

$x \to -\infty \Rightarrow y \to -\infty$

$x \to +\infty \Rightarrow y \to +\infty$

and with change of variables:

$$(\Delta_n(ax),\varphi)=\int_{-\infty}^{+\infty}\Delta_n(y)\varphi\left(\frac{y}{|a|}\right)\frac{1}{|a|}\,dy=\frac{1}{|a|}\int_{-\infty}^{+\infty}\Delta_n(y)\,\varphi\left(\frac{y}{|a|}\right)dy$$

and therefore:

$$\int_{-\infty}^{+\infty}\delta(ax)\varphi(x)\,dx=\lim_{n\in\mathbb{N}^*}(\Delta_n(ax),\varphi)=\lim_{n\in\mathbb{N}^*}\left[\frac{1}{|a|}\int_{-\infty}^{+\infty}\Delta_n(y)\varphi\left(\frac{y}{|a|}\right)dy\right]$$

$$=\frac{1}{|a|}\lim_{n\in\mathbb{N}^*}\int_{-\infty}^{+\infty}\Delta_n(y)\,\varphi\left(\frac{y}{|a|}\right)dy=$$

$$=\frac{1}{|a|}\int_{-\infty}^{+\infty}\delta(y)\,\varphi\left(\frac{y}{|a|}\right)dy=\frac{1}{|a|}\,\varphi\left(\frac{0}{|a|}\right)$$

$$=\frac{\varphi(0)}{|a|}=\frac{1}{|a|}\int_{-\infty}^{+\infty}\delta(x)\varphi(x)\,dx=$$

$$=\int_{-\infty}^{+\infty}\left[\frac{1}{|a|}\,\delta(x)\right]\varphi(x)\,dx$$

It follows that

$$\forall\varphi\in\mathcal{D}(\mathbb{R}):\ \int_{-\infty}^{+\infty}\delta(ax)\varphi(x)\,dx=\int_{-\infty}^{+\infty}\left[\frac{1}{|a|}\,\delta(x)\right]\varphi(x)\,dx$$

$$\Rightarrow\ \delta(ax)=\frac{1}{|a|}\,\delta(x).$$

Derivative of distributions

Following the example of the Dirac delta function, we introduce the following general concept of the derivative of a distribution as follows:

Def: Let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution and let $k \in \mathbb{N}^*$. We define the k^{th} derivative $F^{(k)}$ of F as follows:
$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : (F^{(k)}, \varphi) = (-1)^k (F, \varphi^{(k)})$$

Representing F in terms of a generalized function $F(x)$, the above definition can be equivalently be rewritten as

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_{-\infty}^{+\infty} F^{(k)}(x)\varphi(x)dx = (-1)^k \int_{-\infty}^{+\infty} F(x)\varphi^{(k)}(x)dx$$

To ensure the self-consistency of this definition, we have to ensure that if F is a regular distribution, in which case $F(x)$ is an ordinary function, the above equation holds. Using integration by parts and proof by induction, we can show that indeed it holds. For singular distributions defined via a sequence of locally integrable functions, we can show that

Prop: Let $f \in \mathrm{Seq}(C^\infty(\mathbb{R}))$ be a sequence of locally integrable functions and let $F \in \mathcal{D}'(\mathbb{R})$ be a distribution. Then:
$$\forall k \in \mathbb{N}^* : (f_n \xrightarrow{\mathcal{D}(\mathbb{R})} F \Longrightarrow f_n^{(k)} \xrightarrow{\mathcal{D}(\mathbb{R})} F^{(k)})$$

This proposition ensures the self-consistency between the above definition of distributional derivative and the standard definition of the derivative of a function from Calculus.

▶ Properties of distributional derivatives.

Distributional derivatives continue to satisfy some standard differentiation rules: addition rule, product rule, scalar product rule:

$$\boxed{\begin{aligned}&\forall F, G \in \mathcal{D}'(\mathbb{R}) : (F(x)+G(x))' = F'(x)+G'(x)\\ &\forall g \in C^\infty(\mathbb{R}) : \forall F \in \mathcal{D}'(\mathbb{R}) : (g(x)F(x))' = g'(x)F(x)+g(x)F'(x)\\ &\forall \lambda \in \mathbb{R} : \forall F \in \mathcal{D}'(\mathbb{R}) : (\lambda F(x))' = \lambda F'(x).\end{aligned}}$$

Proof

a) Let $F, G \in \mathcal{D}'(\mathbb{R})$ be given, and let $\varphi \in \mathcal{D}(\mathbb{R})$ be given. Then,

$$\begin{aligned}((F+G)', \varphi) &= (-1)(F+G, \varphi') =\\ &= (-1)[(F, \varphi') + (G, \varphi')] =\\ &= (-1)(F, \varphi') + (-1)(G, \varphi') =\\ &= (F', \varphi) + (G', \varphi) = (F'+G', \varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R})\end{aligned}$$

It follows that $(F+G)' = F'+G'$.

b) Let $F \in \mathcal{D}'(\mathbb{R})$ and $g \in C^\infty(\mathbb{R})$ be given. Then

$$\begin{aligned}\forall \varphi \in \mathcal{D}(\mathbb{R}) : ((gF)', \varphi) &= (-1)(gF, \varphi') = (-1)(F, g\varphi') =\\ &= (-1)(F, (g\varphi)' - g'\varphi) =\end{aligned}$$

$$= (-1)(F,(g\varphi)') - (-1)(F, g'\varphi) =$$
$$= (F', g\varphi) - (-1)(g'F, \varphi) =$$
$$= (gF', \varphi) + (g'F, \varphi) = (gF' + g'F, \varphi)$$
$$\Rightarrow (gF)' = g'F + gF'.$$

c) Let $\lambda \in \mathbb{R}$ and $F \in \mathcal{D}'(\mathbb{R})$ be given. Then

$$((\lambda F)', \varphi) = (-1)(\lambda F, \varphi') = (-1)(F, \lambda\varphi') = (-1)\lambda(F, \varphi')$$
$$= \lambda(F', \varphi) = (F', \lambda\varphi) = (\lambda F', \varphi), \ \forall \varphi \in \mathcal{D}(\mathbb{R}) \Rightarrow$$
$$\Rightarrow (\lambda F)' = \lambda F'.$$

Derivatives of Dirac delta functions

Derivatives of the Dirac delta function are defined via the previously stated distributional derivative definition which immediately yields:

$$\boxed{\begin{aligned}\forall k \in \mathbb{N}^* : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (\delta^{(k)}(x), \varphi(x)) &= \int_{-\infty}^{+\infty} \delta^{(k)}(x)\varphi(x)\,dx \\ &= (-1)^k \int_{-\infty}^{+\infty} \delta(x)\varphi^{(k)}(x)\,dx = (-1)^k \varphi^{(k)}(0)\end{aligned}}$$

For the shifted delta function k^{th} derivative $\delta^{(k)}(x-a)$ we have:

$$\boxed{\begin{aligned}\forall k \in \mathbb{N}^* : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (\delta^{(k)}(x-a), \varphi(x)) &= \int_{-\infty}^{+\infty} \delta^{(k)}(x-a)\varphi(x)\,dx \\ &= (-1)^k \int_{-\infty}^{+\infty} \delta(x-a)\varphi^{(k)}(x)\,dx \\ &= (-1)^k \varphi^{(k)}(a)\end{aligned}}$$

Using the previously defined Gaussian distributions and the theory of distributional derivatives, in general we have:

$$\boxed{\forall k \in \mathbb{N}^* : \Delta_n^{(k)}(ax+b) \xrightarrow{\mathcal{D}(\mathbb{R})} \delta^{(k)}(ax+b)}$$

with $a \in \mathbb{R} - \{0\}$ and $b \in \mathbb{R}$.
Since we have previously shown that

$$\delta(ax+b) = \frac{1}{|a|}\,\delta\!\left(x+\frac{b}{a}\right)$$

differentiating both sides with a distributional derivative and using the scalar-multiplication rule gives the following more general result:

$$\boxed{\forall k \in \mathbb{N}: \ \delta^{(k)}(ax+b) = \frac{1}{|a|}\,\delta^{(k)}\!\left(x+\frac{b}{a}\right)}$$

EXAMPLE

Evaluate the integral $I = \int_{-\infty}^{+\infty} x^2 e^x \delta''(x-1)\,dx$

Solution

$$I = \int_{-\infty}^{+\infty} x^2 e^x \delta''(x-1)\,dx = (-1)^2 \int_{-\infty}^{+\infty} (x^2 e^x)'' \delta(x-1)\,dx$$

We note that

$$\begin{aligned}(x^2e^x)'' &= \left((x^2)'e^x + x^2(e^x)'\right)' = \left(2xe^x + x^2e^x\right)' \\ &= \left[e^x(2x+x^2)\right]' = (e^x)'(2x+x^2) + e^x(2x+x^2)' = \\ &= e^x(2x+x^2) + e^x(2+2x) = \\ &= e^x(2x+x^2+2+2x) = e^x(x^2+4x+2).\end{aligned}$$

and therefore

$$I = \int_{-\infty}^{+\infty} e^x(x^2+4x+2)\,\delta(x-1)\,dx = e^1(1^2+4\cdot 1+2) = 7e.$$

EXERCISES

(2) Evaluate the following integrals:

a) $I = \int_{-\infty}^{+\infty} x(x^2-1)^3 \delta''(x-1)\, dx$

b) $I = \int_{-\infty}^{+\infty} x^2 \exp(-x^2) \left[\delta'(x-2) + \delta''(x-2)\right] dx$

c) $I = \int_{-\infty}^{+\infty} \mathrm{Arctan}(x)\, \delta''(x-\sqrt{2})\, dx$

d) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \;\; xy \exp(xy)\, \delta'(x-1)\, \delta'(y-1)$

e) $I = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \;\; \sin(xy)\, \delta''(x-\pi/4) \left[\delta'(y-1) + \delta''(y-1)\right]$

▼ Algebra with delta functions

Expressions with function-distribution products involving the Dirac delta functions or their derivatives can be simplified using the following fundamental properties.

- 1 We begin by showing that:

$$\boxed{\forall f \in C^{\infty}(\mathbb{R}): \; f(x)\delta(x-a) = f(a)\delta(x-a)}$$

- 2 Taking the distributional derivative on both sides and using the distributional differentiation rules gives the following identities:

$$\boxed{\begin{aligned}
&\forall f \in C^{\infty}(\mathbb{R}): \; f(x)\delta'(x-a) = f(a)\delta'(x-a) - f'(a)\delta(x-a)\\
&\forall f \in C^{\infty}(\mathbb{R}): \; f(x)\delta''(x-a) = f(a)\delta''(x-a) - 2f'(a)\delta'(x-a) + f''(a)\delta(x-a)\\
&\forall f \in C^{\infty}(\mathbb{R}): \; f(x)\delta'''(x-a) = f(a)\delta'''(x-a) - 3f'(a)\delta''(x-a)\\
&\qquad\qquad\qquad\qquad\qquad + 3f''(a)\delta'(x-a) - f'''(a)\delta(x-a)
\end{aligned}}$$

- 3 The general result, established using proof by induction, is given by

$$\boxed{\begin{aligned}
&\forall f \in C^{\infty}(\mathbb{R}): \; f(x)\delta^{(n)}(x-a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f^{(k)}(a)\,\delta^{(n-k)}(x-a)\\
&\text{with } \forall n,k \in \mathbb{N}: \; \binom{n}{k} = \frac{n!}{k!(n-k)!}
\end{aligned}}$$

Recall that the ~~Pascal~~ binomial coefficients satisfy the Pascal identities:

$$\boxed{\begin{array}{l} \forall n \in \mathbb{N}^* : \binom{n}{0} = \binom{n}{n} = 1 \\ \forall n, k \in \mathbb{N}^* : \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \end{array}}$$

and can be calculated via the Pascal triangle, where each coefficient is equal to the sum of the coefficient directly above it plus the coefficient located above and one step to the left.

$n=1:\quad 1 \quad 1$

$n=2:\quad 1 \quad 2 \quad 1$

$n=3:\quad 1 \quad 3 \quad 3 \quad 1$

$n=4:\quad 1 \quad 4 \quad 6 \quad 4 \quad 1$

$n=5:\quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$

We now state the proof for the main result.

<u>Proof</u>

We use proof by induction. For $n=0$, let $f \in C^\infty(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{R})$ be given. Then:

$$\begin{aligned} (f(x)\delta(x-a), \varphi(x)) &= (\delta(x-a), f(x)\varphi(x)) = f(a)\varphi(a) = \\ &= f(a)(\delta(x-a), \varphi(x)) = (\delta(x-a), f(a)\varphi(x)) \\ &= (f(a)\delta(x-a), \varphi(x)) \end{aligned}$$

It follows that

$$\forall f \in C^\infty(\mathbb{R}) : \forall \varphi \in \mathcal{D}(\mathbb{R}) : (f(x)\delta(x-a), \varphi(x)) = (f(a)\delta(x-a), \varphi(x))$$

$$\Rightarrow \forall f \in C^\infty(\mathbb{R}) : f(x)\delta(x-a) = f(a)\delta(x-a).$$

For $n=m$, we assume that

$$f(x)\delta^{(m)}(x-a) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k)}(x-a)$$

For $n=m+1$, we have:

$$f(x)\delta^{(m+1)}(x-a) = (d/dx)[f(x)\delta^{(m)}(x-a)] - f'(x)\delta^{(m)}(x-a) =$$

$$= \frac{d}{dx}\left[\sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\delta^{(m-k)}(x-a)\right]$$

$$- \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k+1)}(a)\,\delta^{(m-k)}(x-a) =$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a)$$

$$- \sum_{k=0}^{m} (-1)^k \binom{m}{k} f^{(k+1)}(a)\,\delta^{(m-k)}(x-a) =$$

$$= f(a)\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a)$$

$$- \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} f^{(k+1)}(a)\,\delta^{(m-k)}(x-a) - (-1)^m f^{(m+1)}(a)\,\delta(x-a)$$

$$= f(a)\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \binom{m}{k} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a)$$

$$- \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k-1} f^{(k)}(a)\,\delta^{(m-k+1)}(x-a) - (-1)^m f^{(m+1)}(a)\,\delta(x-a)$$

$$= f(a)\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(k)}(a)\,\delta^{(m+1-k)}(x-a)$$

$$- (-1)^m f^{(m+1)}(a)\,\delta(x-a) =$$

$$= f(a)\,\delta^{(m+1)}(x-a) + \sum_{k=1}^{m} (-1)^k \binom{m+1}{k} f^{(k)}(a)\,\delta^{(m+1-k)}(x-a)$$

$$+ (-1)^m f^{(m+1)}(a)\,\delta(x-a) =$$

$$= \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} f^{(k)}(a)\,\delta^{(m+1-k)}(x-a)$$

By induction, this concludes the argument. $\square$

EXAMPLE

Simplify the generalized function
$f(x) = x^3 e^x [\delta''(x-1) + 3\delta'(x-1)]$

Solution

Define $g(x) = x^3 e^x$ and note that

$$g'(x) = (x^3)' e^x + x^3 (e^x)' = 3x^2 e^x + x^3 e^x = (x^3 + 3x^2) e^x$$

$$\begin{aligned} g''(x) &= (x^3+3x^2)' e^x + (x^3+3x^2)(e^x)' = \\ &= (3x^2+6x) e^x + (x^3+3x^2) e^x = \\ &= (3x^2+6x+x^3+3x^2) e^x = (x^3+6x^2+6x) e^x \end{aligned}$$

and it follows that

$$g(1) = 1^3 e^1 = e$$

$$g'(1) = (1^3 + 3\cdot 1^2) e^1 = (1+3)e = 4e$$

$$g''(1) = (1^3 + 6\cdot 1^2 + 6\cdot 1) e^1 = (1+6+6)e = 13e$$

Consequently, $f(x)$ simplifies to:

$$\begin{aligned} f(x) &= g(x)\delta''(x-1) + 3g(x)\delta'(x-1) \\ &= [g(1)\delta''(x-1) - 2g'(1)\delta'(x-1) + g''(1)\delta(x-1)] + 3[g(1)\delta'(x-1) - g'(1)\delta(x-1)] \\ &= g(1)\delta''(x-1) + [-2g'(1) + 3g(1)]\delta'(x-1) + [g''(1) - 3g'(1)]\delta(x-1) \\ &= e\delta''(x-1) + [-2(4e)+3e]\delta'(x-1) + [13e - 3(4e)]\delta(x-1) \\ &= e\delta''(x-1) + (-8e+3e)\delta'(x-1) + (13e-12e)\delta(x-1) \\ &= e\delta''(x-1) - 5e\delta'(x-1) + e\delta(x-1). \end{aligned}$$

EXERCISE

③ Simplify the following generalized functions.

a) $f(x) = (x^2 \sin x)\,\delta(x - \pi/4)$

b) $f(x) = \sin^3(x)\,\delta'(x - \pi/3)$

c) $f(x) = x^2 e^x\,\delta'(x-1)$

d) $f(x) = (x+1)^2 e^x\,\delta''(x)$

e) $f(x) = (x-2)^3 (2x-1)^2\,\delta''(x-3)$

f) $f(x) = 3x\sqrt{x^2+1}\,\left[2\delta'(x-1) + \delta''(x-2)\right]$

g) $f(x) = x\,\mathrm{Arctan}(x)\left[2\delta'(x) - \delta(x-1)\right]$

h) $f(x) = x^n\,\delta^{(m)}(x)$ with $n, m \in \mathbb{N}^*$

i) $f(x) = (\sin x)\,\delta^{(n)}(x)$

▼ The Heaviside distribution

The Heaviside distribution is an example of a regular distribution. Given the previously defined sequence of gaussian distributions $\Delta_n(x)$ we define

$$\forall x \in \mathbb{R}: \; E_n(x) = \int_{-\infty}^{x} \Delta_n(t)\,dt = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{x} \exp(-nt^2)\,dt$$

The Heaviside function $H(x)$ is a regular distribution defined as:

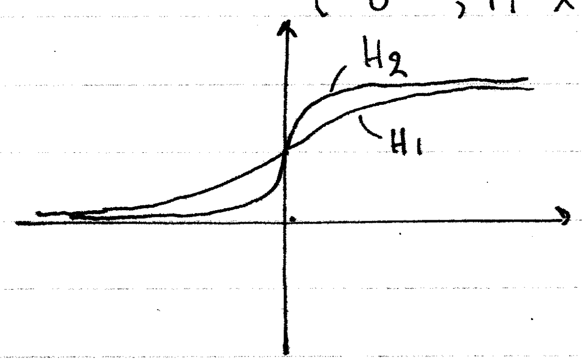
$$E_n(x) \xrightarrow{\mathcal{D}(\mathbb{R})} H(x)$$

and we can show that

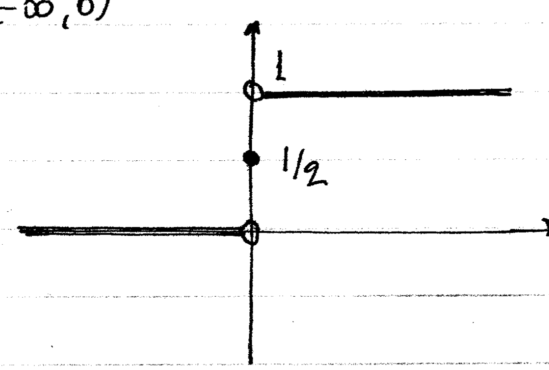
$$\forall \varphi \in \mathcal{D}(\mathbb{R}): \quad (H(x), \varphi(x)) = \int_{-\infty}^{+\infty} H(x)\varphi(x)\,dx = \int_{0}^{+\infty} \varphi(x)\,dx$$

with the representation function $H(x)$ given by

$$\forall x \in \mathbb{R}: \quad H(x) = \begin{cases} 1 & , \text{ if } x \in (0, +\infty) \\ 1/2 & , \text{ if } x = 0 \\ 0 & , \text{ if } x \in (-\infty, 0) \end{cases}$$



Graph of $E_1(x)$, $E_2(x)$, ...



Graph of $H(x)$.

• $H(x)$ is not differentiable on $x=0$. However, since $(d/dx)\,En(x) = \Delta n(x)$, it follows that, in the sense of distributions, the derivative of the distribution generated by $H(x)$ satisfies:

$$\boxed{(d/dx)H(x) = \delta(x)}$$

• Shifted Heaviside distributions can be defined via

$$En(x-a) \xrightarrow{\mathcal{X}(\mathbb{R})} H(x-a) \ , \ \forall a \in \mathbb{R}$$

Applied on a test function, the Heaviside distribution gives:

$$\boxed{\forall \varphi \in \mathcal{A}(\mathbb{R}) : (H(x-a), \varphi(x)) = \int_a^{+\infty} \varphi(x)\,dx}$$

The distributional derivative of $H(x-a)$ is given by:

$$\boxed{(d/dx)H(x-a) = \delta(x-a)}$$

● <u>Distributional derivative of piecewise discontinuous functions</u>

We consider a potentially piecewise discontinuous function of the form

$$f(x) = \begin{cases} f_0(x) \ , & \text{if } x < a_1 \\ f_1(x) \ , & \text{if } a_1 < x < a_2 \\ f_2(x) \ , & \text{if } a_2 < x < a_3 \\ \vdots & \\ f_n(x) \ , & \text{if } a_n < x \end{cases}$$

This function may not necessarily be differentiable or even continuous at the points $a_1, a_2, \dots, a_n$. However, it induces a distribution that can be written in terms of the Heaviside distribution as:

$$f(x) = f_0(x) + [f_1(x) - f_0(x)]\,H(x-a_1) + [f_2(x) - f_1(x)]\,H(x-a_2) + \\ + \dots + [f_n(x) - f_{n-1}(x)]\,H(x-a_n)$$
$$= f_0(x) + \sum_{k=1}^{n} [f_k(x) - f_{k-1}(x)]\,H(x-a_k)$$

It should be noted that the values of the original function at the points $a_1, a_2, \dots, a_n$ have no effect in the above result. Although the original functions will be different functions if they disagree at the isolated points $a_1, a_2, \dots, a_n$, all such functions will induce a unique regular distribution, given by the above equation. Consequently, the distributional derivative of $f(x)$ is given by

$$f'(x) = f_0'(x) + \sum_{k=1}^{n} (d/dx)\left\{[f_k(x) - f_{k-1}(x)]\,H(x-a_k)\right\}$$
$$= f_0'(x) + \sum_{k=1}^{n} [f_k'(x) - f_{k-1}'(x)]\,H(x-a_k) + \\ + \sum_{k=1}^{n} [f_k(x) - f_{k-1}(x)]\,\delta(x-a_k)$$
$$= f_0'(x) + \sum_{k=1}^{n} [f_k'(x) - f_{k-1}'(x)]\,H(x-a_k) + \\ + \sum_{k=1}^{n} [f_k(a_k) - f_{k-1}(a_k)]\,\delta(x-a_k)$$

Higher distributional derivatives can be taken that will result in additional terms involving derivatives of delta functions. All delta terms can be and should be simplified so that they have integer coefficients, as shown in the example below.

EXAMPLES

Find the distributional derivatives $f'(x)$, $f''(x)$ of the distribution induced by:

$$f(x) = \begin{cases} x^2+x & , \text{ if } x<1 \\ x^2+3x & , \text{ if } 1<x<2 \\ x^3+x^2 & , \text{ if } 2<x \end{cases}$$

Solution

We note that

$$f(x) = (x^2+x) + \left[(x^2+3x) - (x^2+x)\right]H(x-1) + \left[(x^3+x^2) - (x^2+3x)\right]H(x-2)$$
$$= (x^2+x) + (x^2+3x-x^2-x)H(x-1) + (x^3+x^2-x^2-3x)H(x-2)$$
$$= (x^2+x) + 2xH(x-1) + (x^3-3x)H(x-2)$$

and therefore:

$$f'(x) = (x^2+x)' + (2x)'H(x-1) + 2x\delta(x-1) + (x^3-3x)'H(x-2) + (x^3-3x)\delta(x-2)$$
$$= 2x+1 + 2H(x-1) + 2\cdot 1\delta(x-1) + (3x^2-3)H(x-2) + (2^3-3\cdot 2)\delta(x-2)$$
$$= (2x+1) + 2H(x-1) + (3x^2-3)H(x-2) + 2\delta(x-1) + 2\delta(x-2)$$

and

$$f''(x) = (2x+1)' + 2\delta(x-1) + (3x^2-3)'H(x-2) + (3x^2-3)\delta(x-2)$$
$$+ 2\delta'(x-1) + 2\delta'(x-2)$$
$$= 2 + 6xH(x-2) + 2\delta(x-1) + (3\cdot 2^2-3)\delta(x-2) + 2\delta'(x-1) +$$
$$+ 2\delta'(x-2)$$
$$= 2 + 6xH(x-2) + 2\delta(x-1) + 9\delta(x-2) + 2\delta'(x-1) + 2\delta'(x-2).$$

EXERCISES

④ Evaluate the distributional derivatives $f'(x), f''(x), f'''(x)$ for the following discontinuous functions.

a) $f(x) = \begin{cases} x^3+2x^2-1 & , \; x<1 \\ x^4+x+1 & , \; x>1 \end{cases}$

b) $f(x) = \begin{cases} x^2 e^x & , \; x<2 \\ x^3 e^x & , \; x>2 \end{cases}$

c) $f(x) = \begin{cases} \exp(-x^2) & , \; x<0 \\ \exp(-x^3) & , \; x>0 \end{cases}$

d) $f(x) = \begin{cases} \operatorname{Arctan} x & , \; x<\sqrt{3} \\ \operatorname{Arctan}(1/x) & , \; x>\sqrt{3} \end{cases}$

e) $f(x) = \begin{cases} \sin x+\cos x & , \; x<\pi/6 \\ \cos^2 x & , \; \pi/6<x<\pi/3 \\ \sin^2 x & , \; \pi/3<x \end{cases}$

f) $f(x) = \begin{cases} x^2-\sin(\pi x) & , \; x<1/3 \\ x\cos(\pi x/2) & , \; 1/3<x<1 \\ x\sin(\pi x/2) & , \; 1<x \end{cases}$

Side limit evaluation of generalized functions

In general, in spite of the notation, a distribution $F(x)$ cannot be evaluated for specific values of x. However, if we restrict the space of distributions $\mathcal{D}'(\mathbb{R})$ to a smaller subspace, then we can assign to them side-values x^+, x^- as follows:

Def: We define the space $\Delta^\infty(\mathbb{R})$ of distributions $F \in \mathcal{D}'(\mathbb{R})$ that can be written as:

$$F(x) = f(x) + \sum_{n\in A} g_n(x) H(x-p_n) + \sum_{n\in B} a_n \delta^{(b_n)}(x-q_n)$$

with $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ being finite or countable sets such that

$$\begin{cases} f \in C^\infty(\mathbb{R}) \\ \forall n\in A : (g_n \in C^\infty(\mathbb{R}) \land p_n \in \mathbb{R}) \\ \forall n \in B : (a_n, q_n \in \mathbb{R} \land b_n \in \mathbb{N}) \end{cases}$$

It can be shown that $\Delta^\infty(\mathbb{R})$ is closed with respect to most operations of interest:

Prop: $\forall F, G \in \Delta^\infty(\mathbb{R}) : (F+G) \in \Delta^\infty(\mathbb{R})$
$\forall F \in \Delta^\infty(\mathbb{R}) : \forall g \in C^\infty(\mathbb{R}) : gF \in \Delta^\infty(\mathbb{R})$
$\forall F \in \Delta^\infty(\mathbb{R}) : F' \in \Delta^\infty(\mathbb{R})$

<u>Remark</u> : If $\max_{n \in B} b_n = N-1$ with $N \in \mathbb{N}^*$, then we say that F is N^{th}-order singular and denote $\Delta^N(\mathbb{R})$ as the subset of all N^{th}-order singular distributions of $\Delta^\infty(\mathbb{R})$. Likewise, if $B = \emptyset$, then F will be a regular distribution, we say that it is 0^{th}-order singular and we denote the space of all 0^{th}-order singular distributions of $\Delta^\infty(\mathbb{R})$ as $\Delta^0(\mathbb{R})$. It is important to emphasize that regretably, the notations $\Delta^0(\mathbb{R})$, $\Delta^N(\mathbb{R})$, $\Delta^\infty(\mathbb{R})$ are not standard.

● Side-values to distributions in $\Delta^\infty(\mathbb{R})$

Given $x \in \mathbb{R}$, we assign the values x^+ and x^- to distributions in $\Delta^\infty(\mathbb{R})$ according to the following rules:

a) $\forall f \in C^\infty(\mathbb{R}) : f(x^+) = f(x^-) = f(x)$

b) $\begin{cases} \forall x \in (-\infty, 0) : H(x^+) = H(x^-) = 0 \\ \forall x \in (0, +\infty) : H(x^+) = H(x^-) = 1 \\ H(0^+) = 1 \wedge H(0^-) = 0 \end{cases}$

c) $\forall x \in \mathbb{R} : \delta(x^+) = \delta(x^-) = 0$

d) $\forall x \in \mathbb{R} : \forall n \in \mathbb{N}^* : \delta^{(n)}(x^+) = \delta^{(n)}(x^-) = 0$

Given a distribution $F \in \Delta^\infty(\mathbb{R})$, the expansion of $F(x)$ in conjunction with the above definitions uniquely defines $F(x^+)$ and $F(x^-)$ for all $x \in \mathbb{R}$.

Generalized integrals on $\Delta^\infty(\mathbb{R})$

In general, with distributions from $\mathcal{D}'(\mathbb{R})$ all integrals are defined on the $(-\infty, +\infty)$ interval. A generalized definition of the integral for distributions on $\Delta^\infty(\mathbb{R})$ is possible as follows:

•1 Let $a, p \in \mathbb{R}$ be given. We define:

$$a \in (p, +\infty) \Rightarrow \int_{p^-}^{a^-} \delta(x-p)\,dx = \int_{p^-}^{a^+} \delta(x-p)\,dx = \int_{p^-}^{+\infty} \delta(x-p)\,dx = 1$$

$$a \in (p, +\infty) \Rightarrow \int_{p^+}^{a^-} \delta(x-p)\,dx = \int_{p^+}^{a^+} \delta(x-p)\,dx = \int_{p^+}^{+\infty} \delta(x-p)\,dx = 0$$

$$a \in (-\infty, p) \Rightarrow \int_{a^+}^{p^+} \delta(x-p)\,dx = \int_{a^-}^{p^+} \delta(x-p)\,dx = \int_{-\infty}^{p^+} \delta(x-p)\,dx = 1$$

$$a \in (-\infty, p) \Rightarrow \int_{a^+}^{p^-} \delta(x-p)\,dx = \int_{a^-}^{p^-} \delta(x-p)\,dx = \int_{-\infty}^{p^-} \delta(x-p)\,dx = 0$$

$$\int_{p^-}^{p^+} \delta(x-p)\,dx = 1$$

•2 For derivatives of the Dirac delta functions such as $\delta'(x-p), \delta''(x-p), \ldots, \delta^{(n)}(x-p), \ldots$, all of the above integrals are zero.

•3 Integrals involving H-terms can be evaluated as basic Riemann integrals.

•4 For a general distribution $F \in \Delta^\infty(\mathbb{R})$ of the form

$$F(x) = f(x) + \sum_{n \in A} g_n(x) H(x-p_n) + \sum_{n \in B} a_n \delta^{(b_n)}(x-q_n)$$

integrals can be defined as linear combinations of the above cases.

A remarkable result about this generalized integral is that it satisfies the following generalized fundamental theorem of Calculus.

Thm: Let $a,b\in\mathbb{R}$ with $a<b$. Then:

$$\forall F\in\Delta^{\infty}(\mathbb{R}): \int_{a^+}^{b^+} F'(x)\,dx = F(b^+)-F(a^+)$$

$$\forall F\in\Delta^{\infty}(\mathbb{R}): \int_{a^+}^{b^-} F'(x)\,dx = F(b^-)-F(a^+)$$

$$\forall F\in\Delta^{\infty}(\mathbb{R}): \int_{a^-}^{b^+} F'(x)\,dx = F(b^+)-F(a^-)$$

$$\forall F\in\Delta^{\infty}(\mathbb{R}): \int_{a^-}^{b^-} F'(x)\,dx = F(b^-)-F(a^-)$$

▶ <u>Extension to improper integrals</u>

We can extend the above theorem to improper integrals by defining:

$$\begin{cases} H(+\infty)=1 \land H(-\infty)=0 \\ \delta(+\infty)=\delta(-\infty)=0 \\ \forall n\in\mathbb{N}^{*}: \delta^{(n)}(+\infty)=\delta^{(n)}(-\infty)=0 \\ \forall f\in C^{\infty}(\mathbb{R}): \left(f(+\infty)=\lim\limits_{x\to+\infty} f(x) \land f(-\infty)=\lim\limits_{x\to-\infty} f(x)\right) \end{cases}$$

assuming that the limits exist. Note that the convergence of improper integrals is not assured and should be investigated on a case by case basis.

EXAMPLE

Evaluate the integral $I = \int_{0^-}^{+\infty} (x^2+1)^2 \delta''(x)\,dx$

Solution

Define $g(x) = (x^2+1)^2$, $\forall x \in \mathbb{R}$. Then:

$$g'(x) = [(x^2+1)^2]' = 2(x^2+1)(x^2+1)' = 2(x^2+1)(2x) = 4x(x^2+1)$$
$$= 4x^3 + 4x \quad , \forall x \in \mathbb{R}$$

and

$$g''(x) = (4x^3+4x)' = 12x^2 + 4$$

and it follows that

$$I = \int_{0^-}^{+\infty} (x^2+1)^2 \delta''(x)\,dx =$$

$$= \int_{0^-}^{+\infty} [g(0)\delta''(x) - 2g'(0)\delta'(x) + g''(0)\delta(x)]\,dx =$$

$$= g(0)\int_{0^-}^{+\infty} \delta''(x)\,dx - 2g'(0)\int_{0^-}^{+\infty} \delta'(x)\,dx + g''(0)\int_{0^-}^{+\infty} \delta(x)\,dx$$

$$= g(0)\cdot 0 - 2g'(0)\cdot 0 + g''(0)\cdot 1 = g''(0) = 12\cdot 0^2 + 4 = 4.$$

EXERCISES

(5) Evaluate the following integrals by first simplifying the integrands.

a) $I = \int_{-\infty}^{1^+} e^x [\delta(x) + \delta(x-1)]\, dx$

b) $I = \int_{1^+}^{3^-} x^2 e^x [\delta'(x-1) + \delta'(x) + \delta'(x-2)]\, dx$

c) $I = \int_{0^-}^{\pi/4^+} \cos^2 x\, [\delta''(x) + 3\delta''(x - \pi/4)]\, dx$

d) $I = \int_{0^-}^{0^+} (x+3)^2 (2x-1)^3\, \delta''(x)\, dx$

e) $I = \int_{0^+}^{(\pi/2)^+} \sin x\, [1 - \cos x]\, [\delta''(x) + \delta''(x - \pi/4) + \delta''(x - \pi/2)]\, dx$

f) $I = \int_{0^+}^{+\infty} \exp(-x^2)\, \delta'''(x-1)\, dx$

g) $I = \int_{-1^-}^{+1^-} x^2 (x-2)^3 [\delta''(x-1) + \delta''(x+1)]\, dx$

h) $I = \int_{0^-}^{1^+} x^2 \sqrt{3x+1}\, [\delta'(x) + \delta'(x-1)]\, dx$

i) $I = \int_{0^+}^{1^-} (x+1)\sqrt{2-x^2}\, \delta'(x - 1/2)\, dx$

Distributions and Green's Functions

Integrodifferential form

Consider an inhomogeneous linear differential equation of the form

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = f(x) \qquad (1)$$

with $a_0, a_1, \ldots, a_n \in C^0(I)$ and $f \in C^0(I)$ with $I \subseteq \mathbb{R}$ some interval. Using the Dirac delta function and its derivatives, we define a generalized function $L(x,t)$ via:

$$L(x,t) = \sum_{k=0}^{n-1} (-1)^k a_k(x)\, \delta^{(k)}(t-x) \qquad (2)$$

Then we may rewrite the ODE above as

$$\int_I L(x,t)\, y(t)\, dt = f(x) \qquad (3)$$

Note the similarity in structure to a linear system of equations from linear algebra. Here, the generalized function $L(x,t)$ represents the linear differential operator $L: C^n(I) \to C^0(I)$ associated with Eq. (1), and it is called the <u>kernel</u> of the linear ODE. Eq. (3) is the <u>integrodifferential form</u> of the linear ODE given by Eq. (1).

EXAMPLE

Consider the linear ODE:

$$y''(x) + x y'(x) - (x^2-1)\, y(x) = f(x)$$

The corresponding kernel $L(x,t)$ is given by

$$L(x,t) = (-1)^2 \delta''(t-x) + (-1)^1 x \delta'(t-x) - (x^2-1)\delta(t-x)$$
$$= \delta''(t-x) - x\delta'(t-x) - (x^2-1)\delta(t-x)$$

and the integro differential form of the ODE is given by

$$\int_{-\infty}^{+\infty} \left[\delta''(t-x) - x\delta'(t-x) - (x^2-1)\delta(t-x)\right] y(t)\, dt = f(x)$$

General theory of Green's functions

Def: Consider an ODE of the form

$$\int_I L(x,t)\,y(t)\,dt = f(x)$$

with $I \subseteq \mathbb{R}$ some interval. A Green's function $G(x,\xi)$ is any function that satisfies the equation

$$\int_I L(x,t)\,G(t,\xi)\,dt = \delta(x-\xi)$$

in the sense of distributions.

Remark: The Green's function does not have to be unique. Usually, it will have free parameters that can be determined by applying an initial condition or some boundary conditions. Using a linear algebra analogy, $G(x,\xi)$ can be thought of as representing an "inverse" of the operator represented by $L(x,t)$, except that due to a null space of homogeneous solutions, the inverse operation is not unique.

Thm: Let $G(x,\xi)$ be a Green's function of a linear ODE $Ly = f$ defined on some interval $I \subseteq \mathbb{R}$. Then a particular solution of the ODE is given by:

$$y_p(x) = \int_I G(x,\xi)\,f(\xi)\,d\xi$$

<u>Proof</u>

Since:

$$\int_I L(x,t)\, y_p(t)\, dt = \int_I L(x,t)\left[\int_I G(t,\xi) f(\xi)\, d\xi\right] dt =$$

$$= \int_I dt \int_I d\xi\; L(x,t)\, G(t,\xi)\, f(\xi) =$$

$$= \int_I d\xi\, f(\xi) \left[\int_I dt\; L(x,t)\, G(t,\xi)\right]$$

$$= \int_I d\xi\, f(\xi)\, \delta(x-\xi) = f(x)$$

it follows that $y_p(x)$ is a solution of the linear ODE

$$\int_I L(x,t)\, y(t)\, dt = f(x) \qquad \square$$

- The Green's function can be found using the following theorem. Transform methods, e.g. the Laplace transform are an alternate technique which we shall discuss later.

Thm: Consider the linear ODE $Ly = f$ defined on an interval $I \subseteq \mathbb{R}$ with $L : C^n(I) \to C^0(I)$ defined as:
$\forall y \in C^n(I) : Ly = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$
with $a_0, a_1, \ldots, a_{n-1} \in C^0(I)$. Let $G(x, \xi)$ be a Green's function of the operator L. We assume that:

$$\text{null}(L) = \text{span}\{y_1, y_2, \ldots, y_n\}$$

Then it follows that for a given $\xi \in I$, $G(x, \xi)$ is given by

$$G(x,\xi) = \begin{cases} A_1(\xi) y_1(x) + A_2(\xi) y_2(x) + \cdots + A_n(\xi) y_n(x), & \text{if } x < \xi \\ B_1(\xi) y_1(x) + B_2(\xi) y_2(x) + \cdots + B_n(\xi) y_n(x), & \text{if } x > \xi \end{cases}$$

with $A_1, A_2, \ldots, A_n \in C^n(I)$ and $B_1, B_2, \ldots, B_n \in C^n(I)$ such that it satisfies the following conditions:

(a) $G(x,\xi), \partial G(x,\xi)/\partial x, \ldots, \partial^{n-2} G(x,\xi)/\partial x^{n-2}$ are continuous on $x = \xi$.

(b) $$\lim_{x \to \xi^+} \frac{\partial^{n-1} G(x,\xi)}{\partial x^{n-1}} - \lim_{x \to \xi^-} \frac{\partial^{n-1} G(x,\xi)}{\partial x^{n-1}} = 1$$

Proof

Since $G(x,\xi)$ is a Green's function of the linear operator L, it follows that $LG(x,\xi) = \delta(x-\xi)$. Localizing for $x < \xi$ and for $x > \xi$, we have:

For $x < \xi$:

$$LG(x,\xi) = 0 \Leftrightarrow G(x,\xi) = A_1(\xi) y_1(x) + A_2(\xi) y_2(x) + \cdots + A_n(\xi) y_n(x)$$

For $x > \xi$:

$$LG(x,\xi) = 0 \Leftrightarrow G(x,\xi) = B_1(\xi) y_1(x) + B_2(\xi) y_2(x) + \cdots + B_n(\xi) y_n(x)$$

It follows that:

$$G(x,\xi) = \begin{cases} A_1(\xi)\, y_1(x) + A_2(\xi)\, y_2(x) + \cdots + A_n(\xi)\, y_n(x)\,, & \text{if } x<\xi \\ B_1(\xi)\, y_1(x) + B_2(\xi)\, y_2(x) + \cdots + B_n(\xi)\, y_n(x)\,, & \text{if } x>\xi \end{cases}$$

To establish the conditions (a) and (b) we note that $G(x,\xi)$ satisfies the following equation, in the sense of distributions:

$$\frac{\partial^n G(x,\xi)}{\partial x^n} + a_{n-1}(x)\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} + \cdots + a_1(x)\frac{\partial G(x,\xi)}{\partial x}$$

$$+ a_0(x)\, G(x,\xi) = \delta(x-\xi) \qquad (1)$$

(a) To show that $G(x,\xi), \partial G(x,\xi)/\partial x, \ldots, \partial^{n-2}G(x,\xi)/\partial x^{n-2}$ are continuous on $x=\xi$, we assume that one of them is not continuous on $x=\xi$. Then $\partial^{n-1}G(x,\xi)/\partial x^{n-1}$ is at least a 1st-order singular distribution and $\partial^n G(x,\xi)/\partial x^n$ is therefore at least a 2nd-order singular distribution. It follows that the left-hand-side of Eq.(1) is at least a 2nd-order singular distribution. This is a contradiction because the right-hand-side is a 1st-order singular distribution. Thus condition (a) is proved.

(b) We define

$$F(x,\xi) = \frac{\partial^n G(x,\xi)}{\partial x^n} - \delta(x-\xi) = \sum_{k=0}^{n-1} -a_k(x)\frac{\partial^k G(x,\xi)}{\partial x^k}$$

$$= -a_{n-1}(x)\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - \sum_{k=0}^{n-2} a_k(x)\frac{\partial^k G(x,\xi)}{\partial x^k}$$

From (a) we know that the 2nd term (the sum from $k=0$ to $k=n-2$) is continuous at $x=\xi$. We also know that

$$\frac{\partial^n G(x,\xi)}{\partial x^n} + a_{n-1}(x)\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}}$$

has to be a 1st-order singular distribution since the right-hand-side of Eq.(1) is 1st-order singular. It follows that:

$\partial^n G(x,\xi)/\partial x^n$ 1st-order singular at $x=\xi \Rightarrow$

$\Rightarrow \partial^{n-1}G(x,\xi)/\partial x^{n-1}$ not continuous and regular at $x=\xi \Rightarrow$

$\Rightarrow a_{n-1}(x)\partial^{n-1}G(x,\xi)/\partial x^{n-1}$ regular and not continuous at $x=\xi \Rightarrow$

$\Rightarrow F(x,\xi)$ regular and not continuous at $x=\xi$. (2)

We write:

$$F(x,\xi) = \frac{\partial^n G(x,\xi)}{\partial x^n} - \delta(x-\xi) = \frac{\partial}{\partial x}\left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)\right]$$

$$= \frac{\partial f(x,\xi)}{\partial x}$$

with $f(x,\xi) = \dfrac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)$

From Eq.(2) it follows that

(2) $\Rightarrow f(x,\xi)$ continuous on $x=\xi \Rightarrow \lim\limits_{x\to\xi^+} f(x,\xi) = \lim\limits_{x\to\xi^-} f(x,\xi) \Rightarrow$

$$\Rightarrow \lim_{x\to\xi^+}\left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)\right] = \lim_{x\to\xi^-}\left[\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - H(x-\xi)\right]$$

$$\Rightarrow \lim_{x\to\xi^+}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - 1 = \lim_{x\to\xi^-}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} \Rightarrow$$

$$\Rightarrow \lim_{x\to\xi^+}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} - \lim_{x\to\xi^-}\frac{\partial^{n-1}G(x,\xi)}{\partial x^{n-1}} = 1 \qquad \square$$

Application to the initial value problem

We can now apply the Green's function theory to the initial value problem of a linear differential equation. To obtain a unique solution for the Green's function we make, in the proof below, the causality assumption that $G(x,\xi)=0$ for $x<\xi$ (i.e. the future has no effect on the past).

Thm: Consider the linear ODE $Ly=f$, defined on an interval $I=[a,b]\subseteq\mathbb{R}$ with $L: C^n(I)\to C^0(I)$ given by:
$\forall y\in C^n(I): Ly = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y$
with $a_0, a_1, \dots, a_{n-1}\in C^0(I)$ and $f\in C^0(I)$.
We assume that $\text{null}(L)=\text{span}\{y_1, y_2, \dots, y_n\}$ for $y_1, y_2, \dots, y_n\in C^n(I)$. Then, the corresponding Green's function is given by

$$G(x,\xi) = \begin{cases} \sum_{k=1}^{n} B_k(\xi) y_k(x) & , \text{if } x\geq\xi \\ 0 & , \text{if } x<\xi \end{cases}$$

where $B_1, \dots, B_n$ are given by:

$$(B_1(\xi), B_2(\xi), \dots, B_n(\xi)) = W[y_1, y_2, \dots, y_n](\xi)^{-1}(0,0,\dots,0,1)$$

A corresponding particular solution is:

$$y_p(x) = \sum_{k=1}^{n} y_k(x)\left[\int_a^x f(t) B_k(t)\,dt\right], \quad \forall x\in[a,b].$$

Proof

From the previous theorem, the general form of the Green's function is:

$$G(x,\xi) = \begin{cases} \sum_{k=1}^{n} A_k(\xi) y_k(x) \ , & \text{if } x<\xi \\ \sum_{k=1}^{n} B_k(\xi) y_k(x) \ , & \text{if } x>\xi \end{cases}$$

We note that

$$\forall m \in [0,n-2]\cap\mathbb{N} : \partial^m G(x,\xi)/\partial x^m \text{ continuous at } x=\xi \Leftrightarrow$$

$$\Leftrightarrow \forall m\in[0,n-2]\cap\mathbb{N} : \lim_{x\to\xi^+} \frac{\partial^m G(x,\xi)}{\partial x^m} = \lim_{x\to\xi^-} \frac{\partial^m G(x,\xi)}{\partial x^m} \Leftrightarrow$$

$$\Leftrightarrow \forall m\in[0,n-2]\cap\mathbb{N} : \sum_{k=1}^{n} B_k(\xi) y_k^{(m)}(\xi) = \sum_{k=1}^{n} A_k(\xi) y_k^{(m)}(\xi) \Leftrightarrow$$

$$\Leftrightarrow \forall m\in[0,n-2]\cap\mathbb{N} : \sum_{k=1}^{n} [B_k(\xi)-A_k(\xi)] y_k^{(m)}(\xi) = 0 \quad (1)$$

and

$$\lim_{x\to\xi^+} \frac{\partial^{n-1} G(x,\xi)}{\partial x^{n-1}} - \lim_{x\to\xi^-} \frac{\partial^{n-1} G(x,\xi)}{\partial x^{n-1}} = 1 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} B_k(\xi) y_k^{(n-1)}(\xi) - \sum_{k=1}^{n} A_k(\xi) y_k^{(n-1)}(\xi) = 1 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} [B_k(\xi)-A_k(\xi)] y_k^{(n-1)}(\xi) = 1 \quad (2)$$

To enforce uniqueness we introduce the causality assumption that $\forall k\in[n] : A_k(\xi)=0$. Then, from Eq.(1) and Eq.(2), we have:

$$\begin{cases} \forall m\in[0,n-2]\cap\mathbb{N} : \sum_{k=1}^{n} B_k(\xi) y_k^{(m)}(\xi) = 0 \\ \sum_{k=1}^{n} B_k(\xi) y_k^{(n-1)}(\xi) = 1 \end{cases} \Leftrightarrow$$

$$\begin{bmatrix} y_1(\xi) & y_2(\xi) & \cdots & y_n(\xi) \\ y_1'(\xi) & y_2'(\xi) & \cdots & y_n'(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(\xi) & y_2^{(n-2)}(\xi) & \cdots & y_n^{(n-2)}(\xi) \\ y_1^{(n-1)}(\xi) & y_2^{(n-2)}(\xi) & \cdots & y_n^{(n-2)}(\xi) \end{bmatrix} \begin{bmatrix} B_1(\xi) \\ B_2(\xi) \\ \vdots \\ B_{n-1}(\xi) \\ B_n(\xi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \iff$$

$$\iff W[y_1, \dots, y_n](\xi)(B_1(\xi), B_2(\xi), \dots, B_n(\xi)) = (0, 0, \dots, 0, 1)$$

$$\iff (B_1(\xi), B_2(\xi), \dots, B_n(\xi)) = W[y_1, \dots, y_n](\xi)^{-1}(0, 0, \dots, 0, 1)$$

The corresponding particular solution is:

$$y_p(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi = \int_a^x \left[\sum_{k=1}^n B_k(\xi) y_k(x) \right] f(\xi)\, d\xi$$

$$= \sum_{k=1}^n \int_a^x B_k(\xi) y_k(x) f(\xi)\, d\xi =$$

$$= \sum_{k=1}^n y_k(x) \left[\int_a^x B_k(\xi) f(\xi)\, d\xi \right] \qquad \square$$

<u>Remark</u>: The particular solution $y_p(x)$ given above is the exact solution to the initial value problem

$$\begin{cases} \forall x \in [a,b] : (Ly)(x) = f(x) \\ y(a) = y'(a) = y''(a) = \cdots = y^{(n-1)}(a) = 0 \end{cases}$$

in which the system is initialized from an initial state of rest. The general solution is:

$$\forall x \in [a,b] : y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + \int_a^x G(x,\xi) f(\xi)\, d\xi$$

for more general initial conditions.

Remark: For the special case of a second-order linear ODE, the Green's function simplifies to

$$G(x,\xi) = \frac{\begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1(x) & y_2(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} \quad \text{for } x \geq \xi$$

and $G(x,\xi) = 0$ for $x < \xi$, and the solution to the initial value problem

$$\begin{cases} y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = f(x) \\ y(a) = 0 \wedge y'(a) = 0 \end{cases}$$

is given by

$$y(x) = \int_a^x G(x,\xi) f(\xi)\, d\xi.$$

EXERCISES

(10) Find the kernel $L(x,t)$ for the following linear differential equations in order to rewrite them in the form: $\int_{-\infty}^{+\infty} L(x,t)y(t)\,dt = f(x)$.

a) $y''(x) + ay'(x) + by(x) = f(x)$
b) $xy''(x) + (x^2-1)y'(x) + x^2y(x) = f(x)$
c) $[(2x+1)y']' + x^3y(x) = f(x)$
d) $(x^2-1)y'''(x) + 3xy'(x) - y(x) = f(x)$

(11) Let $G(x,\xi)$ be the Green's function to the linear ODE $Ly = f$ with $L: C^n(\mathbb{R}) \to C^0(\mathbb{R})$ defined as
$\forall y \in C^n(\mathbb{R}):\ Ly = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y$
with $a_0, a_1, \dots, a_{n-1} \in C^0(\mathbb{R})$ corresponding to the initial condition $y(a) = y'(a) = y''(a) = \dots = y^{(n-1)}(a) = 0$ of initial rest.
a) If y_1 is the solution to $Ly_1 = f_1$ initialized from rest at $x=a$ and y_2 is the solution to $Ly_2 = f_2$ initialized from rest, then show that the unique solution to the initial value problem

$$\begin{cases} Ly = \lambda f_1 + \mu f_2 \\ y(a) = y'(a) = \dots = y^{(n-1)}(a) = 0 \end{cases}$$

is $y(x) = \lambda y_1(x) + \mu y_2(x)$.

b) Show that $G(x,x)=0$ and also that

$$\forall k \in [n-2]: \left.\frac{\partial^k G(x,\xi)}{\partial x^k}\right|_{\xi = x} = 0$$

(12) Find the Green's functions for the following linear ODEs satisfying the causality condition:

a) $y''(x) + 3y'(x) + 2y(x) = f(x)$

b) $y''(x) + 6y'(x) + 9y(x) = f(x)$

c) $x^2 y''(x) + xy'(x) - 2y(x) = f(x)$

d) $y'''(x) - y'(x) = f(x)$

e) $y'''(x) - y''(x) - y'(x) + y(x) = f(x)$

f) $y'''(x) - 6y'(x) + 5y(x) = f(x)$

g) $y^{(4)}(x) + \cancel{y'''}(x) + y''(x) + y'(x) = f(x)$

ODE 7: Laplace Transforms

LAPLACE TRANSFORMS

Definition of Laplace transform

Def: Let $f:(0,+\infty)\longrightarrow \mathbb{R}$ be a function. We define the Laplace transform $F(s)=\mathcal{L}(f(t))$ of $f(t)$ in terms of the following improper integral, if it converges,

$$F(s)=\int_0^{+\infty} e^{-st} f(t)\,dt \equiv \mathcal{L}(f(t))$$

Remarks

(a) The domain of $F(s)$ depends on the convergence of the Laplace integral, which in term depends on the function $f(t)$.
(b) It is possible to use a distribution for $f(t)$. Then $F(s)$ will sstill be a regular function. The theory of Laplace transforms of distributions requires some additional care.
(c) By convention, the original function is denoted as a function of t and represented with a lower-case letter. The transform is denoted as $F(s)=\mathcal{L}(f(t))$ as a function of s and represented with upper-case letter.

Convergence of the Laplace transform

We establish a sufficient (but not necessary) condition for the convergence of the Laplace integral as follows:

<u>Def</u>: Let $f:[0,+\infty)\to\mathbb{R}$ be a function. We say that

(a) f piecewise continuous on $(0,+\infty) \Leftrightarrow$

$$\Leftrightarrow \exists t_1,t_2,\ldots,t_n \in (0,+\infty): \begin{cases} 0<t_1<t_2<\cdots<t_n \\ f \text{ continuous on } [0,t_1)\cup(t_n,+\infty) \\ \forall k\in\mathbb{N}\cap[1,n-1]: f \text{ continuous on } (t_k,t_{k+1}) \end{cases}$$

(b) f has exponential order $\gamma \Leftrightarrow$

$$\Leftrightarrow \exists M>0: \exists \delta>0: \forall t\in(\delta,+\infty): |\exp(-\gamma t)f(t)| \leq M$$

<u>notation</u>: For convenience we introduce the following non-standard notation:

$f\in E_\gamma(\mathbb{R}_+) \Leftrightarrow f$ has exponential order γ

$f\in PC^0(\mathbb{R}_+) \Leftrightarrow f$ is piecewise continuous on $[0,+\infty)$

and compare with

$f\in C^0(\mathbb{R}_+) \Leftrightarrow f$ continuous on $[0,+\infty)$

We note that $\mathbb{R}_+ = [0,+\infty)$.

<u>Thm</u>: Let $f:[0,+\infty)\to\mathbb{R}$ be a function and let $F(s)=\mathcal{L}(f(t))$. Then:

$$f\in E_\gamma(\mathbb{R}_+)\cap PC^0(\mathbb{R}_+) \Rightarrow \forall s\in(\gamma,+\infty): F(s)=\int_0^{+\infty} e^{-st}f(t)\,dt \text{ converges.}$$

<u>Proof</u>

Since $f\in E_\gamma(\mathbb{R}_+)\cap PC^0(\mathbb{R}_+) \Rightarrow f\in E_\gamma(\mathbb{R}_+) \Rightarrow$

$$\Rightarrow \exists M>0: \exists \delta>0: \forall t\in(\delta,+\infty): |\exp(-\gamma t)f(t)| < M$$

Choose some $M>0$ and $\delta>0$ such that
$\forall t\in(\delta,+\infty): |\exp(-\gamma t)f(t)|\leqslant M$
It follows that

$$\forall x\in(\delta,+\infty): \int_\delta^x |e^{-st}f(t)|\,dt = \int_\delta^x |e^{(\gamma-s)t}e^{-\gamma t}f(t)|\,dt =$$

$$= \int_\delta^x e^{(\gamma-s)t}|e^{-\gamma t}f(t)|\,dt \leqslant \int_\delta^x Me^{(\gamma-s)t}\,dt$$

$$= M\frac{\exp((\gamma-s)x)-\exp((\gamma-s)\delta)}{\gamma-s} \qquad (1)$$

Let $s\in(\gamma,+\infty)$ be given. Then we note that

$$s\in(\gamma,+\infty) \Rightarrow s>\gamma \Rightarrow \gamma-s<0 \Rightarrow \lim_{x\to+\infty}\exp((\gamma-s)x)=0 \Rightarrow$$

$$\Rightarrow \int_\delta^{+\infty} Me^{(\gamma-s)t}\,dt = \lim_{x\to+\infty}\left[M\frac{\exp((\gamma-s)x)-\exp((\gamma-s)\delta)}{\gamma-s}\right]$$

$$= \frac{-M\exp((\gamma-s)\delta)}{\gamma-s} \Rightarrow$$

$$\Rightarrow \int_\delta^{+\infty} Me^{(\gamma-s)t}\,dt \text{ converges.} \qquad (2)$$

From Eq.(1) and Eq.(2), via the comparison test it follows that

$$\int_\delta^{+\infty} |e^{-st}f(t)|\,dt \text{ converges.} \qquad (3)$$

From Eq.(3), via the absolute convergence test, it follows that

$$\int_\delta^{+\infty} e^{-st}f(t)\,dt \text{ converges} \Rightarrow \int_0^{+\infty} e^{-st}f(t)\,dt \text{ converges.} \qquad \Box$$

Immediate consequences

a) Linearity of Laplace transform

It is easy, although tedious to show that
$f, g \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \Rightarrow \forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 f + \lambda_2 g) \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+)$
It follows that if the Laplace integral converges for $f(t)$ and $g(t)$ it also converges for $\lambda_1 f(t) + \lambda_2 g(t)$ and therefore
$\forall \lambda_1, \lambda_2 \in \mathbb{R} : \mathcal{L}[\lambda_1 f(t) + \lambda_2 g(t)] = \lambda_1 \mathcal{L}(f(t)) + \lambda_2 \mathcal{L}(g(t))$
Using proof by induction this generalizes as follows:

$$\boxed{\begin{aligned}&f_1, f_2, \ldots, f_n \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \Rightarrow \\ &\Rightarrow \forall \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : \mathcal{L}\left[\sum_{k=1}^{n} \lambda_k f_k(t)\right] = \sum_{k=1}^{n} \lambda_k \mathcal{L}(f_k(t))\end{aligned}}$$

b) Uniform convergence of the Laplace integral

From the above convergence proof, it is also established that the Laplace integral, under the above conditions, converges both absolutely and uniformly with respect to $s \in (\gamma, +\infty)$. From the theory of uniform convergent integrals it follows that the Laplace integral can be exchanged with:

(1) A limit on s:

$$\boxed{\lim_{s \to \sigma} \int_0^{+\infty} e^{-st} f(t)\, dt = \int_0^{+\infty} \lim_{s \to \sigma} \left[e^{-st} f(t)\right] dt}$$

with $\sigma \in (\gamma, +\infty)$ or $\sigma = +\infty$.

(2) A derivative with respect to s for $s \in (\gamma, +\infty)$

$$\boxed{\forall s \in (\gamma, +\infty): \frac{d}{ds}\int_0^{+\infty} e^{-st} f(t)\,dt = \int_0^{+\infty} \frac{d}{ds}\left[e^{-st} f(t)\right] dt}$$

(3) An integral with respect to s:

$$\boxed{\forall s_1, s_2 \in (\gamma, +\infty): \int_{s_1}^{s_2} ds \int_0^{+\infty} dt\; e^{-st} f(t) = \int_0^{+\infty} dt \int_{s_1}^{s_2} ds\; e^{-st} f(t)}$$

In general, these operations are not allowed with respect to an arbitrary improper integral over $(0, +\infty)$. However, they ARE ALWAYS allowed with the Laplace integral as long as s satisfies the sufficient convergence condition $s > \gamma$.

Laplace transforms of elementary functions

From the definition we can show that

$$\mathcal{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, \quad \forall a \in \mathbb{R} - (-1)\mathbb{N}^*$$

$\hookrightarrow$ special cases: $\mathcal{L}(1) = \frac{1}{s}$; $\mathcal{L}(t) = \frac{1}{s^2}$

$$\forall n \in \mathbb{N}^* : \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}, \quad \forall a \in \mathbb{R} \qquad \mathcal{L}(\sinh(at)) = \frac{a}{s^2-a^2}, \quad \forall a \in \mathbb{R}$$

$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}, \quad \forall a \in \mathbb{R} \qquad \mathcal{L}(\cosh(at)) = \frac{s}{s^2-a^2}, \quad \forall a \in \mathbb{R}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad \forall a \in \mathbb{R}$$

Using fundamental properties of Laplace transforms we can calculate the transforms of more complicated functions.

EXAMPLE

Show that $\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$

Proof

We note that

$$I(T) = \int_0^T \sin(at)e^{-st}\,dt = \int_0^T \sin(at)\left(\frac{e^{-st}}{-s}\right)'dt =$$

$$= \left[\frac{-\sin(at)e^{-st}}{s}\right]_0^T - \int_0^T (\sin(at))'\frac{e^{-st}}{-s}\,dt =$$

$$= \frac{-\sin(aT)e^{-sT}+\sin 0\cdot e^0}{s} - \int_0^T a\cos(at)\frac{e^{-st}}{-s}\,dt$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\int_0^T \cos(at)e^{-st}\,dt$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\int_0^T \cos(at)\left(\frac{e^{-st}}{-s}\right)'dt =$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\left[\left[\frac{\cos(at)e^{-st}}{-s}\right]_0^T - \int_0^T (\cos(at))'\frac{e^{-st}}{-s}\,dt\right]$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s}\,\frac{\cos(aT)e^{-sT}-\cos 0\, e^0}{-s} + \frac{a}{s}\int_0^T \frac{(-a\sin(at))e^{-st}}{s}\,dt$$

$$= \frac{-\sin(aT)e^{-sT}}{s} + \frac{a}{s^2}\left[1-\cos(aT)e^{-sT}\right] - \frac{a^2}{s^2}\int_0^T \sin(at)e^{-st}\,dt$$

$$= \frac{a}{s^2} - \frac{\left[a\cos(aT)+s\sin(aT)\right]e^{-sT}}{s^2} - \frac{a^2}{s^2}I(T) \quad (1)$$

From the zero bounded theorem we note that:

Define $b(T) = \dfrac{a\cos(aT) + s\sin(aT)}{s^2}$, $\forall\, T \in [0, +\infty)$
and note that

$$\forall T \in [0,+\infty) : |b(T)| = \left| \frac{a\cos(aT) + s\sin(aT)}{s^2} \right| = \frac{|a\cos(aT) + s\sin(aT)|}{s^2} \leqslant$$

$$\leqslant \frac{|a\cos(aT)| + |s\sin(aT)|}{s^2} = \frac{|a|\cdot|\cos(aT)| + |s|\cdot|\sin(aT)|}{s^2}$$

$$\leqslant \frac{|a| + |s|}{s^2} \Rightarrow b \text{ bounded on } [0,+\infty) \qquad (2)$$

and $\lim\limits_{T\to+\infty} e^{-sT} = 0$ for $s > 0$. (3)

From Eq.(2) and Eq.(3):

$$\lim_{T\to+\infty} \frac{[a\cos(aT) + s\sin(aT)]\, e^{-sT}}{s^2} = 0$$

and from Eq.(1), taking the limit $T \to +\infty$:

$$\lim_{T\to+\infty} I(T) = \frac{a}{s^2} - \frac{a^2}{s^2} \lim_{T\to+\infty} I(T) \Leftrightarrow \left(1 + \frac{a^2}{s^2}\right) \lim_{T\to+\infty} I(T) = \frac{a}{s^2}$$

$$\Leftrightarrow \frac{s^2 + a^2}{s^2} \lim_{T\to+\infty} I(T) = \frac{a}{s^2} \Leftrightarrow \lim_{T\to+\infty} I(T) = \frac{a}{s^2 + a^2}$$

and therefore

$$\mathcal{L}(\sin(at)) = \int_0^{+\infty} \sin(at) e^{-st}\, dt = \lim_{T\to+\infty} I(T) = \frac{a}{s^2 + a^2}$$

EXERCISES

① Use the definition of the Laplace transform to show that

a) $\mathcal{L}(t^a) = \dfrac{\Gamma(a+1)}{s^{a+1}}$ b) $\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}$

c) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$ d) $\mathcal{L}(\sinh(at)) = \dfrac{a}{s^2-a^2}$

e) $\mathcal{L}(\cosh(at)) = \dfrac{s}{s^2-a^2}$

② Find the Laplace transform of the following functions.

a) $f(t) = (t-2)^2$ b) $f(t) = (t-1)^2(t+1)^2$

c) $f(t) = \sin(2t)\cos(3t)$ d) $f(t) = \cos(t)\cos(3t)$

e) $f(t) = \sin(4t)\sin(3t)$ f) $f(t) = \sin^2(5t)$

g) $f(t) = \cos^2(3t)$ h) $f(t) = \sin(2t)[\sin(2t) - \cos(2t)]$

(Hint: For problems (b),…,(h) use trigonometric identities from Precalculus to eliminate the products).

i) $f(t) = 2\cosh(3t) - 3\sinh(3t)$

j) $f(t) = (\sin t + \cos t)^2$ k) $f(t) = \cosh^2(3t)$

Operational properties of Laplace Transforms

The following operational properties of Laplace transforms also follow from the definition and the uniform convergence of the Laplace transform.

Thm: Let $f \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+)$ and assume that $\mathcal{L}(f(t)) = F(s)$. Then, it follows that

$$\mathcal{L}[e^{at} f(t)] = F(s-a), \quad \forall a \in \mathbb{R}$$

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as} F(s), \quad \forall a \in (0,+\infty)$$

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right), \quad \forall a \in (0,+\infty)$$

$$\lim_{t \to 0^+} \frac{f(t)}{t} \in \mathbb{R} \Rightarrow \mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{+\infty} F(\sigma)\,d\sigma$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s), \quad \forall n \in \mathbb{N}^*$$

Evaluating Laplace Transforms

Method: We use a sequence of implications to build up the function $f(t)$ and its transform using the properties of the Laplace transform and the Laplace transform of fundamental functions.

EXAMPLES

a) Find $\mathcal{L}(f(t))$ for $f(t) = e^{2t}\sin^3 t$.

Solution

Since $\sin(3t) = -4\sin^3 t + 3\sin t \Rightarrow$

$$\Rightarrow \sin^3 t = (1/4)[3\sin t - \sin(3t)] \Rightarrow$$

$$\to \mathcal{L}(\sin^3 t) = \mathcal{L}\left[(1/4)[3\sin t - \sin(3t)]\right]$$

$$= (1/4)[3\mathcal{L}(\sin t) - \mathcal{L}(\sin(3t))]$$

$$= (1/4)\left[3\frac{1}{s^2+1} - \frac{3}{s^2+3^2}\right] =$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] = \frac{3}{4}\left[\frac{(s^2+9)-(s^2+1)}{(s^2+1)(s^2+9)}\right]$$

$$= \frac{3}{4}\,\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)} = \frac{3}{4}\,\frac{8}{(s^2+1)(s^2+9)} =$$

$$= \frac{6}{(s^2+1)(s^2+9)} \Rightarrow$$

$$\to \mathcal{L}(e^{2t}\sin^3 t) = \frac{6}{[(s-2)^2+1][(s-2)^2+9]} =$$

$$= \frac{6}{(s^2-4s+4+1)(s^2-4s+4+9)} = \frac{6}{(s^2-4s+5)(s^2-4s+13)}$$

b) Find $\mathcal{L}(f(t))$ for $f(t) = t\cos(3t+\pi/6)$.

Solution

We note that

$$f(t) = t\cos(3t+\pi/6) = t\left[\cos(3t)\cos(\pi/6) - \sin(3t)\sin(\pi/6)\right] =$$
$$= t\left[(\sqrt{3}/2)\cos(3t) - (1/2)\sin(3t)\right] =$$
$$= (t/2)\left[\sqrt{3}\cos(3t) - \sin(3t)\right]$$

and

$$\mathcal{L}\left[\sqrt{3}\cos(3t) - \sin(3t)\right] = \sqrt{3}\,\mathcal{L}(\cos(3t)) - \mathcal{L}(\sin(3t)) =$$
$$= \sqrt{3}\,\frac{s}{s^2+3^2} - \frac{3}{s^2+3^2} = \frac{s\sqrt{3}-3}{s^2+9} \Rightarrow$$

$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}\left\{(t/2)\left[\sqrt{3}\cos(3t) - \sin(3t)\right]\right\} =$$
$$= (1/2)(-1)\frac{d}{ds}\left(\frac{s\sqrt{3}-3}{s^2+9}\right) =$$
$$= \frac{-1}{2}\,\frac{(s\sqrt{3}-3)'(s^2+9) - (s\sqrt{3}-3)(s^2+9)'}{(s^2+9)^2} =$$
$$= \frac{-1}{2}\,\frac{\sqrt{3}(s^2+9) - (2s)(s\sqrt{3}-3)}{(s^2+9)^2} =$$
$$= \frac{-1}{2}\,\frac{s^2\sqrt{3} + 9\sqrt{3} - s^2(2\sqrt{3}) + 6s}{(s^2+9)^2} =$$
$$= \frac{-1}{2}\,\frac{-s^2\sqrt{3} + 6s + 9\sqrt{3}}{(s^2+9)^2} = \frac{s^2\sqrt{3} - 6s - 9\sqrt{3}}{2(s^2+9)^2}$$

c) Find $\mathcal{L}(f(t))$ for $f(t) = \dfrac{e^t \sin(2t)}{t}$

Solution

Since,

$$\mathcal{L}(\sin(2t)) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4} \qquad (1)$$

and

$$\lim_{t\to 0^+} \frac{\sin(2t)}{t} = 2 \lim_{t\to 0^+} \frac{\sin(2t)}{2t} = 2\cdot 1 = 2 \in \mathbb{R} \qquad (2)$$

it follows, from Eq. (1) and Eq. (2), that

$$\mathcal{L}\left(\frac{\sin(2t)}{t}\right) = \int_s^{+\infty} \frac{2}{\sigma^2+4}\, d\sigma = 2\int_0^{+\infty} \frac{d\sigma}{\sigma^2+2^2} =$$

$$= 2\left[\frac{1}{2}\operatorname{Arctan}\left(\frac{\sigma}{2}\right)\right]_s^{+\infty} = \left[\operatorname{Arctan}\left(\frac{\sigma}{2}\right)\right]_s^{+\infty} =$$

$$= \lim_{\sigma\to+\infty}\left[\operatorname{Arctan}(\sigma/2)\right] - \operatorname{Arctan}(s/2) =$$

$$= \frac{\pi}{2} - \operatorname{Arctan}\left(\frac{s}{2}\right) = \operatorname{Arccot}\left(\frac{s}{2}\right) = \operatorname{Arctan}\left(\frac{2}{s}\right) \Rightarrow$$

$$\Rightarrow \mathcal{L}\left(\frac{e^t \sin(2t)}{t}\right) = \operatorname{Arctan}\left(\frac{2}{s-1}\right).$$

$\to$ In this argument we have used the following trigonometric identities

$$\boxed{\begin{array}{l} \forall x \in \mathbb{R}: \operatorname{Arctan}(x) + \operatorname{Arccot}(x) = \pi/2 \\ \forall x \in \mathbb{R}^*: \operatorname{Arccot}(x) = \operatorname{Arctan}(1/x) \end{array}}$$

Laplace transform of piecewise defined functions

Method: To evaluate the Laplace transform of a function of the form

$$f(t) = \begin{cases} f_0(t), & \text{if } 0 \leq t < a_1 \\ f_1(t), & \text{if } a_1 < t < a_2 \\ f_2(t), & \text{if } a_2 < t < a_3 \\ \vdots \\ f_n(t), & \text{if } a_n < t \end{cases}$$

•1 We rewrite $f(t)$ in terms of the Heavyside function as:

$$\begin{aligned} f(t) &= f_0(t) + [f_1(t) - f_0(t)]H(t-a_1) + [f_2(t) - f_1(t)]H(t-a_2) \\ &\quad + \cdots + [f_n(t) - f_{n-1}(t)]H(t-a_n) \\ &= f_0(t) + \sum_{k=1}^{n} [f_k(t) - f_{k-1}(t)]H(t-a_k) \end{aligned}$$

•2 Define functions $g_k(t)$ such that

$\forall k \in [n]: g_k(t-a_k) = f_k(t) - f_{k-1}(t)$

The needed definition is:

$\forall k \in [n]: g_k(t) = f_k(t+a_k) - f_{k-1}(t+a_k)$

•3 Find the Laplace transforms $\mathcal{L}(g_k(t)) = G_k(s)$

•4 It follows that

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}\left[f_0(t) + \sum_{k=1}^{n} g_k(t-a_k)H(t-a_k) \right] = \\ &= \mathcal{L}[f_0(t)] + \sum_{k=1}^{n} \mathcal{L}[g_k(t-a_k)H(t-a_k)] \\ &= F_0(s) + \sum_{k=1}^{n} \exp(-a_k s)\, G_k(s). \end{aligned}$$

EXAMPLE

Find $\mathcal{L}(f(t))$ for $f(t) = \begin{cases} \sin t & , \text{ if } 0<t<\pi \\ e^t & , \text{ if } \pi<t \end{cases}$

Solution

We rewrite $f(t)$ as:

$$f(t) = \sin t + [e^t - \sin t]\, H(t-\pi) =$$
$$= \sin t + [\exp((t-\pi)+\pi) - \sin((t-\pi)+\pi)]\, H(t-\pi)$$
$$= \sin t + g(t-\pi)\, H(t-\pi)$$

with

$$g(t) = \exp(t+\pi) - \sin(t+\pi) = e^{\pi} e^{t} - (-1)\sin t =$$
$$= e^{\pi} e^{t} + \sin t.$$

It follows that

$$\mathcal{L}(g(t)) = \mathcal{L}(e^{\pi} e^{t} + \sin t) = e^{\pi}\mathcal{L}(e^t) + \mathcal{L}(\sin t) =$$
$$= e^{\pi}\left(\frac{1}{s-1}\right) + \frac{1}{s^2+1} = \frac{e^{\pi}(s^2+1) + (s-1)}{(s-1)(s^2+1)}$$
$$= \frac{e^{\pi}s^2 + e^{\pi} + s - 1}{(s-1)(s^2+1)} = \frac{e^{\pi}s^2 + s + (e^{\pi}-1)}{(s-1)(s^2+1)} \Rightarrow$$

$$\Rightarrow \mathcal{L}(g(t-\pi)H(t-\pi)) = e^{-\pi s}\, \frac{e^{\pi}s^2 + s + (e^{\pi}-1)}{(s-1)(s^2+1)} \Rightarrow$$

$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(\sin t + g(t-\pi)H(t-\pi)) =$$
$$= \mathcal{L}(\sin t) + \mathcal{L}(g(t-\pi)H(t-\pi))$$
$$= \frac{1}{s^2+1} + \frac{e^{-\pi s}[e^{\pi}s^2 + s + (e^{\pi}-1)]}{(s-1)(s^2+1)}$$

EXERCISES

③ Evaluate the Laplace transform for the following functions.

a) $f(t) = t^2 e^{3t}$ b) $f(t) = (t-2)^2 \sin t$

c) $f(t) = e^{-t}\cos(3t+\pi/6)$ d) $f(t) = e^{-4t}\sin(3t)\cos t$

e) $f(t) = e^{2t}\cos^2(t+\pi/3)$ f) $f(t) = e^{-3t}\cosh(2t)$

g) $f(t) = e^{-2t}[2\sinh(t) - 3\cosh(2t)]$

h) $f(t) = t^3\sinh(2t)$ i) $f(t) = (\sin^2 t)H(t-\pi/4)$

j) $f(t) = te^{2t}H(t-1)$ k) $f(t) = \begin{cases} te^{2t}, & \text{if } t\in[0,3] \\ e^{2t}, & \text{if } t\in(3,+\infty) \end{cases}$

l) $f(t) = \begin{cases} \sin t, & \text{if } t\in(0,\pi/3) \\ t\cos t, & \text{if } t\in(\pi/3,+\infty) \end{cases}$

m) $f(t) = \dfrac{\sin(3t)}{t}$ n) $f(t) = \dfrac{1-\cos(2t)}{3t}$

o) $f(t) = \dfrac{1-\exp(3t)}{t}$ p) $f(t) = \dfrac{\sinh(2t)}{5t}$

q) $f(t) = \dfrac{1-\cosh(4t)}{4t}$

Laplace transforms of functions defined as series

Given a function $f(t)$ defined as a power series

$$f(t) = \sum_{n=0}^{+\infty} a_n t^n$$

due to the uniform convergence of the power series, the Laplace integral can be done term by term and we get:

$$F(s) = \mathcal{L}(f(t)) = \int_0^{+\infty} e^{-st}\left[\sum_{n=0}^{+\infty} a_n t^n\right] dt = \sum_{n=0}^{+\infty} a_n \left[\int_0^{+\infty} e^{-st} t^n \, dt\right]$$

$$= \sum_{n=0}^{+\infty} a_n \mathcal{L}(t^n) = \sum_{n=0}^{+\infty} a_n \frac{\Gamma(n+1)}{s^{n+1}} = \sum_{n=0}^{+\infty} \frac{n!\, a_n}{s^{n+1}}$$

The same idea can be applied on Frobenius series, restricted at $t > 0$ such that for

$$f(t) = t^\lambda \sum_{n=0}^{+\infty} a_n t^n = \sum_{n=0}^{+\infty} a_n t^{n+\lambda}$$

the corresponding Laplace transform of $f(t)$ is given by

$$F(s) = \mathcal{L}(f(t)) = \sum_{n=0}^{+\infty} \frac{a_n \Gamma(n+\lambda+1)}{s^{n+\lambda+1}}$$

EXAMPLE

Show that $\mathcal{L}(J_0(t)) = \dfrac{1}{\sqrt{s^2+1}}$

Solution

$$J_0(t) = \left(\frac{t}{2}\right)^0 \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,\Gamma(n+1+0)} \left(\frac{t}{2}\right)^{2n} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{t}{2}\right)^{2n} =$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}\,(n!)^2}\, t^{2n} \Rightarrow$$

$$\Rightarrow \mathcal{L}(J_0(t)) = \mathcal{L}\left[\sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}\,(n!)^2}\, t^{2n}\right] = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}\,(n!)^2}\, \mathcal{L}(t^{2n})$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^{2n}\,(n!)^2}\, \frac{(2n)!}{s^{2n+1}} = \frac{1}{s} \sum_{n=0}^{+\infty} \frac{(-1)^n\,(2n)!}{(2n)!!\,(2n)!!}\, \frac{1}{s^{2n}}$$

$$= \frac{1}{s}\left[1 + \sum_{n=1}^{+\infty} \frac{(-1)^n\,(2n-1)!!}{(2n)!!} \left(\frac{1}{s^2}\right)^n\right] =$$

$$= \frac{1}{s}\left[1 + \sum_{n=1}^{+\infty} \binom{-1/2}{n} \left(\frac{1}{s^2}\right)^n\right] =$$

$$= \frac{1}{s} \sum_{n=0}^{+\infty} \binom{-1/2}{n} \left(\frac{1}{s^2}\right)^n =$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{s}\left(\frac{s^2+1}{s^2}\right)^{-1/2} =$$

$$= \frac{1}{s}\sqrt{\frac{s^2}{s^2+1}} = \sqrt{\frac{s^2}{s^2(s^2+1)}} = \sqrt{\frac{1}{s^2+1}} =$$

$$= \frac{1}{\sqrt{s^2+1}}$$

↳ Note that the key step is to use the binomial series expansion in reverse. We also recall from Calculus 2 that

$$\forall n \in \mathbb{N}^*: \binom{a}{n} = \prod_{k=1}^{n} \left(\frac{a+1-k}{k} \right)$$

and therefore

$$\forall n \in \mathbb{N}^*: \binom{-1/2}{n} = \prod_{k=1}^{n} \left(\frac{-1/2+1-k}{k} \right) = \prod_{k=1}^{n} \left(\frac{-1+2-2k}{2k} \right) =$$

$$= \prod_{k=1}^{n} \frac{1-2k}{2k} = (-1)^n \prod_{k=1}^{n} \frac{(2k-1)}{2k} =$$

$$= (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

EXERCISES

(4) Use power series to show that

a) $\mathcal{L}(J_0(at)) = \dfrac{1}{\sqrt{s^2+a^2}}$

b) $\mathcal{L}(\sin(\sqrt{t})) = \dfrac{\sqrt{\pi}}{2s\sqrt{s}} \exp\left(\dfrac{-1}{4s}\right)$

c) $\mathcal{L}\left(\dfrac{\cos(\sqrt{t})}{\sqrt{t}}\right) = \exp\left(\dfrac{-1}{4s}\right)\sqrt{\dfrac{\pi}{s}}$

d) $\mathcal{L}\left(\dfrac{1}{\sqrt{\pi t}} \exp\left(\dfrac{-a^2}{4t}\right)\right) = \dfrac{\exp(-a\sqrt{s})}{\sqrt{s}}$

e) $\mathcal{L}(J_0(\sqrt{t})) = \dfrac{\exp(-s/4)}{s}$

f) $\mathcal{L}\left(\dfrac{\exp(-at)}{\sqrt{\pi t}}\right) = \dfrac{1}{\sqrt{s+a}}$

(5) Although these results can be established by evaluating the Laplace integral, provide an alternate proof using power series

a) $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

b) $\mathcal{L}(\sin(at)) = \dfrac{a}{s^2+a^2}$

c) $\mathcal{L}(\cos(at)) = \dfrac{s}{s^2+a^2}$

(6) Consider the error function defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-\tau^2)\, d\tau$$

Use power series to show that

$$\mathcal{L}(\operatorname{erf}(\sqrt{t})) = \frac{1}{s\sqrt{s+1}}$$

(7) Use the results from exercises 4, 5, 6 to find the Laplace transform of

a) $f(t) = t^2 J_0(3t)$

b) $f(t) = t \sin(\sqrt{t})$

c) $f(t) = \sqrt{t} \cos(\sqrt{t})$

d) $f(t) = t^3 \operatorname{erf}(\sqrt{t})$

e) $f(t) = te^{2t} \operatorname{erf}(\sqrt{t})$

f) $f(t) = te^{-t} J_0(t)$

Fundamental properties of Laplace transforms

Techniques based on Laplace transforms are founded on the following fundamental properties of Laplace transforms

① → Lerch's theorem

$$\boxed{\forall f,g \in PC^0(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) : \left(\mathcal{L}(f) = \mathcal{L}(g) \Rightarrow f = g \right)}$$

Remark: This theorem shows that two functions f,g that are piecewise continuous on $\mathbb{R}_+$ and of exponential order γ that have equal Laplace transforms have to be themselves equal. It follows that given $F(s) = \mathcal{L}(f(t))$ we can define the inverse Laplace transform operation $\mathcal{L}^{-1}$ such that given $F(s)$ we can find the original unique function $f(t)$ as $f(t) = \mathcal{L}^{-1}(F(s))$. Using complex analysis it is possible to represent the inverse Laplace transform in terms of an integral in the complex plane, however we will not need this representation in our work below.

② → Laplace transform of derivatives

$$\boxed{\begin{cases} f \text{ differentiable on } \mathbb{R}_+ \\ f \in E_\gamma(\mathbb{R}_+) \wedge f' \in PC^0(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \end{cases} \Rightarrow \mathcal{L}(f'(t)) = sF(s) - f(0)}$$

Remarks: It is necessary to assume that $f' \in PC^0(\mathbb{R}_+)$ but we do not need to assume $f' \in E_\gamma(\mathbb{R}_+)$. The theorem can be proved using integration by parts.

The theorem generalizes for the second derivative $f''(t)$ and for the n^{th}-derivative $f^{(n)}(t)$ as follows:

Thm:

$$\left.\begin{array}{l} f \text{ twice differentiable on } \mathbb{R}_+ \\ f'' \in PC^0(\mathbb{R}_+) \wedge f, f' \in E_\gamma(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \end{array}\right\} \Rightarrow \mathcal{L}(f''(t)) = s^2 F(s) - s f(0) - f'(0)$$

Thm

$$\left.\begin{array}{l} f \text{ n-times differentiable on } \mathbb{R}_+ \\ f^{(n)} \in PC^0(\mathbb{R}_+) \\ \forall k \in [0, n-1] \cap \mathbb{N} : f^{(k)} \in E_\gamma(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \mathcal{L}(f^{(n)}(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

These theorems are immediate consequences and the second one can be proved via induction. Another immediate consequence is the following operational property:

Thm

$$\left.\begin{array}{l} f\in PC^0(\mathbb{R}_+)\cap E_\gamma(\mathbb{R}_+) \\ \mathcal{L}(f(t)) = F(s) \end{array}\right\} \Rightarrow \mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

3 → Limit properties of Laplace transforms

An immediate consequence of the definition and the uniform convergence of the Laplace integral with respect to s is the following result

Prop: $$\left.\begin{array}{l} f\in PC^0(\mathbb{R}_+)\cap E_\gamma(\mathbb{R}_+) \\ \mathcal{L}(f(t)) = F(s) \end{array}\right\} \Rightarrow \lim_{s\to+\infty} F(s) = 0$$

Using the previous theorem on the Laplace transform of the derivative of a function we can also prove the following theorem:

Thm : Assume that

$$\left\{\begin{array}{l} f\in E_\gamma(\mathbb{R}_+) \wedge f \text{ differentiable on } \mathbb{R}_+ \\ f' \in PC^0(\mathbb{R}_+) \\ \mathcal{L}(f(t)) = F(s) \end{array}\right.$$

Then it follows that:

a) $\lim_{s\to+\infty} sF(s) = f(0)$

b) $\lim_{s\to 0} sF(s) \in \mathbb{R} \Rightarrow \lim_{t\to+\infty} f(t) = \lim_{s\to 0^+} sF(s)$

EXERCISES

(8) Use the following steps to derive the Laplace transforms of the general Bessel function $J_n(t)$ with $n \in \mathbb{N}$.

a) Use the power series definition of $J_0(t)$ and $J_1(t)$ to show that $J_1'(t) = -J_0(t)$.

b) Show that $\mathcal{L}(J_1(t)) = \dfrac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}}$

c) Use the identity $2J_n'(t) = J_{n-1}(t) - J_{n+1}(t)$ and proof by induction to show that
$$\mathcal{L}(J_n(t)) = \frac{(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}}$$

d) Use the above results to show that $\lim\limits_{t \to +\infty} J_n(t) = 0$

(9) Alternate proof of Laplace transform of $f(t) = \sin(at)$

(a) Show that $f(t) = \sin(at)$ is the unique solution of the following initial value problem
$$\begin{cases} f''(t) + a^2 f(t) = 0 \\ f(0) = 0 \wedge f'(0) = a \end{cases}$$

(b) Use the previous solution to show that
$$\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$$

c) Take the derivative of $\sin(at)$ to show that
$$\mathcal{L}(\cos(at)) = \frac{s}{s^2+a^2}$$

(10) Alternative proof of Laplace transform of $f(t)=e^{at}$.

a) Show that $f(t)=e^{at}$ is the unique solution of the initial value problem

$$\begin{cases} f'(t)-af(t)=0 \\ f(0)=1 \end{cases}$$

b) Use (a) to show that $\mathcal{L}(e^{at})=\dfrac{1}{s-a}$

(11) Use the theorem about the Laplace transform of the derivative of a function to show that

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right)=\frac{F(s)}{s}$$

(12) The error function $\operatorname{erf}(t)$ and the complementary error function $\operatorname{erfc}(t)$ are defined as

$$\forall t\in\mathbb{R}:\ \operatorname{erf}(t)=\frac{2}{\sqrt{\pi}}\int_0^t \exp(-\tau^2)\,d\tau$$

$$\forall t\in\mathbb{R}:\ \operatorname{erfc}(t)=\frac{2}{\sqrt{\pi}}\int_t^{+\infty} \exp(-\tau^2)\,d\tau = 1-\operatorname{erf}(t)$$

a) Use the Laplace integral to show that

$$\mathcal{L}(\exp(-t^2))=\exp\left(\frac{s^2}{4}\right)\operatorname{erfc}\left(\frac{s}{2}\right)$$

(Hint: it will be necessary to complete squares and use the method of substitution)

b) Show that $\mathcal{L}(\operatorname{erf}(t))=\dfrac{1}{s}\exp\left(\dfrac{s^2}{4}\right)\operatorname{erfc}\left(\dfrac{s}{2}\right)$

Laplace transforms of functions defined as integrals

Such functions can be differentiated via the 1st Fundamental Theorem of Calculus, so we can find the Laplace transform of their derivative. If a differential equation is obtained, we can get initial conditions via the initial-value theorem or the final-value theorem. With this technique we can find the Laplace transform for the following special functions:

1) Sine integral

$$\boxed{\mathrm{Si}(t)=\int_0^t \frac{\sin\tau}{\tau}\,d\tau \Rightarrow \mathcal{L}(\mathrm{Si}(t))=\frac{1}{s}\operatorname{Arctan}\left(\frac{1}{s}\right)}$$

2) Cosine integral

$$\boxed{\mathrm{Ci}(t)=\int_t^{+\infty} \frac{\cos\tau}{\tau}\,d\tau \Rightarrow \mathcal{L}(\mathrm{Ci}(t))=\frac{\ln(s^2+1)}{2s}}$$

3) Exponential integral

$$\boxed{\mathrm{Ei}(t)=\int_t^{+\infty} \frac{\exp(-\tau)}{\tau}\,d\tau \Rightarrow \mathcal{L}(\mathrm{Ei}(t))=\frac{\ln(s+1)}{s}}$$

EXAMPLES

Show that for the cosine integral function $Ci(t) = \int_t^{+\infty} \frac{\cos\tau}{\tau}\, d\tau$ its Laplace transform is

$$\mathcal{L}(Ci(t)) = \frac{\ln(s^2+1)}{2s}$$

Solution

Let $F(s) = \mathcal{L}(Ci(t))$. Since,

$$Ci'(t) = \frac{d}{dt}\int_t^{+\infty} \frac{\cos\tau}{\tau}\, d\tau = \frac{-\cos t}{t} \Rightarrow tCi'(t) = -\cos t$$

and

$$\mathcal{L}(-\cos t) = -\mathcal{L}(\cos t) = \frac{-s}{s^2+1^2} = \frac{-s}{s^2+1}$$

and

$$\mathcal{L}(Ci'(t)) = s\mathcal{L}(Ci(t)) - Ci(0) = sF(s) - Ci(0) \Rightarrow$$

$$\Rightarrow \mathcal{L}(tCi'(t)) = (-1)(d/ds)\mathcal{L}(Ci'(t)) = (-1)(d/ds)(sF(s) - Ci(0))$$

$$= -(d/ds)(sF(s))$$

it follows that

$$\mathcal{L}(tCi'(t)) = \mathcal{L}(-\cos t) \Leftrightarrow -(d/ds)(sF(s)) = \frac{-s}{s^2+1} \Leftrightarrow$$

$$\Leftrightarrow (d/ds)(sF(s)) = \frac{s}{s^2+1} \Leftrightarrow$$

$$\Leftrightarrow sF(s) = \int \frac{s\,ds}{s^2+1} = \frac{1}{2}\int \frac{2s\,ds}{s^2+1} = \frac{1}{2}\int \frac{(s^2+1)'}{s^2+1}\, ds =$$

$$= \frac{\ln|s^2+1|}{2} + c = \frac{\ln(s^2+1)}{2} + c$$

For $s \to 0^+$, we have

$$\lim_{s\to 0^+} sF(s) = \lim_{s\to 0^+}\left[\frac{\ln(s^2+1)}{2} + c\right] = \frac{\ln(0+1)}{2} + c = c \Rightarrow$$

$$\Rightarrow\ c = \lim_{s\to 0^+} sF(s) = \lim_{t\to+\infty} Ci(t) = \lim_{t\to+\infty}\int_t^{+\infty}\frac{\cos\tau}{\tau}\,d\tau = 0$$

and therefore

$$sF(s) = \frac{\ln(s^2+1)}{2} \Rightarrow F(s) = \frac{\ln(s^2+1)}{2s}$$

EXERCISES

(13) Use the properties of Laplace transforms to show that:

a) $\mathcal{L}(Si(t)) = \dfrac{1}{s} \operatorname{Arctan}\left(\dfrac{1}{s}\right)$

b) $\mathcal{L}(Ei(t)) = \dfrac{\ln(s+1)}{s}$

(14) Use the power-series method to show that

$\mathcal{L}(Si(t)) = \dfrac{1}{s} \operatorname{Arctan}\left(\dfrac{1}{s}\right)$

(15) Show that $\displaystyle\int_0^{+\infty} \frac{\sin \tau}{\tau}\, d\tau = \frac{\pi}{2}$

(16) Use the previous results and the properties of Laplace transforms to show the following generalizations

a) $\mathcal{L}(Ci(at)) = \dfrac{1}{2s} \ln\left(\dfrac{s^2+a^2}{a^2}\right)$

b) $\mathcal{L}(Si(at)) = \dfrac{1}{s} \operatorname{Arctan}\left(\dfrac{a}{s}\right)$

c) $\mathcal{L}(Ei(at)) = \dfrac{1}{s} \ln\left(\dfrac{s+a}{a}\right)$

Application to differential equations

The main idea for solving differential equations using the Laplace transform technique is that a wider range of ordinary differential equation initial value problems, to be solved on $t \in [0, +\infty)$ can be transformed to an algebraic problem. and the Laplace transform of the solution can be found via basic algebra. The challenge then is to apply an inverse Laplace transform and find the actual solution to the initial value problem.

→ Methodology: Evaluating inverse Laplace transforms.

If $F(s) = \mathcal{L}(f(t))$, then the following properties are useful in the evaluation of inverse Laplace transforms

$$\boxed{\begin{aligned} &\mathcal{L}^{-1}[F(s-a)] = e^{at} f(t) \\ &\mathcal{L}^{-1}[e^{-as} F(s)] = f(t-a) H(t-a) \\ &\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right) \end{aligned}}$$

To evaluate the inverse Laplace transform of functions of the form $P(s)/Q(s)$ where P, Q are polynomials we use partial fraction decomposition in conjunction with the following known inverse Laplace transforms:

$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$	$\mathcal{L}^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{t^{n-1}e^{at}}{(n-1)!}$

If we encounter irreducible quadratic factors then the following additional inverse Laplace transforms can be very useful, when combined with s-shifting.

$$\boxed{\begin{aligned} \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] &= \sin(at) \longrightarrow \mathcal{L}^{-1}\left[\frac{a}{(s-b)^2+a^2}\right] = e^{bt}\sin(at) \\ \mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] &= \cos(at) \longrightarrow \mathcal{L}^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right] = e^{bt}\cos(at) \end{aligned}}$$

More complicated cases can be handled by computer algebra systems or by inverse Laplace transform tables.

EXAMPLES

a) Use Laplace transforms to solve the initial value problem

$$\begin{cases} y''(t) - 3y'(t) + 2y(t) = 4e^{2t} \\ y(0) = -3 \wedge y'(0) = 5 \end{cases}$$

Solution

Define $Y(s) = \mathcal{L}(y(t))$. Then

$$\mathcal{L}[y''(t) - 3y'(t) + 2y(t)] = \mathcal{L}[y''(t)] - 3\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] =$$

$$= [s^2 Y(s) - s y(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s)$$

$$= s^2 Y(s) - (-3)s - 5 - 3sY(s) + 3(-3) + 2Y(s)$$

$$= s^2 Y(s) + 3s - 5 - 3sY(s) - 9 + 2Y(s)$$

$$= (s^2 - 3s + 2) Y(s) + (3s - 5 - 9) =$$

$$= (s^2 - 3s + 2) Y(s) + (3s - 14)$$

and

$$\mathcal{L}(4e^{2t}) = 4\mathcal{L}(e^{2t}) = 4 \cdot \frac{1}{s-2} = \frac{4}{s-2}$$

so it follows that

$$\begin{cases} y''(t) - 3y'(t) + 2y(t) = 4e^{2t} \\ y(0) = -3 \wedge y'(0) = 5 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow (s^2 - 3s + 2) Y(s) + (3s - 14) = \frac{4}{s-2} \Leftrightarrow$$

$$\Leftrightarrow (s^2 - 3s + 2) Y(s) = \frac{4}{s-2} - (3s - 14) = \frac{4 - (3s-14)(s-2)}{s-2}$$

$$= \frac{4 - (3s^2 - 6s - 14s + 28)}{s-2} = \frac{4 - 3s^2 + 6s + 14s - 28}{s-2}$$

$$= \frac{-3s^2+(6+14)s+(4-28)}{s-2} = \frac{-3s^2+20s-24}{s-2} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{-3s^2+20s-24}{(s^2-3s+2)(s-2)} = \frac{-3s^2+20s-24}{(s-2)(s-1)(s-2)} =$$

$$= \frac{-3s^2+20s-24}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{(s-2)^2} + \frac{C}{s-2} \qquad (1)$$

with

$$A = \left.\frac{-3s^2+20s-24}{(s-2)^2}\right|_{s=1} = \frac{-3+20-24}{(1-2)^2} = -3+20-24 = -7$$

$$B = \left.\frac{-3s^2+20s-24}{s-1}\right|_{s=2} = \frac{-3\cdot 2^2+20\cdot 2-24}{2-1} =$$

$$= -3\cdot 4+40-24 = -12+40-24 = 40-36 = 4$$

▶ To find C, we multiply both sides with s and take the limit $s \to +\infty$:

$$\lim_{s\to+\infty}\left[\frac{As}{s-1} + \frac{Bs}{(s-2)^2} + \frac{Cs}{s-2}\right] = \lim_{s\to+\infty} \frac{s[-3s^2+20s-24]}{(s-1)(s-2)^2}$$

$$\Leftrightarrow A+0+C = -3 \Leftrightarrow -7+C = -3 \Leftrightarrow C = 7-3 = 4$$

It follows that

$$Y(s) = \frac{-7}{s-1} + \frac{4}{(s-2)^2} + \frac{4}{s-2}$$

Since:

$$\mathcal{L}^{-1}\left[\frac{-7}{s-1} + \frac{4}{s-2}\right] = -7\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] =$$

$$= -7e^t + 4e^{2t}$$

and $\mathcal{L}^{-1}\left(\frac{4}{s^2}\right) = 4\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = 4t \Rightarrow \mathcal{L}^{-1}\left[\frac{4}{(s-2)^2}\right] = 4te^{2t}$

it follows that

$$y(t) = -7e^t + 4e^{2t} + 4t\,e^{2t} = -7e^t + 4(t+1)e^{2t}.$$

ODEs forced with generalized functions

A big advantage of the Laplace transform method is that it can be used to solve problems with discontinuous forcing or problems where the forcing is a generalized function. This requires extending the definition of the Laplace transform to generalized functions.

Def: Let $F \in \mathcal{D}^{\infty}(\mathbb{R})$ be a distribution with expansion

$$F(x) = f(x) + \sum_{n \in A} g_n(x) H(x - p_n) + \sum_{n \in B} a_n \delta^{(b_n)}(x - q_n)$$

with:

$$\begin{cases} A \subseteq \mathbb{N} \wedge B \subseteq \mathbb{N} \wedge f \in C^{\infty}(\mathbb{R}) \\ \forall n \in A : (g_n \in C^{\infty}(A) \wedge p_n \in \mathbb{R}) \\ \forall n \in B : (a_n, q_n \in \mathbb{R} \wedge b_n \in \mathbb{N}) \end{cases}$$

We say that F is of exponential order (notation: $F \in \mathcal{E}\mathcal{D}^{\infty}(\mathbb{R}_+)$) if and only if all of the following conditions are satisfied

a) f is of exponential order (as a function)

b) $\forall n \in A$: g_n is of exponential order

c) b_n bounded sequence

d) B not finite $\Rightarrow \lim\limits_{n \in \mathbb{N}} q_n = +\infty$

e) $\exists M > 0 : \forall n \in B : |a_n| \leq \exp(M q_n)$

<u>Def</u>: Given a distribution $f \in E\Delta^{\infty}_{+}(\mathbb{R})$ we define the $\mathcal{L}_+$ and $\mathcal{L}_-$ Laplace transforms as follows:

$$\mathcal{L}_+[f(t)] = \int_{0^+}^{+\infty} f(t)e^{-st}\,dt$$

$$\mathcal{L}_-[f(t)] = \int_{0^-}^{+\infty} f(t)e^{-st}\,dt$$

<u>Remark</u>: It can be shown that for any distribution $f \in E\Delta^{\infty}_{+}(\mathbb{R})$ we have:

- <u>For $\mathcal{L}_+$ transforms:</u>

  Given $F(s) = \mathcal{L}_+(f(t))$ we have:

  a) $\mathcal{L}_+[f'(t)] = sF(s) - f(0^+)$

  $\mathcal{L}_+[f''(t)] = s^2F(s) - sf(0^+) - f'(0^+)$

  $\forall n \in \mathbb{N}^* : \mathcal{L}_+[f^{(n)}(t)] = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0^+)$

  b) $\lim_{s\to+\infty} sF(s) = f(0^+)$

- <u>For $\mathcal{L}_-$ transforms</u>

  Given $F(s) = \mathcal{L}_-(f(t))$ we have

  a) $\mathcal{L}_-[f'(t)] = sF(s) - f(0^-)$

  $\mathcal{L}_-[f''(t)] = s^2F(s) - sf(0^-) - f'(0^-)$

  $\forall n \in \mathbb{N}^* : \mathcal{L}_-[f^{(n)}(t)] = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0^-)$

  b) $\lim_{s\to+\infty} sF(s) = f(0^+) \longleftarrow (!!!)$

- In practice, it is recommended to use the $\mathcal{L}_-$ transform because all other properties of Laplace transforms apply with no further modifications over the broader space $E\Delta^\infty_+(\mathbb{R})$ of generalized functions of exponential order.

- Laplace transforms of delta functions:

$\mathcal{L}_-[\delta(t)] = 1$ $\mathcal{L}_-[\delta^{(n)}(t)] = s^n$	$\mathcal{L}_-[\delta(t-a)] = e^{-as}$ $\mathcal{L}_-[\delta^{(n)}(t-a)] = e^{-as}\, s^n$

- Another advantage of the $\mathcal{L}_-$ transform is that for problems where the forcing function includes $\delta(t)$ or terms like $\delta^{(n)}(t)$, it makes physical sense to give the initial condition at 0^- instead of at 0^+.

- For problems that are forced with $H(t-a)$ or $\delta^{(n)}(t-a)$ terms, it is usually necessary to use the following property to find the corresponding inverse Laplace transform:

$$\boxed{F(s) = \mathcal{L}(f(t)) \Rightarrow \mathcal{L}^{-1}[e^{-as} F(s)] = f(t-a)\,H(t-a)}$$

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y''(t)+y'(t)-2y(t) = \delta(t)+\delta(t-2) \\ y(0^-)=0 \wedge y'(0^-)=0 \end{cases}$$

Solution

Define $F(s)=\mathcal{L}_-(y(t))$ and note that

$\mathcal{L}_-(y'(t)) = sY(s)-y(0^-) = sY(s)$

$\mathcal{L}_-(y''(t)) = s^2Y(s)-sy(0^-)-y'(0^-) = s^2Y(s)-s\cdot 0-0 = s^2Y(s)$

$\mathcal{L}_-(\delta(t)+\delta(t-2)) = 1+e^{-2s}$

It follows that

$$\begin{cases} y''(t)+y'(t)-2y(t) = \delta(t)+\delta(t-2) \\ y(0^-)=0 \wedge y'(0^-)=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow s^2Y(s)+sY(s)-2Y(s) = 1+e^{-2s} \Leftrightarrow$$

$$\Leftrightarrow (s^2+s-2)Y(s) = 1+e^{-2s} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{1+e^{-2s}}{s^2+s-2} = \frac{1+e^{-2s}}{(s+2)(s-1)} =$$

$$= (1+e^{-2s})\left[\frac{A}{s+2}+\frac{B}{s-1}\right]$$

with $A = \left.\frac{1}{s-1}\right|_{s=-2} = \frac{1}{(-2)-1} = \frac{-1}{3}$

$B = \left.\frac{1}{s+2}\right|_{s=1} = \frac{1}{1+2} = \frac{1}{3}$

It follows that

$$Y(s) = (1+e^{-2s})\left[\frac{-1/3}{s+2} + \frac{1/3}{s-1}\right] =$$

$$= \frac{1+e^{-2s}}{3}\left[\frac{-1}{s+2} + \frac{1}{s-1}\right]$$

Since

$$\mathcal{L}^{-1}\left[\frac{-1}{s+2} + \frac{1}{s-1}\right] = -e^{-2t} + e^{t} = e^{t}(1-e^{-3t})$$

it follows that

$$y(t) = \mathcal{L}^{-1}\left[\frac{1+e^{-2s}}{3}\left(\frac{-1}{s+2} + \frac{1}{s-1}\right)\right] =$$

$$= (1/3)e^{t}(1-e^{-3t}) + (1/3)H(t-2)\,e^{t-2}\left(1-e^{-3(t-2)}\right)$$

$$= (1/3)e^{t}(1-e^{-3t}) + (1/3)H(t-2)\,\frac{e^{t}}{e^{2}}\left[1-e^{6}e^{-3t}\right]$$

$$= \frac{1}{3}\left[e^{t}(1-e^{-3t}) + \frac{e^{t}(1-e^{6}e^{-3t})}{e^{2}}\,H(t-2)\right]$$

b) Solve the initial value problem

$$\begin{cases} y''(t) + 4y'(t) + 4y(t) = H(t-2) \\ y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases}$$

Solution

Let $Y(s) = \mathcal{L}_-[y(t)]$ and note that

$$\mathcal{L}_-[y''(t) + 4y'(t) + 4y(t)] =$$
$$= [s^2 Y(s) - s y(0^-) - y'(0^-)] + 4[sY(s) - y(0^-)] + 4Y(s)$$
$$= [s^2 Y(s) - s] + 4[sY(s) - 1] + 4Y(s) =$$
$$= s^2 Y(s) - s + 4sY(s) - 4 + 4Y(s) =$$
$$= (s^2 + 4s + 4)Y(s) - (s+4) = (s+2)^2 Y(s) - (s+4)$$

and

$$\mathcal{L}_-(H(t-2)) = \frac{e^{-2s}}{s}$$

It follows that

$$\begin{cases} y''(t) + 4y'(t) + 4y(t) = H(t-2) \\ y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases} \Leftrightarrow (s+2)^2 Y(s) - (s+4) = \frac{e^{-2s}}{s}$$

$$\Leftrightarrow Y(s) \cdot (s+2)^2 = (s+4) + \frac{e^{-2s}}{s} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{s+4}{(s+2)^2} + \frac{e^{-2s}}{s(s+2)^2}$$

For the partial fraction decompositions, we note that

$$\frac{s+4}{(s+2)^2} = \frac{A}{(s+2)^2} + \frac{B}{s+2}$$

with $A = (s+4)|_{s=-2} = -2+4 = 2$. To find B, multiply both sides with s and take the limit $s \to +\infty$:

$$\lim_{s\to+\infty} \frac{s(s+4)}{(s+2)^2} = \lim_{s\to+\infty} \frac{As}{(s+2)^2} + \lim_{s\to+\infty} \frac{Bs}{s+2} \Leftrightarrow$$

$\Leftrightarrow 1 = 0 + B \Leftrightarrow B = 1$

We also note that

$$\frac{1}{s(s+2)^2} = \frac{C}{s} + \frac{D}{(s+2)^2} + \frac{E}{s+2}$$

with

$$C = \left.\frac{1}{(s+2)^2}\right|_{s=0} = \frac{1}{(0+2)^2} = \frac{1}{4}$$

$$D = \left.\frac{1}{s}\right|_{s=-2} = \frac{1}{-2} = \frac{-1}{2}$$

To find E, multiply both sides with s and take the limit $s\to+\infty$:

$$\lim_{s\to+\infty} \frac{Cs}{s} + \lim_{s\to+\infty} \frac{Ds}{(s+2)^2} + \lim_{s\to+\infty} \frac{Es}{s+2} = \lim_{s\to+\infty} \frac{s}{s(s+2)^2}$$

$\Leftrightarrow C + 0D + E = 0 \Leftrightarrow C + E = 0 \Leftrightarrow E = -C = -1/4$

From the above results:

$$Y(s) = \frac{2}{(s+2)^2} + \frac{1}{s+2} + e^{-2s}\left[\frac{1}{4s} - \frac{1}{2(s+2)^2} - \frac{1}{4(s+2)}\right]$$

To find $y(t)$ we note that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+2)^2} + \frac{1}{s+2}\right] = 2\,\frac{t^{2-1}e^{-2t}}{(2-1)!} + e^{-2t} =$$

$$= 2te^{-2t} + e^{-2t} = (2t+1)e^{-2t}$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{4s}-\frac{1}{2(s+2)^2}-\frac{1}{4(s+2)}\right]=\frac{1}{4}-\frac{1}{2}\frac{t^{2-1}e^{-2t}}{(2-1)!}-\frac{1}{4}e^{-2t}$$

$$=(1/4)-(1/2)te^{-2t}-(1/4)e^{-2t}=$$

$$=(1/4)\left[1-2te^{-2t}-e^{-2t}\right]=(1/4)\left[1-(2t+1)e^{-2t}\right]$$

and therefore

$$y(t)=\mathcal{L}^{-1}\left[\frac{2}{(s+2)^2}+\frac{1}{s+2}+e^{-2s}\left(\frac{1}{4s}-\frac{1}{2(s+2)^2}-\frac{1}{4(s+2)}\right)\right]$$

$$=(2t+1)e^{-2t}+(1/4)\left[1-(2(t-2)+1)e^{-2(t-2)}\right]H(t-2)$$

$$=(2t+1)e^{-2t}+(1/4)\left(1-(2t-4+1)e^{-2t}e^{4}\right)H(t-2)$$

$$=(2t+1)e^{-2t}+(1/4)\left(1-(2t-3)e^{4}e^{-2t}\right)H(t-2)$$

EXERCISES

(17) Use Laplace transforms to follow the following initial value problem

a) $\begin{cases} y'(t) - ay(t) = 0 & \text{with } a, b \in \mathbb{R} \\ y(0) = b \end{cases}$

b) $\begin{cases} y''(t) - 5y'(t) + 6y(t) = e^{-t} \\ y(0) = 1 \wedge y'(0) = 3 \end{cases}$

c) $\begin{cases} y''(t) + y(t) = \sin^2 t \\ y(0) = 1 \wedge y'(0) = 0 \end{cases}$ d) $\begin{cases} y''(t) + 4y'(t) + 3y(t) = H(t-1) \\ y(0^-) = 0 \wedge y'(0^-) = 0 \end{cases}$

e) $\begin{cases} y''(t) + 7y'(t) + 12y(t) = H(t-1) + H(t-2) \\ y(0^-) = 0 \wedge y'(0^-) = 1 \end{cases}$

f) $\begin{cases} y''(t) + 6y'(t) + 9y(t) = t \\ y(0) = 0 \wedge y'(0) = 0 \end{cases}$

g) $\begin{cases} y''(t) + 10y'(t) + 25y(t) = \delta(t) \\ y(0^-) = 0 \wedge y'(0^-) = 0 \end{cases}$

h) $\begin{cases} y''(t) + 11y'(t) + 30y(t) = \delta(t) + \delta'(t-3) \\ y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases}$

i) $\begin{cases} y'''(t) - 2y''(t) + y'(t) - 2y(t) = \delta''(t-a) & \text{with } a > 0 \\ y(0^-) = 0 \wedge y'(0^-) = 0 \wedge y''(0^-) = 1 \end{cases}$

j) $\begin{cases} y'''(t) + 3y''(t) + 3y'(t) + y(t) = \delta''(t) + H(t-1) \\ y(0^-) = 0 \wedge y'(0^-) = 1 \wedge y''(0^-) = 1 \end{cases}$

k) $\begin{cases} y''''(t) - a^4 y(t) = \delta'(t) + \delta'(t-3) \\ y(0^-) = 1 \wedge y'(0^-) = y''(0^-) = y'''(0^-) = 0 \end{cases}$

k) $\begin{cases} y''''(t) - a^4 y(t) = \delta'(t) + \delta'(t-3) \\ y(0^-) = 1 \wedge y'(0^-) = y''(0^-) = y'''(0^-) = 0 \end{cases}$

l) $\begin{cases} y''''(t) - a^4 y(t) = \delta''(t) + H(t-2) \\ y(0^-) = y'(0^-) = y''(0^-) = 0 \wedge y'''(0^-) = 1 \end{cases}$

Systems of linear ODEs

c) Solve the linear system
$$\begin{cases} x'(t) = 2x(t) - 3y(t) \\ y'(t) = y(t) - 2x(t) \end{cases}$$
with $x(0) = 8 \wedge y(0) = 3$.

<u>Solution</u>

Let $X(s) = \mathcal{L}(x(t))$ and $Y(s) = \mathcal{L}(y(t))$. Then:

$$\begin{cases} x'(t) = 2x(t) - 3y(t) \\ y'(t) = y(t) - 2x(t) \end{cases} \Leftrightarrow \begin{cases} sX(s) - x(0) = 2X(s) - 3Y(s) \\ sY(s) - y(0) = Y(s) - 2X(s) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} sX(s) - 8 - 2X(s) + 3Y(s) = 0 \\ sY(s) - 3 - Y(s) + 2X(s) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (s-2)X(s) + 3Y(s) = 8 \\ (s-1)Y(s) + 2X(s) = 3 \end{cases} \Leftrightarrow \begin{cases} (s-2)X(s) + 3Y(s) = 8 \\ 2X(s) + (s-1)Y(s) = 3 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 3 \end{bmatrix} =$$

$$= \frac{1}{(s-2)(s-1) - 3\cdot 2} \begin{bmatrix} s-1 & -3 \\ -2 & s-2 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$= \frac{1}{s^2 - 3s + 2 - 6} \begin{bmatrix} (s-1)8 - 3\cdot 3 \\ -2\cdot 8 + 3(s-2) \end{bmatrix}$$

$$= \frac{1}{s^2 - 3s - 4} \begin{bmatrix} 8s - 8 - 9 \\ 3s - 6 - 16 \end{bmatrix} = \frac{1}{s^2 - 3s - 4} \begin{bmatrix} 8s - 17 \\ 3s - 22 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s-4)} \begin{bmatrix} 8s - 17 \\ 3s - 22 \end{bmatrix}$$

$$\Longleftrightarrow X(s) = \frac{8s-17}{(s+1)(s-4)} \wedge Y(s) = \frac{3s-22}{(s+1)(s-4)}.$$

With partial fraction decomposition, we have:

$$X(s) = \frac{8s-17}{(s+1)(s-4)} = \frac{A}{s+1} + \frac{B}{s-4}$$

$$Y(s) = \frac{3s-22}{(s+1)(s-4)} = \frac{C}{s+1} + \frac{D}{s-4}$$

with

$$A = \left.\frac{8s-17}{s-4}\right|_{s=-1} = \frac{8(-1)-17}{(-1)-4} = \frac{-8-17}{-1-4} = \frac{-25}{-5} = 5$$

$$B = \left.\frac{8s-17}{s+1}\right|_{s=4} = \frac{8\cdot 4-17}{4+1} = \frac{32-17}{5} = \frac{15}{5} = 3$$

$$C = \left.\frac{3s-22}{s-4}\right|_{s=-1} = \frac{3(-1)-22}{(-1)-4} = \frac{-3-22}{-5} = \frac{-25}{-5} = 5$$

$$D = \left.\frac{3s-22}{s+1}\right|_{s=4} = \frac{3\cdot 4-22}{4+1} = \frac{12-22}{5} = \frac{-10}{5} = -2$$

and therefore:

$$\begin{cases} X(s) = \dfrac{5}{s+1} + \dfrac{3}{s-4} \\ Y(s) = \dfrac{5}{s+1} + \dfrac{-2}{s-4} \end{cases} \Longleftrightarrow \begin{cases} x(t) = 5e^{-t} + 3e^{4t} \\ y(t) = 5e^{-t} - 2e^{4t} \end{cases}$$

EXERCISES

(18) Use Laplace transforms to solve the following systems of ordinary differential equations

(a) $\begin{cases} x'(t) = x(t) - a y(t) \\ y'(t) = a x(t) + y(t) \\ x(0) = 1 \wedge y(0) = 0 \end{cases}$

b) $\begin{cases} x''(t) + y'(t) + 3x(t) = e^{-t} \\ y''(t) - x'(t) + 2y(t) = \cos(3t) \\ x(0) = 1 \wedge x'(0) = 0 \wedge y(0) = 0 \wedge y'(0) = 1 \end{cases}$

c) $\begin{cases} x'(t) - y(t) = \delta'(t) \\ y'(t) - x(t) = \delta(t-2) \\ x(0^-) = x'(0^-) = y(0^-) = y'(0^-) = 0 \end{cases}$

d) $\begin{cases} x''(t) + y(t) = \delta(t) \\ x''(t) - y'(t) = H(t-1) \\ x(0^-) = 1 \wedge x'(0^-) = 0 \wedge y(0^-) = 1 \wedge y'(0^-) = 0 \end{cases}$

(19) Linear damped oscillator.

Consider the linear damped oscillator governed by the following initial value problem

$$\begin{cases} m x''(t) + b x'(t) + K x(t) = 0 \\ x(0) = x_0 \wedge x'(0) = u_0 \end{cases}$$

a) Show that the Laplace transform $X(s) = \mathcal{L}(x(t))$ of the unique solution to the initial value problem is given by

$$X(s) = \frac{(s+a)x_0}{(s+a)^2 + (w^2 - a^2)} + \frac{u_0 + a x_0}{(s+a)^2 + (w^2 - a^2)}$$

with $a = b/(2m)$ and $w = k/m$.

b) Show that the solution $x(t)$ is given according to the following 3 cases:

Case I: If $w^2 - a^2 > 0$, (damped oscillatory case)

$$x(t) = x_0 e^{-at} \cos\left(t\sqrt{w^2 - a^2}\right) + \frac{u_0 + a x_0}{\sqrt{w^2 - a^2}} e^{-at} \sin\left(t\sqrt{w^2 - a^2}\right)$$

Case II: If $w^2 - a^2 = 0$, (critically damped case)

$$x(t) = x_0 e^{-at} + (u_0 + a x_0) t e^{-at}$$

Case III: If $w^2 - a^2 < 0$, (overdamped case)

$$x(t) = x_0 \cosh\left(t\sqrt{a^2 - w^2}\right) + \frac{u_0 + a x_0}{\sqrt{a^2 - w^2}} \sinh\left(t\sqrt{a^2 - w^2}\right)$$

20. Consider the initial value problem

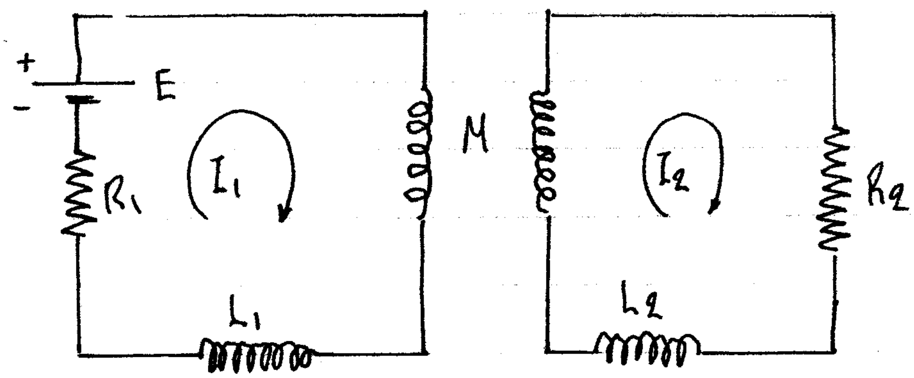
$$\begin{cases} x''(t) + k_1^2 y(t) = 0 \\ y''(t) + k_2^2 x(t) = 0 \\ x(0) = a \wedge y(0) = b \wedge x'(0) = y'(0) = 0 \end{cases}$$

Use Laplace transforms to show that $x(t)$ and $y(t)$ are given by

$$\begin{cases} x(t) = \left(\dfrac{a k_2 + b k_1}{2 k_2}\right) \cos\left(t\sqrt{k_1 k_2}\right) + \left(\dfrac{a k_2 - b k_1}{2 k_2}\right) \cosh\left(t\sqrt{k_1 k_2}\right) \\ y(t) = \left(\dfrac{a k_2 + b k_1}{2 k_1}\right) \cos\left(t\sqrt{k_1 k_2}\right) - \left(\dfrac{a k_2 - b k_1}{2 k_1}\right) \cosh\left(t\sqrt{k_1 k_2}\right) \end{cases}$$

(21) Inductively coupled circuits.

We consider two inductively coupled circuits of the form:



The currents satisfy the following system of differential equations:

$$\begin{cases} L_1 \dfrac{dI_1}{dt} + R_1 I_1 + M \dfrac{dI_2}{dt} = E \\ L_2 \dfrac{dI_2}{dt} + R_2 I_2 + M \dfrac{dI_1}{dt} = 0 \end{cases}$$

a) Using initial condition $I_1(0) = I_2(0) = 0$, show, using Laplace transforms, that $I_1(t)$ and $I_2(t)$ will satisfy

$$I_1(t) = \frac{EL_2}{L_1L_2 - M^2} \frac{e^{a_1 t} - e^{a_2 t}}{a_1 - a_2} + \frac{ER_2}{a_1 - a_2}\left(\frac{e^{a_1 t}}{a_1} - \frac{e^{a_2 t}}{a_2}\right) + \frac{E}{R_1}$$

$$I_2(t) = \frac{EM}{L_1L_2 - M^2} \frac{e^{a_1 t} - e^{a_2 t}}{a_2 - a_1}$$

where a_1, a_2 are the roots of the equation

$(L_1L_2 - M^2)a^2 + (L_1R_2 + L_2R_1)a + R_1R_2 = 0$

b) What happens when $L_1L_2 = M^2$?

▼ Laplace transform of a convolution

Def: Let $f, g \in PC(\mathbb{R}_+)$ be two piecewise-continuous functions. We define the convolution $f*g$ as:

$$\forall t \in [0,+\infty) : (f*g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

Remark: It can be shown that convolution satisfies the associative and commutative properties:

$\forall f, g \in PC(\mathbb{R}_+) : f*g = g*f$

$\forall f, g, h \in PC(\mathbb{R}_+) : f*(g*h) = (f*g)*h$

↳ The Laplace transform of a convolution is given by the following theorem:

Thm:

$$\left.\begin{array}{l} f, g \in PC(\mathbb{R}_+) \cap E_\gamma(\mathbb{R}_+) \\ F(s) = \mathcal{L}(f(t)) \\ G(s) = \mathcal{L}(g(t)) \end{array}\right\} \Rightarrow \mathcal{L}((f*g)(t)) = F(s)G(s)$$

Methodology: The convolution theorem can help with

(a) Inverse Laplace transforms of products.

(b) ODEs with general forcing term

(c) Integral and integrodifferential equations.

EXAMPLES

a) Solve the initial value problem

$$\begin{cases} y''(t) + \omega^2 y(t) = f(t) \\ y(0) = y_0 \wedge y'(0) = y_1 \end{cases}$$

using Laplace transforms, with $\omega \in (0, +\infty)$ and $y_0, y_1 \in \mathbb{R}$.

Solution

Define $Y(s) = \mathcal{L}(y(t))$, and note that

$$\mathcal{L}[y''(t) + \omega^2 y(t)] = s^2 Y(s) - s\, y(0) - y'(0) + \omega^2 Y(s)$$
$$= (s^2 + \omega^2) Y(s) - y_0 s - y_1 .$$

It follows that, with the definition $F(s) = \mathcal{L}(f(t))$,

$$y''(t) + \omega^2 y(t) = f(t) \Leftrightarrow (s^2 + \omega^2) Y(s) - y_0 s - y_1 = F(s).$$
$$\Leftrightarrow (s^2 + \omega^2) Y(s) = y_0 s + y_1 + F(s) \Leftrightarrow$$
$$\Leftrightarrow Y(s) = \frac{y_0 s + y_1 + F(s)}{s^2 + \omega^2} = y_0 \frac{s}{s^2 + \omega^2} + \frac{y_1}{\omega} \frac{\omega}{s^2 + \omega^2} + \frac{1}{\omega} \frac{\omega F(s)}{s^2 + \omega^2} \quad (1)$$

Since $\mathcal{L}^{-1}\left(\frac{s}{s^2 + \omega^2}\right) = \cos(\omega t)$ and

$$\mathcal{L}^{-1}\left(\frac{\omega}{s^2 + \omega^2}\right) = \sin(\omega t)$$

$$\text{Eq. (1)} \Leftrightarrow y(t) = y_0 \cos(\omega t) + (y_1/\omega) \sin(\omega t) + \frac{1}{\omega} \int_0^t d\tau\, f(\tau) \sin(\omega(t - \tau))$$

b) Evaluate the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] \quad \text{with } a>0.$$

Solution

We note that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \frac{1}{a}\sin(at) \equiv f(t) \Rightarrow$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = (f*f)(t) = \int_0^t f(\tau)f(t-\tau)\,d\tau =$$

$$= \int_0^t \left[\frac{\sin(a\tau)}{a}\right]\left[\frac{\sin(a(t-\tau))}{a}\right] d\tau$$

$$= \frac{1}{a^2}\int_0^t \sin(a\tau)\sin(at-a\tau)\,d\tau$$

$$= \frac{1}{a^2}\int_0^t (1/2)\left[\cos(a\tau-(at-a\tau)) - \cos(a\tau+(at-a\tau))\right] d\tau$$

$$= \frac{1}{a^2}\int_0^t (1/2)\left[\cos(a\tau-at+a\tau) - \cos(a\tau+at-a\tau)\right] d\tau$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2a\tau-at) - \cos(at)\right] d\tau$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2a\tau-at)}{2a} - \tau\cos(at)\right]_{\tau=0}^{\tau=t} =$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2at-at)}{2a} - t\cos(at)\right] - \frac{1}{2a^2}\left[\frac{\sin(0-at)}{2a} - 0\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(at)}{2a} - t\cos(at)\right] - \frac{1}{2a^2}\left[\frac{-\sin(at)}{2a}\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(at)}{2a} - t\cos(at) + \frac{\sin(at)}{2a}\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin(at)}{a} - t\cos(at)\right] = \frac{\sin(at) - at\cos(at)}{2a^3}$$

c) Solve the integral equation

$$\int_0^t \frac{y(a)}{\sqrt{t-a}}\, da = t^n$$

with $n \in \mathbb{N}^*$.

Solution

Define $Y(s) = \mathcal{L}(y(t))$. We note that

$$\mathcal{L}\left(\int_0^t \frac{y(a)}{\sqrt{t-a}}\, da\right) = \mathcal{L}\left(y(t) * (1/\sqrt{t})\right) = \mathcal{L}(y(t))\,\mathcal{L}(t^{-1/2})$$

$$= Y(s)\frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \Gamma(1/2)\frac{Y(s)}{s^{1/2}} =$$

$$= \frac{Y(s)\sqrt{\pi}}{s^{1/2}}$$

and $\mathcal{L}(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$ (because $n \in \mathbb{N}^*$)

and therefore,

$$\int_0^t \frac{y(a)}{\sqrt{t-a}}\, da = t^n \Leftrightarrow \frac{Y(s)\sqrt{\pi}}{s^{1/2}} = \frac{n!}{s^{n+1}} \Leftrightarrow$$

$$\Leftrightarrow Y(s) = \frac{(n!)\, s^{1/2}}{s^{n+1}\sqrt{\pi}} = \frac{(n!)}{s^{n+1/2}\sqrt{\pi}} = \frac{(n!)}{\Gamma(n+1/2)\sqrt{\pi}}\,\frac{\Gamma(n+1/2)}{s^{n+1/2}}$$

$$= \frac{(n!)}{\Gamma(n+1/2)\sqrt{\pi}}\,\frac{\Gamma(n-1/2+1)}{s^{n-1/2+1}} \Leftrightarrow$$

$$\Leftrightarrow y(t) = \mathcal{L}^{-1}\left[\frac{n!}{\Gamma(n+1/2)\sqrt{\pi}}\,\frac{\Gamma(n-1/2+1)}{s^{n-1/2+1}}\right] = \frac{n!}{\Gamma(n+1/2)\sqrt{\pi}}\,\mathcal{L}^{-1}\left[\frac{\Gamma(n-1/2+1)}{s^{n-1/2+1}}\right]$$

$$= \frac{n!}{\Gamma(n+1/2)\sqrt{\pi}}\, t^{n-1/2} = \frac{n!\, t^{n-1}\sqrt{t}}{\Gamma(n+1/2)\sqrt{\pi}}$$

We simplify further noting that

$$\Gamma(n+1/2) = \Gamma(n+1-1/2) = \Gamma(1/2)\prod_{k=1}^{n}(k-1/2) =$$

$$= \sqrt{\pi}\prod_{k=1}^{n}\frac{2k-1}{2} = \frac{\sqrt{\pi}}{2^n}\prod_{k=1}^{n}(2k-1) =$$

$$= \frac{\sqrt{\pi}\,(2n-1)!!}{2^n}$$

and therefore, we have:

$$y(t) = \frac{n!\,t^{n-1}\sqrt{t}}{\Gamma(n+1/2)\sqrt{\pi}} = \frac{n!\,t^{n-1}\sqrt{t}}{\left[\frac{\sqrt{\pi}\,(2n-1)!!}{2^n}\right]\sqrt{\pi}} =$$

$$= \frac{2^n\,n!}{\pi\,(2n-1)!!}\,t^{n-1}\sqrt{t}$$

EXERCISES

(22) Use the convolution theorem to evaluate the following inverse Laplace transforms:

a) $\mathcal{L}^{-1}\left[\dfrac{s}{(s^2-a^2)(s-b)}\right] = \dfrac{1}{2}\left[\dfrac{e^{at}}{a-b} - \dfrac{e^{-at}}{a+b} - \dfrac{2b\,e^{bt}}{a^2-b^2}\right]$

b) $\mathcal{L}^{-1}\left[\dfrac{1}{(s-1)\sqrt{s}}\right] = e^x \operatorname{erf}(\sqrt{x})$

c) $\mathcal{L}^{-1}\left[\dfrac{s\exp(-\pi s/2)}{(s^2+1)(s^2+9)}\right] = \dfrac{H(t-\pi/2)}{8}\left[\sin(3t)+\sin t\right]$

d) $\mathcal{L}^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right] = \dfrac{\sin(at)+at\cos(at)}{2a}$

e) $\mathcal{L}^{-1}\left[\dfrac{s}{(s^2-a^2)^2}\right] = \dfrac{t\sinh(at)}{2a}$

(23) Use the Laplace transform in conjunction with the convolution theorem to solve the following initial value problems or integrodifferential equations under the forcing functions $f(t)$ and $g(t)$ (wherever it applies).

a) $\begin{cases} x'(t) = x(t) + y(t) + f(t) \\ y'(t) = x(t) - y(t) \\ x(0) = y(0) = 0 \end{cases}$

b) $\begin{cases} x'(t) = 2x(t) - y(t) + f(t) \\ y'(t) = x(t) - y(t) + g(t) \\ x(0) = y(0) = 1. \end{cases}$

c) $x(t) + \displaystyle\int_0^t (t-a)\,x(a)\,da = f(t)$

d) $x(t) = f(t) - \int_0^t \sin(a)\, x(t-a)\, da$

e) $x(t) + \int_0^t x(a)\, da = 1$

f) $x(t) = \cos t + \int_0^t e^{-a} x(t-a)\, da$

g) $\begin{cases} x'(t) = 1 - \sin t - \int_0^t x(a)\, da \\ x(0) = 0 \end{cases}$

h) $x(t) = f(t) + \int_0^t a\, x(t-a)\, da$

i) $\int_0^t x(a) x(t-a)\, da = 2x(t) + t - 2$

j) $\int_0^t x(a) \sin(t-a)\, da = x(t)$

24 Show that the integrodifferential equation

$$\int_0^t \frac{y(a)}{(t-a)^n}\, da = f(t)$$

with $0 < a < 1$ and $f(0) = 0$ has the solution

$$y(t) = \frac{\sin(n\pi)}{\pi} \int_0^t f'(a)(t-a)^{n-1}\, da$$