Lecture Notes on Mathematical Modeling

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MM1: Introduction

INTRODUCTION

Autonomous dynamical systems

- An autonomous dynamical system is a system of n differential equations of the form:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3, \dots, x_n) \\ \dot{x}_2 = f_2(x_1, x_2, x_3, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, x_2, x_3, \dots, x_n) \end{cases}$$

▶ notation: $\dot{x}_K = dx_K/dt = x_K'(t)$.

- The system can be also rewritten as:
$$\dot{x} = f(x)$$
with $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

- We assume that an initial value condition is given at $t=0$: $x(0) = x_0$, with $x_0 \in \mathbb{R}^n$.

- Classification of autonomous systems

a) Linear Autonomous systems: These are systems where $f(x) = Ax$ with $A \in GL(n, \mathbb{R})$. Note that

$GL(n,\mathbb{R})$ is the set of all nonsingular $n\times n$ matrices.

b) Nonlinear autonomous systems: These are systems where $f(x)$ is nonlinear.

- Jacobian matrix

The Jacobian matrix of the autonomous system $\dot{x} = f(x)$ is defined as

$$\boxed{[Df]_{ab} = \frac{\partial f_a}{\partial x_b}}$$

Note that for a linear system with $f(x) = Ax$ we have $Df = A$.

- Systems reducible to autonomous

a) High-order ODEs $\longrightarrow$ $\boxed{y^{(n)} = F(y, y', y'', \dots, y^{(n-1)})}$

We let: $x_1 = y,\ x_2 = y',\ x_3 = y'', \dots, x_n = y^{(n-1)}$.

EXAMPLE

$\ddot{x} - b\dot{x} + kx = 0$ (linear oscillator)

Let $x_1 = x$ and $x_2 = \dot{x} \longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = bx_2 - kx_1 \end{cases}$

b) Time-dependent system

A time-dependent system of the form

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2, \dots, x_n) \\ \dot{x}_2 = f_2(t, x_1, x_2, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(t, x_1, x_2, \dots, x_n) \end{cases}$$

can be rewritten as an autonomous system by setting $x_0 = t$. Then:

$$\begin{cases} \dot{x}_0 = 1 \\ \dot{x}_1 = f_1(x_0, x_1, x_2, \dots, x_n) \\ \dot{x}_2 = f_2(x_0, x_1, x_2, \dots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_0, x_1, x_2, \dots, x_n) \end{cases}$$

EXERCISES

① Rewrite the following ODEs as autonomous systems:

a) $\ddot{y} + a\dot{y} + by = 0$

b) $\dddot{y} - a\dot{y} = b\ddot{y}$

c) $\ddot{y} = y\dot{y}$

d) $\dddot{y} = \dot{y}\,(y + \ddot{y})$

e) $\ddot{y} + (1+t^2)\dot{y} + 2ty = 0$

f) $\ddot{y} - \dot{y} + ty = 0$

g) $\ddddot{y} + 3t\dot{y} - t^2 y = 0$

h) $\ddot{y} - ty = 0.$

② Evaluate the Jacobian matrix for the following autonomous systems

a) $\begin{cases} \dot{x}_1 = 3x_1 - 2x_2 \\ \dot{x}_2 = 5x_1 + 3x_2 \end{cases}$

b) $\begin{cases} \dot{x}_1 = 2x_1 - x_2(x_1 - x_2) \\ \dot{x}_2 = -x_2 + x_1(2x_2 - x_1) \end{cases}$

c) $\begin{cases} \dot{x}_1 = x_1^2(x_1 - x_2)^3 \\ \dot{x}_2 = x_2^3(x_1 - x_2)^2 \end{cases}$

d) $\begin{cases} \dot{x}_1 = \cos x_1 \sin x_2 \\ \dot{x}_2 = \cos x_2 \sin x_1 \end{cases}$

Existence and Uniqueness

Consider the problem

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

with $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and $f: \mathbb{R}^n \to \mathbb{R}^n$

We also define

$$\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

with $x = (x_1, x_2, \dots, x_n)$ a vector in $\mathbb{R}^n$.

- Definition:

f Lipschitz continuous in $A \Leftrightarrow$

$\Leftrightarrow \exists L > 0 : \forall x, y \in A : \|f(x) - f(y)\| \leq L\|x - y\|$

with L = Lipschitz constant of f.

If $L < 1 \Rightarrow f$ is a contraction

- Proposition: Let $f: \mathbb{R}^n \to \mathbb{R}^m$. Then

a) $\left.\begin{array}{l} f \text{ differentiable in } A \\ \nabla f \text{ bounded in } A \end{array}\right\} \Rightarrow f$ Lipschitz continuous in A.

b) $\left.\begin{array}{l} f \text{ differentiable in } A \\ f \text{ Lipschitz continuous in } A \end{array}\right\} \Rightarrow \nabla f$ bounded in A

c) $\left.\begin{array}{l} f \text{ differentiable in } A \\ \nabla f \text{ not bounded in } A \end{array}\right\} \Rightarrow f$ not Lipschitz continuous in A.

↳ Note that (c) is the contrapositive of (b).

• <u>Theorem</u> : Assume that

a) f Lipschitz continuous in $B(x_0, \delta)$ with Lipschitz constant L, where $\delta > 0$ and $B(x_0,\delta) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \delta\}$

b) $\forall x \in B(x_0, \delta) : \|f(x)\| \leq M$

Then $\dot{x} = f(x)$ with $x(t_0) = x_0$ has a unique solution for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ as long as $0 < \varepsilon < \min(1/L, \delta/M)$

• <u>Theorem</u> : Assume that

a) f Lipschitz continuous in $B(x_0, \delta)$ with Lipschitz constant L

b) y solution for $x(t_0) = y_0$ and z solution for $x(t_0) = z_0$ defined for $t \in [t_0, t_1]$

Then :

$$\boxed{\forall t \in [t_0, t_1] : \|y(t) - z(t)\| \leq \|y_0 - z_0\| e^{L(t-t_0)}}$$

↳ The divergence between two solutions with nearby initial conditions do not grow apart at a faster than exponential rate.

↳ Note that the existence of the unique solution is not guaranteed for infinite time.

EXAMPLES

a) Show that $f(x) = 2x+3$, $\forall x \in \mathbb{R}$ is Lipschitz continuous in $\mathbb{R}$.

Solution

1st method: By definition.

Let $x, y \in \mathbb{R}$ be given. Then

$$|f(x) - f(y)| = |(2x+3) - (2y+3)| = |2x+3-2y-3| =$$
$$= |2x-2y| = |2(x-y)| = |2||x-y|$$
$$= 2|x-y| \Rightarrow |f(x)-f(y)| \leq 2|x-y|.$$

It follows that

$\forall x, y \in \mathbb{R}: |f(x)-f(y)| \leq 2|x-y| \Rightarrow$

$\Rightarrow f$ Lipschitz continuous in $\mathbb{R}$.

2nd method: By proposition

f differentiable in $\mathbb{R}$ with $f'(x) = (2x+3)' = 2$, $\forall x \in \mathbb{R}$ (1)

Since:

$\forall x \in \mathbb{R}: (|f'(x)| = |2| = 2) \Rightarrow \forall x \in \mathbb{R}: (|f'(x)| \leq 2) \Rightarrow$

$\Rightarrow f'$ bounded in $\mathbb{R}$ (2)

From (1) and (2) it follows that f is Lipschitz continuous in $\mathbb{R}$.

b) Show that $f(x) = x^{2/3}$, $\forall x \in (0, +\infty)$ is not Lipschitz continuous on $(0, +\infty)$

Solution

f differentiable in $(0,+\infty)$ with

$$f'(x) = (x^{2/3})' = (2/3)\,x^{2/3-1} = (2/3)\,x^{-1/3} =$$
$$= \frac{2}{3\sqrt[3]{x}}\,, \quad \forall x \in (0,+\infty) \qquad (1)$$

However, since:

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{2}{3\sqrt[3]{x}} = +\infty \Rightarrow$$

$\Rightarrow f'$ not bounded in $(0,+\infty)$ (2)

From (1) and (2), it follows that f not Lipschitz continuous in $(0,+\infty)$.

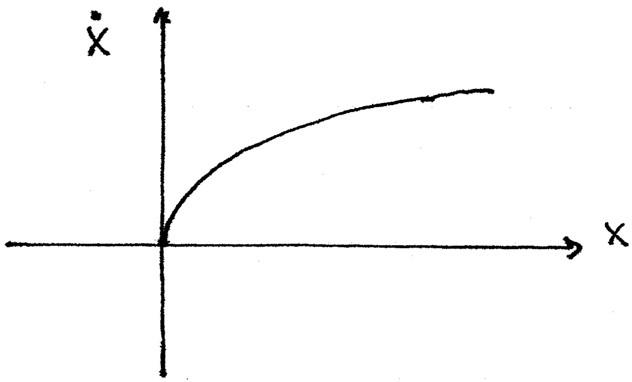
Examples on existence and uniqueness

c) $\dot{x} = 1+x^2$

- We can show, using standard ODE techniques, that $\dot{x} = 1+x^2 \Leftrightarrow x(t) = \tan(t+c)$ with c dependant on the initial condition. Obviously, a solution <u>exists</u>. Furthermore, the "if and only if" ($\Leftrightarrow$) ensures the <u>uniqueness</u> of this solution. However, we note that the solution has a singularity occuring at finite time when $t+c = k\pi + \pi/2$, consequently the existence and uniqueness holds for finite time only.

d) $\begin{cases} \dot{x} = 3x^{2/3} \\ x(0) = 0 \end{cases}$

has two solutions: $x(t) = 0$ and $x(t) = t^3$ (!!)
thus we have existence but not uniqueness.
Note that $f(x) = 3x^{2/3}$ is continuous but not Lipschitz continuous.



The solution $x(t) = 0$ is VERY unstable because the slope of the function $f(x) = 3x^{2/3}$ is infinite at $x = 0$. This instability manifests in the existence of a second solution, such as $x(t) = t^3$ (there is in fact an infinite set of such solutions). From a physical standpoint, $x(t) = 0$ is the solution one would expect if we initialize with $x(0) = 0$. The lack of uniqueness indicates that the system could spontaneously break into the second solution $x(t) = t^3$ at time $t = 0$ if there is even an infinitesimal deviation in the initial condition, thus giving a 3rd solution:

$$x(t) = \begin{cases} t^3 & , \text{ if } t \geq 0 \\ 0 & , \text{ if } t < 0 \end{cases}$$

that combines the previous two solutions. Note that this spontaneous break can just as well occur at any other time t_0. (see homework).

EXERCISES

③ Show that the following functions are Lipschitz continuous in $\mathbb{R}$:

a) $f(x) = ax + b$ with $a \neq 0$.

b) $f(x) = \sin x$

c) $f(x) = |x|$ ⟶ Note that it is not differentiable in $\mathbb{R}$. (fails at $x = 0$)

d) $f(x) = \sqrt{x^2 + a}$ with $a > 0$

④ Show that the following functions are not Lipschitz continuous.

a) $f(x) = x^2$ in $\mathbb{R}^2$

b) $f(x) = \sqrt{x}$ in $[0, +\infty)$

c) $f(x) = x^a$ in $[0, +\infty)$ with $0 < a < 1$.

⑤ Consider the system $\dot{x} = x^a$ with $a \in (0,1)$ and with initial condition $x(0) = 0$. Show that this system has an infinite set of solutions.

⑥ a) Consider the system $\dot{x} = x^a$ with $a > 1$ and $x(0) = x_0 > 0$. Show that $x(t)$ becomes infinite at finite time. What happens when $a = 1$.

b) Show that the solution of $\dot{x} = x^a + b$ with $a > 1$ and $b > 0$ and $x(0) = x_0 > 0$ also becomes infinite at finite time.

Fixed points and stability

- Consider the autonomous system $\dot{x} = f(x)$, with $f: \mathbb{R}^n \to \mathbb{R}^n$. We say that
  x_0 is a fixed point $\Leftrightarrow f(x_0) = 0$

- If x_0 is a fixed point, then $\dot{x} = f(x)$ with initial condition $x(t_0) = x_0$ has solution $x(t) = x_0$. Thus if we start at a fixed point, we will stay at the fixed point. The question of stability concerns what happens when we start with an initial condition near a fixed point.

- Stability of fixed points

Let $x_0 \in \mathbb{R}^n$ be a fixed point of $\dot{x} = f(x)$.

(1) x_0 Lyapunov stable $\Leftrightarrow$

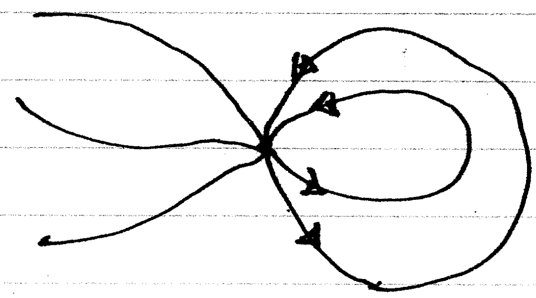
$$\forall \varepsilon > 0 : \exists \delta > 0 : \left(\|x(t_0) - x_0\| < \delta \Rightarrow \right.$$
$$\left. \Rightarrow (\forall t > t_0 : \|x(t) - x_0\| < \varepsilon) \right)$$

(2) x_0 attracting $\Leftrightarrow$

$$\exists \delta > 0 : \left(\|x(t_0) - x_0\| < \delta \Rightarrow \lim_{t \to +\infty} x(t) = x_0 \right)$$

$\hookrightarrow$ In a Lyapunov stable fixed point, solutions that start near the fixed point will stay near the fixed point. In an attracting fixed point, solutions that start near the fixed point will eventually converge into the fixed point.

↳ Note that it is possible for a fixed point to be attractive without being Lyapunov stable, as in the following example:



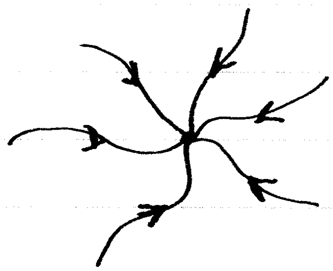
This occurs when there are trajectories that start near the fixed point, then wander far away from the fixed point before returning back to the fixed point for a final approach. This remark motivates the following additional definitions:

③ x_0 asymptotically stable $\Leftrightarrow \begin{cases} x_0 \text{ Lyapunov stable} \\ x_0 \text{ attracting.} \end{cases}$

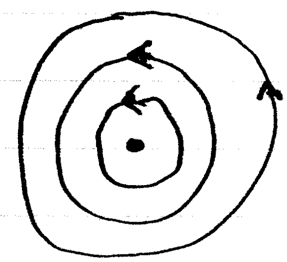
④ x_0 neutrally stable $\Leftrightarrow \begin{cases} x_0 \text{ Lyapunov stable} \\ x_0 \text{ not attracting} \end{cases}$

⑤ x_0 unstable $\Leftrightarrow \begin{cases} x_0 \text{ not Lyapunov stable} \\ x_0 \text{ not attracting.} \end{cases}$

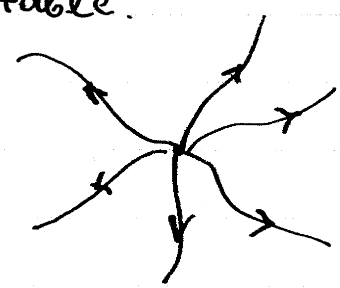
Examples of definitions



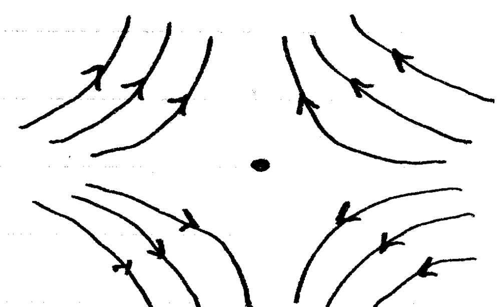
asymptotically stable.



neutrally stable.



unstable (source)



unstable (saddle point)

The distinction between sources and saddle points will be explained later.

⑥ x_0 is <u>exponentially stable</u> if and only if

a) x_0 is asymptotically stable AND

b) $\exists\, a, b, \delta \in (0, +\infty) : \big(\|x(t_0) - x_0\| < \delta \Rightarrow$

$\quad \Rightarrow \big(\forall t > t_0 : \|x(t) - x_0\| \leqslant a e^{-bt} \|x(t_0) - x_0\| \big)\big)$

▼ Lyapunov functions

Let $\dot{x} = f(x)$ be an autonomous system with $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let x_0 be a fixed point such that $f(x_0) = 0$.

Def: We say that a function $V: A \longrightarrow \mathbb{R}$ with $A \subseteq \mathbb{R}^n$ an open set is a Lyapunov function if it satisfies the following conditions:

a) $V(x_0) = 0$

b) $V(x) > 0, \ \forall x \in A - \{x_0\}$

c) V continuous in A

d) $x(t_0) \in A \Rightarrow \forall t > t_0 : V(x(t)) \leq V(x(t_0))$

- The domain A of the Lyapunov function is called a trapping region of the autonomous system.

Thm: (1st Lyapunov Theorem)

If

a) $f(x_0) = 0$

b) There is a Lyapunov function $V: A \to \mathbb{R}$ with $V(x_0) = 0$

Then $x = x_0$ is Lyapunov stable.

Thm : (2nd Lyapunov Theorem)

If :

a) $f(x_0) = 0$ with $x_0 \in A$.

b) There is a Lyapunov function $V : A \to \mathbb{R}$ with $V(x_0) = 0$

* c) $x(0) \in A \Rightarrow \forall t > t_0 : V(x(t)) < V(x(t_0))$ (!)

Then $x = x_0$ is asymptotically stable.

EXERCISES

(7) Consider the system

$$\dot{p} = f(p,q)$$
$$\dot{q} = g(p,q)$$

with $f(p,q) = \partial H(p,q)/\partial q$

$g(p,q) = -\partial H(p,q)/\partial p$.

If (p_0,q_0) is a fixed point and if

$H(p,q) > 0$, $\forall (p,q) \in A - \{(p_0,q_0)\}$

with A an open set that contains (p_0,q_0),

then show that (p_0,q_0) is Lyapunov stable.

↳ Systems of this type are called Hamiltonian systems.

MM2: 1D autonomous systems

1D AUTONOMOUS SYSTEMS

Stability analysis for 1d systems

- We recall from analysis strong differentiability:

<u>Def</u>: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$. We say that f is strongly differentiable at $x_0 \in A$ if and only if there is a function $g: A \to \mathbb{R}$ such that

$$\begin{cases} \forall x \in A: f(x) = f(x_0) + (x - x_0) f'(x_0) + |x| g(x) \\ \lim_{x \to x_0} g(x) = 0 \end{cases}$$

<u>Prop</u>: Let $f: A \to \mathbb{R}$ be a function with $A \subseteq \mathbb{R}$ and let $x_0 \in A$.

$$\left. \begin{array}{l} f \text{ differentiable at } x_0 \\ f' \text{ continuous at } x_0 \end{array} \right\} \Rightarrow f \text{ strongly differentiable at } x_0$$

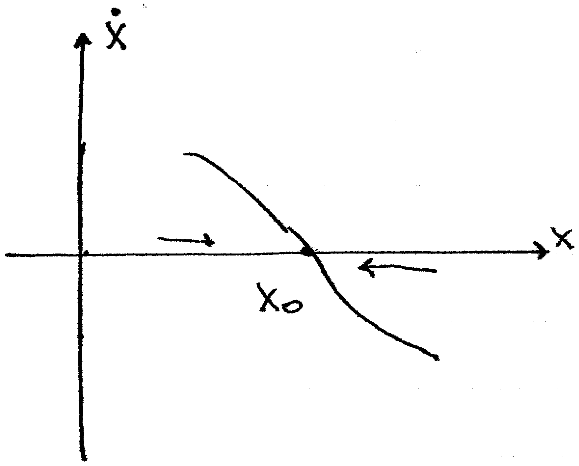
- The stability of 1d autonomous dynamical systems is determined via the following theorem.

<u>Thm</u>: Consider the system $\dot{x} = f(x)$ with $f: \mathbb{R} \to \mathbb{R}$ a function which is strongly differentiable on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a fixed point with $f(x_0) = 0$. Then:

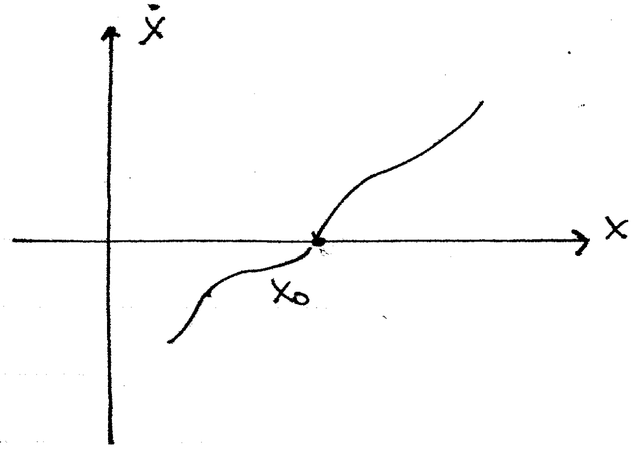
a) $f'(x_0) < 0 \Rightarrow x_0$ asymptotically stable

b) $f'(x_0) > 0 \Rightarrow x_0$ unstable.

↳ The theorem is interpreted according to the following phase portraits:



x_0 asymptotically stable



x_0 unstable

Proof

Define $y(t) = x(t) - x_0 \Rightarrow x(t) = y(t) + x_0 \Rightarrow$

$$\Rightarrow \dot{y} = \dot{x} = f(x) = f(x_0 + y) = f(x_0) + yf'(x_0) + |y|g(y) =$$
$$= yf'(x_0) + |y|g(y)$$

with $\lim_{y \to 0} g(y) = 0$, since f is strongly differentiable in x_0. It follows that

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in (-\delta, 0) \cup (0, \delta) : |g(y)| < \varepsilon$$

Let $\varepsilon = (1/2)|f'(x_0)|$ and let $\delta > 0$ be the corresponding δ such that $\forall y \in (-\delta, 0) \cup (0, \delta) : |g(y)| < \varepsilon$. We see that:

$$|\dot{y} - yf'(x_0)| = ||y|g(y)| = |y||g(y)| < |y|\,\varepsilon = |y|(1/2)|f'(x_0)| .$$
$$= (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow |\dot{y} - yf'(x_0)| < (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow -(1/2)|yf'(x_0)| < \dot{y} - yf'(x_0) < (1/2)|yf'(x_0)| \Rightarrow$$
$$\Rightarrow yf'(x_0) - (1/2)|yf'(x_0)| < \dot{y} < yf'(x_0) + (1/2)|yf'(x_0)|$$

First, we note that:

a) If $yf'(x_0) > 0 \Rightarrow \dot{y} > yf'(x_0) - (1/2)\,|yf'(x_0)| =$
$= yf'(x_0) - (1/2)\,yf'(x_0) =$
$= (1/2)\,yf'(x_0) \Rightarrow \dot{y} > (1/2)\,yf'(x_0)$

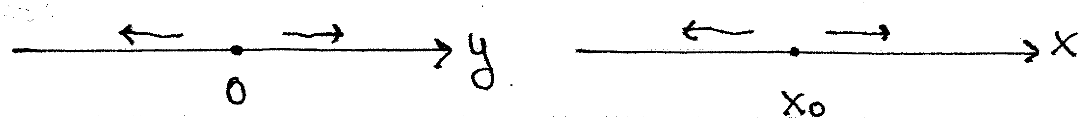
b) If $yf'(x_0) < 0 \Rightarrow$
$\Rightarrow \dot{y} < yf'(x_0) + (1/2)\,|yf'(x_0)| = yf'(x_0) - (1/2)\,yf'(x_0)$
$= (1/2)\,yf'(x_0) \Rightarrow \dot{y} < (1/2)\,yf'(x_0)$.

We have thus shown that for $y \in (-\delta, 0) \cup (0, \delta)$:
$yf'(x_0) > 0 \Rightarrow \dot{y} > (1/2)\,yf'(x_0)$
$yf'(x_0) < 0 \Rightarrow \dot{y} < (1/2)\,yf'(x_0)$

We now distinguish between the following cases:

<u>Case 1</u>: Assume that $f'(x_0) > 0$. Then for
$y \in (0, \delta) \Rightarrow yf'(x_0) > 0 \Rightarrow \dot{y} > (1/2)\,yf'(x_0) > 0 \Rightarrow$
$\Rightarrow y(t)$ increasing.
$y \in (-\delta, 0) \Rightarrow yf'(x_0) < 0 \Rightarrow \dot{y} < (1/2)\,yf'(x_0) < 0 \Rightarrow$
$\Rightarrow y(t)$ decreasing.

It follows that the fixed point x_0 is unstable:



<u>Case 2</u>: Assume that $f'(x_0) < 0$. Then for
$y \in (0, \delta) \Rightarrow yf'(x_0) < 0 \Rightarrow \dot{y} < (1/2)\,yf'(x_0) < 0 \Rightarrow$
$\Rightarrow y(t)$ decreasing $\Rightarrow$ Lyapunov stability.

Since $y = 0$ is a fixed point, it follows that if we initialize at ~~$y(0) \in (-\delta, 0)$~~ $y(0) \in (0, \delta)$, then $y(t) \geq 0$ and furthermore $y(0)\exp((1/2)f'(x_0)t) \geq y(t) \geq 0 \Rightarrow$
$\Rightarrow \lim\limits_{t \to +\infty} y(t) = 0 \Rightarrow$ fixed point is attracting

Likewise, for

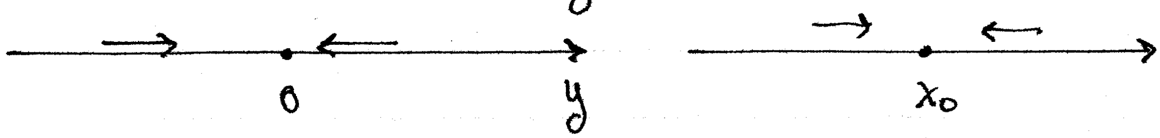
$y \in (-\delta, 0) \Rightarrow y f'(x_0) > 0 \Rightarrow \dot{y} > (1/2) y f'(x_0) > 0 \Rightarrow$

$\Rightarrow y(t)$ increasing $\Rightarrow$ Lyapunov stability

and similarly we can show that

$y(0) \exp((1/2) f'(x_0) t) \leqslant y(t) \leqslant 0 \Rightarrow \lim\limits_{t \to +\infty} y(t) = 0 \Rightarrow$

$\Rightarrow$ fixed point is attracting.



In both cases, initializing at $y(0) \in (-\delta, 0) \cup (0, \delta)$ yields both Lyapunov stability and the attracting property, therefore the fixed point x_0 is asymptotically stable.

EXAMPLES

a) $\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases}$ with $a > 0$ (Exponential growth model)

- Exact solution $x(t) = x_0 \exp(at)$
- Fixed points.

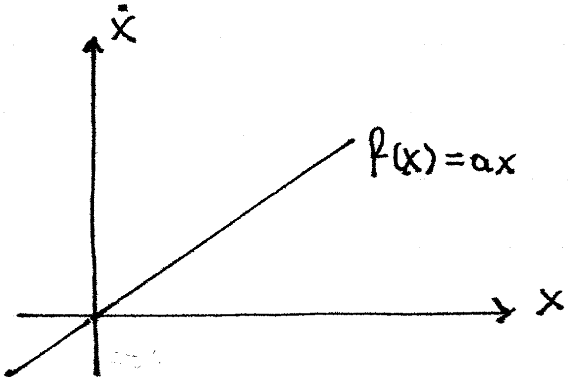
Let $f(x) = ax$. Then

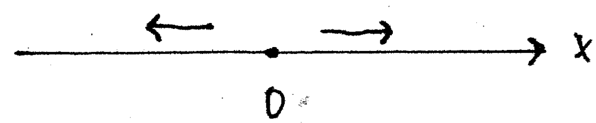
x fixed point $\Leftrightarrow f(x) = 0 \Leftrightarrow ax = 0 \Leftrightarrow \underline{x = 0}$

- Stability.

$f'(x) = (ax)' = a$

At $x = 0$: $f'(0) = a > 0 \Rightarrow x = 0$ is an unstable fixed point.





b) $\begin{cases} \dot{x} = (a/b)x(b - x) \\ x(0) = x_0 \end{cases}$ with $a > 0$ and $b > 0$ (Logistic Model)

Here a = growth rate
b = carrying capacity.

- Fixed points.

Let $f(x) = (a/b)\,x\,(b - x)$.

x fixed point $\Leftrightarrow f(x)=0 \Leftrightarrow (a/b)x(b-x)=0 \Leftrightarrow$
$\Leftrightarrow x(b-x)=0 \Leftrightarrow x=0 \vee x=b$

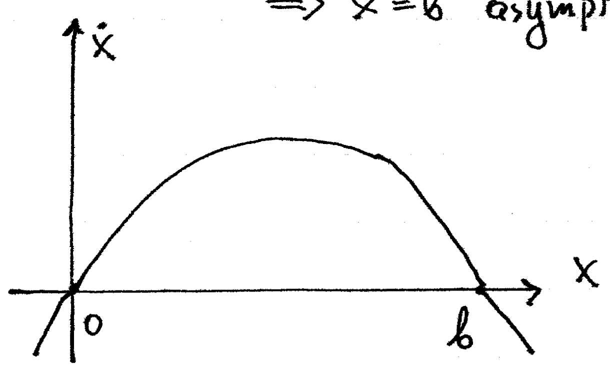
▶ Stability:

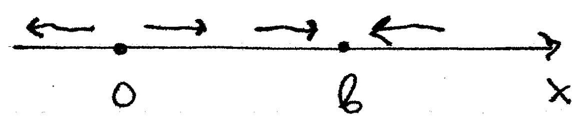
$$f'(x)=\frac{a}{b}\frac{d}{dx}x(b-x)=\frac{a}{b}\frac{d}{dx}(bx-x^2)=$$

$$=\frac{a}{b}(b-2x)=a-\frac{2ax}{b}$$

For $x=0$: $f'(0)=a>0 \Rightarrow x=0$ unstable.
For $x=b$: $f'(b)=a-\frac{2ab}{b}=a-2a=-a<0 \Rightarrow$

$\Rightarrow x=b$ asymptotically stable.





↦ The stability theorem given above may fail if at a fixed point $x=x_0$ we have $f'(x_0)=0$. An alternative method for determining fixed point stability useful in such situations is the construction of a sign table.

c) $\dot{x} = 2x(x-1)^2(x-2)^3$

▶ Fixed points

Let $f(x) = 2x(x-1)^2(x-2)^3$. Then

x fixed point $\Leftrightarrow f(x)=0 \Leftrightarrow 2x(x-1)^2(x-2)^3 = 0 \Leftrightarrow$

$\Leftrightarrow 2x = 0 \vee (x-1)^2 = 0 \vee (x-2)^3 = 0 \Leftrightarrow$

$\Leftrightarrow x=0 \vee x=1 \vee x=2$.

▶ Stability

x		0		1		2	
$2x$	$-$	0	$+$		$+$		$+$
$(x-1)^2$	$+$		$+$	0	$+$		$+$
$(x-2)^3$	$-$		$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$-$	0	$+$
	$\rightarrow$		$\leftarrow$		$\leftarrow$		$\rightarrow$
		stab.		unst.		unst.	

Thus $x=0$ is asymptotically stable and $x=1$ and $x=2$ are unstable.

EXERCISES

(1) Find the fixed-points and determine their stability for the following systems:

a) $\dot{x} = x^2 - 9$

b) $\dot{x} = x - x^3$

c) $\dot{x} = x^2(6-x)$

d) $\dot{x} = x(1-x)(2-x)$

e) $\dot{x} = (x+1)^3(x-3)^2$

f) $\dot{x} = (2x-1)^2(3x+1)^4$

g) $\dot{x} = 1 - 2\cos x$

h) $\dot{x} = e^{-x}\sin x$

i) $\dot{x} = 2\sin x + \sin 2x$

j) $\dot{x} = \cos x + 2\sin x$

k) $\dot{x} = -2x\ln(3x)$

▼ Potential and 1d systems

Consider a 1d autonomous system $\dot{x} = f(x)$ with f continuous in $\mathbb{R}$. Then we may define a potential function

$$V(x) = \int_x^c f(t)\,dt \Rightarrow f(x) = -\frac{dV(x)}{dx} = -V'(x).$$

It follows that

$$\frac{dx}{dt} = -V'(x).$$

- Let $x(t)$ be a solution of the autonomous system. We will show that $V(x(t))$ decreases with time, that is the system evolves towards lower potentials. Formally:

$$\boxed{t_1 < t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2))}$$

Proof

$$\frac{d}{dt} V(x(t)) = V'(x(t)) \frac{dx(t)}{dt} = V'(x(t)) f(x(t)) =$$

$$= V'(x(t))[-V'(x(t))] = -[V'(x(t))]^2 \Rightarrow$$

$$\Rightarrow V(x(t_2)) - V(x(t_1)) = \int_{t_1}^{t_2} \left[\frac{d}{dt} V(x(t))\right] dt =$$

$$= \int_{t_1}^{t_2} -[V'(x(t))]^2 dt \leqslant 0 \Rightarrow$$

$$\Rightarrow V(x(t_1)) \geqslant V(x(t_2)). \qquad \square$$

Remarks

a) Fixed points occur at the min/max points of the potential function $V(x)$.
b) Stable fixed points occur at the min points of $V(x)$.
c) Unstable fixed points occur at the max points of $V(x)$.

$\mapsto$ No periodic solutions

- A 1d autonomous system $\dot{x} = f(x)$ never has any periodic solution that is not constant for all time.

Proof

Let $x(t)$ be a solution of $\dot{x} = f(x)$ such that
$x(t) = x(t+T), \forall t \in \mathbb{R}$. (1)
Let $V(x)$ be the potential function. Then
$(1) \Rightarrow V(x(t)) = V(x(t+T)), \forall t \in \mathbb{R} \Rightarrow$

$$\Rightarrow \int_t^{t+T} [V'(x(\tau))]^2 d\tau = 0, \forall t \in \mathbb{R} \Rightarrow$$

$$\Rightarrow V'(x(\tau)) = 0 \quad , \forall \tau \in [t, t+T] \, , \forall t \in \mathbb{R} \Rightarrow$$
$$\Rightarrow V'(x(t)) = 0 \quad , \forall t \in \mathbb{R}$$
$$\Rightarrow dx(t)/dt = 0 \, , \forall t \in \mathbb{R} \Rightarrow x(t) \text{ constant } \square .$$

EXERCISES

(2) Show that the system $\dot{x} = x - \sin t$ with $x(0) = 1/2$ admits the periodic solution $x(t) = (1/2)(\sin t + \cos t)$. How does this reconcile with our claim that 1d autonomous systems do not admit periodic solutions?

(3) Likewise show that the system $\ddot{x} = -\omega^2 x$ with $x(0) = 0$ admits the periodic solution $x(t) = \quad \sin(\omega t)$. Again, how do we reconcile this paradox?

(4) Find the fixed-points and determine their stability for 1d autonomous systems with potentials given by

a) $V(x) = x^4 - 3x^3$

b) $V(x) = (x^2 - 1)^2$

c) $V(x) = x + 4\sin x$

d) $V(x) = e^{-x}\cos x$

e) $V(x) = (x^2 - 2)e^{-x}$

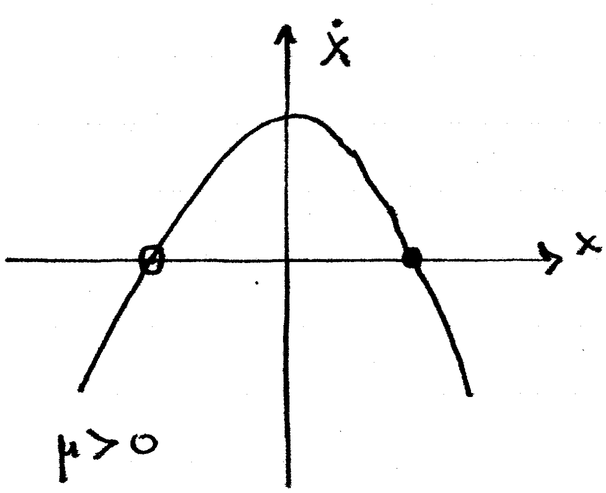
Local Bifurcations with 1d systems

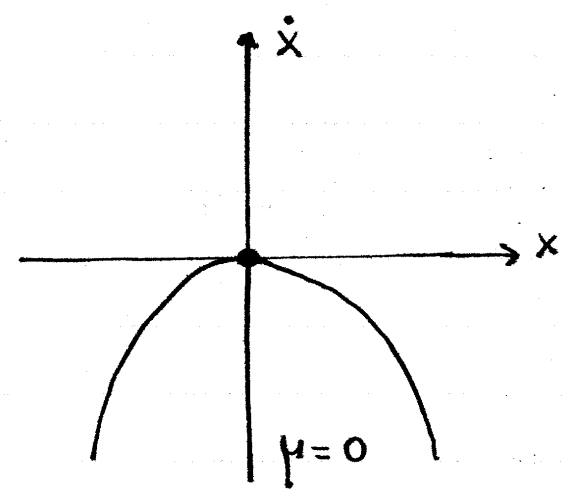
- In general, bifurcations fall under two general categories
  (a) Local Bifurcations
  (b) Global Bifurcations
  However 1d systems admit only local bifurcations.

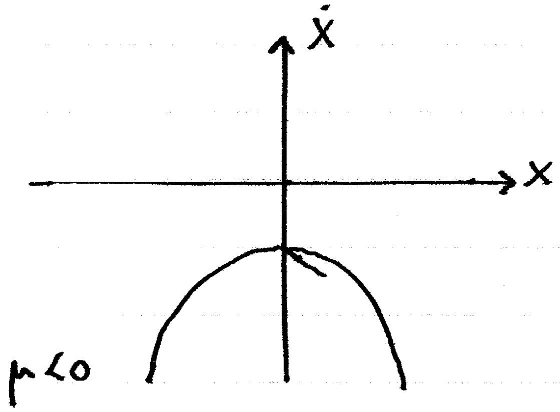
- Consider the 1d autonomous system $\dot{x} = f(x, \mu)$ with $\mu \in \mathbb{R}$ a parameter. A local bifurcation occurs when the number of fixed points changes as we vary the value of the parameter μ. The three most common types of local bifurcations are:

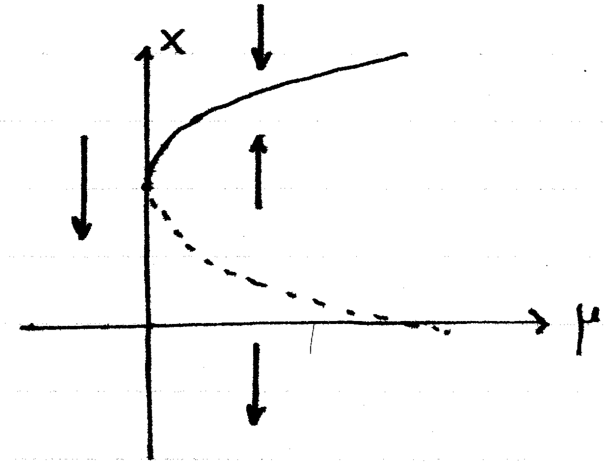
(1) Saddle-node bifurcation $\longrightarrow$ $\boxed{\dot{x} = \mu - x^2}$

Two fixed-points with opposite stability properties collide into a saddle point which then vanishes:







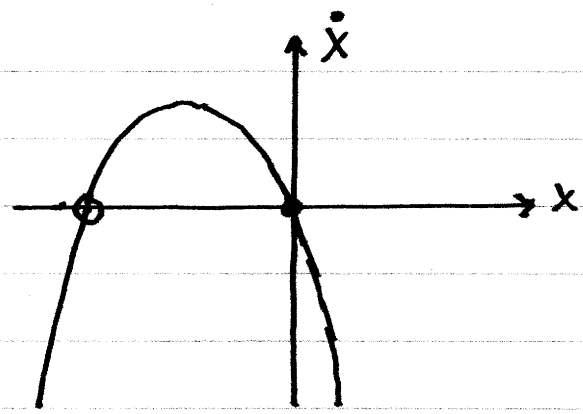


(Bifurcation diagram)

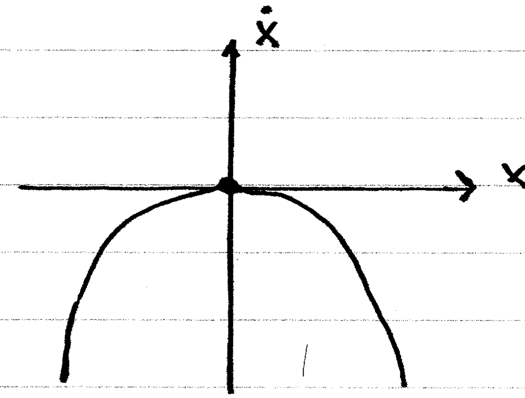
↳ A bifurcation diagram shows the motion of the fixed-points on the x-axis as a function of the parameter μ. We use a solid line to denote the motion of a stable fixed-point and a dotted-line to show the motion of an unstable fixed-point.

(2) → <u>Transcritical bifurcation</u> → $\boxed{\dot{x} = \mu x - x^2}$

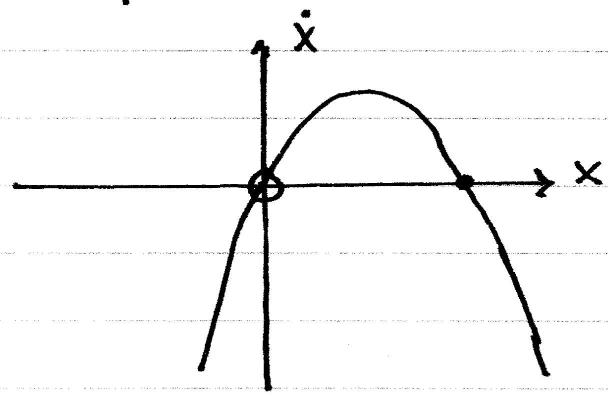
Two fixed points with opposite stability properties collide into a saddle-point which breaks up into two fixed points again with opposite stability properties but also with their stability properties exchanged. One fixed point is independent of μ.



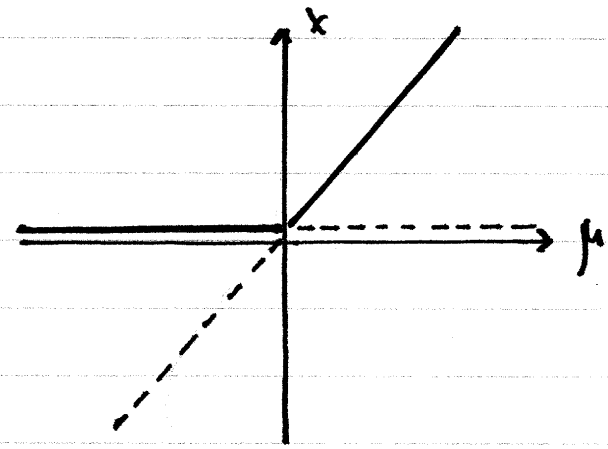
$\mu < 0$



$\mu = 0$



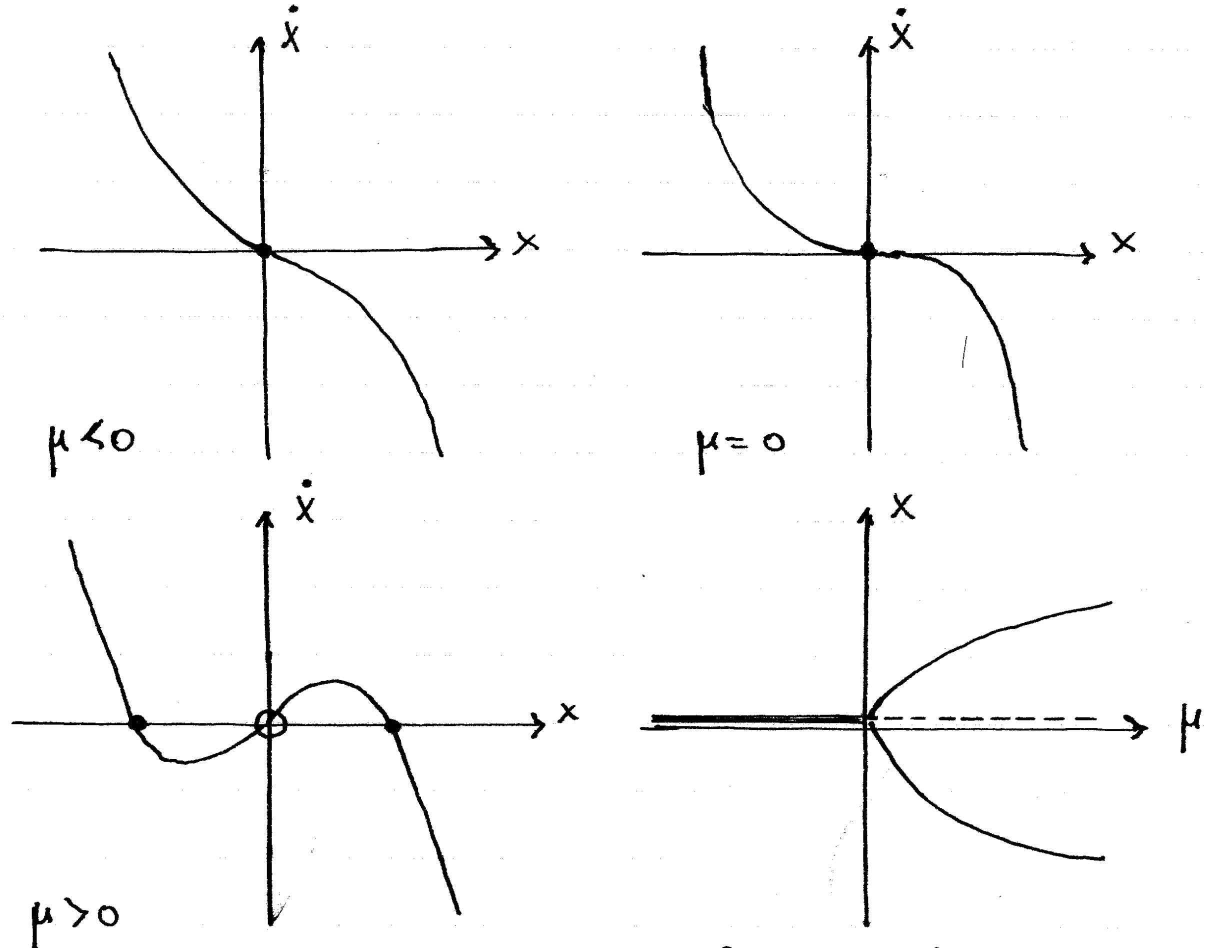
$\mu > 0$



Bifurcation diagram.

③ → <u>Pitchfork bifurcation</u> → $\boxed{\dot{x} = \mu x - x^3}$

In a pitchfork bifurcation, a fixed-point breaks into 3 fixed-points. The inner fixed-point has opposite stability property with respect to the original fixed-point. The 2 outer fixed-points have the same stability property as the original fixed-point. We call this bifurcation a pitchfork bifurcation because the bifurcation diagram resembles a pitchfork. The inner fixed-point is independant of the parameter μ.



Bifurcation diagram

Tangency condition

From the examples above we see that bifurcations occur always when the graph of $f(x)$ is tangent to the x-axis. It follows that candidate points (x_0, μ_0) for bifurcation events can be located by solving the system of equations:

$$\boxed{\begin{cases} f(x_0, \mu_0) = 0 \\ f_x(x_0, \mu_0) = 0 \end{cases}}$$

Here, the subscripts represent partial derivatives, thus $f_x = \dfrac{\partial f}{\partial x}$.

Sufficient conditions

Once we identify a candidate for a bifurcation event at (x_0, μ_0) it can be classified by confirming the corresponding sufficient conditions. The sufficient conditions for the bifurcations considered above are:

Saddle-node bifurcation	Transcritical bifurcation	Pitchfork bifurcation.
$f(x_0, \mu_0) = 0$	$f(x_0, \mu_0) = 0$	$f(x_0, \mu_0) = 0$
$f_x(x_0, \mu_0) = 0$	$f_x(x_0, \mu_0) = 0$	$f_x(x_0, \mu_0) = 0$
$f_\mu(x_0, \mu_0) \neq 0$	$f_\mu(x_0, \mu_0) = 0$	$f_\mu(x_0, \mu_0) = 0$
$f_{xx}(x_0, \mu_0) \neq 0$	$f_{xx}(x_0, \mu_0) \neq 0$	$f_{xx}(x_0, \mu_0) = 0$
	$f_{x\mu}(x_0, \mu_0) \neq 0$	$f_{x\mu}(x_0, \mu_0) \neq 0$
		$f_{xxx}(x_0, \mu_0) \neq 0$

Procedure

To identify and classify bifurcation events (x_0, μ_0) we work as follows:

- 1 Solve the equations

$$f(x,\mu) = 0$$
$$f_x(x,\mu) = 0$$

to identify candidates (x_0, μ_0).

- 2 Calculate $f_\mu(x_0, \mu_0)$.
  1) If $f_\mu(x_0,\mu_0) \neq 0$, then check that $f_{xx}(x_0,\mu_0) \neq 0$. If so, then $(x_0, \mu_0) \leftarrow$ saddle-node bifurcation
  2) If $f_\mu(x_0,\mu_0) = 0$, then check that $f_{x\mu}(x_0,\mu_0) \neq 0$. Then, if:
     a) $f_{xx}(x_0,\mu_0) \neq 0 \leftarrow$ trancritical bifurcation
     b) $f_{xx}(x_0,\mu_0) = 0 \leftarrow$, then check that $f_{xxx}(x_0,\mu_0) \neq 0 \leftarrow$ pitchfork bifurcation.

- 3 For a saddle-node bifurcation we have 2 fixed points
  a) For $\mu > \mu_0$, if $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) < 0$
  b) For $\mu < \mu_0$, if $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) > 0$
  (see exercise 9)

- 4 For a pitchfork bifurcation we have 3 fixed points
  a) For $\mu > \mu_0$, if $f_{xxx}(x_0,\mu_0) f_{x\mu}(x_0,\mu_0) < 0$
  b) For $\mu < \mu_0$, if $f_{xxx}(x_0,\mu_0) f_{x\mu}(x_0,\mu_0) > 0$
  (see exercise 11).

EXAMPLES

a) Saddle-Node Bifurcation: $\dot{x} = \mu - x - e^{-x}$

Let $f(x,\mu) = \mu - x - e^{-x} \Rightarrow f_x(x,\mu) = -1 + e^{-x}$.

$$\begin{cases} f(x,\mu)=0 \\ f_x(x,\mu)=0 \end{cases} \Leftrightarrow \begin{cases} \mu - x - e^{-x} = 0 \\ -1 + e^{-x} = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - x - e^{-x} = 0 \\ e^{-x} = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu - 0 - e^{-0} = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu - 1 = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu = 1 \\ x = 0 \end{cases}$$

thus possible bifurcation at $(x_0, \mu_0) = (0,1)$

$f_\mu(x,\mu) = 1 \Rightarrow f_\mu(x_0,\mu_0) = 1 \neq 0 \qquad (1)$

$f_{xx}(x,\mu) = -e^{-x} \Rightarrow f_{xx}(x_0,\mu_0) = -e^{-0} = -1 \neq 0 \quad (2)$

From (1) and (2): saddle-node bifurcation at $(x_0,\mu_0) = (0,1)$

Since $f_{xx}(x_0,\mu_0) f_\mu(x_0,\mu_0) = 1 \cdot (-1) = -1 < 0 \Rightarrow$

$\Rightarrow$ two fixed points for $\mu > 1$ and no fixed points for $\mu < 1$.

b) Transcritical Bifurcation: $\dot{x} = \mu \ln x + x - 1$

Let $f(x,\mu) = \mu \ln x + x - 1 \Rightarrow f_x(x,\mu) = \frac{\mu}{x} + 1$

$$\begin{cases} f(x,\mu)=0 \\ f_x(x,\mu)=0 \end{cases} \Leftrightarrow \begin{cases} \mu \ln x + x - 1 = 0 \\ \mu/x + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} -x \ln x + x - 1 = 0 \\ \mu = -x \end{cases}$$

Let $g(x) = -x\ln x + x - 1$. Note the obvious solution $x=1$ since $g(1) = -1\ln 1 + 1 - 1 = -0 + 0 = 0$. We now show the solution is unique.

$$g'(x) = -(x)'\ln x - x(\ln x)' + 1 = -\ln x - x\frac{1}{x} + 1 =$$
$$= -\ln x - 1 + 1 = -\ln x$$

It follows that $g \nearrow (0,1)$ and $g \searrow (1,+\infty)$ thus $\forall x \in (0,1) \cup (1,+\infty) : g(x) < 0$.

We conclude that the solution $x=1$ is unique and therefore a bifurcation may occur when $(x_0, \mu_0) = (1,-1)$. Note that

$$f_\mu(x,\mu) = \ln x \Rightarrow f_\mu(1,-1) = \ln 1 = 0$$

$$f_{x\mu}(x,\mu) = \frac{1}{x} \Rightarrow f_{x\mu}(1,-1) = \frac{1}{1} = 1 \neq 0$$

$$f_{xx}(x,\mu) = -\frac{\mu}{x^2} \Rightarrow f_{xx}(1,-1) = -\frac{-1}{1^2} = 1 \neq 0$$

It follows that there is a transcritical bifurcation at $(x_0, \mu_0) = (1,-1)$.

c) <u>Pitchfork bifurcation</u> : $\dot{x} = -x + \mu \tanh x$

Let $f(x,\mu) = -x + \mu \tanh x \Rightarrow$
$\Rightarrow f_x(x,\mu) = -1 + \mu(1 - \tanh^2 x)$

$$\begin{cases} f(x,\mu) = 0 \\ f_x(x,\mu) = 0 \end{cases} \Leftrightarrow \begin{cases} -x + \mu \tanh x = 0 \\ -1 + \mu(1 - \tanh^2 x) = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu \tanh x = x \\ -1 + \mu - (\mu \tanh x)\tanh x = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \mu \tanh x = x \\ -1 + \mu - x\tanh x = 0 \end{cases} \Leftrightarrow \begin{cases} \mu \tanh x = x \\ \mu = 1 + x\tanh x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1+x\tanh x)\tanh x = x & (1) \\ \mu = 1 + x\tanh x \end{cases}$$

Since $\tanh 0 = 0$, $x=0$ is an obvious solution of (1).
We now show that this solution is unique.
Let $g(x) = (1+x\tanh x)\tanh x - x =$

$$= \tanh x + x\tanh^2 x - x \Rightarrow$$

$$\Rightarrow g'(x) = (1-\tanh^2 x) + \tanh^2 x + x(2\tanh x)(1-\tanh^2 x) - 1$$

$$= 1 - \tanh^2 x + \tanh^2 x + 2x\tanh x - 2x\tanh^3 x - 1$$

$$= 2x\tanh x - 2x\tanh^3 x =$$

$$= 2x\tanh x\,(1-\tanh^2 x)$$

Note that $-1 < \tanh x < 1 \Rightarrow 1 - \tanh^2 x > 0$ thus

x		0	
$2x$	$-$	0	$+$
$\tanh x$	$-$	0	$+$
$1-\tanh^2 x$	$+$		$+$
$g'(x)$	$+$	0	$+$
$g(x)$	$\nearrow$		$\nearrow$

Since $g \nearrow (-\infty, 0)$ and $g \nearrow (0, +\infty)$ and $g(0) = 0$, it follows that $x=0$ is a unique solution of $g(x) = 0$.

For $x = 0 \Rightarrow \mu = 1 + 0\tanh 0 = 1$ thus there is a bifurcation at $(x_0, \mu_0) = (0, 1)$.

Now, we note that:

$f_\mu(x,\mu) = \tanh x \Rightarrow f_\mu(0,1) = \tanh 0 = 0$

$f_{x\mu}(x,\mu) = 1 - \tanh^2 x \Rightarrow f_{x\mu}(0,1) = 1 - \tanh^2 0 = 1 - 0 = 1 \neq 0$

$$f_{xx}(x,\mu) = -\mu \frac{\partial}{\partial x} \tanh^2 x =$$

$$= -2\mu \tanh x \cdot (1 - \tanh^2 x) \Rightarrow$$

$\Rightarrow f_{xx}(0,1) = -2 \cdot 1 \cdot 0 \cdot (1-0) = 0$, thus we rule out transcritical.

$$f_{xxx}(x,\mu) = \frac{\partial}{\partial x}\left[-2\mu \tanh x + 2\mu \tanh^3 x\right] =$$

$$= -2\mu(1 - \tanh^2 x) + 6\mu \tanh^2 x (1 - \tanh^2 x)$$

$$= 2\mu(1 - \tanh^2 x)[-1 + 3\tanh^2 x] \Rightarrow$$

$$\Rightarrow f_{xxx}(0,1) = 2 \cdot 1 \cdot (1-0)[-1 + 3 \cdot 0] =$$

$$= 2 \cdot 1 \cdot 1 \cdot (-1) = -2 \neq 0.$$

It follows that $(x_0, \mu_0) = (0,1)$ are pitchfork bifurcation. Since

$f_{xxx}(0,1) f_{x\mu}(0,1) = (-2) \cdot 1 = -2 < 0 \Rightarrow$

$\Rightarrow$ there are 3 fixed points for $\mu > 1$.

EXERCISES

(5) Show that the following systems undergo a saddle-node bifurcation. Find the value $\mu = \mu_0$ where such a bifurcation occurs

a) $\dot{x} = 1 + \mu x + x^2$

b) $\dot{x} = \mu + x - \ln(1+x)$

c) $\dot{x} = \mu + \frac{x}{2} - \frac{x}{1+x}$

d) $\dot{x} = 1 - \frac{\mu x^2}{2} + \frac{x^4}{4}$

(6) Likewise, show that the following systems undergo a transcritical bifurcation. Find the value $\mu = \mu_0$ where the bifurcation occurs.

a) $\dot{x} = \mu x + x^2$

b) $\dot{x} = \mu x - \ln(1+x)$

c) $\dot{x} = \mu x - x(1-x)$

d) $\dot{x} = x(\mu - e^x)$

e) $\dot{x} = \mu x + x^4$

(7) Likewise, show that the following systems undergo a pitchfork bifurcation. Find the value $\mu = \mu_0$ where the bifurcation occurs.

a) $\dot{x} = \mu x + x^3$

b) $\dot{x} = \mu x - x^3$

c) $\dot{x} = \mu x - \sinh(x)$

d) $\dot{x} = x + \frac{\mu x}{1+x^2}$

e) $\dot{x} = \mu x + \sin x$

More on sufficient conditions for bifurcation events

We will now derive the sufficient conditions for classifying bifurcation events. The proofs are based on the implicit function theorem.

Implicit function theorem

First we define the ball $B((x_0,y_0),\varepsilon)$ as:

$$B((x_0,y_0),\varepsilon)=\{(x,y)\in\mathbb{R}^2 \,|\, (x-x_0)^2+(y-y_0)^2<\varepsilon^2\}$$

The implicit function theorem states:

<u>Thm</u>: Assume that the function $f: A\to\mathbb{R}$ with $A\subseteq\mathbb{R}^2$ satisfies

a) $f(x_0,y_0)=0$
b) $\forall (x,y)\in B((x_0,y_0),\varepsilon): f_y(x,y)\neq 0$
c) f_x, f_y continuous at $B((x_0,y_0),\varepsilon)$

Then, there is a <u>unique</u> function g such that $\forall (x,y)\in B((x_0,y_0),\varepsilon): f(x,g(x))=0$

Note that condition (b) can be weakened to $f_y(x_0,y_0)\neq 0$. Then, combined with (c) it follows that there is an ε for which both (b) and (c) are satisfied.

① → Saddle-node bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0$ (1)

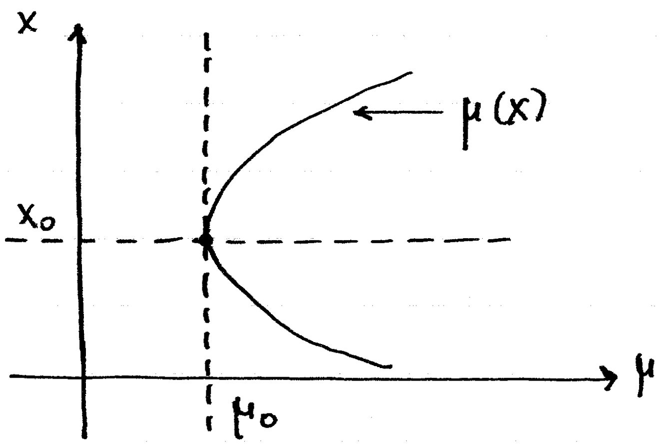
$f_x(x_0, \mu_0) = 0$ (2)

$f_\mu(x_0, \mu_0) \neq 0$ (3)

$f_{xx}(x_0, \mu_0) \neq 0$ (4)

- Analysis

The typical bifurcation diagram for a saddle-node bifurcation is shown below:



We see that we have to show that there is a unique function $\mu(x)$ such that

$f(x, \mu(x)) = 0, \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

with

$$\mu'(x_0) = \frac{d}{dx}\mu(x_0) = 0$$

$$\mu''(x_0) = \frac{d^2}{dx^2}\mu(x_0) \neq 0$$

The condition $\mu'(x_0) = 0$ ensures that the bifurcation curve is tangent to $\mu = \mu_0$. The condition $\mu''(x_0) \neq 0$ ensures that x_0 is a minimum or maximum so that the bifurcation curve $\mu(x)$ remains on the same half-plane defined by $\mu = \mu_0$.

- Construction: Since $f(x_0, y_0) = 0$ and $f_\mu(x_0, \mu_0) \neq 0$, it follows that the implicit function theorem applies and therefore there is a unique function $\mu(x)$ such that

$$\forall x \in (x_0-\varepsilon, x_0+\varepsilon) : f(x,\mu(x)) = 0 \qquad (5)$$

Thus $\mu(x)$ is hereby constructed.

- <u>Proof</u>: We will now show that $\mu'(x_0)=0$ and $\mu''(x_0)\neq 0$.

Differentiating (5) with respect to x gives:

$$f_x(x,\mu(x)) + f_\mu(x,\mu(x))\,\mu'(x) = 0 \qquad (6)$$

For $x = x_0$:

$f_x(x_0,\mu(x_0)) = f_x(x_0,\mu_0) = 0$ and

$f_\mu(x_0,\mu(x_0)) = f_\mu(x_0,\mu_0) \neq 0$

thus:

$$\mu'(x_0) = \frac{-f_x(x_0,\mu_0)}{f_\mu(x_0,\mu_0)} = 0 \qquad (7)$$

Differentiating (6) one more time with respect to x gives:

$$f_{xx} + f_{x\mu}\cdot\mu' + (f_{\mu x} + f_{\mu\mu}\cdot\mu')\mu' + f_\mu\cdot\mu'' = 0 \Rightarrow$$

$$\Rightarrow f_{xx} + (2f_{x\mu} + f_{\mu\mu}\cdot\mu')\mu' + f_\mu\cdot\mu'' = 0$$

evaluated at $(x,\mu(x))$. For $x=x_0$, $\mu'(x_0)=0$, and therefore:

$$f_{xx}(x_0,\mu_0) + f_\mu(x_0,\mu_0)\cdot\mu''(x_0) = 0$$

Since $f_{xx}(x_0,\mu_0)\neq 0$ and $f_\mu(x_0,\mu_0)\neq 0$, it follows that

$$\mu''(x_0) = \frac{-f_{xx}(x_0,\mu_0)}{f_\mu(x_0,\mu_0)} \neq 0.$$

• <u>Stability</u>: We will now show that the two fixed-points that emerge one one of the two half-planes defined by $\mu=\mu_0$ on the bifurcation diagram have opposite stability.

From (6):

$f_x(x,\mu(x)) = -f_\mu(x,\mu(x))\mu'(x)$, $\forall x \in (x_0-\varepsilon, x_0+\varepsilon)$

Since $f_\mu(x_0,\mu_0)\neq 0$, we can choose $\varepsilon>0$ small enough so that

$\forall x \in (x_0-\varepsilon, x_0+\varepsilon): f_\mu(x,\mu(x))\neq 0$

Thus $f_\mu(x,\mu(x))$ maintains its sign in $x\in(x_0-\varepsilon, x_0+\varepsilon)$.

Since $\mu''(x_0)\neq 0$ and $\mu'(x_0)=0$, we expect that $\mu'(x_0)$ changes sign from $x\in(x_0-\varepsilon, x_0)$ to $x\in(x_0, x_0+\varepsilon)$. Thus, so does $f_x(x,\mu(x))$ and it follows that the two fixedpoints, when they exist, have opposite stability.

② → Transcritical Bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0$ (1)

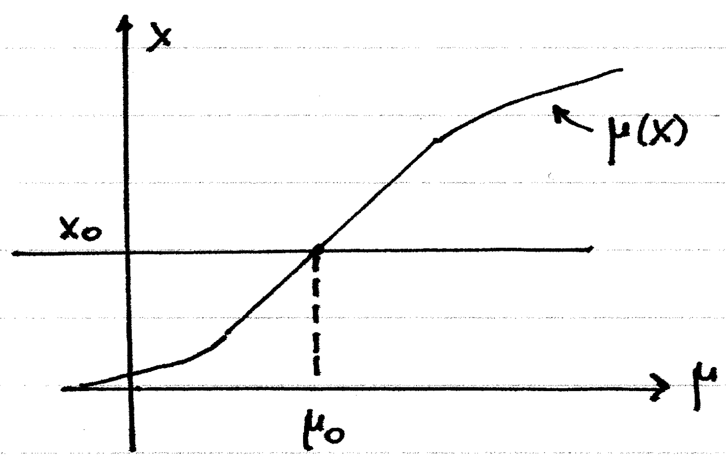
$f_x(x_0, \mu_0) = 0$ (2)

$f_\mu(x_0, \mu_0) = 0$ (3)

$f_{xx}(x_0, \mu_0) \neq 0$ (4)

$f_{x\mu}(x_0, \mu_0) \neq 0$ (5)

The typical bifurcation diagram for a transcritical bifurcation is shown below:



• Analysis : We see that there are two bifurcation curves passing through (x_0, μ_0) :

a) The line (ℓ): $x = x_0$ (independent of the parameter μ)

b) The line (ℓ_2): $\mu = \mu(x)$ passing from one half-plane to the other, separated by $\mu = \mu_0$, with $\mu_0 = \mu(x_0)$.

It follows that :

$f(x_0, \mu) = 0 \quad , \forall \mu \in (\mu_0 - \varepsilon_1, \mu_0 + \varepsilon_1)$
$f(x, \mu(x)) = 0 , \ \forall x \in (x_0 - \varepsilon_2, x_0 + \varepsilon_2)$
Note that to get two distinct curves pass through (x_0, μ_0) it is necessary to violate the implicit function theorem. Since $f(x_0, \mu_0) = 0$, to violate the theorem we require that $f_\mu(x_0, \mu_0) = 0$.
Let us now define

$$F(x, \mu) = \begin{cases} f(x, \mu)/(x - x_0) & , x \neq x_0 \\ f_x(x, \mu) & , x = x_0 \end{cases}$$

It follows that $f(x, \mu) = (x - x_0) F(x, \mu)$, thus we assume that $x = x_0$ is a bifurcation curve. We also note that $F(x, \mu)$ retains continuity because

$$\lim_{x \to x_0} F(x, \mu) = \lim_{x \to x_0} \frac{f(x, \mu)}{x - x_0} \overset{\text{L'Hospital}}{=} \lim_{x \to x_0} \frac{f_x(x, \mu)}{1 - 0} =$$

$$= \lim_{x \to x_0} f_x(x, \mu) = f_x(x_0, \mu) = F(x_0, \mu).$$

Note that L'Hospital applies since $\lim_{x \to x_0} f(x, \mu) = f(x_0, \mu) = 0$.

We will now show that $F(x, \mu)$ has a unique curve passing through (x_0, μ_0) and accross $\mu = \mu_0$.

- <u>Construction</u>: We note that

$$F(x_0, \mu_0) = f_x(x_0, \mu_0) = 0 \quad \text{and}$$
$$F_\mu(x_0, \mu_0) = f_{x\mu}(x_0, \mu_0) \neq 0$$

therefore the implicit function theorem applies.
It follows that there is a unique function $\mu(x)$ such that $F(x,\mu(x))=0$ for all x near x_0.
$x=\mu(x)$ is a bifurcation curve since
$f(x,\mu(x)) = (x-x_0)F(x,\mu(x)) = (x-x_0)\cdot 0 = 0$

- Proof : We will now show that the curve $x=\mu(x)$ passes accross $\mu=\mu_0$. To do that, it is sufficient to show that $\mu'(x_0)\neq 0$.

Since $F(x,\mu(x))=0 \Rightarrow$
$\Rightarrow F_x(x,\mu(x)) + F_\mu(x,\mu(x))\mu'(x) = 0 \Rightarrow$
$$\Rightarrow \mu'(x_0) = \frac{-F_x(x_0,\mu(x_0))}{F_\mu(x_0,\mu(x_0))} = \frac{-f_{xx}(x_0,\mu(x_0))}{f_{x\mu}(x_0,\mu(x_0))}$$
Since $f_{xx}(x_0,\mu_0)\neq 0$ and $f_{x\mu}(x_0,\mu_0)\neq 0$, $\mu'(x_0)$ is well-defined and $\mu'(x_0)\neq 0$.
It follows that $x=\mu(x)$ does not have a min or max at $x=x_0$, thus it will go accross the line $\mu=\mu_0$.

- Stability: We will now show that both bifurcation lines $(\ell_1): x=x_0$ and $(\ell_2): x=\mu(x)$ change stability upon crossing the point (x_0,μ_0).

a) For the line $(\ell_1): x=x_0$:
$$f_x(x_0,\mu) = f_x(x_0,\mu_0) + \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm = $$
$$= \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$$

Since $f_{x\mu}(x_0,\mu_0) \neq 0 \Rightarrow$
$\Rightarrow \exists \varepsilon > 0 : \forall \mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon) : f_{x\mu}(x_0,\mu) \neq 0$
Thus $f_{x\mu}(x_0,\mu)$ maintains its sign in $(\mu_0-\varepsilon, \mu_0+\varepsilon)$ therefore $f_x(x_0,\mu)$ changes sign from $\mu > \mu_0$ to $\mu < \mu_0$.

b) For the line $(\ell_2) : x = \mu(x)$

$$f_x(x,\mu(x)) = \frac{\partial}{\partial x}\Big[(x-x_0)F(x,\mu(x))\Big] =$$
$$= F(x,\mu(x)) + (x-x_0)F_x(x,\mu(x))$$
$$= (x-x_0)F_x(x,\mu(x))$$

Here we have used $F(x,\mu(x)) = 0$.
At $x = x_0$: $F_x(x_0,\mu(x_0)) = f_{xx}(x_0,\mu(x_0)) =$
$$= f_{xx}(x_0,\mu_0) \neq 0 \Rightarrow$$
$\Rightarrow \exists \varepsilon > 0 : \forall x \in (x_0-\varepsilon, x_0+\varepsilon) : F_x(x,\mu(x)) \neq 0$
Thus $F_x(x,\mu(x))$ does not change sign in $x \in (x_0-\varepsilon, x_0+\varepsilon)$ but $x - x_0$ does change from negative to positive. It follows that $f_x(x,\mu(x))$ changes sign accross $x = x_0$
From (a) and (b) above we conclude that since for both curves f_x changes sign accross the point (x_0,μ_0), the stability for both curves also changes.

③ → Pitchfork bifurcation conditions

Let us assume that

$f(x_0, \mu_0) = 0$ (1)

$f_x(x_0, \mu_0) = 0$ (2)

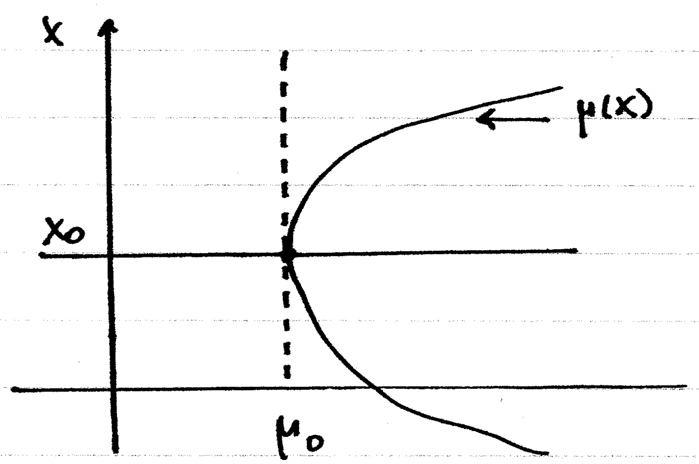
$f_\mu(x_0, \mu_0) = 0$ (3)

$f_{xx}(x_0, \mu_0) = 0$ (4)

$f_{x\mu}(x_0, \mu_0) \neq 0$ (5)

$f_{xxx}(x_0, \mu_0) \neq 0$ (6)

The typical bifurcation diagram for a pitchfork bifurcation is shown below:



• Analysis : The bifurcation diagram has two lines:

(a) The line (l_1): $x = x_0$ which is independent of μ.

(b) The curve (l_2): $\mu = \mu(x)$ which is tangent to the line (l): $\mu = \mu_0$. It follows that μ must satisfy $\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0$.

Both lines intersect at (x_0, μ_0).

Again, in order to have two curves passing through

(x_0, μ_0) it is necessary to violate the implicit function theorem. Since $f(x_0, \mu_0) = 0$, it is thus necessary to have $f_\mu(x_0, \mu_0) = 0$.
Again, let us define

$$F(x, \mu) = \begin{cases} f(x,\mu)/(x - x_0) & , \text{ if } x \neq x_0 \\ f_x(x_0, \mu) & , \text{ if } x = x_0 \end{cases}$$

Similarly with our transcritical bifurcation argument, it follows that

$$f(x, \mu) = (x - x_0) F(x, \mu)$$

$$\lim_{x \to x_0} F(x, \mu) = F(x_0, \mu).$$

Thus $(\ell_1): x = x_0$ is by definition a bifurcation line.

- <u>Construction</u>: We note that

$$F(x_0, \mu_0) = f_x(x_0, \mu_0) = 0$$

$$F_\mu(x_0, \mu_0) = f_{x\mu}(x_0, \mu_0) \neq 0$$

It follows that the implicit function theorem applies and thus there is a unique function $\mu(x)$ such that $F(x, \mu(x)) = 0$. It follows that

$$f(x, \mu(x)) = (x - x_0) F(x, \mu(x)) = (x - x_0) \cdot 0 = 0$$

Thus $\mu(x)$ has been constructed.

- <u>Proof</u>: We will now show that $\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0$.
  Using a calculation similar to the one

we did for the saddle-node proof, it follows that

$$\mu'(x_0) = \frac{-F_x(x_0,\mu_0)}{F_\mu(x_0,\mu_0)} = \frac{-f_{xx}(x_0,\mu_0)}{f_{x\mu}(x_0,\mu_0)} = 0$$

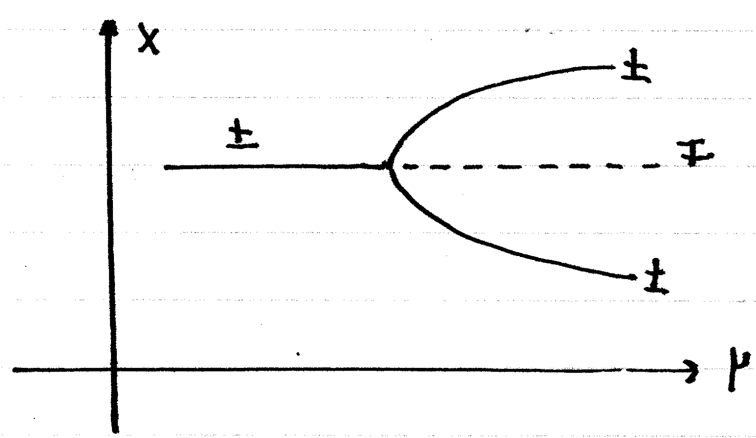
because $f_{xx}(x_0,\mu_0) = 0$

and therefore

$$\mu''(x_0) = \frac{-F_{xx}(x_0,\mu_0)}{F_\mu(x_0,\mu_0)} = \frac{-f_{xxx}(x_0,\mu_0)}{f_{x\mu}(x_0,\mu_0)} \neq 0$$

because $f_{xxx}(x_0,\mu_0) \neq 0$.

- <u>Stability</u>: We will now show that the inner fixed point changes stability accross the point (x_0,μ_0). We will also show that the outer fixed points after, the pitchfork occurs, have the same stability with each other as well as with the inner fixe dpoint BEFORE the fixed point. This is all shown in the diagram below:



a) For the line $x = x_0$:

$$f_x(x_0, \mu) = f_x(x_0, \mu_0) + \int_{\mu_0}^{\mu} f_{x\mu}(x_0, m)\,dm =$$

$$= \int_{\mu_0}^{\mu} f_{x\mu}(x_0, m)\,dm$$

Since $f_{x\mu}(x_0, \mu_0) \neq 0 \Rightarrow$

$\Rightarrow f_{x\mu}(x_0, m)$ does not change sign accross $m = \mu_0$

$\Rightarrow f_x(x, \mu)$ changes sign accross $\mu = \mu_0$

$\Rightarrow$ The fixed point on the line $x = x_0$ changes stability.

b) For the line $\mu = \mu(x)$:

$$f_x(x, \mu(x)) = \frac{\partial}{\partial x}\Big[(x - x_0) F(x, \mu(x))\Big] =$$

$$= F(x, \mu(x)) + (x - x_0) F_x(x, \mu(x))$$

$$= (x - x_0) F_x(x, \mu(x)) =$$

$$= (x - x_0)\left[-F_\mu(x, \mu(x))\, \mu'(x)\right]$$

Here we used the identity

$$F_x(x, \mu(x)) + F_\mu(x, \mu(x))\, \mu'(x) = 0$$

We note that accross $x = x_0$:

$x - x_0$ changes sign, and

$\mu'(x_0) = 0$ and $\mu''(x_0) \neq 0 \Rightarrow \mu'(x_0)$ changes sign

and $F_\mu(x_0, \mu(x_0)) = F_\mu(x_0, \mu_0) = f_{x\mu}(x_0, \mu_0) \neq 0 \Rightarrow$

$\Rightarrow F_\mu(x,\mu(x))$ does not change sign.
Thus $f_x(x,\mu(x))$ does not change sign. It follows that the two outer fixed points have the same stability.

c) We now compare the stability of the outer fixed points with the inner fixed point. Recall that f_x for these fixed points is:

inner point: $f_x(x_0,\mu) = \int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm$

outer points: $f_x(x,\mu(x)) = -\mu'(x)(x-x_0)F_\mu(x,\mu(x))$

We assume, with no loss of generality, that $\mu''(x_0)>0$. This implies that $\mu(x)$ has a minimum at $x=x_0$, so the 3 fixed points occur when $\mu>\mu_0$. We may thus assume that $\underline{\mu>\mu_0}$. It also follows that when x is near x_0, $\mu'(x)$ is increasing, and therefore:

$x-x_0<0 \Rightarrow \mu'(x)<0$

$x-x_0>0 \Rightarrow \mu'(x)>0$

Thus: $\mu'(x)(x-x_0)>0$ when x is near x_0.

It follows that:

$f_x(x,\mu(x))$ opposite sign as $F_\mu(x,\mu(x))$
same sign as $F_\mu(x_0,\mu_0)$ (x near x_0)
same sign as $f_{x\mu}(x_0,\mu_0)$
same sign as $f_{x\mu}(x_0,m)$ (m near μ_0)
same sign as $\int_{\mu_0}^{\mu} f_{x\mu}(x_0,m)\,dm = f_x(x_0,\mu)$
(use $\mu>\mu_0$).

Thus $f_x(x,\mu(x))$ has opposite sign from $f_x(x_0,\mu)$, thus outer and inner points have opposite stability.

EXERCISES

(8) Identify the bifurcations that the following systems undergo, find the parameter values $\mu = \mu_0$ where the bifurcations occur, and classify them as saddle-node, transcritical, or pitchfork.

a) $\dot{x} = \mu \sin x - \sin 2x$

b) $\dot{x} = \mu + \cos x + \cos 2x$

c) $\dot{x} = \mu + \sin x + \cos 2x$

d) $\dot{x} = \dfrac{\sin x}{\mu + \cos x}$

e) $\dot{x} = \dfrac{\sin x}{\mu + \sin x}$

(9) Consider a system with a saddle-node bifurcation satisfying the relevant sufficient conditions at (x_0, μ_0). Show that

a) If $f_{xx}(x_0, \mu_0) f_\mu (x_0, \mu_0) < 0$, then we have 2 fixed-points for $\mu > \mu_0$.

b) If $f_{xx}(x_0, \mu_0) f_\mu (x_0, \mu_0) > 0$, then we have 2 fixed-points for $\mu < \mu_0$.

c) Discuss the stability of the two fixed points for the above cases. Distinguish the case $f_\mu (x_0, \mu_0) > 0$ vs. $f_\mu (x_0, \mu_0) < 0$.

(10) Consider a system with a transcritical bifurcation at (x_0, μ_0) that satisfies the relevant sufficient conditions. Assume $x = x_0$ is a fixed point for all μ. Show that:

a) If $f_{x\mu}(x_0,\mu_0)>0$, then $x=x_0$ transitions from unstable to stable with increasing μ.

b) If $f_{x\mu}(x_0,\mu_0)<0$, then $x=x_0$ transitions from stable to unstable with increasing μ.

c) If $f_{xx}(x_0,\mu_0)>0$, then the other fixed point transitions from unstable to stable with increasing x.

d) If $f_{xx}(x_0,\mu_0)<0$, then the other fixed point transitions from stable to unstable with increasing x.

(11) Consider a system with a pitchfork bifurcation at (x_0,μ_0) that satisfies the relevant sufficient conditions. Show that

a) If $f_{xxx}(x_0,\mu_0)f_{x\mu}(x_0,\mu_0)>0$, then there are 3 fixed points at $\mu<\mu_0$

b) If $f_{xxx}(x_0,\mu_0)f_{x\mu}(x_0,\mu_0)<0$, then there are 3 fixed points at $\mu>\mu_0$.

c) Discuss the stability of the fixed points for the above two cases. Distinguish the case $f_{x\mu}(x_0,\mu_0)>0$ vs. $f_{x\mu}(x_0,\mu_0)<0$.

MM3: Linear autonomous systems

LINEAR AUTONOMOUS SYSTEMS

- A linear autonomous system is a system of ordinary differential equations of the form
$$\dot{x} = Ax$$
with $x \in \mathbb{R}^n$ a vector and $A \in M_n(\mathbb{R})$ an $n \times n$ matrix. In detail:

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ \dot{x}_n = A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n \end{cases}$$

Exact solutions

- An exact solution can be written in terms of the matrix exponential.

<u>Def</u>: $$\boxed{\exp(A) = \sum_{n=0}^{+\infty} \frac{A^n}{n!}} \quad (\text{with } A^0 = I)$$

→ <u>Properties</u>:
- $AB = BA \Rightarrow \exp(A+B) = \exp(A)\exp(B)$
- $[\exp(A)]^{-1} = \exp(-A)$
- $\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$

- The solution of $\dot{x} = Ax$ with $x(0) = x_0$ is

$$\boxed{x(t) = \exp(tA)\, x(0)}$$

- <u>Eigenvalues and eigenvectors</u>

<u>Def</u> : $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in M_n(\mathbb{R})$ with eigenvector $x \in \mathbb{C}^n$ if and only if $Ax = \lambda x$.

▷ <u>notation</u> : $\lambda(A)$ = the set of all eigenvalues of A.

<u>Thm</u> : $\boxed{\lambda \in \lambda(A) \Leftrightarrow \det(A - \lambda I) = 0}$

- We note that $p(\lambda) = \det(A - \lambda I)$ is a polynomial called the <u>characteristic polynomial</u> of A.
- Assume that A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with corresponding eigenvectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$. Then,
  a) The eigenvalues $v_1, v_2, \dots, v_n$ are linearly independent. Thus any $x \in \mathbb{R}^n$ can be written as:
  $$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$
  with $c_1, c_2, \dots, c_n$ constant.
  b) For $P = [v_1 v_2 \cdots v_n]$, A can be written as
  $$A = P \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) P^{-1}.$$
  with

$$\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

c) If the initial condition of $\dot{x} = Ax$ satisfies

$$x(0) = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

then

$$\boxed{x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n.}$$

<u>Proof</u>

$$x(t) = \exp(tA)\, x(0) = \exp(tA)\left[c_1 v_1 + c_2 v_2 + \cdots + c_n v_n\right]$$

$$= \sum_{a=1}^{n} c_a \exp(tA)\, v_a = \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{1}{k!} (tA)^k\right] v_a$$

$$= \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{t^k}{k!} (A^k v_a)\right] =$$

$$= \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{t^k}{k!} \lambda_a^k v_a\right] =$$

$$= \sum_{a=1}^{n} c_a \left[\sum_{k=0}^{+\infty} \frac{(\lambda_a t)^k}{k!}\right] v_a = \sum_{a=1}^{n} c_a e^{\lambda_a t} v_a \quad \square$$

We see that when the eigenvalues are all distinct, we can find the exact solution without calculating the matrix exponential.

• Matrix Exponential - 2x2 case

Let $A \in M_2(\mathbb{R})$ be a 2x2 matrix with eigenvalues λ_1, λ_2.

a) If $\lambda_1 \neq \lambda_2$, then

$$\boxed{\exp(tA) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} A}$$

b) If $\lambda_1 = \lambda_2 = \lambda$, then

$$\boxed{\exp(tA) = e^{\lambda t}(1 - \lambda t) I + t e^{\lambda t} A}$$

EXERCISES

① Write the exact solution for the following systems

a) $\begin{cases} \dot{x}_1 = 4x_1 + x_2 \\ \dot{x}_2 = -2x_1 + x_2 \end{cases}$

b) $\begin{cases} \dot{x}_1 = -5x_1 - x_2 \\ \dot{x}_2 = x_1 - 3x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_1 + x_2 \end{cases}$

d) $\begin{cases} \dot{x}_1 = 2x_1 + x_2 + 3x_3 \\ \dot{x}_2 = x_1 + 2x_2 + 3x_3 \\ \dot{x}_3 = 3x_1 + 3x_2 + 20x_3 \end{cases}$

② Show that

$$\exp\left(\vartheta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

③ Let $R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$

Show that $R(\vartheta)$ has real-only eigenvalues if and only if $\sin\vartheta = 0$.

④ Let $A \in M_n(\mathbb{R})$ with $A^2 = I$.
Show that: $\lambda \in \lambda(A) \Rightarrow \lambda = 1$ or $\lambda = -1$

⑤ Let $A \in M_n(\mathbb{R})$ with $\det A \neq 0$. Show that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}. Can a non-singular matrix have $\lambda = 0$ as an eigenvalue?

▼ Lyapunov function for $\dot{x} = Ax$

- Consider the linear autonomous system $\dot{x} = Ax$. If $\det A \neq 0$, then $Ax = 0 \Leftrightarrow x = 0$. Thus $x = 0$ is the unique fixed point. It's stability can be investigated by constructing an appropriate Lyapunov function.

↳ Definition of $V(x)$

Let $x, y \in \mathbb{C}^n$ with $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. We define the inner product:

$$\boxed{\langle x | y \rangle = \overline{x_1}\, y_1 + \overline{x_2}\, y_2 + \dots + \overline{x_n}\, y_n}$$

The bar (e.g. $\bar{x}$) represents the complex conjugate. We note that

$$|\langle x | y \rangle|^2 = \langle x | y \rangle \langle y | x \rangle .$$

For the matrix $A = [A_{ab}]$ we define the Hermitian matrix $A^H = [\overline{A_{ba}}]$. It can then be shown that

$$\langle x | Ay \rangle = \langle A^H x | y \rangle$$
$$\langle Ax | y \rangle = \langle x | A^H y \rangle$$

Let λ_a be the eigenvalues of A with eigenvectors u_a for $a \in \{1, 2, 3, \dots, n\}$. Also, let $\overline{\lambda_a}$ be the

eigenvalues of A^H with eigenvectors v_a.
We define the Lyapunov function $V(x)$ as:

$$\boxed{V(x) = \sum_a b_a |\langle v_a | x \rangle|^2}$$

Here $b_a > 0$ are arbitrary positive constants.
The sum runs from $a = 1, 2, 3, \ldots, n$.
By definition, it is easy to see that

$V(0) = 0$

$x \neq 0 \Rightarrow V(x) > 0$.

↳ Stability theorem

$$\boxed{\begin{array}{l} \mathrm{Re}(\lambda_a) \leqslant 0, \ \forall a \Rightarrow x = 0 \text{ is Lyapunov stable} \\ \mathrm{Re}(\lambda_a) < 0, \ \forall a \Rightarrow x = 0 \text{ is asymptotically stable} \end{array}}$$

Proof

We note that

$$\langle v_a | Ax \rangle = \langle A^H v_a | x \rangle = \langle \overline{\lambda_a} v_a | x \rangle = $$
$$= \lambda_a \langle v_a | x \rangle$$

and

$$\langle Ax | v_a \rangle = \langle x | A^H v_a \rangle = \langle x | \overline{\lambda_a} v_a \rangle =$$
$$= \overline{\lambda_a} \langle x | v_a \rangle$$

It follows that:

$$\frac{dV}{dt} = \frac{d}{dt}\sum_a b_a |\langle v_a | x \rangle|^2 =$$

$$= \frac{d}{dt}\sum_a b_a \langle v_a | x \rangle \langle x | v_a \rangle =$$

$$= \sum_a \left[b_a \left(\frac{d}{dt} \langle v_a | x \rangle \right) \langle x | v_a \rangle + b_a \langle v_a | x \rangle \left(\frac{d}{dt} \langle x | v_a \rangle \right) \right]$$

$$= \sum_a b_a \left[\langle v_a | Ax \rangle \langle x | v_a \rangle + \langle v_a | x \rangle \langle Ax | v_a \rangle \right] =$$

$$= \sum_a b_a \left[\lambda_a \langle v_a | x \rangle \langle x | v_a \rangle + \langle v_a | x \rangle \left(\overline{\lambda_a} \langle x | v_a \rangle \right) \right]$$

$$= \sum_a b_a (\lambda_a + \overline{\lambda_a}) \langle v_a | x \rangle \langle x | v_a \rangle =$$

$$= \sum_a 2 b_a \, \mathrm{Re}(\lambda_a) |\langle v_a | x \rangle|^2$$

For $x \neq 0$, $|\langle v_a | x \rangle|^2 > 0$, and by definition $b_a > 0$ for all a. Recall that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$

a) If $\mathrm{Re}(\lambda_a) \leq 0 \Rightarrow dV/dt \leq 0 \Rightarrow$

$\Rightarrow x = 0$ Lyapunov stable.

b) If $\mathrm{Re}(\lambda_a) < 0 \Rightarrow dV/dt < 0 \Rightarrow$

$\Rightarrow x = 0$ asymptotically stable. $\square$

↳ A matrix A whose eigenvalues satisfy $\mathrm{Re}(\lambda_a) < 0$, $\forall a$ is called <u>negative-definite</u>

Assuming $A \in M_n(\mathbb{R})$, it can be shown that A negative-definite $\Rightarrow \forall x \in \mathbb{R}^n : \langle x | Ax \rangle < 0$

EXERCISES

(6) a) Show that if $A+A^H$ is negative-definite then $V(x) = \langle x|x\rangle$ is a Lyapunov function of the system $\dot{x} = Ax$.

b) Consider the 2×2 case:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$$

Show that if $a+d<0$ and $4ad > (b+c)^2$ then $A+A^H$ is negative-definite.

(7) Consider the system

$$\begin{cases} \dot{x}_1 = \mu x_1 - x_2 \\ \dot{x}_2 = x_1 + (\mu+1)x_2 \end{cases}$$

a) Show that $(x_1, x_2) = (0,0)$ is the unique fixed-point of the system for all $\mu \in \mathbb{R}$.

b) Show that if $2\mu+1<0$, then $(x_1,x_2) = (0,0)$ is an asymptotically stable fixed-point.

c) What happens when $2\mu+1=0$.

▼ The 2×2 linear autonomous system

Consider the 2×2 linear autonomous system:

$$\begin{cases} \dot{x}_1 = ax_1 + bx_2 \\ \dot{x}_2 = cx_1 + dx_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The eigenvalues λ_1, λ_2 of A are found by solving the equation:

$$\begin{aligned} \det(A - \lambda I) = 0 &\Leftrightarrow (a-\lambda)(d-\lambda) - bc = 0 \Leftrightarrow \\ &\Leftrightarrow \lambda^2 - (a+d)\lambda + (ad - bc) = 0 \\ &\Leftrightarrow \lambda^2 - \tau\lambda + D = 0 \end{aligned}$$

with: $\tau = \operatorname{tr} A = a + d = \lambda_1 + \lambda_2$

$D = \det A = ad - bc = \lambda_1 \lambda_2$

The solution reads:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

The general solution of the system reads

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

with v_1, v_2 the eigenvectors corresponding to the eigenvalues λ_1, λ_2.

We note that an eigenvalue λ that satisfies

a) $Re(\lambda) < 0 \longrightarrow$ Gives a contribution that vanishes thus approaching the fixed-point.

b) $Re(\lambda) > 0 \longrightarrow$ Gives a contribution that diverges away from the fixed-point.

c) $Re(\lambda) = 0 \longrightarrow$ Gives a contribution that neither approaches nor diverges from the fixed-point.

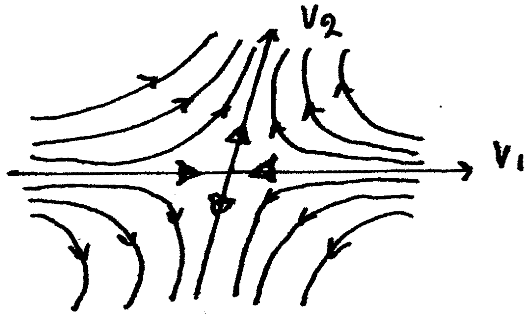
d) $Im(\lambda) \neq 0 \longrightarrow$ Gives a contribution that spirals around the fixed point.

e) $Im(\lambda) = 0 \longrightarrow$ Gives a contribution that does not spiral around the fixed point.

Based on that, we classify the $(0,0)$ fixed point as follows:

Classification of fixed-points in 2d

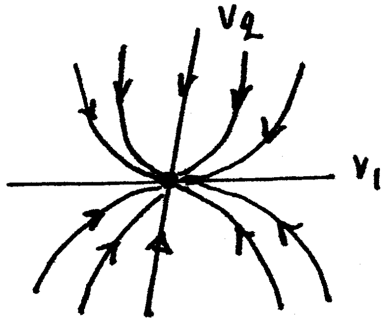
1) Saddle node :



- Eigenvalue condition: $\lambda_1 \lambda_2 < 0$
- τ-D condition: $D < 0$
- Unstable

The shape of the saddle node is controlled by the eigenvectors v_1, v_2.

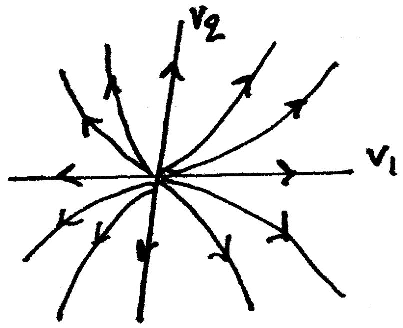
2) Sink :



- Eigenvalue condition
  $\lambda_1, \lambda_2 \in \mathbb{R} \wedge \lambda_1 < 0 \wedge \lambda_2 < 0$
- τ-D condition:
  $D > 0 \wedge \tau^2 - 4D > 0 \wedge \tau < 0$
- Exponentially stable

(v_1 slow, v_2 fast : $|\lambda_1| < |\lambda_2|$)

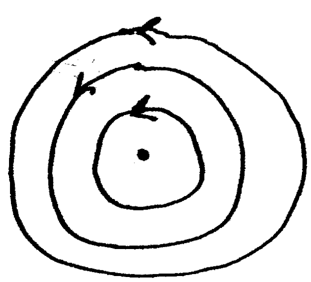
3) Source :



- Eigenvalue condition:
  $\lambda_1, \lambda_2 \in \mathbb{R} \wedge \lambda_1 > 0 \wedge \lambda_2 > 0$
- τ-D condition:
  $D > 0 \wedge \tau^2 - 4D > 0 \wedge \tau > 0$
- Unstable

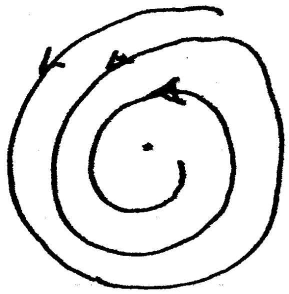
(v_1 fast , v_2 slow : $\lambda_1 > \lambda_2 > 0$)

4) Center :



- Eigenvalue condition
  $\begin{cases} Re(\lambda_1) = Re(\lambda_2) = 0 \\ Im(\lambda_1) \neq 0 \wedge Im(\lambda_2) \neq 0 \end{cases}$
- τ-D condition
  $D > 0 \wedge \tau = 0$
- Neutrally stable
  (i.e. Lyapunov stable but not attracting)
- Note that $D > 0 \wedge \tau = 0 \Rightarrow \Delta < 0$

5) **Stable spiral:**



- Eigenvalue condition

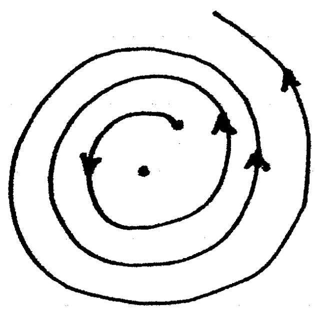
$$\begin{cases} \mathrm{Re}(\lambda_1) < 0 \wedge \mathrm{Re}(\lambda_2) < 0 \\ \mathrm{Im}(\lambda_1) \neq 0 \wedge \mathrm{Im}(\lambda_2) \neq 0 \end{cases}$$

- τ-D condition

$D > 0 \wedge \tau^2 - 4D < 0 \wedge \tau < 0$

- Exponentially stable

6) **Unstable spiral:**



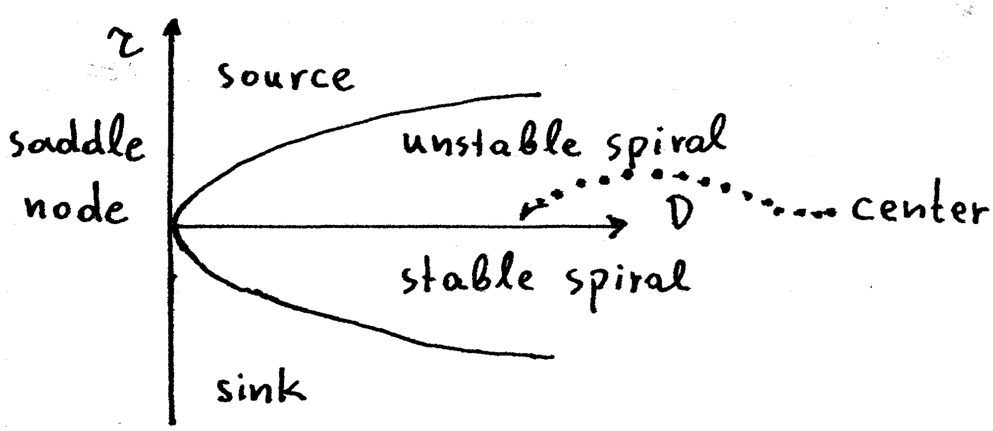
- Eigenvalue condition

$$\begin{cases} \mathrm{Re}(\lambda_1) > 0 \wedge \mathrm{Re}(\lambda_2) > 0 \\ \mathrm{Im}(\lambda_1) \neq 0 \wedge \mathrm{Im}(\lambda_2) \neq 0 \end{cases}$$

- τ-D condition

$D > 0 \wedge \tau^2 - 4D < 0 \wedge \tau > 0$

Summary of τ-D conditions



$D < 0$: saddle point

$D > 0$: $\tau = 0$: center

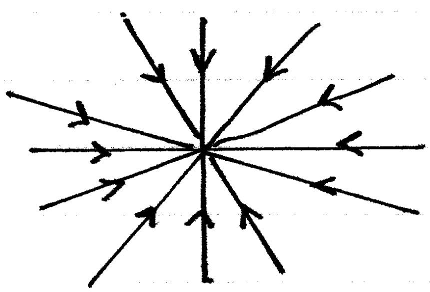
$\tau^2 - 4D > 0$: source ($\tau > 0$), sink ($\tau < 0$)

$\tau^2 - 4D < 0$: spiral, stable ($\tau < 0$) or unstable ($\tau > 0$)

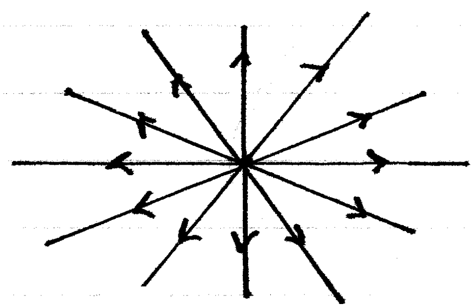
Borderline nodes

Borderline nodes occur when $\lambda_1 = \lambda_2$ which occurs when $\tau^2 - 4D = 0$. Let $E_\lambda = \{ v \in \mathbb{R}^2 \mid Av = \lambda v \}$ be the eigenspace associated with the eigenvalue $\lambda = \lambda_1 = \lambda_2$. We distinguish between two cases: $\dim E_\lambda = 1$ or $\dim E_\lambda = 2$.

7) Stars

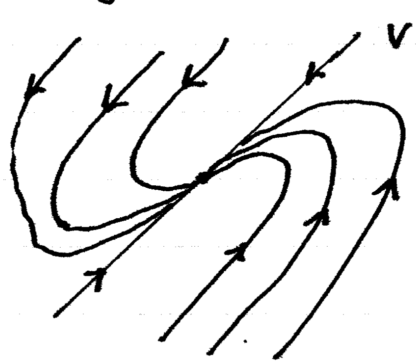


$\lambda_1 = \lambda_2 < 0$
$\dim E_\lambda = 2$

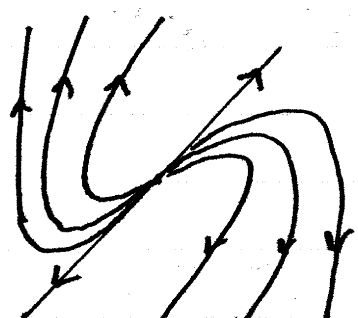


$\lambda_1 = \lambda_2 > 0$
$\dim E_\lambda = 2$

8) Degenerate nodes



$\lambda_1 = \lambda_2 < 0$
$\dim E_\lambda = 1$



$\lambda_1 = \lambda_2 > 0$
$\dim E_\lambda = 1.$

EXAMPLES

a) $\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = 4x_1 - 2x_2 \end{cases} \Leftrightarrow \dfrac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(-2-\lambda) - 4 = -2 - \lambda + 2\lambda + \lambda^2 - 4 =$$

$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0 \Leftrightarrow \underline{\lambda = -3 \vee \lambda = 2}.$$

Since $\begin{cases} \lambda_1, \lambda_2 \in \mathbb{R} \\ \lambda_1\lambda_2 < 0 \end{cases} \Rightarrow (0,0)$ is a saddle-node.

- To draw a phase portrait we need the eigenvectors. In general, for eigenvalue λ
  $Av = \lambda v \Leftrightarrow (A - \lambda I)v = 0 \Leftrightarrow \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $\lambda_1 = 2$:

$\begin{bmatrix} 1-2 & 1 \\ 4 & -2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} -x + y = 0 \\ 4x - 4y = 0 \end{cases} \Leftrightarrow y = x$

$\Leftrightarrow (x,y) = (x,x) = x(1,1)$

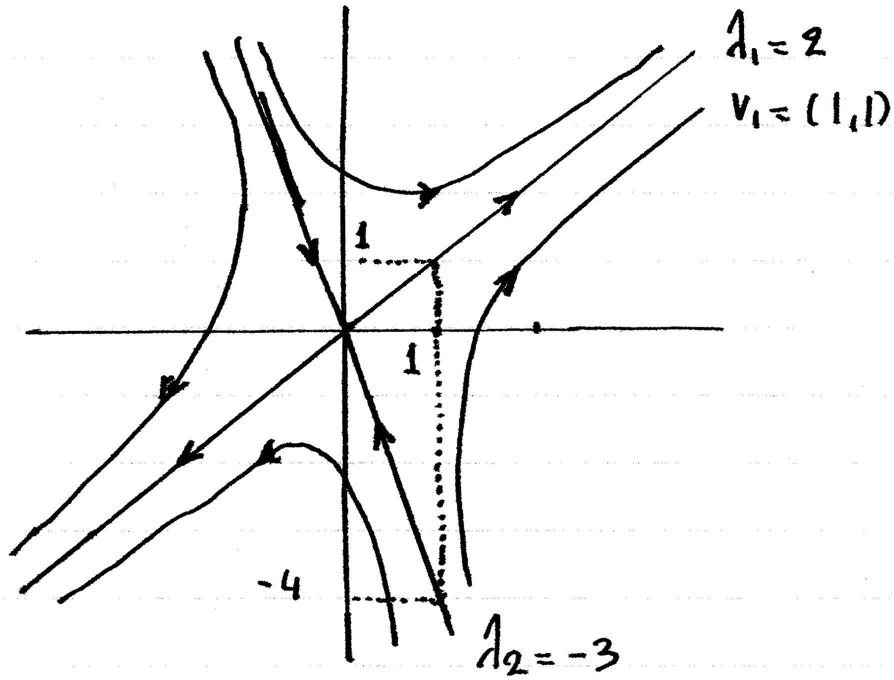
thus $v_1 = (1,1)$.

For $\lambda_2 = -3$:

$\begin{bmatrix} 1-(-3) & 1 \\ 4 & -2-(-3) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 4x + y = 0 \\ 4x + y = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow y = -4x \Leftrightarrow (x,y) = (x,-4x) = x(1,-4)$

thus $v_2 = (1,-4)$



"saddle node"

b) $\begin{cases} \dot{x}_1 = x_1 - 2x_2 \\ \dot{x}_2 = 2x_1 - x_2 \end{cases} \leftarrow A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)(-1-\lambda) + 4 = -1 - \lambda + \lambda + \lambda^2 + 4 =$$

$$= \lambda^2 + 3 = 0 \iff \lambda = \pm i\sqrt{3}$$

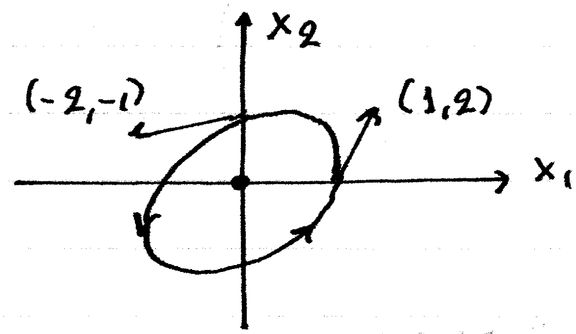
It follows that $(0,0)$ is a center.

- Clockwise or counterclockwise?

The direction of the orbits can be determined by calculating Ax with x a unit vector:

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$



When a linear system has a center, the shape of the orbits can be derived by noting that

$$V(x) = ax_1^2 + bx_1x_2 + cx_2^2$$

with appropriate choice of a, b, c remains constant along the orbits around $(0,0)$.

This $V(x)$ is a Lyapunov function.

For this problem:

$$V(x) = ax_1^2 + bx_1x_2 + cx_2^2 \Rightarrow$$

$$\Rightarrow dV(x)/dt = 2ax_1\dot{x}_1 + b(\dot{x}_1x_2 + x_1\dot{x}_2) + 2cx_2\dot{x}_2 =$$

$$= 2ax_1(x_1 - 2x_2) + b[(x_1 - 2x_2)x_2 + x_1(2x_1 - x_2)] + 2cx_2(2x_1 - x_2)$$

$$= 2ax_1^2 - 4ax_1x_2 + bx_1x_2 - 2bx_2^2 + 2bx_1^2 - bx_1x_2 + 4cx_1x_2 - 2cx_2^2$$

$$= (2a+2b)x_1^2 + (-4a + b - b + 4c)x_1x_2 - 2(b+c)x_2^2 =$$

$$= 2(a+b)x_1^2 + 4(c-a)x_1x_2 - 2(b+c)x_2^2$$

Require:

$$\begin{cases} a+b=0 \\ c-a=0 \\ b+c=0 \end{cases} \Leftrightarrow \begin{cases} c-c=0 \\ a=c \\ b=-c \end{cases} \Leftrightarrow \begin{cases} a=c \\ b=-c \end{cases} \Leftrightarrow (a,b,c) = c(1,-1,1)$$

Choose: $(a,b,c) = (1,-1,1)$, thus

$V(x) = x_1^2 - x_1x_2 + x_2^2$.

Center orbits have equation:

$$\boxed{(C):\ x_1^2 - x_1x_2 + x_2^2 = C_1}$$

c) $\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = 3x_1 \end{cases} \longleftarrow \quad A = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = -\lambda(-1-\lambda) - (-1)\cdot 3$$

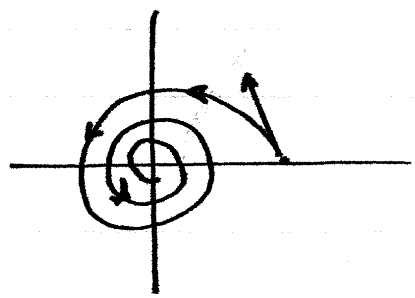
$$\left.\begin{aligned} &= \lambda + \lambda^2 + 3 = \lambda^2 + \lambda + 3 = 0 \\ &\Delta = 1 - 12 = -11 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{11}}{2}$$

Since $\lambda_{1,2}$ are complex and $Re(\lambda_1) = Re(\lambda_2) < 0$ it follows that $(0,0)$ is <u>stable spiral.</u>

Since

$$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

the direction is counterclockwise.



d) $\begin{cases} \dot{x}_1 = 4x_1 - x_2 \\ \dot{x}_2 = -x_1 + 4x_2 \end{cases} \longleftarrow \quad A = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1$$

$$\left.\begin{aligned} &= 16 - 8\lambda + \lambda^2 - 1 = \lambda^2 - 8\lambda + 15 \\ &\Delta = 64 - 4\cdot 15 = 64 - 60 = 4 \end{aligned}\right\} \Rightarrow \lambda_{1,2} = \frac{8 \pm 2}{2} = \begin{cases} 5 \\ 3 \end{cases}$$

thus $(0,0)$ is a source.

- Eigenvalues:

For $\lambda_1 = 3$:

$$Ax = 3x \Leftrightarrow \begin{cases} 4x_1 - x_2 = 3x_1 \\ -x_1 + 4x_2 = 3x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2 \Leftrightarrow (x_1, x_2) = (1,1)x_1$$

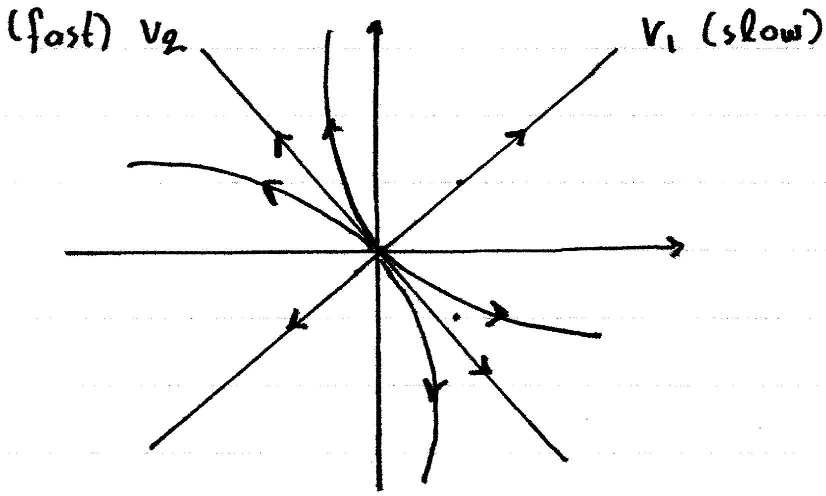
Choose $v_1 = (1,1)$.

For $\lambda_2 = 5$:

$$Ax = 5x \Leftrightarrow \begin{cases} 4x_1 - x_2 = 5x_1 \\ -x_1 + 4x_2 = 5x_2 \end{cases} \Leftrightarrow \begin{cases} -x_1 - x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x_1 = -x_2 \Leftrightarrow (x_1, x_2) = (-1,1)x_2$$

Choose $v_2 = (-1,1)$.



EXERCISES

⑧ Classify the fixed-point of the following systems and draw the phase-portrait.

a) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 3x_2 \end{cases}$

b) $\begin{cases} \dot{x}_1 = 5x_1 + 10x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = 3x_1 - 4x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$

d) $\begin{cases} \dot{x}_1 = -3x_1 + 2x_2 \\ \dot{x}_2 = x_1 - 2x_2 \end{cases}$

e) $\begin{cases} \dot{x}_1 = 5x_1 + 2x_2 \\ \dot{x}_2 = -17x_1 - 5x_2 \end{cases}$

f) $\begin{cases} \dot{x}_1 = -3x_1 + 4x_2 \\ \dot{x}_2 = -2x_1 + 3x_2 \end{cases}$

g) $\begin{cases} \dot{x}_1 = 4x_1 - 3x_2 \\ \dot{x}_2 = 8x_1 - 6x_2 \end{cases}$

h) $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 2x_2 \end{cases}$

⑨ Consider the system

$$\begin{cases} \dot{x}_1 = a x_1 + b x_2 \\ \dot{x}_2 = a x_2 \end{cases}$$

with $a \neq 0$ and $b \neq 0$. Show that $(0,0)$ is a degenerate node.

MM4: Nonlinear autonomous systems

NONLINEAR AUTONOMOUS SYSTEMS

▼ Local analysis of fixed points

Consider the nonlinear autonomous systems
$\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.
Let $x_0 \in \mathbb{R}^n$ be a fixed point with $f(x_0) = 0$.
Let $x_0(t) = x_0$ be a solution with the fixed point as initial condition.

To examine the stability of x_0, we consider the following perturbation around x_0:

$$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$$

with $0 < \varepsilon \ll 1$. It follows that:

$$\dot{x}(t) = \dot{x}_0 + \varepsilon \dot{x}_1(t) + O(\varepsilon^2) = \varepsilon \dot{x}_1(t) + O(\varepsilon^2)$$

$$f(x) = f(x_0 + \varepsilon x_1) = f(x_0) + (\varepsilon\ ) Df(x_0) x_1 + O(\varepsilon^2)$$

$$= \varepsilon Df(x_0) x_1 + O(\varepsilon^2)$$

Equating the ε terms gives the linearization

$$\boxed{\dot{x}_1(t) = Df(x_0) x_1(t)}$$

Here Df is the Jacobian matrix given by

$$\boxed{[Df]_{ab} = \frac{\partial f_a}{\partial x_b}}$$

<u>Def</u> : We say that the fixed point x_0 is a <u>hyperbolic point</u> if and only if

$$\boxed{\forall \lambda \in \lambda(Df(x_0)) : Re(\lambda) \neq 0}$$

- It can be shown that if x_0 is a hyperbolic fixed-point, then the local behavior of the nonlinear systems is topologically equivalent to the local behaviour of the linearized equation $\dot{x} = Df(x_0)\, x$.
- It follows that hyperbolic fixed-points can be classified according to the eigenvalues of the Jacobian matrix $Df(x_0)$.

<u>EXAMPLE</u>

$$\begin{cases} \dot{x}_1 = x_1(3 - x_1 - x_2) \\ \dot{x}_2 = x_2(x_1 - 1) \end{cases}$$

- <u>Fixed points:</u>

$$\begin{cases} x_1(3-x_1-x_2) = 0 \\ x_2(x_1-1) = 0 \end{cases} \Leftrightarrow \begin{cases} x_1(3-x_1) = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} 1\cdot(3-1-x_2) = 0 \\ x_1 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_1 = 3 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_2 = 2 \\ x_1 = 1 \end{cases}$$

thus set of fixed points: $\{(0,0), (3,0), (1,2)\}$.

- Jacobian

$$\frac{\partial f_1}{\partial x_1} = 1\cdot(3-x_1-x_2)+x_1(-1) = 3-2x_1-x_2$$

$$\frac{\partial f_1}{\partial x_2} = -x_1$$

$$\frac{\partial f_2}{\partial x_1} = x_2$$

$$\frac{\partial f_2}{\partial x_2} = x_1-1$$

thus $Df(x_1,x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$

$$= \begin{bmatrix} 3-2x_1-x_2 & -x_1 \\ x_2 & x_1-1 \end{bmatrix}$$

- At $(0,0)$

$$Df(0,0) = \begin{bmatrix} 3-0-0 & 0 \\ 0 & 0-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 3$ with $v_1 = (1,0)$

$\lambda_2 = -1$ with $v_2 = (0,1)$

thus $(0,0)$ is a saddle point.

- At $(1,2)$

$$Df(1,2) = \begin{bmatrix} 3-2\cdot1-2 & -1 \\ 2 & 1-1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$p(\lambda)=\det(Df(1,2)-\lambda I)=\begin{vmatrix}-1-\lambda & -1\\ 2 & -\lambda\end{vmatrix}=$$

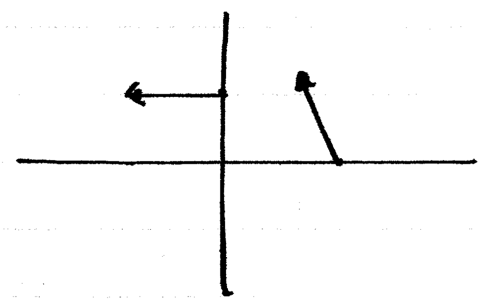
$$=(-1-\lambda)(-\lambda)-(-1)\cdot 2=\lambda(\lambda+1)+2=$$

$$\left.\begin{array}{l}=\lambda^2+\lambda+2\\ \Delta=1^2-4\cdot 1\cdot 2=-7\end{array}\right\}\Rightarrow \lambda_{1,2}=\frac{-1\pm i\sqrt{7}}{2}$$

Note that

$$\begin{bmatrix}-1 & -1\\ 2 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}-1\\ 2\end{bmatrix}$$

$$\begin{bmatrix}-1 & -1\\ 2 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}-1\\ 0\end{bmatrix}$$



Thus $(1,2)$ is a counterclockwise stable spiral.

- **At $(3,0)$**

$$Df(3,0)=\begin{bmatrix}3-2\cdot 3-0 & -3\\ 0 & 3-1\end{bmatrix}=\begin{bmatrix}-3 & -3\\ 0 & 2\end{bmatrix}$$

$$p(\lambda)=\det(Df(3,0)-\lambda I)=\begin{vmatrix}-3-\lambda & -3\\ 0 & 2-\lambda\end{vmatrix}=$$

$$=(-3-\lambda)(2-\lambda)-(-3)\cdot 0=(\lambda+3)(\lambda-2)=0\Leftrightarrow$$

$\Leftrightarrow \lambda_1=-3$ or $\lambda_2=2$. $\leftarrow$ $(3,0)$ is a saddle point.

Eigenvectors:

a) For $\underline{\lambda_1=-3}$:

$$Ax=-3x\Leftrightarrow\begin{cases}-3x_1-3x_2=-3x_1\\ 2x_2=-3x_2\end{cases}\Leftrightarrow\begin{cases}-3x_2=0\\ 5x_2=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow x_2=0\Leftrightarrow (x_1,x_2)=(1,0)\,x_1\leftarrow \underline{v_1=(1,0)}$$

b) For $\underline{\lambda_2 = 2}$:

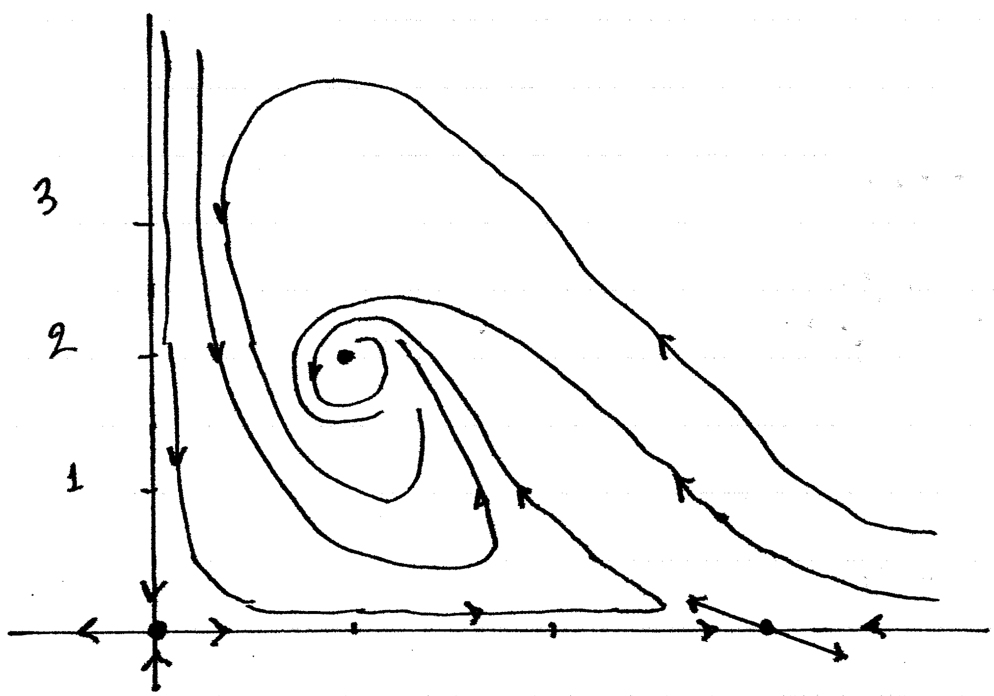
$$Ax = 2x \Leftrightarrow \begin{cases} -3x_1 - 3x_2 = 2x_1 \\ 2x_2 = 2x_2 \end{cases} \Leftrightarrow -5x_1 - 3x_2 = 0 \Leftrightarrow$$

$$\Leftrightarrow x_2 = \frac{-5}{3} x_1 \Leftrightarrow (x_1, x_2) = (1, \frac{-5}{3}) x_1$$

thus choose $v_2 = (3, -5)$.

• <u>Phase Portrait</u>

(0,0) saddle point with $\lambda_1 = 3$, $v_1 = (1,0)$, $\lambda_2 = -1$, $v_2 = (0,1)$
(1,2) counterclockwise stable spiral
(3,0) saddle point with $\lambda_1 = -3$, $v_1 = (1,0)$, $\lambda_2 = 2$, $v_2 = (3,-5)$



EXERCISES

(1) Classify the fixed points for the following systems and attempt to draw the phase portrait.

a) $\begin{cases} \dot{x}_1 = x_1 - x_2 \\ \dot{x}_2 = x_1^2 - 4 \end{cases}$

b) $\begin{cases} \dot{x}_1 = 1 + x_2 - e^{-x_1} \\ \dot{x}_2 = x_1^3 - x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = x_2 + x_1 - x_1^3 \\ \dot{x}_2 = -x_2 \end{cases}$

d) $\begin{cases} \dot{x}_1 = x_1 x_2 - 1 \\ \dot{x}_2 = x_1 - x_2^3 \end{cases}$

e) $\begin{cases} \dot{x}_1 = x_1 (3 - 2x_1 - 2x_2) \\ \dot{x}_2 = x_2 (2 - x_1 - x_2) \end{cases}$

f) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = x_1 - x_1^3 \end{cases}$

g) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = \cos x_1 \end{cases}$

Nonlinear Centers

- Fixed points which, according to local linear analysis, appear to be centers are NOT hyperbolic. It follows that the original nonlinear system may or may not be a center. To determine whether a fixed point with
$$\exists \lambda \in \lambda(Df(x_0)) : Re(\lambda) = 0$$
is or is not a center, we rely on the following methods:

① → Conversion to polar coordinates

A two-dimensional autonomous system of the form
$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$
can be rewritten in polar coordinates (r, ϑ) with $x_1 = r\cos\vartheta$ and $x_2 = r\sin\vartheta$ using the following identities:

$$\boxed{\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}} \quad \boxed{\dot{\vartheta} = \frac{x_1\dot{x}_2 - \dot{x}_1 x_2}{r^2}}$$

Proof

$$x_1^2+x_2^2 = r^2\cos^2\vartheta + r^2\sin^2\vartheta = r^2(\cos^2\vartheta+\sin^2\vartheta) = r^2 \Rightarrow$$
$$\Rightarrow 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2r\dot{r} \Rightarrow \dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$$

Since $\begin{cases} x_1 = r\cos\vartheta \\ x_2 = r\sin\vartheta \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta \\ \dot{x}_2 = \dot{r}\sin\vartheta + r\dot{\vartheta}\cos\vartheta \end{cases} \Rightarrow$

$$\Rightarrow x_1\dot{x}_2 - \dot{x}_1x_2 = (r\cos\vartheta)(\dot{r}\sin\vartheta + r\dot{\vartheta}\cos\vartheta) - (\dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta)(r\sin\vartheta)$$
$$= r\dot{r}\cos\vartheta\sin\vartheta + r^2\dot{\vartheta}\cos^2\vartheta - r\dot{r}\cos\vartheta\sin\vartheta + r^2\dot{\vartheta}\sin^2\vartheta =$$
$$= r^2\dot{\vartheta}\cos^2\vartheta + r^2\dot{\vartheta}\sin^2\vartheta = r^2\dot{\vartheta}(\cos^2\vartheta+\sin^2\vartheta) = r^2\dot{\vartheta} \Rightarrow$$
$$\Rightarrow \dot{\vartheta} = \frac{x_1\dot{x}_2 - \dot{x}_1x_2}{r^2}$$

EXAMPLE

$$\begin{cases} \dot{x}_1 = -x_2 + ax_1(x_1^2+x_2^2) = f_1(x_1,x_2) \\ \dot{x}_2 = x_1 + ax_2(x_1^2+x_2^2) = f_2(x_1,x_2) \end{cases}$$

Solution

Obvious fixed point at $(x_1,x_2) = (0,0)$

Jacobian

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 3ax_1^2 + ax_2^2 \\ \frac{\partial f_1}{\partial x_2} &= -1 + 2ax_1x_2 \\ \frac{\partial f_2}{\partial x_1} &= 1 + 2ax_1x_2 \\ \frac{\partial f_2}{\partial x_2} &= ax_1^2 + 3ax_2^2 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow Df(x_1,x_2) = \begin{bmatrix} 3ax_1^2 + ax_2^2 & -1 + 2ax_1x_2 \\ 1 + 2ax_1x_2 & ax_1^2 + 3ax_2^2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow Df(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - (-1)\cdot 1 = \lambda^2 + 1 \Rightarrow$$

$$\Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \leftarrow \text{ a center ?}$$

• Convert to polar coordinates:

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2 =$$

$$= x_1[-x_2 + ax_1(x_1^2+x_2^2)] + x_2[x_1 + ax_2(x_1^2+x_2^2)]$$

$$= -x_1x_2 + ax_1^2r^2 + x_1x_2 + ax_2^2r^2 =$$

$$= ax_1^2r^2 + ax_2^2r^2 = ar^2(x_1^2+x_2^2) = ar^4 \Rightarrow$$

$$\Rightarrow \underline{\dot{r} = ar^3}, \text{ and}$$

$$r^2\dot{\theta} = x_1\dot{x}_2 - \dot{x}_1x_2 =$$

$$= x_1[x_1 + ax_2(x_1^2+x_2^2)] - [-x_2 + ax_1(x_1^2+x_2^2)]x_2 =$$

$$= x_1^2 + ax_1x_2r^2 + x_2^2 - ax_1x_2r^2 =$$

$$= x_1^2 + x_2^2 = r^2 \Rightarrow \underline{\dot{\theta} = 1}$$

Thus $\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases}$.

For $a = 0$: $\dot{r} = 0$ and $\dot{\theta} = 1 \Rightarrow (0,0)$ is a center.

For $a > 0$: $\dot{r} > 0$ and $\dot{\theta} = 1 \Rightarrow$

$\Rightarrow (0,0)$ is unstable counterclockwise spiral.

For $a<0$: $\dot{r}<0$ and $\dot{\vartheta}=1 \Rightarrow$
$\Rightarrow (0,0)$ is a counterclockwise stable spiral.

② → Conservative systems

Consider a general autonomous system $\dot{x}=f(x)$ with $x\in\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^n$.

Def: We say that the system is conservative if and only if

a) There is a function $V:\mathbb{R}^n\to\mathbb{R}$ such that $(d/dt)V(x(t))=0$

b) $\forall A\subset\mathbb{R}^n: (A \text{ open set} \Rightarrow V \text{ non-constant in } A)$.

Def: $x_0\in\mathbb{R}^n$ is an isolated fixed-point if and only if

a) $f(x_0)=0$

b) $\exists\varepsilon>0:\forall x\in\mathbb{R}^n: (0<\|x-x_0\|<\varepsilon \Rightarrow f(x)\neq 0)$.

Def: $x(t)$ is a closed orbit if and only if $\forall t>0:\exists\tau>0: x(t+\tau)=x(t)$.

Thm: Assume that

a) $x_0\in\mathbb{R}^n$ is an isolated fixed-point

b) f is continuously differentiable in $\mathbb{R}^n$

c) the system is conservative with $(d/dt)V(x(t))=0$

d) x_0 is a local min or max of $V(x)$.

Then,

$\exists \varepsilon > 0 : (\|x(0) - x_0\| < \varepsilon \Rightarrow x(t)$ is a closed orbit$)$.

Prop : If $\dot{x} = f(x)$ is conservative then it has no attracting fixed points.

Proof

Let $x_0 \in \mathbb{R}^n$ be an attracting fixed point. Let A be the basin of attraction of x_0 such that

$$\forall y \in A : \left(x(0) = y \Rightarrow \lim_{t \to +\infty} x(t) = x_0\right).$$

Let $y \in A$ be given and choose $x(0) = y$. Then

$V(y) = V(x(0)) = V(x(t))$, $\forall t > 0 \Rightarrow$

$$\Rightarrow V(y) = \lim_{t \to +\infty} V(x(t)) = V\left(\lim_{t \to +\infty} x(t)\right) = V(x_0), \forall y \in A$$

$\Rightarrow V$ constant in $A \leftarrow$ contradiction.

Thus x_0 cannot be an attracting fixed point. $\square$

$\hookrightarrow$ Thus to show that a system is NOT conservative it is sufficient to show that it has an attracting fixed point.

- Constructing $V(x)$ is easy for systems of the following forms:

Form 1 : $\boxed{\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1)\end{cases}} \longleftarrow \boxed{\ddot{x} = f(x)}$

Consider:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1\dot{x}_1 + x_2 f(x_1) = x_1\dot{x}_1 + \dot{x}_1 f(x_1) =$$
$$= \dot{x}_1(x_1 + f(x_1)) \Rightarrow$$
$$\Rightarrow -f(x_1)\dot{x}_1 + x_2\dot{x}_2 = 0 \longleftarrow$$ easily integrated to yield $V(x)$.

EXAMPLE

$$\begin{cases}\dot{x}_1 = -x_2 - x_2^3 \\ \dot{x}_2 = x_1\end{cases}$$

- Fixed points

$$\begin{cases}-x_2 - x_2^3 = 0 \\ x_1 = 0\end{cases} \Leftrightarrow \begin{cases}x_2(1+x_2^2) = 0 \\ x_1 = 0\end{cases} \Leftrightarrow$$
$$\Leftrightarrow \begin{cases}x_2 = 0 \\ x_1 = 0\end{cases} \vee \begin{cases}1 + x_2^2 = 0 \\ x_1 = 0\end{cases} \Leftrightarrow \begin{cases}x_1 = 0 \\ x_2 = 0\end{cases}$$
$$\Leftrightarrow (x_1, x_2) = (0, 0)$$

- Local linear analysis

$$Df(x_1, x_2) = \begin{bmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2\end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -1-3x_2^2 \\ 1 & 0 \end{bmatrix} \Rightarrow Df(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= (-\lambda)(-\lambda) - (-1)\cdot 1 = \lambda^2 + 1 \Rightarrow$$

$\Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \Rightarrow (0,0)$ is a <u>linear center.</u>

- However we still have to prove that it is a nonlinear center.

- <u>Nonlinear center.</u>

Let:

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_2 - x_2^3) + x_2 x_1 =$$

$$= -x_1x_2 - x_1x_2^3 + x_1x_2 = -x_1x_2^3 =$$

$$= -x_2^3\dot{x}_2 \Rightarrow$$

$$\Rightarrow x_1\dot{x}_1 + (x_2 + x_2^3)\dot{x}_2 = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left[\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_2^4}{4}\right] = 0$$

$$\Rightarrow \underline{2x_1^2 + 2x_2^2 + x_2^4 = c} \qquad (1)$$

For $V(x_1, x_2) = 2x_1^2 + 2x_2^2 + x_2^4$ we have $V(0,0) = 0$ and $V(x_1, x_2) > 0$, $\forall (x_1, x_2) \in \mathbb{R}^2 - \{(0,0)\}$, thus $(0,0)$ is a local minimum. It follows that $(0,0)$ is a nonlinear center. The closed trajectories are given by (1).

Form 2 : $\boxed{\begin{aligned}\dot{x}_1 &= f(x_1)\,g_1(x_2)\\ \dot{x}_2 &= f(x_2)\,g_2(x_1)\end{aligned}}$.

Consider:

$$\frac{dV}{dt} = \frac{g_2(x_1)}{f(x_1)}\dot{x}_1 - \frac{g_1(x_2)}{f(x_2)}\dot{x}_2 =$$

$$= \frac{g_2(x_1)}{f(x_1)} f(x_1) g_1(x_2) - \frac{g_1(x_2)}{f(x_2)} f(x_2) g_2(x_1) =$$

$$= g_2(x_1) g_1(x_2) - g_1(x_2) g_2(x_1) = 0$$

which can then be easily integrated to yield the Lyapunov function.

EXAMPLE

$$\begin{cases}\dot{x}_1 = x_1 - x_1 x_2\\ \dot{x}_2 = -x_2 + x_1 x_2\end{cases}$$

- Fixed points

$$\begin{cases} x_1 - x_1 x_2 = 0\\ -x_2 + x_1 x_2 = 0\end{cases} \Leftrightarrow \begin{cases} x_1(1-x_2) = 0\\ x_2(x_1 - 1) = 0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 0\\ x_2(0-1) = 0\end{cases} \vee \begin{cases} x_2 = 1\\ 1\cdot(x_1 - 1) = 0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = 0\\ x_2 = 0\end{cases} \vee \begin{cases} x_1 = 1\\ x_2 = 1\end{cases} \leftarrow \text{Fixed points: } \{(0,0),(1,1)\}.$$

- <u>Local linear analysis</u>

$$Df(x_1,x_2) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} =$$
$$= \begin{bmatrix} 1-x_2 & -x_1 \\ x_2 & x_1-1 \end{bmatrix}$$

- At $(0,0)$:

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(Df(0,0)) = \{+1,-1\} \Rightarrow$$

$\Rightarrow (0,0)$ is a saddle point.

- At $(1,1)$

$$Df(1,1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(1,1) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} =$$
$$= (-\lambda)(-\lambda) - 1\cdot(-1) = \lambda^2+1 \Rightarrow$$

$\Rightarrow \lambda(Df(1,1)) = \{+i, -i\} \Rightarrow (1,1)$ is a linear center.

- Nonlinear center.

We now show that $(1,1)$ is a nonlinear center.

Construct a Lyapunov function:

Note that:

$$\dot{x}_1 = x_1(1-x_2)$$
$$\dot{x}_2 = x_2(x_1-1)$$

so we define:

$$\frac{dV}{dt} = \frac{x_1 - 1}{x_1}\dot{x}_1 - \frac{1-x_2}{x_2}\dot{x}_2 =$$

$$= \frac{x_1-1}{x_1} x_1(1-x_2) - \frac{1-x_2}{x_2} x_2(x_1-1) =$$

$$= (x_1-1)(1-x_2) - (1-x_2)(x_1-1) = 0 \Rightarrow$$

$$\Rightarrow \left(1-\frac{1}{x_1}\right)\dot{x}_1 + \left(1-\frac{1}{x_2}\right)\dot{x}_2 = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left[x_1 - \ln x_1 + x_2 - \ln x_2\right] = 0 \Rightarrow$$

$$\Rightarrow \underline{x_1 + x_2 - \ln(x_1 x_2) = c} \leftarrow \text{shape of trajectories.}$$

We now show that $(1,1)$ is an extremum by calculating the Hessian:
Let $f(x_1, x_2) = x_1 + x_2 - \ln(x_1 x_2)$. Then

$$\nabla f(x_1, x_2) = (\partial f/\partial x_1, \partial f/\partial x_2) =$$
$$= (1 - 1/x_1, 1 - 1/x_2) \Rightarrow$$

$$\Rightarrow \nabla f(1,1) = (1-1, 1-1) = (0,0).$$

Since:

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(1-\frac{1}{x_1}\right) = \frac{1}{x_1^2}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(1-\frac{1}{x_2}\right) = \frac{1}{x_2^2}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(1-\frac{1}{x_2}\right) = 0$$

the Hessian reads:

$$\Delta(x_1,x_2) = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 =$$

$$= \frac{1}{x_1^2}\frac{1}{x_2^2} - 0^2 = \left(\frac{1}{x_1 x_2}\right)^2 \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \Delta(1,1) = 1 > 0 \\ \dfrac{\partial^2 f(1,1)}{\partial x_1^2} = \dfrac{1}{1^2} = 1 > 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ $(1,1)$ is a local min of
$f(x_1,x_2) = x_1 + x_2 - \ln(x_1 x_2)$

It follows that $(1,1)$ is a nonlinear center.

$\hookrightarrow$ Recall that for

$$\Delta = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2$$

we have the following sufficient conditions:

a) $\left.\begin{array}{l} \Delta(x_0) > 0 \\ \partial^2 f(x_0)/\partial x_1^2 > 0 \end{array}\right\} \Rightarrow x_0 \in \mathbb{R}^2$ is a local min

b) $\left.\begin{array}{l} \Delta(x_0) > 0 \\ \partial^2 f(x_0)/\partial x_1^2 < 0 \end{array}\right\} \Rightarrow x_0 \in \mathbb{R}^2$ is a local max.

EXERCISES

(2) Show that the following systems are conservative, locate and classify all fixed points. Draw a phase portrait.

a) $\ddot{x} = x^3 - x$

b) $\ddot{x} = x - x^2$

c) $\ddot{x} = 1 - e^x$

d) $\ddot{x} = (x-a)(x^2-a)$

(3) Similarly for the following systems:

a) $\begin{cases} \dot{x} = -kxy \\ \dot{y} = kxy - \ell y \end{cases}$
with $k>0$, $\ell>0$

b) $\dot{x} = x - xy$
$\dot{y} = \mu xy - \mu y$

(4) Consider the system

$$\begin{cases} \dot{x} = xy \\ \dot{y} = -x^2 \end{cases}$$

a) Show that $V(x,y) = x^2 + y^2$ is conserved.

b) Show that $(x,y) = (0,0)$ is a fixed point but not an isolated fixed point.

c) Show that V has a minimum at $(0,0)$ but $(0,0)$ is not a nonlinear center.

3 → Reversible systems

- Consider the system $\dot{x} = f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$

Def: We say that a mapping $P : \mathbb{R}^n \to \mathbb{R}^n$ is an involution if and only if
$\forall x \in \mathbb{R}^n : P(P(x)) = x$.

Def: We say that the system $\dot{x} = f(x)$ is reversible if and only if there is an involution P such that

$$\frac{d}{dt} P(x) = -f(P(x))$$

- A reversible system is invariant under the transformation

$$t \to -t$$
$$x \to P(x)$$

- We define the symmetry section of the involution P as:

$$\mathrm{Fix}(P) = \{x \in \mathbb{R}^n \mid P(x) = x\}$$

Thm: Assume that the system $\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$ is reversible under the involution P. Then, if

$$\left.\begin{array}{l} x_0 \in \mathrm{Fix}(P) \\ x_0 \text{ linear center} \end{array}\right\} \Rightarrow x_0 \text{ nonlinear center.}$$

We confine our attention to the two-dimensional system

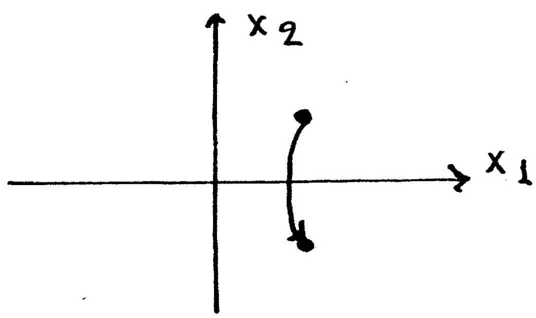
$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases} \quad (1)$$

▶ <u>Reflection around x-axis</u>

Assume that

$$f(x_1, -x_2) = -f(x_1, +x_2), \quad \forall x_1, x_2 \in \mathbb{R}$$
$$g(x_1, -x_2) = g(x_1, x_2), \quad \forall x_1, x_2 \in \mathbb{R}$$

Then, the system (1) is reversible under the involution $(x_1, -x_2) = P(x_1, x_2)$



We note that the symmetry section is $\mathrm{Fix}(P) = \{(x,0) \mid x \in \mathbb{R}\}$.

<u>Proof</u>

Let $x = (x_1, x_2)$ and $F(x) = (f(x_1, x_2), g(x_1, x_2))$.
Then:

$$\frac{d}{dt} P(x) = \frac{d}{dt}(x_1, -x_2) = (\dot{x}_1, -\dot{x}_2) =$$
$$= (f(x_1, x_2), -g(x_1, x_2)) =$$
$$= (-f(x_1, -x_2), -g(x_1, -x_2)) =$$
$$= -(f(x_1, -x_2), g(x_1, -x_2)) = -F(P(x)) \quad \square$$

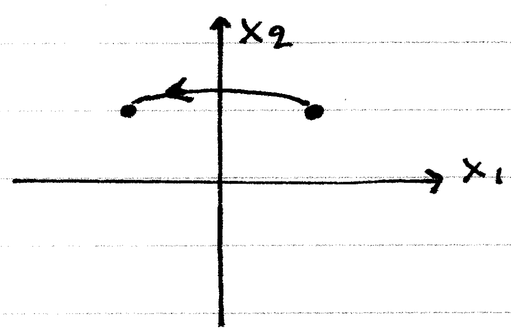
▶ Reflection around y-axis

Assume that

$$f(-x_1,x_2)=f(x_1,x_2) \quad , \forall x_1,x_2 \in \mathbb{R}$$
$$g(-x_1,x_2)=-g(x_1,x_2), \forall x_1,x_2 \in \mathbb{R}$$

then (1) is reversible under the involution

$$P(x_1,x_2)=(-x_1,x_2)$$



We note that the symmetry section is

$$\text{Fix}(P)=\{(0,y) \mid y\in\mathbb{R}\}$$

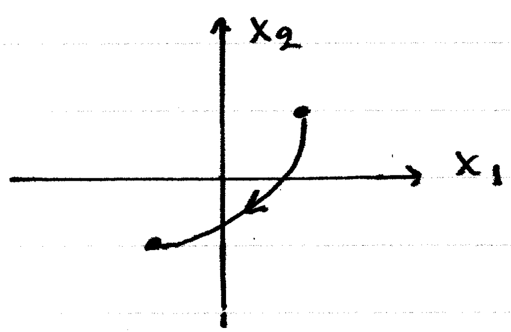
▶ Reflection around x-axis and y-axis

Assume that

$$f(-x_1,-x_2)=f(x_1,x_2) \quad , \forall x_1,x_2 \in \mathbb{R}$$
$$g(-x_1,-x_2)=g(x_1,x_2) \quad , \forall x_1,x_2 \in \mathbb{R}$$

then (1) is reversible under the involution

$$P(x_1,x_2)=(-x_1,-x_2)$$



We note that the symmetry section is:

$$\text{Fix}(P)=\{(0,0)\}.$$

EXAMPLE

$$\begin{cases} \dot{x}_1 = x_2 - x_2^3 \\ \dot{x}_2 = -x_1 - x_2^2 \end{cases} \longleftarrow \text{Classification of fixed points.}$$

Proof

Let $f(x_1,x_2) = x_2 - x_2^3$ and $g(x_1,x_2) = -x_1 - x_2^2$.

- Fixed points:

$$\begin{cases} f(x_1,x_2) = 0 \\ g(x_1,x_2) = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 - x_2^3 = 0 \\ -x_1 - x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2(1-x_2)(1+x_2) = 0 \\ x_1 = -x_2^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \vee \begin{cases} x_1 = -1 \\ x_2 = 1 \end{cases} \vee \begin{cases} x_1 = -1 \\ x_2 = -1 \end{cases}$$

- Jacobian

$$Df(x_1,x_2) = \begin{bmatrix} 0 & 1-3x_2^2 \\ -1 & -2x_2 \end{bmatrix}$$

- At $(x_1,x_2) = (0,0)$

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(0,0) - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - (-1) =$$

$$= \lambda^2 + 1 \Rightarrow \lambda(Df(0,0)) = \{+i, -i\} \Rightarrow$$

$\Rightarrow (0,0)$ is a linear center.

Since $f(x_1,-x_2) = (-x_2) - (-x_2)^3 = -(x_2 - x_2^3) =$

$$= -f(x_1,x_2)$$

and

$g(x_1,-x_2) = -x_1-(-x_2)^2 = -x_1 - x_2^2 = g(x_1,x_2)$

it follows that the system is reversible. Thus, since

$$\left.\begin{array}{l}(0,0) \text{ is a linear center} \\ (0,0)\in \mathrm{Fix}(P) = \{(x,0) \mid x\in\mathbb{R}\}\end{array}\right\} \Rightarrow (0,0) \text{ is a nonlinear center.}$$

- At $(x_1,x_2) = (-1,1)$

$$Df(-1,1) = \begin{bmatrix} 0 & 1-3\cdot 1^2 \\ -1 & -2\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(-1,1)-\lambda I) = \begin{vmatrix} -\lambda & -2 \\ -1 & -\lambda-2 \end{vmatrix} =$$

$$= (-\lambda)(-\lambda-2)-(-1)(-2) = \lambda(\lambda+2)-2 = \lambda^2+2\lambda-2.$$

$$\Delta = 2^2-4\cdot 1\cdot(-2) = 4+8 = 12 = 4\cdot 3 \Rightarrow \lambda_{1,2} = \frac{-2\pm 2\sqrt{3}}{2} = -1\pm\sqrt{3}$$

$$\Rightarrow \lambda(Df(-1,1)) = \{-1-\sqrt{3}, -1+\sqrt{3}\} \Rightarrow (-1,1) \text{ is a saddle point.}$$

- At $(x_1,x_2) = (-1,-1)$

$$Df(-1,-1) = \begin{bmatrix} 0 & 1-3(-1)^2 \\ -1 & -2(-1) \end{bmatrix} = \begin{bmatrix} 0 & 1-3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(-1,-1)-\lambda I) = \begin{vmatrix} -\lambda & -2 \\ -1 & -\lambda+2 \end{vmatrix} = (-\lambda)(-\lambda+2)-(-1)(-2)$$

$$= \lambda^2-2\lambda-2$$

$$\Delta = (-2)^2-4\cdot 1\cdot(-2) = 4+8 = 12 = 4\cdot 3 \Rightarrow \lambda_{1,2} = \frac{-(-2)\pm 2\sqrt{3}}{2} = 1\pm\sqrt{3}$$

$$\Rightarrow \lambda(Df(-1,-1)) = \{1+\sqrt{3}, 1-\sqrt{3}\} \Rightarrow$$

$\Rightarrow (-1,-1)$ is a saddle point.

EXAMPLE

$$\begin{cases} \dot{x}_1 = 2\cos x_1 + \cos x_2 \\ \dot{x}_2 = 2\cos x_2 + \cos x_1 \end{cases}$$ ← Show that system is reversible but not conservative.

- **Reversability.**

Let $f(x_1, x_2) = 2\cos x_1 + \cos x_2$ and

$g(x_1, x_2) = 2\cos x_2 + \cos x_1$.

Since:

$$f(-x_1, -x_2) = 2\cos(-x_1) + \cos(-x_2) =$$
$$= 2\cos x_1 + \cos x_2 = f(x_1, x_2)$$

and

$$g(-x_1, -x_2) = 2\cos(-x_2) + \cos(-x_1) =$$
$$= 2\cos x_2 + \cos x_1 = g(x_1, x_2)$$

thus the system is reversible with respect to the involution $P(x_1, x_2) = (-x_1, -x_2)$

- **Not conservative**

It is sufficient to show that the system has an attracting fixed point.

We first find the fixed points of the system:

$$\begin{cases} 2\cos x_1 + \cos x_2 = 0 \\ 2\cos x_2 + \cos x_1 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \cos x_1 \\ \cos x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cos x_1 = 0 \\ \cos x_2 = 0 \end{cases} \Leftrightarrow \exists \kappa, \lambda \in \mathbb{Z} : \begin{cases} x_1 = \kappa\pi + \pi/2 \\ x_2 = \lambda\pi + \pi/2 \end{cases}$$

The Jacobian of the system reads:

$$Df(x_1, x_2) = \begin{bmatrix} -2\sin x_1 & -\sin x_2 \\ -\sin x_1 & -2\sin x_2 \end{bmatrix}$$

At $(x_1, x_2) = (\pi/2, \pi/2)$:

$$Df(\pi/2, \pi/2) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(Df(\pi/2, \pi/2) - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{vmatrix}$$

$$= (-2-\lambda)^2 - (-1)^2 = (\lambda+2)^2 - 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow (\lambda+2)^2 = 1 \Leftrightarrow \lambda + 2 = \pm 1 \Leftrightarrow \lambda = -2 \pm 1 = \begin{cases} -3 \\ -1 \end{cases}$$

thus $\lambda(Df(\pi/2, \pi/2)) = \{-3, -1\} \Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is a sink $\Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is asymptotically stable $\Rightarrow$

$\Rightarrow (\pi/2, \pi/2)$ is attracting $\Rightarrow$

$\Rightarrow$ the system is not conservative.

EXERCISES

(5) Show that the following systems are reversible and then find and classify all fixed points.

a) $\begin{cases} \dot{x}_1 = x_2(1-x_1^2) \\ \dot{x}_2 = 1-x_2^2 \end{cases}$

b) $\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \cos x_2 \end{cases}$

c) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = \sin x_1 \end{cases}$

d) $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = x_2^2 - x_1 \end{cases}$

e) $\begin{cases} \dot{x}_1 = x_2 - x_2^3 \\ \dot{x}_2 = x_1 \cos x_2 \end{cases}$

f) $\ddot{x} + (\dot{x})^2 + x = 3$

g) $\ddot{x} + x\dot{x} + x = 0$

Index theory

Index theory is a global method that provides global information about the phase portrait of a two-dimensional autonomous system.

Definition of the index

Consider the two-dimensional autonomous system

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$

We note that at (x_1, x_2), the angle φ of the vector $(\dot{x}_1, \dot{x}_2)$ is given by

$$\varphi(x_1, x_2) = \mathrm{Arctan}\left(\frac{g(x_1, x_2)}{f(x_1, x_2)} \right)$$

Let C be a simple closed curve. We define the index $I(C)$ of C as:

$$\boxed{I(C) = \oint_C \frac{d\varphi(x_1, x_2)}{2\pi}}$$

Explicit form of the index integral

We note that:

$$d\varphi = d\left(\mathrm{Arctan}\,(g/f)\right) = \frac{1}{1+(g/f)^2}\, d\left(\frac{g}{f}\right) =$$

$$= \frac{1}{1+(g/f)^2}\,\frac{f\,dg - g\,df}{f^2} =$$

$$= \frac{f\,dg - g\,df}{f^2+g^2} \Rightarrow$$

$$\Rightarrow I(C) = \oint_C \frac{d\varphi}{2\pi} = \oint_C \frac{f\,dg - g\,df}{2\pi(f^2+g^2)}$$

Let $C : \rho(t) \in \mathbb{R}^2$, $t \in [0,1]$ be a parameterization of the curve C. Then, the differentials df and dg are given by:

$$df = [\dot{\rho}(t) \cdot \nabla f(\rho(t))]\,dt$$
$$dg = [\dot{\rho}(t) \cdot \nabla g(\rho(t))]\,dt$$

It follows that:

$$I(C) = \oint_C \frac{f\,dg - g\,df}{f^2+g^2} =$$

$$= \int_0^1 dt\, \frac{f(\rho(t))\nabla g(\rho(t)) \cdot \dot{\rho}(t) - g(\rho(t))\nabla f(\rho(t)) \cdot \dot{\rho}(t)}{2\pi\left[f^2(\rho(t)) + g^2(\rho(t))\right]}$$

$$= \int_0^1 dt\, \dot{\rho}(t) \cdot \left[\frac{f(\rho(t))\nabla g(\rho(t)) - g(\rho(t))\nabla f(\rho(t))}{2\pi\left[f^2(\rho(t)) + g^2(\rho(t))\right]}\right]$$

- Properties of the index

(1) $\boxed{I(C) \in \mathbb{Z}}$ (i.e. $I(C)$ is an integer).

Proof

Going around the curve C, both initial and final value of φ point in the same direction, therefore the variation $\Delta\varphi$ of the angle must be a multiple of 2π. It follows that

$$\Delta\varphi = \oint_C d\varphi = 2k\pi \ , \ \text{with } k \in \mathbb{Z} \Rightarrow$$

$$\Rightarrow I(C) = \frac{1}{2\pi} \oint_C d\varphi = \frac{1}{2\pi} \cdot (2k\pi) = k \in \mathbb{Z} \ \square$$

(2) Assume that there are no fixed points in the interior of a simple closed curve C. Then $I(C) = 0$.

Proof

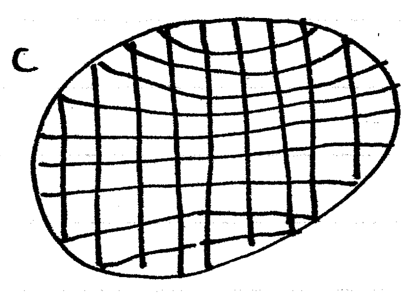


Fig. 1

We divide the interior of the curve C into a mesh of N closed simple curves γ_k with $k \in [N]$. We assume that the loops γ_k are small enough so that the maximum angle variation around γ_K does not exceed $\pi/2$. This is possible only because there are no fixed points in the interior of any γ_K (see fig. 2)

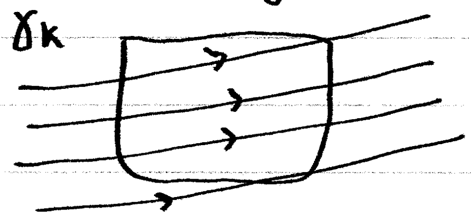


Fig. 2

It follows that

$$\forall K \in [N] : \oint_{\gamma_K} d\varphi = 0$$

and therefore

$$I(C) = \frac{1}{2\pi} \oint_C d\varphi = \frac{1}{2\pi} \left[\sum_{K=1}^{N} \oint_{\gamma_K} d\varphi \right] = 0 \quad \square$$

③ Invariance with contour deformation

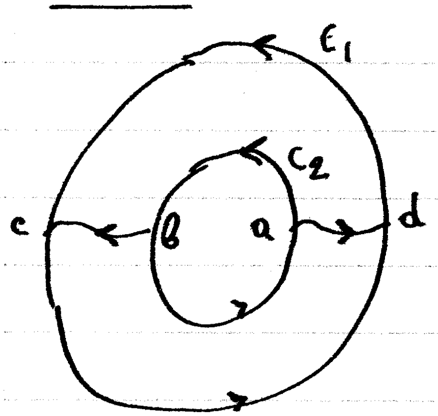
Def : Let C_1, C_2 be two simple closed curves with
$C_1 : \rho_1(t) \in \mathbb{R}^2,\ t \in [0,1]$ and
$C_2 : \rho_2(t) \in \mathbb{R}^2,\ t \in [0,1]$.
We say that $C_1 \sim C_2$ if and only if there is a mapping $\rho : [0,1]^2 \to \mathbb{R}^2$ such that
a) $\forall t \in [0,1] : (\rho(t,0) = \rho_1(t) \wedge \rho(t,1) = \rho_2(t))$
b) ρ continuous at $[0,1]^2$

c) $\forall (t,a) \in [0,1]^2$: $\rho(t,a)$ not a fixed point.

↳ $C_1 \sim C_2$ means that C_1 can be continuously deformed into C_2 without crossing any fixed points.

- $\boxed{C_1 \sim C_2 \Rightarrow I(C_1) = I(C_2)}$

Proof



Define:

C_{bc} : path from b to c

C_{ad} : path from a to d

C_{ab} : counterclockwise path from a to b

C_{ba} : counterclockwise path from b to a

C_{cd} : counterclockwise path from c to d.

C_{dc} : counterclockwise path from d to c.

We also let $-C$ represent the path C with its direction reversed. (e.g. $-C_{ab}$ vs. C_{ba}).

Now consider the paths Γ_1 and Γ_2 defined as:

$\Gamma_1 = C_{ad} \cup C_{dc} \cup (-C_{bc}) \cup (-C_{ab})$

$\Gamma_2 = C_{bc} \cup C_{cd} \cup (-C_{ad}) \cup (-C_{ba})$

There are no fixed points in the interiors of Γ_1 and Γ_2, therefore $I(\Gamma_1) = 0$ and $I(\Gamma_2) = 0$.

We note that

$$2\pi I(\Gamma_1) = \int_{C_{ad}} d\varphi + \int_{C_{dc}} d\varphi + \int_{-C_{bc}} d\varphi + \int_{-C_{ab}} d\varphi =$$

$$= \int_{C_{ad}} d\varphi + \int_{C_{dc}} d\varphi - \int_{C_{bc}} d\varphi - \int_{C_{ab}} d\varphi \quad (1)$$

and

$$2\pi I(\Gamma_2) = \int_{C_{bc}} d\varphi + \int_{C_{cd}} d\varphi + \int_{-C_{ad}} d\varphi + \int_{-C_{ba}} d\varphi =$$

$$= \int_{C_{bc}} d\varphi + \int_{C_{cd}} d\varphi - \int_{C_{ad}} d\varphi - \int_{C_{ba}} d\varphi \quad (2)$$

Adding (1) and (2) gives: the cancellations: C_{bc}, C_{ad}

$$2\pi[I(\Gamma_1) + I(\Gamma_2)] = \int_{C_{cd}} d\varphi + \int_{C_{dc}} d\varphi - \int_{C_{ab}} d\varphi - \int_{C_{ba}} d\varphi =$$

$$= \oint_{C_1} d\varphi - \oint_{C_2} d\varphi = 2\pi[I(C_1) - I(C_2)]$$

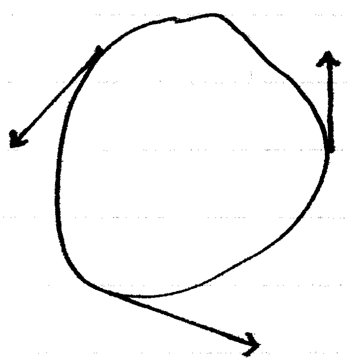
$\Rightarrow I(C_1) - I(C_2) = I(\Gamma_1) + I(\Gamma_2) = 0 + 0 = 0 \Rightarrow$

$\Rightarrow I(C_1) = I(C_2)$. $\square$

(4) Index of closed orbits

- $\boxed{\text{If } C \text{ is a closed orbit of the system then } I(C) = 1}$

Proof: If C is a closed orbit of the system, then it is easy to see that the vector $(\dot{x}_1, \dot{x}_2)$ is tangent to C for all points of C. Thus, the total change in the angle φ is $\Delta\varphi = 2\pi$. It follows that



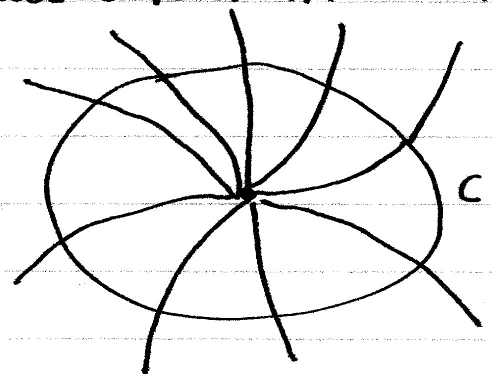
$$I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{2\pi}{2\pi} = 1 \quad \square$$

- Index of a fixed point

Def: Let $x_0 \in \mathbb{R}^2$ be a fixed point. Let C be a counterclockwise curve whose interior contains the fixed point x_0, but no other fixed points. We define the index $I(x_0)$ of the fixed point x_0 as $I(x_0) = I(C)$

↳ We note that from property 3 above, $I(x_0)$ is independent of our choice of C, subject to the stated constraints.

<u>Thm</u>: Let $x_0 \in \mathbb{R}^2$ be a fixed point such that trajectories radiate from or towards x_0 in all directions. Then $I(x_0) = +1$.



<u>Proof</u>

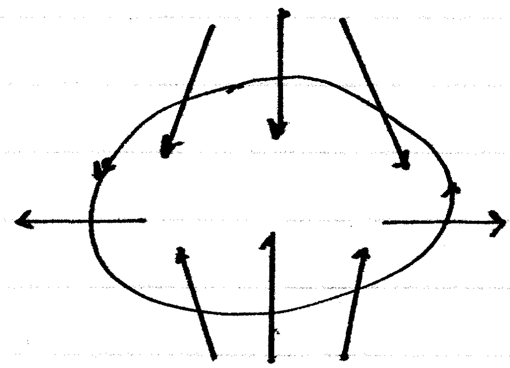
Consider a small enough loop C around x_0 constructed so that it is perpendicular to every trajectory it intersects. Then the total change in the angle around C is $\Delta\varphi = 2\pi$. It follows that:

$$I(x_0) = I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{2\pi}{2\pi} = 1. \quad \square$$

$\hookrightarrow$ It follows that the following fixed points have $I(x_0) = 1$:

a) sources c) stable spirals e) degenerate nodes
b) sinks d) unstable spirals f) stars.

Thm : $\boxed{\text{Let } x_0 \in \mathbb{R}^2 \text{ be a \underline{saddle node}. Then } I(x_0) = -1.}$



Proof

Let C be a small loop around the saddle node x_0. The angle φ varies clockwise around C with $\Delta\varphi = -2\pi$ (see fig.) It follows that

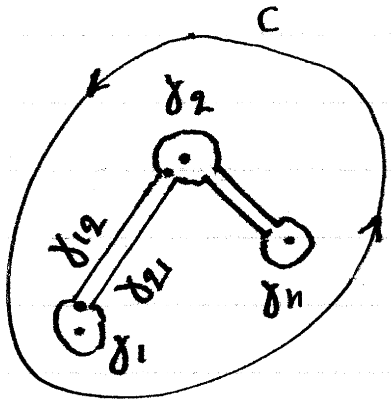
$$I(x_0) = I(C) = \frac{1}{2\pi}\oint_C d\varphi = \frac{\Delta\varphi}{2\pi} = \frac{-2\pi}{2\pi} = -1 \quad \square$$

- <u>Index of a curve surrounding fixed points.</u>

Thm : Let C be a simple closed curve containing the fixed points $x_1, x_2, \dots, x_n$. Then:

$$\boxed{I(C) = \sum_{a=1}^{n} I(x_a)}$$

Proof



We deform continuously C into a curve $\Gamma \sim C$ such that Γ consists of small loops γ_a around the fixed points x_a and connecting bridges γ_{ab} connecting x_a to x_b as shown in the figure. We further assume that the gap between γ_{ab} to γ_{ba} tends to zero. That implies that $\gamma_{ab} = -\gamma_{ba}$ and γ_a are closed. It follows that

$$I(C) = I(\Gamma) = \frac{1}{2\pi}\oint_\Gamma d\varphi =$$

$$= \frac{1}{2\pi}\left[\sum_{a=1}^{n}\oint_{\gamma_a} d\varphi + \sum_{a=1}^{n-1}\int_{\gamma_{a,a+1}} d\varphi + \sum_{a=1}^{n}\int_{\gamma_{a+1,a}} d\varphi\right]$$

$$= \frac{1}{2\pi}\left[\sum_{a=1}^{n}\oint_{\gamma_a} d\varphi + \sum_{a=1}^{n-1}\int_{\gamma_{a,a+1}} d\varphi - \sum_{a=1}^{n-1}\int_{\gamma_{a,a+1}} d\varphi\right]$$

$$= \frac{1}{2\pi}\left[\sum_{a=1}^{n}\oint_{\gamma_a} d\varphi\right] = \sum_{a=1}^{n}\left[\frac{1}{2\pi}\oint_{\gamma_a} d\varphi\right] =$$

$$= \sum_{a=1}^{n} I(x_a) \qquad \square$$

<u>Corollary</u>: Let C be a closed trajectory enclosing the fixed points $x_1, x_2, \ldots, x_n$. Then

$$\sum_{a=1}^{n} I(x_a) = +1$$

<u>Proof</u>

Since C is a closed trajectory, from property 4, we have $I(C) = +1$. Thus, from the theorem:

$$\sum_{a=1}^{n} I(x_a) = I(C) = +1 \qquad \square$$

EXAMPLES

a) Show that the system

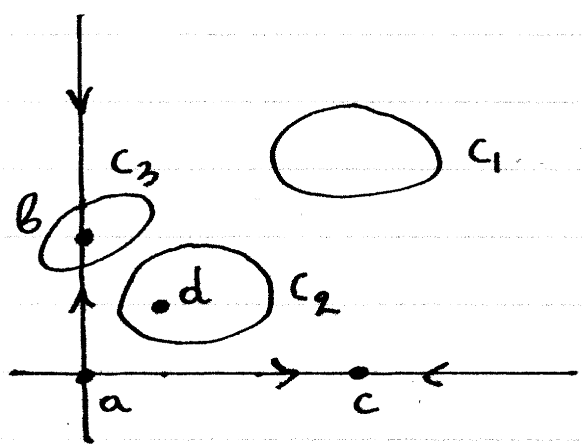
$$\begin{cases} \dot{x}_1 = x_1(3 - x_1 - 2x_2) \\ \dot{x}_2 = x_2(2 - x_1 - x_2) \end{cases}$$

does not have any closed trajectories.

Solution

It can be shown that this system has the following fixed points:

$a = (0,0)$ unstable node $\Rightarrow I(a) = 1$
$b = (0,2)$ stable node $\Rightarrow I(b) = 1$
$c = (3,0)$ stable node $\Rightarrow I(c) = 1$
$d = (1,1)$ saddle node $\Rightarrow I(d) = -1$.



- Let C_1 be a curve enclosing no fixed points. Then
  $I(C_1) = 0 \neq 1 \Rightarrow$
  $\Rightarrow C_1$ not a trajectory.

- Let C_2 be any curve that encloses only the fixed point $d = (1,1)$. Then
  $I(C_2) = I(d) = -1 \neq +1 \Rightarrow C_2$ not a trajectory.

- Let C_3 be any curve enclosing a or b or c or any combination of these three fixed points. Then C_3 will intersect at least the x_1-axis or the x_2-axis (or both). Since both the x_1-axis and the x_2-axis are trajectories, and trajectories cannot intersect, it follows that C_3 is not a trajectory. □

↳ We see that trajectories that cannot be ruled out by index theory, can be eliminated, sometimes, by the constraint that two trajectories cannot intersect.

b) Show that the system

$$\begin{cases} \dot{x}_1 = x_1 e^{-x_1} \\ \dot{x}_2 = 1 + x_1 + x_2^2 \end{cases}$$

does not have any closed trajectories.

<u>Solution</u>

Fixed points:

$$\begin{cases} x_1 e^{-x_1} = 0 \\ 1 + x_1 + x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ 1 + x_1 + x_2^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ 1 + x_2^2 = 0 \end{cases}$$

Since $1 + x_2^2 = 0$ is inconsistent, there are no

fixed points.
Now, let C be any closed curve. Then
$I(C) = 0 \neq 1 \Rightarrow C$ not a trajectory.

EXERCISES

(10) Show that the following systems do not have any closed trajectories.

a) $\begin{cases} \dot{x}_1 = x_1(4 - x_2 - x_1^2) \\ \dot{x}_2 = x_2(x_1 - 1) \end{cases}$

b) $\begin{cases} \dot{x}_1 = x_1^2 + x_2^2 \\ \dot{x}_2 = x_1 - 2 \end{cases}$

(11) A system has three closed trajectories C_1, C_2, C_3, all counterclockwise, with C_2, C_3 enclosed by C_1. We also know that C_2 is not enclosed by C_3 and vice versa.
Show that there is at least one fixed point enclosed by C_1, but not enclosed by C_2 and C_3.

(12) Consider the parameterized system
$\begin{cases} \dot{x}_1 = f(x_1, x_2, a) \\ \dot{x}_2 = g(x_1, x_2, a) \end{cases}$
As we vary a from a_0 to a_1 this system undergoes one or more local bifurcations. Show that the sum $J(a)$ of all indices of all fixed points is constant with respect to a.

MM5: Limit cycles

LIMIT CYCLES

- A limit cycle is a closed orbit that functions as a stable or unstable fixed point.
- Recall that $x(t)$ is a closed orbit if and only if:

$$\boxed{\exists T>0 : \forall t \in \mathbb{R} : x(t+T) = x(t).}$$

Ruling out closed orbits

We use the following techniques to rule out the existence of closed orbits.

1 → Index theory.

We have already shown previously that index theory can be used to rule out the existence of closed orbits.

2 → Gradient systems

- A gradient system is a system of the form $\dot{x} = -\nabla V$ with $x(t) \in \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$.
- A gradient system has no closed orbits.

Proof

Assume $x(t)$ is a closed orbit with $x(T) = x(0)$.
Then $V(x(T)) = V(x(0)) \Rightarrow$

$$\Rightarrow V(x(T)) - V(x(0)) = \int_0^T \frac{dV}{dt}\, dt =$$

$$= \int_0^T (\nabla V) \cdot (\dot{x}(t))\, dt = \int_0^T (-\dot{x}(t)) \cdot (\dot{x}(t))\, dt =$$

$$= \int_0^T - \|\dot{x}(t)\|^2\, dt = 0 \Rightarrow$$

$\Rightarrow \forall t \in (0,T) : \dot{x}(t) = 0 \Rightarrow \forall t \in (0,T) : x(t) = x(0)$
$\Rightarrow x(0)$ fixed point. $\Rightarrow x(t)$ not an orbit. $\square$

EXAMPLE

Show that

$$\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = x_1 \cos x_2 \end{cases}$$

has no closed orbits.

Solution

Let $f_1(x_1, x_2) = \sin x_2$ and $f_2(x_1, x_2) = x_1 \cos x_2$
It follows that:

$$\int f_1(x_1,x_2)dx_1 = \int \sin x_2 \, dx_1 = x_1 \sin x_2 + c$$

$$\int f_2(x_1,x_2)\, dx_2 = \int x_1 \cos x_2 dx_2 = x_1 \int \cos x_2 dx_2 =$$

$$= x_1 \sin x_2 + c$$

Choose $c=0$. For $V(x_1,x_2) = -x_1 \sin x_2 \Rightarrow$

$\Rightarrow \nabla V(x_1,x_2) = \quad -(f_1(x_1,x_2), f_2(x_1,x_2))$

$\Rightarrow$ the system is a gradient system

$\Rightarrow$ there are no closed orbits.

EXERCISES

(1) Show that the following systems have no closed orbits by showing that they are gradient systems.

a) $\begin{cases} \dot{x}_1 = x_2^2 + x_2 \cos x_1 \\ \dot{x}_2 = 2x_1x_2 + \sin x_1 \end{cases}$

b) $\begin{cases} \dot{x}_1 = 3x_1^2 - 1 - e^{x_2} \\ \dot{x}_2 = -2x_1 e^{2x_2} \end{cases}$

c) $\begin{cases} \dot{x}_1 = -2x_1 e^{x_1^2 + x_2^2} \\ \dot{x}_2 = -2x_2 e^{x_1^2 + x_2^2} \end{cases}$

3 → Lyapunov functions

- Consider the system $\dot{x} = f(x)$. Assume that there is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

  a) $V(x_0) = 0$

  b) $V(x) > 0 \, , \forall x \in \mathbb{R}^n - \{x_0\}$

  c) $x(t) \neq x_0 \Rightarrow (d/dt) V(x(t)) < 0$.

Then the system has no closed orbits.

Proof

Assume that $x(t)$ is a periodic orbit with $x(0) \neq x_0$. It follows that

$$\exists T > 0 : \forall n \in \mathbb{N} : x(nT) = x(0) \neq x_0 . \quad (1)$$

From (a), (b), (c) it follows that x_0 is an asymptotically stable fixed-point with attracting basin $\mathbb{R}^n$, and therefore

$$\lim_{t \to +\infty} x(t) = x_0 \qquad (2)$$

From (1) and (2):

$$x(0) = \lim_{n \in \mathbb{N}} x(nT) = \lim_{t \to +\infty} x(t) = x_0 \Rightarrow$$

$\Rightarrow x(0) = x_0 \leftarrow$ contradiction.

EXAMPLES

a) Show that the system

$$\begin{cases} \dot{x}_1 = -x_1 + 4x_2 \\ \dot{x}_2 = -x_1 - x_2^3 \end{cases}$$

has no closed orbits.

Solution

We try $V(x_1, x_2) = x_1^2 + a x_2^2$. Then

a) $V(0,0) = 0$ and

b) $V(x_1, x_2) > 0, \ \forall (x_1, x_2) \in \mathbb{R}^2 - \{(0,0)\}$ and

c) $dV/dt = 2x_1\dot{x}_1 + 2ax_2\dot{x}_2 =$
$$= 2x_1(-x_1 + 4x_2) + 2ax_2(-x_1 - x_2^3) =$$
$$= -2x_1^2 + 8x_1x_2 - 2ax_1x_2 - 2ax_2^4 =$$
$$= -2x_1^2 - 2ax_2^4 + 2(4-a)x_1x_2.$$

For $a = 4$: $V(x_1, x_2) = x_1^2 + 4x_2^2$ and
$dV/dt = -2x_1^2 - 2ax_2^4 < 0, \ \forall (x_1, x_2) \in \mathbb{R}^2 - \{(0,0)\}$.
From (a), (b), (c) it follows that the system has no closed orbits.

b) Show that the system $\ddot{x} + (\dot{x})^3 + x = 0$ has no closed orbits.

Solution

We try $V(x, \dot{x}) = x^2 + \dot{x}^2$. We note that

a) $V(0,0) = 0$

b) $V(x,\dot{x}) > 0$ if $(x,\dot{x}) \neq 0$

c) $dV/dt = 2x\dot{x} + 2\dot{x}\ddot{x} = 2\dot{x}(x+\ddot{x}) =$

$$= 2\dot{x}\left(-(\dot{x})^3\right) = -2(\dot{x})^4 < 0$$

for $\dot{x} \neq 0$.

From (a), (b), (c) it follows that the system has no periodic solutions.

EXERCISES

(2) Show that the system

$$\begin{cases} \dot{x}_1 = x_2 - x_1^3 \\ \dot{x}_2 = -x_1 - x_2^3 \end{cases}$$

has no closed orbits.
(Hint: Use $V(x_1, x_2) = ax_1 + bx_2$)

(3) Show that the system

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_2^3 - 2x_2^4 \\ \dot{x}_2 = -x_1 - x_2 + x_1 x_2 \end{cases}$$

has no closed orbits.
(Hint: Use $V(x_1, x_2) = x_1^m + ax_2^n$)

4 → Dulac's Criterion

Consider the system $\dot{x} = f(x)$ with $f: \mathbb{R}^2 \to \mathbb{R}^2$.
We assume that

a) f continously differentiable in $A \subseteq \mathbb{R}^2$

b) A simply connected

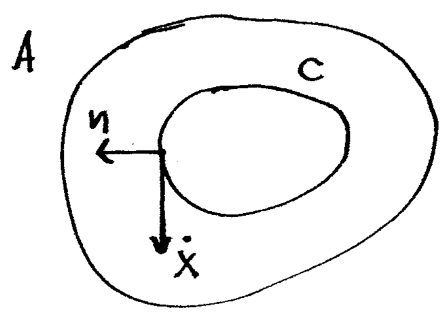
c) $g: A \to \mathbb{R}$ continuously differentiable in A

* d) $\nabla \cdot (g(x) f(x))$ does not change sign in A
(i.e. $\nabla \cdot (g(x) f(x)) > 0, \forall x \in A$ OR
$\nabla \cdot (g(x) f(x)) < 0, \forall x \in A$)

Then, the system has no closed orbits in A.

Proof

Assume that there is a closed orbit C within A.



Since $\dot{x}$ is always tangent to C, then if n is a vector normal to C, then $\dot{x} \cdot n = 0$.
Let B be the interior of C.
Then from (d):

$$I = \iint_B \nabla \cdot (g(x) f(x))\, dx \neq 0. \qquad (1)$$

On the other hand:

$$I = \iint_B \nabla \cdot (f(x) g(x))\, dx = \iint_B \nabla \cdot (g(x)\dot{x})\, dx =$$

$$= \oint_C [g(x)\dot{x}] \cdot n\, dl = \oint_C g(x)(\dot{x} \cdot n)\, dl = 0 \quad (2)$$

since everywhere $\dot{x} \cdot n = 0$. However, (1) and (2) contradict thus there are no closed orbits.

→ There is no general method for finding the function $g(x_1, x_2)$. We usually try out the following possibilities:

$g(x_1, x_2) = 1$

$g(x_1, x_2) = \dfrac{1}{x_1^a x_2^b}$

$g(x_1, x_2) = \exp(a x_1)$

$g(x_1, x_2) = \exp(b x_2)$

EXAMPLES

a) Show that the system

$$\begin{cases} \dot{x}_1 = x_1(2 - x_1 - x_2) \\ \dot{x}_2 = x_2(4x_1 - x_1^2 - 3) \end{cases}$$

has no closed orbits in the first quadrant.

Solution

Let $A=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1>0 \wedge x_2>0\}$

Choose $g(x)=\dfrac{1}{x^a y^b}$. Then

$$\nabla\cdot(g\dot{x})=\frac{\partial}{\partial x_1}\left[\frac{x_1(2-x_1-x_2)}{x_1^a x_2^b}\right]+\frac{\partial}{\partial x_2}\left[\frac{x_2(4x_1-x_1^2-3)}{x_1^a x_2^b}\right]$$

$$=\frac{1}{x_2^b}\frac{\partial}{\partial x_1}\left[2x_1^{1-a}-x_1^{2-a}-x_2x_1^{1-a}\right]+\frac{\partial}{\partial x_2}\left[x_2^{1-b}\frac{4x_1-x_1^2-3}{x_1^a}\right]$$

$$=\frac{1}{x_2^b}\left[(2-x_2)(1-a)x_1^{-a}-(2-a)x_1^{1-a}\right]+\frac{4x_1-x_1^2-3}{x_1^a}(1-b)x^{-b}$$

For $b=1$: the 2nd term vanishes.

For $a=1$: the 1st term loses x_1 dependence.

Then:

$$\nabla\cdot(g\dot{x})=\frac{1}{x_2}\left[0-(2-1)x_1^{1-1}\right]=\frac{-1}{x_2}<0,\ \forall\,(x_1,x_2)\in A.$$

b) Show that the system

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-x_1-x_2+x_1^2+x_2^2\end{cases}$$

has no closed orbits.

Solution

For $g(x)=e^{ax_1}$:

$$\nabla\cdot(g\dot{x}) = \frac{\partial}{\partial x_1}\left[e^{ax_1}x_2\right] + \frac{\partial}{\partial x_2}\left[e^{ax_1}(-x_1-x_2+x_1^2+x_2^2)\right]$$

$$= ax_2e^{ax_1} + e^{ax_1}(-0-1+0+2x_2) =$$

$$= e^{ax_1}\left[ax_2-1+2x_2\right] =$$

$$= e^{ax_1}\left[(a+2)x_2-1\right]$$

For $a=-2$: $\nabla\cdot(g\dot{x}) = e^{-2x_1}\cdot(0-1) =$

$$= -e^{-2x_1} < 0\ ,\ \forall (x_1,x_2)\in\mathbb{R}^2$$

$\Rightarrow$ no closed orbits. $\square$

EXERCISES

(4) Show that the system

$$\begin{cases} \dot{x}_1 = a_1 x_1 - b_1 x_1^2 - c_1 x_1 x_2 \\ \dot{x}_2 = a_2 x_2 - b_2 x_2^2 - c_2 x_1 x_2 \end{cases}$$

with $a_1, a_2, b_1, b_2, c_1, c_2 \in (0, +\infty)$ has no closed orbits in the first quadrant.

(Use: $g(x_1, x_2) = \dfrac{1}{x_1 x_2}$)

(5) Show that the system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 - 1 \\ \dot{x}_2 = x_2 (x_1 - 1) \end{cases}$$

has no closed orbits in the first quadrant.

(Use: $g(x_1, x_2) = \dfrac{1}{x_1^2 x_2}$)

Definition of orbital stability

- Limit cycles are isolated periodic orbits that act as a global fixed point that may be stable or unstable. We are therefore interested in defining the concept of orbital stability for limit cycles.
- Consider the system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$. Let $x(t|x_0, t_0)$ be the solution with initial condition $x(t_0) = x_0$. We define the orbit:
$$\Gamma_+(x_0, t_0) = \{x(t|x_0, t_0) \mid t > t_0\}.$$
- Point-set distance.
Let $p \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$. We define the distance $d(p, A)$ of p from A as:
$$d(p, A) = \inf_{x \in A} \|x - p\|.$$

Stability definitions

① $\Gamma_+(x_0, t_0)$ orbitally stable $\Leftrightarrow$
$$\boxed{(\forall \varepsilon > 0 : \exists \delta > 0 : \forall y_0 \in B(x_0, \delta) : \forall t > t_0 : d(x(t|y_0, t_0), \Gamma_+(x_0, t_0)) < \varepsilon)}$$

(2) $\boxed{\begin{array}{l}\Gamma_+(x_0,t_0) \text{ \underline{orbitally attracting}} \Leftrightarrow \\ (\exists \delta > 0 : \forall y_0 \in B(x_0,\delta) : \\ \quad : \lim\limits_{t\to+\infty} d(x(t\,|\,y_0,t_0), \Gamma_+(x_0,t_0)) = 0\end{array}}$

(3) $\boxed{\begin{array}{l}\Gamma_+(x_0,t_0) \text{ \underline{asymptotically orbitally stable}} \Leftrightarrow \\ \begin{cases}\Gamma_+(x_0,t_0) \text{ orbitally stable} \\ \Gamma_+(x_0,t_0) \text{ orbitally attracting.}\end{cases}\end{array}}$

(4) $\boxed{\begin{array}{l}\Gamma_+(x_0,t_0) \text{ \underline{neutrally orbitally stable}} \Leftrightarrow \\ \Leftrightarrow \begin{cases}\Gamma_+(x_0,t_0) \text{ orbitally stable} \\ \Gamma_+(x_0,t_0) \text{ NOT orbitally attracting.}\end{cases}\end{array}}$

(5) $\boxed{\begin{array}{l}\Gamma_+(x_0,t_0) \text{ \underline{orbitally unstable}} \Leftrightarrow \\ \Leftrightarrow \begin{cases}\Gamma_+(x_0,t_0) \text{ NOT orbitally stable} \\ \Gamma_+(x_0,t_0) \text{ NOT orbitally attracting.}\end{cases}\end{array}}$

● Stability regions

- Let C be an orbitally attracting limit cycle. We define the <u>basin of attraction</u> $\mathcal{B}(C)$ of C as:

$$\boxed{\mathcal{B}(C) = \{y_0 \in \mathbb{R}^n \mid \lim_{t\to+\infty} d(x(t\,|\,y_0,0), C) = 0\}}$$

- Let $A \subseteq \mathbb{R}^n$ be a simple bounded region.
  We say that

$$\boxed{A \text{ trapping region} \Leftrightarrow \forall y_0 \in A : \forall t > 0 : x(t \mid y_0, 0) \in A}$$

The Poincare - Bendixson theorem

Consider the two-dimensional autonomous system:

$$\begin{cases} \dot{x}_1 = f(x_1, x_2) \\ \dot{x}_2 = g(x_1, x_2) \end{cases}$$

and let $x(t) = (x_1(t), x_2(t))$.

Now we introduce the following notation:

- Trajectories through $x(0)$:

  $\Gamma(x_0, S) = \{x(t) \mid x(0) = x_0 \wedge t \in S\}$ with $S \subseteq \mathbb{R}$.

  $\Gamma(x_0) = \Gamma(x_0, \mathbb{R})$

  $\Gamma_+(x_0, t) = \Gamma(x_0, (t, +\infty))$, $\Gamma_+(x_0) = \Gamma(x_0, (0, +\infty))$

- Closure of a set.

  Let $A \subseteq \mathbb{R}^2$ be an open set and let ∂A be it's boundary set. We define the closure of A as:

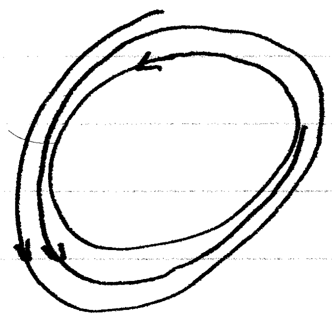
  $cl(A) = A \cup \partial A$.

- ω limit set

  Let $x_0 \in \mathbb{R}^2$ be given. The ω limit set of x_0 is defined as:

$$\boxed{\omega(x_0) = \bigcap_{t>0} cl(\Gamma_+(x_0, t))}$$

▸ <u>interpretation</u> : Consider a periodic orbit C that tends to attract neighbohring orbits. Then $\Gamma_+(x_0, t)$ looks like a thick ring around C that becomes increasingly thinner when $t \to +\infty$. Thus, for x_0 being the starting point of a trajectory attracted to C, $\omega(x_0)$ is equal to the limit cycle C itself.



Note that if $\Gamma(x_0)$ is itself a periodic orbit, then $\Gamma(x_0) = \omega(x_0)$.
Furthermore, if $\Gamma(x_0)$ approaches a fixed point y, then $\omega(x_0) = \{y\}$.

<u>Thm</u> : (<u>Poincare - Bendixson</u>)

> Assume that
> a) $A \subseteq \mathbb{R}^2$ is closed and bounded
> b) A contains no fixed points
> c) $\exists\, x_0 \in A : \Gamma_+(x_0) \subseteq A$
> Then : $\omega(x_0)$ is a periodic orbit.

↳ It follows that either $\Gamma_+(x_0)$ itself is a periodic orbit, in which case $\Gamma_+(x_0) = \omega(x_0)$ or $\Gamma_+(x_0)$ approaches the periodic limit cycle $\omega(x_0)$.

- The immediate consequence of the Poincare-Bendixson theorem is that the only "structures" that can attract orbits in two-dimensional systems are fixed points or periodic limit cycles. In higher dimensional systems, orbits can also be attracted by strange attractors.
- The main difficulty with applying this theorem is in establishing the condition

$$\exists x_0 \in A : \Gamma_+(x_0) \subseteq A$$

One way is by locating a <u>trapping region</u>:

<u>Def</u>: Let $A \subseteq \mathbb{R}^2$ be a bounded set. with $N(x,y) \in \mathbb{R}^2$ an outward normal vector defined on ∂A (the boundary of A). We say that A is a <u>trapping region</u> if and only if:

$$N(x,y) \cdot (f(x,y), g(x,y)) < 0, \ \forall (x,y) \in \partial A$$

<u>Prop</u>: If A is a trapping region then:

$$\forall x_0 \in A : \Gamma_+(x_0) \subseteq A$$

↳ In other words, in a trapping region <u>all</u> trajectories inside the region do <u>not</u> leave the region.

EXAMPLE

a) Show that the system

$$\begin{cases} \dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 + x_2 - x_2(x_1^2 + x_2^2). \end{cases}$$

has a limit cycle.

Solution

▶ Fixed points.

We note that $(x_1, x_2) = (0,0)$ is an obvious fixed point. We will now show it is unique. Define:

$$f_1(x_1, x_2) = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$

$$f_2(x_1, x_2) = -x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

Using polar coordinates: $x_1 = R\cos\vartheta \wedge x_2 = R\sin\vartheta$ it follows that $x_1^2 + x_2^2 = R^2(\cos^2\vartheta + \sin^2\vartheta) = R^2$ and therefore

$$f_1(x_1, x_2) = R\cos\vartheta + R\sin\vartheta - R\cos\vartheta \cdot R^2 =$$

$$= (R - R^3)\cos\vartheta + R\sin\vartheta = R[(1-R^2)\cos\vartheta + \sin\vartheta]$$

$$f_2(x_1, x_2) = -R\cos\vartheta + R\sin\vartheta - R\sin\vartheta \cdot R^2 = -R\cos\vartheta + (R - R^3)\sin\vartheta$$

$$= R[-\cos\vartheta + (1-R^2)\sin\vartheta].$$

Since $R = 0 \Leftrightarrow (x_1, x_2) = (0,0)$, we assume with no loss of generality that $R \neq 0$. Then:

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases} \Leftrightarrow \begin{cases} R[(1-R^2)\cos\vartheta + \sin\vartheta] = 0 \\ R[-\cos\vartheta + (1-R^2)\sin\vartheta] = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (1-R^2)\cos\vartheta + \sin\vartheta = 0 \\ -\cos\vartheta + (1-R^2)\sin\vartheta = 0 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} 1-R^2 & 1 \\ -1 & 1-R^2 \end{bmatrix} \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (1)$$

Since $D = \begin{vmatrix} 1-R^2 & 1 \\ -1 & 1-R^2 \end{vmatrix} = (1-R^2)^2 - (-1) = (1-R^2)^2 + 1 \geq 1 > 0$

it follows that

$(1) \Leftrightarrow \begin{cases} \cos\vartheta = 0 \\ \sin\vartheta = 0 \end{cases} \Leftrightarrow \cos^2\vartheta + \sin^2\vartheta = 0 \longleftarrow$ contradiction

Thus, the fixed point $(x_1, x_2) = (0,0)$ is unique

▶ Trapping region

We construct a trapping region bound by two concentric circles centered at $(0,0)$ with radius R_1 and R_2 with $R_1 < R_2$. For a general circle with radius R and a normal vector $N(x_1,x_2)$ oriented outward, we define

$N(x_1,x_2) = (x_1, x_2)$ with $x_1^2 + x_2^2 = R^2$.

It follows that

$$N(x_1,x_2) \cdot (f_1(x_1,x_2), f_2(x_1,x_2)) = (x_1,x_2) \cdot (f_1(x_1,x_2), f_2(x_1,x_2))$$
$$= x_1 f_1(x_1,x_2) + x_2 f_2(x_1,x_2) =$$
$$= x_1[x_1 + x_2 - x_1(x_1^2+x_2^2)] + x_2[-x_1 + x_2 - x_2(x_1^2+x_2^2)]$$
$$= x_1^2 + x_1x_2 - x_1^2R^2 - x_1x_2 + x_2^2 - x_2^2R^2 =$$
$$= (x_1^2+x_2^2) - R^2(x_1^2+x_2^2) = R^2 - R^2R^2 = R^2(1-R^2)$$

For $R_2 > 1$, we have $N(x_1,x_2) \cdot (f_1(x_1,x_2), f_2(x_1,x_2)) < 0$, thus the normal vector $N(x_1,x_2)$ points inward to the big circle, and therefore the trapping region.

For $R_1 < 1$, we have $N(x_1,x_2) \cdot (f_1(x_1,x_2), f_2(x_1,x_2)) > 0$, thus the normal vector points outward of the small circle, and

therefore inwards to the trapping region.
Consequently for $R_1 = \sqrt{1/2}$ and $R_2 = \sqrt{2}$, we define
$A = \{ (x_1, x_2) \in \mathbb{R}^2 \mid +1/2 \leq x_1^2 + x_2^2 \leq 2 \}$ and
we have:

$$\begin{cases} A \text{ closed and bounded} \\ A \text{ contains no fixed points} \\ A \text{ is a trapping region.} \end{cases}$$

It follows from the Poincare-Bendixson theorem that
the system has a limit cycle.

Constructing a trapping region

When polar coordinates are not helpful, we can use the nullclines to obtain evidence that there may be a limit cycle and manually construct a trapping region A using the necessary and sufficient condition

$$N(x,y)\cdot(f(x,y),g(x,y))<0\ ,\ \forall (x,y)\in\partial A.$$

EXAMPLE

Construct a trapping region for:

$$\begin{cases} \dot{x}_1 = -x_1 + a x_2 + x_1^2 x_2 & (1)\\ \dot{x}_2 = b - a x_2 - x_1^2 x_2 & (2)\end{cases}$$

with $a>0$ and $b>0$. Show that the system has limit cycle

Solution

- Nullcline analysis

For (1):

$$f(x_1,x_2)\geqslant 0 \Leftrightarrow -x_1 + a x_2 + x_1^2 x_2 \geqslant 0 \Leftrightarrow$$

$$\Leftrightarrow -x_1 + x_2(a+x_1^2)\geqslant 0 \Leftrightarrow x_2(a+x_1^2)\geqslant x_1 \Leftrightarrow$$

$$\Leftrightarrow x_2 \geqslant \frac{x_1}{a+x_1^2}$$

thus $\dot{x}_1>0$ above curve (1)

$\dot{x}_1<0$ below curve (1)

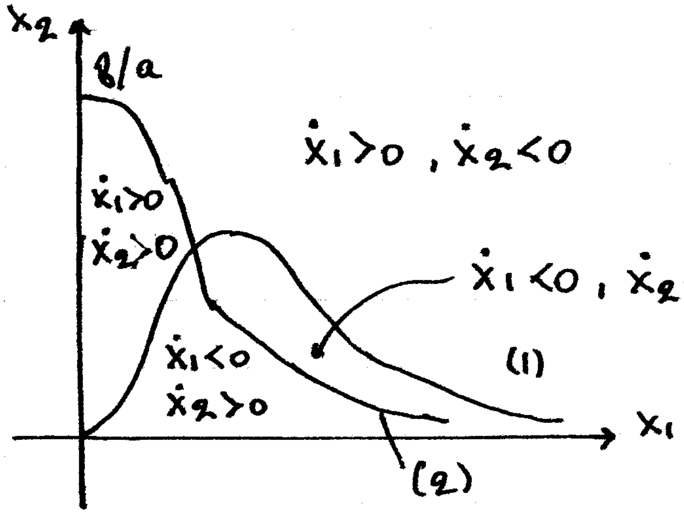
For (2):

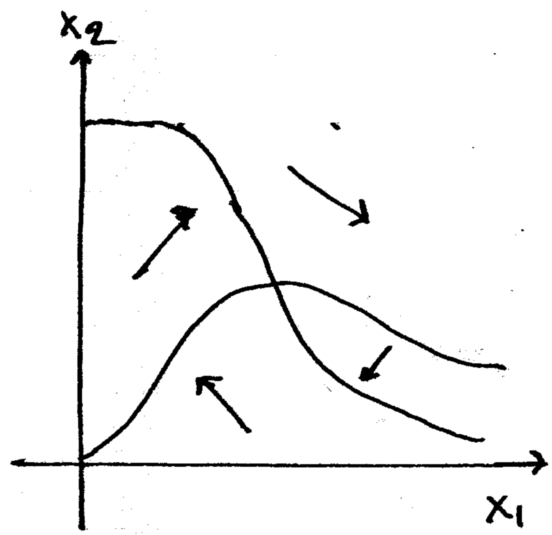
$$g(x_1,x_2)\geq 0 \Leftrightarrow b-ax_2-x_1^2x_2\geq 0 \Leftrightarrow$$
$$\Leftrightarrow b-x_2(a+x_1^2)\geq 0 \Leftrightarrow x_2(a+x_1^2)\leq b \Leftrightarrow$$
$$\Leftrightarrow x_2\leq \frac{b}{a+x_1^2}$$

thus $\dot{x}_2>0$ below curve (2)

$\dot{x}_2<0$ above curve (2).

It follows that:





Thus a limit cycle is indicated.

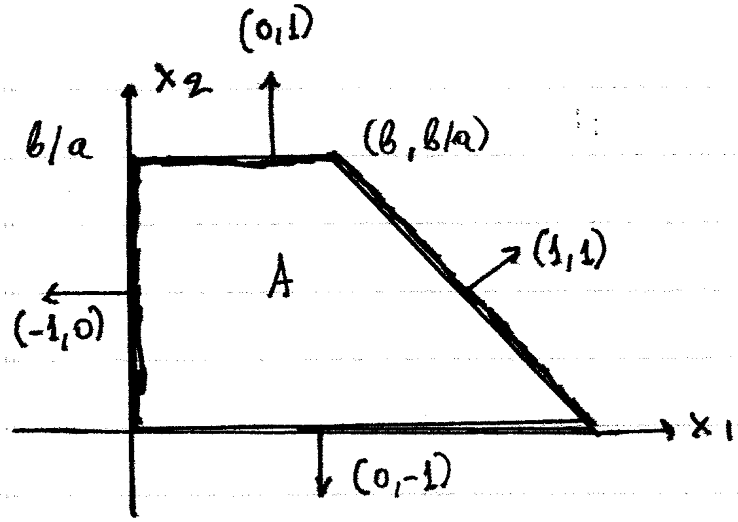
- <u>Trapping Region</u>

a) x_1-axis with $x_1>0$.

At $x_2=0$, normal vector is $(0,-1)$ and

$$(0,-1)\cdot(f(x_1,0),g(x_1,0))=-g(x_1,0)=$$
$$=-(b-a\cdot 0-x_1^2\cdot 0)=-b<0$$

thus x_1-axis is trapping.



The claimed trapping region with the outward normal vectors indicated.

b) At x_2-axis with $x_2 > 0$
At $x_1 = 0$, normal vector is $(-1,0)$ and

$$(-1,0)\cdot(f(0,x_2), g(0,x_2)) = -f(0,x_2) =$$
$$= -(-0 + ax_2 + 0x_2) = -ax_2 < 0 \Rightarrow$$

$\Rightarrow$ x_2-axis is trapping.

c) At infinity: $\begin{cases} \dot{x}_1 \sim x_1^2 x_2 \\ \dot{x}_2 \sim -x_1^2 x_2 \end{cases} \Rightarrow \dfrac{\dot{x}_1}{\dot{x}_2} \sim -1$

Consider any line with slope -1. Thus $n = (1,1)$.
We note that

$$(1,1)\cdot(f(x_1,x_2), g(x_1,x_2)) = f(x_1,x_2) + g(x_1,x_2) =$$
$$= (-x_1 + \underline{\underline{ax_2}} + \underline{x_1^2 x_2}) + (b - \underline{\underline{ax_2}} - \underline{x_1^2 x_2}) =$$
$$= b - x_1 < 0 \quad \text{for} \quad x_1 > b.$$

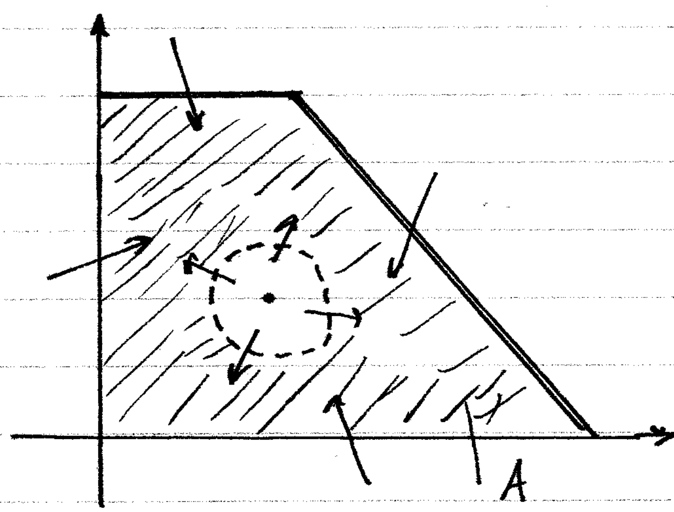
Thus construct a line from $(0, b/a)$ to $(b, b/a)$ and from $(b, b/a)$ to the x_1-axis with slope -1. The latter segment has been shown to be trapping. Now consider the line from $(0, b/a)$ to $(b, b/a)$:

At $x_2 = b/a$ with $0 < x_1 < b$, using $n = (0,1)$

$$(0,1)\cdot(f(x_1, b/a), g(x_1, b/a)) = g(x_1, b/a) =$$
$$= b - a\cdot(b/a) - x_1^2 (b/a) =$$
$$= b - b - x_1^2 b/a = \frac{-b x_1^2}{a} < 0 \Rightarrow \text{also trapping}$$

Thus the claimed region A is a trapping region.

↳ Because the region A contains a fixed point where the two nullclines intersect, we cannot apply the Poincare – Bendixson theorem. However if we show that the fixed point is <u>repelling</u> (i.e. a source, unstable spiral/star, unstable degenerate node), then we can remove a ball around the fixed point and redefine the trapping region as follows:



Thus we will now derive the necessary and sufficient condition for the fixed point to be repelling.

- <u>Fixed point location</u>

$$\begin{cases} f(x_1,x_2)=0 \\ g(x_1,x_2)=0 \end{cases} \Leftrightarrow \begin{cases} -x_1+ax_2+x_1^2x_2=0 \\ b-ax_2-x_1^2x_2=0 \end{cases} \Leftrightarrow$$

Add: $b-x_1=0$

$$\Leftrightarrow \begin{cases} b-x_1=0 \\ b-ax_2-x_1^2x_2=0 \end{cases} \Leftrightarrow \begin{cases} x_1=b \\ b-ax_2-b^2x_2=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1=b \\ b-(a+b^2)x_2=0 \end{cases} \Leftrightarrow \begin{cases} x_1=b \\ x_2=\dfrac{b}{a+b^2} \end{cases}$$

- <u>Jacobian</u>

$$DF(x_1,x_2)=\begin{bmatrix} -1+2x_1x_2 & a+x_1^2 \\ -2x_1x_2 & -(a+x_1^2) \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(b, b/(a+b^2)) = \begin{bmatrix} -1+2b\cdot\dfrac{b}{a+b^2} & a+b^2 \\ -2b\dfrac{b}{a+b^2} & -(a+b^2) \end{bmatrix} =$$

$$= \frac{1}{a+b^2}\begin{bmatrix} -(a+b^2)+2b^2 & (a+b^2)^2 \\ -2b^2 & -(a+b^2)^2 \end{bmatrix} =$$

$$= \frac{1}{a+b^2}\begin{bmatrix} b^2-a & (a+b^2)^2 \\ -2b^2 & -(a+b^2)^2 \end{bmatrix}$$

- Conditions

The fixed point $(b, b/(a+b^2))$ is repelling if and only if $\tau = \lambda_1 + \lambda_2 > 0$ and $D = \lambda_1 \lambda_2 > 0$.

$$\tau = tr(DF(b, b/a)) = \frac{1}{a+b^2}\left[(b^2-a) - (a+b^2)^2\right] =$$

$$= \frac{(b^2-a) - (a^2+2ab^2+b^4)}{(a+b^2)^2} =$$

$$= \frac{b^2 - a - a^2 - 2ab^2 - b^4}{(a+b^2)^2} = \frac{-b^4 + (1-2a)b^2 - (a+a^2)}{(a+b^2)^2}$$

$$D = \det(DF(b, b/(a+b^2))) =$$

$$= \frac{1}{(a+b^2)^2}\left[(b^2-a)\left[-(a+b^2)^2\right] - (-2b^2)(a+b^2)^2\right] =$$

$$= \frac{-(b^2-a)(a+b^2)^2 + 2b^2(a+b^2)^2}{(a+b^2)^2} =$$

$$= \frac{(a+b^2)^2\left[-(b^2-a)+2b^2\right]}{(a+b^2)^2} = -(b^2-a) + 2b^2 =$$

$$= b^2 + a.$$

Note that $a > 0 \wedge b > 0 \Rightarrow D = a + b^2 > 0$

thus the necessary and sufficient condition is:

$$\tau > 0 \iff -b^4 + (1-2a)b^2 - (a+a^2) > 0$$

$$\iff b^4 + (2a-1)b^2 + (a+a^2) < 0 \iff p(b^2) < 0$$

with $p(x) \equiv x^2 + (2a-1)x + (a+a^2)$.

Discriminant:

$$\Delta = (2a-1)^2 - 4\cdot 1\cdot (a^2+a) = 4a^2 - 4a + 1 - 4a^2 - 4a = 1 - 8a$$

For $\Delta \leq 0 \Rightarrow p(b^2) \geq 0,\ \forall b \in \mathbb{R} \Rightarrow \tau \leq 0$

For $\Delta > 0 \iff 1 - 8a > 0 \iff a < 1/8 \iff 0 < a < 1/8$

we have two zeroes:

$$b_{1,2}^2 = \frac{(1-2a) \pm \sqrt{1-8a}}{2}$$

thus: $\tau > 0 \iff p(b^2) < 0 \iff b_1^2 < b^2 < b_2^2 \iff$

$$\iff \frac{1-2a-\sqrt{1-8a}}{2} < b^2 < \frac{1-2a+\sqrt{1-8a}}{2}$$

It follows that the fixed point is repelling when

$$\boxed{\begin{array}{l} 0 < a < 1/8 \\ \dfrac{1-2a-\sqrt{1-8a}}{2} < b^2 < \dfrac{1-2a+\sqrt{1-8a}}{2} \end{array}}$$

EXERCISES

(6) Use the Poincare-Bendixson theorem to show that the following systems have a limit cycles.

a) $\begin{cases} \dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + 5x_2^2) \\ \dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{cases}$

b) $\begin{cases} \dot{x}_1 = x_1 - x_2 - x_1^3 \\ \dot{x}_2 = x_1 + x_2 - x_2^3 \end{cases}$

c) $\begin{cases} \dot{x}_1 = -x_1 - x_2 + x_1(x_1^2 + 2x_2^2) \\ \dot{x}_2 = x_1 - x_2 + x_2(x_1^2 + 2x_2^2) \end{cases}$

(7) Consider the system

$$\begin{cases} \dot{x}_1 = x_1(1 - 4x_1^2 - x_2^2) - (1/2)x_2(1 + x_1) \\ \dot{x}_2 = x_2(1 - 4x_1^2 - x_2^2) + 2x_1(1 + x_1) \end{cases}$$

Use the function $V(x_1, x_2) = (1 - 4x_1^2 - x_2^2)^2$ to show that all trajectories converge to $(c) : 4x_1^2 + x_2^2 = 1$ as $t \to +\infty$.

(8) Consider the nonlinear oscillator

$$\ddot{x} + F(x, \dot{x})\dot{x} + x = 0$$

We assume that $F(x, \dot{x}) < 0$ when $z \leq a$, and $F(x,\dot{x}) > 0$ when $z \geq b$, with $z = x^2 + \dot{x}^2$. Show that the system has a limit cycle with $a < z < b$.

⑨ Consider the system

$$\begin{cases} \dot{x}_1 = x_2 + a x_1 (1 - 2b - x_1^2 - x_2^2) \\ \dot{x}_2 = -x_1 + a x_2 (1 - x_1^2 - x_2^2) \end{cases}$$

a) Write the system in polar coordinates.

b) Determine how many limit cycles exist in this system.

Lienard systems

Consider the system governed by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \quad (1)$$

Equivalently, we may rewrite (1) as an autonomaus system as:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_1)x_2 - g(x_1) \end{cases}$$

Thm : (Lienard's theorem)

Assume that

a) f, g are continuously differentiable in $\mathbb{R}$.

b) f even $(\forall x \in \mathbb{R} : f(-x) = f(x))$

c) g odd $(\forall x \in \mathbb{R} : g(-x) = -g(x))$

d) $g(x) > 0, \forall x \in (0, +\infty)$

e) For $F(x)$ given by:

$$F(x) = \int_0^x f(t)\,dt$$

there is an $a \in (0, +\infty)$ such that:

$$\begin{cases} F(x) < 0, & \forall x \in (0, a) \\ F(a) = 0 \\ F(x) > 0, & \forall x \in (a, +\infty) \end{cases}$$

Then there is a stable limit cycle around the origin (0,0). Furthermore, the limit cycle is <u>unique</u>.

EXAMPLE

a) Show that

$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$ (van der Pol oscillator)

has a limit cycle, for $\mu > 0$.

<u>Solution</u>

Let $f(x) = \mu(x^2-1)$ and $g(x) = x$. We note that:

a) f, g are continuously differentiable

b) $f(-x) = \mu((-x)^2-1) = \mu(x^2-1)$
$= f(x)$, $\forall x \in \mathbb{R} \Rightarrow f$ even

c) $g(-x) = -x = -g(x), \forall x \in \mathbb{R} \Rightarrow g$ odd.

d) $g(x) = x > 0$, $\forall x \in (0, +\infty)$

e) Let

$$F(x) = \int_0^x f(t)\,dt = \int_0^x \mu(t^2-1)\,dt =$$

$$= \mu\left[\frac{x^3}{3} - x\right]_0^x = \mu\left[\frac{x^3}{3} - x\right] =$$

$$= \frac{\mu}{3}(x^3 - 3x) = \frac{1}{3}\mu x(x^2-3)$$

For $a = \sqrt{3}$:

$\forall x \in (0, \sqrt{3})$: $x^2 - 3 < 0 \wedge x > 0 \Rightarrow F(x) < 0$

$$F(\sqrt{3}) = \frac{1}{3}\mu\sqrt{3}\left((\sqrt{3})^2 - 3\right) = 0$$

$$\forall x \in (\sqrt{3}, +\infty):\ x^2 - 3 > 0 \wedge x > 0 \Rightarrow F(x) > 0.$$

From (a), (b), (c), (d), (e) it follows that the system has a stable limit cycle.

EXERCISES

(10) Show that the system
$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \tanh x = 0$$
has a unique stable limit cycle when $\mu > 0$.

(11) Consider the system
$$\ddot{x} + \mu(x^4 - 1)\dot{x} + x = 0$$

a) Show that it has a unique stable limit cycle when $\mu > 0$.

b) If $\mu < 0$, does the system still have a stable limit cycle?

▼ Hopf bifurcation

- Consider the two-dimensional system

$$\begin{cases} \dot{x} = f(x, y, \mu) \\ \dot{y} = g(x, y, \mu) \end{cases} \quad (1)$$

and define

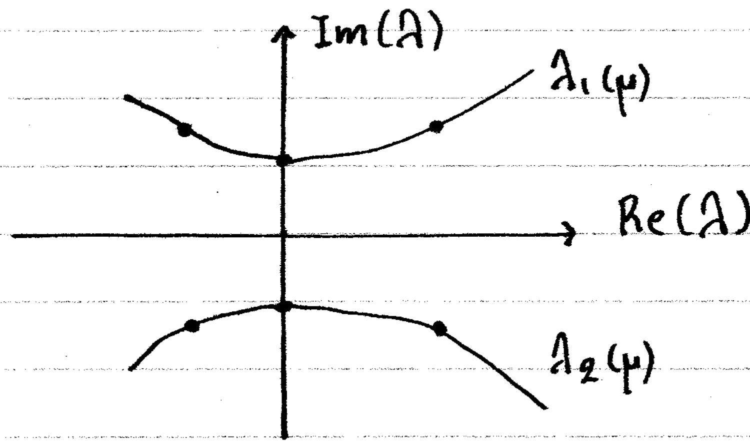
$$F(x, y, \mu) = (f(x, y, \mu), g(x, y, \mu))$$

Let $(x_0(\mu), y_0(\mu))$ be the location of a fixed point, dependant on the parameter μ.
Let $\lambda_1(\mu), \lambda_2(\mu)$ be the eigenvalues of the Jacobian of the above-referenced fixed point:

$$\lambda(DF(x_0(\mu), y_0(\mu))) = \{\lambda_1(\mu), \lambda_2(\mu)\}.$$

- <u>Def</u>: We say that (1) undergoes a <u>Hopf bifurcation</u> at $\mu = \mu_0$ when the following conditions are satisfied:

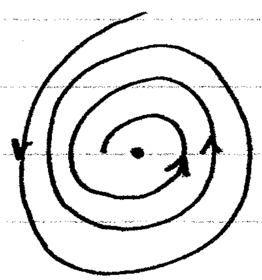
a) $\forall \mu \in (\mu_0 - \varepsilon, \mu_0 + \varepsilon) : \lambda_{1,2}(\mu) = \gamma(\mu) \pm i\omega(\mu)$
b) $\gamma(\mu_0) = 0$
c) $\gamma(\mu) < 0, \ \forall \mu < \mu_0$
d) $\gamma(\mu) > 0, \ \forall \mu > \mu_0$



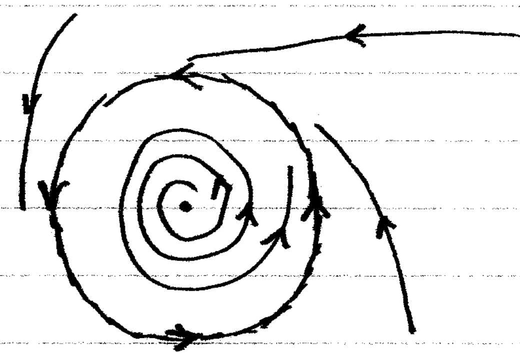
Eigenvalue diagram.

Phenomenology of the Hopf bifurcation

(1) Supercritical Hopf Bifurcation



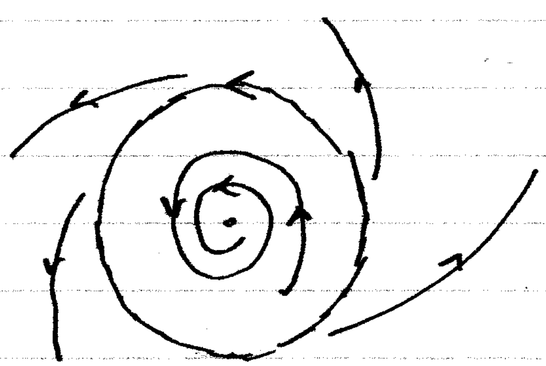
$\mu < \mu_0$



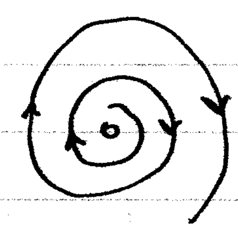
$\mu > \mu_0$

A stable spiral fixed point for $\mu < \mu_0$ becomes an unstable spiral for $\mu > \mu_0$ surrounded by a stable limit cycle which expands with increasing μ.

(2) Subcritical Hopf Bifurcation



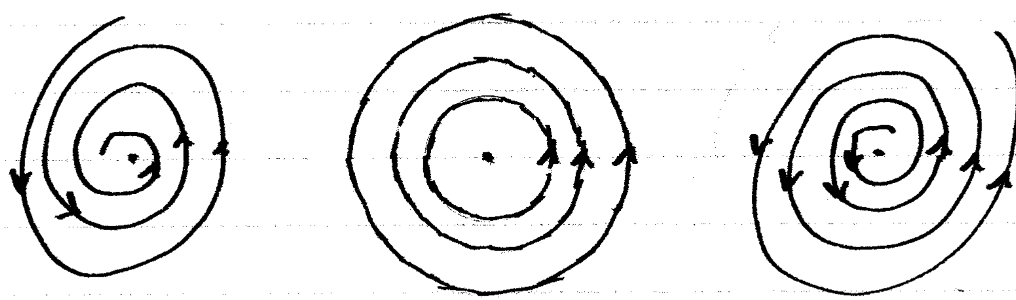
$\mu < \mu_0$



$\mu > \mu_0$

At $\mu < \mu_0$, a stable spiral is bounded by an unstable limit cycle. With increasing μ, the unstable limit cycle becomes smaller. At $\mu = \mu_0$, the cycle collapses onto the fixed point. At $\mu > \mu_0$, the fixed point becomes an unstable spiral.

(3) Degenerate Hopf Bifurcation



$\mu < \mu_0$ $\qquad\qquad$ $\mu = \mu_0$ $\qquad\qquad$ $\mu > \mu_0$

A stable spiral at $\mu < \mu_0$ becomes a nonlinear center at $\mu = \mu_0$ and then an unstable spiral at $\mu > \mu_0$. No limit cycle occurs at either $\mu < \mu_0$ or $\mu > \mu_0$.

Hopf bifurcation prototype

Consider the system:

$$\boxed{\begin{cases} \dot{x}_1 = \mu x_1 - x_2 + \sigma x_1 (x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 + \mu x_2 + \sigma x_2 (x_1^2 + x_2^2) \end{cases}}$$

This system undergoes a Hopf bifurcation at $(0,0)$ when $\mu = 0$.

a) $\sigma = +1 \Rightarrow$ subcritical Hopf bifurcation

b) $\sigma = -1 \Rightarrow$ supercritical Hopf bifurcation

c) $\sigma = 0 \Rightarrow$ degenerate Hopf bifurcation.

Proof

$$DF(0,0) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(0,0) - \lambda I) = \begin{vmatrix} \mu-\lambda & -1 \\ 1 & \mu-\lambda \end{vmatrix} =$$

$$= (\mu-\lambda)(\mu-\lambda) - (-1)\cdot 1 = (-\mu+\lambda)^2 + 1 =$$

$$= (-\mu+\lambda+i)(-\mu+\lambda-i) \Rightarrow$$

$$\Rightarrow \lambda(DF(0,0)) = \{\mu - i, \mu + i\}$$

Thus, there is a Hopf bifurcation at $\mu = 0$.

- <u>Polar representation</u>

$$r^2 = x_1^2 + x_2^2 \Rightarrow$$

$$\Rightarrow 2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 =$$

$$= 2x_1\left[\mu x_1 - x_2 + \sigma x_1(x_1^2+x_2^2)\right] + 2x_2\left[x_1 + \mu x_2 + \sigma x_2(x_1^2+x_2^2)\right]$$

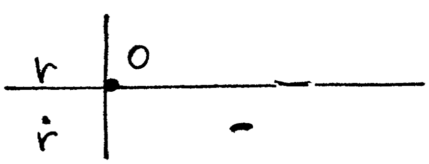
$$= 2\mu x_1^2 - \underline{2x_1x_2} + 2\sigma x_1^2 r^2 + \underline{2x_1x_2} + 2\mu x_2^2 + 2\sigma x_2^2 r^2$$

$$= 2\mu(x_1^2+x_2^2) + 2\sigma r^2(x_1^2+x_2^2) = 2\mu r^2 + 2\sigma r^4 \Rightarrow$$

$$\Rightarrow \dot{r} = \sigma r^3 + \mu r \Rightarrow \dot{r} = r(\sigma r^2 + \mu)$$

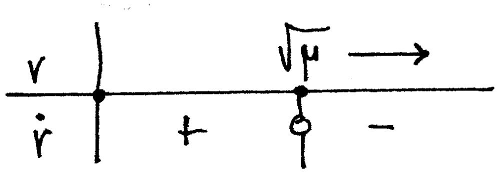
- <u>Analysis</u>

a) For $\sigma = -1$: $\dot{r} = r(-r^2+\mu)$



$\mu < 0$

stable spiral

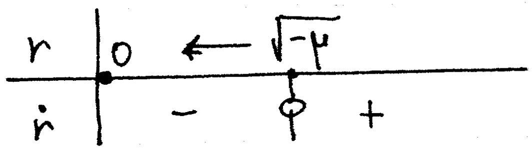


$\mu > 0$

unstable spiral
+ stable limit cycle

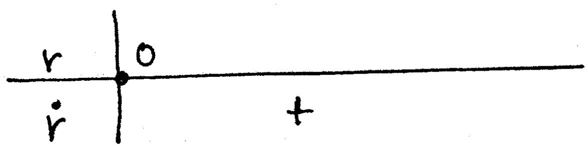
Thus: supercritical Hopf bifurcation

b) For $\sigma = 1$: $\dot{r} = r(r^2 + \mu)$



$\mu < 0$

stable spiral + unstable limit cycle.



$\mu > 0$

unstable spiral

Thus: subcritical Hopf bifurcation

c) For $\sigma = 0$: $\boxed{\dot{r} = +\mu r}$

Note that:

$\mu < 0 \Rightarrow \dot{r} < 0 \Rightarrow$ stable spiral

$\mu = 0 \Rightarrow \dot{r} = 0 \Rightarrow$ nonlinear center

$\mu > 0 \Rightarrow \dot{r} > 0 \Rightarrow$ unstable spiral

Thus: degenerate Hopf bifurcation.

• Classifying Hopf Bifurcations

It is rather easy to show that a system undergoes a Hopf bifurcation, simply by examining the Jacobian's eigenvalues as a function of μ. What is more challenging is to determine whether the Hopf bifurcation is supercritical, subcritical, or degenerate. To do that, we work as follows:

•1 Write the system in the form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{bmatrix}$$

with f, g nonlinear functions such that

$f(x_0, y_0, \mu_0) = 0$ and $g(x_0, y_0, \mu_0) = 0$

•2 Evaluate the 1st Lyapunov coefficient

$$\Lambda_1 = [f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + \frac{1}{\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]$$

Here the subscripts denote partial derivatives and all terms are evaluated at $(x, y, \mu) = (x_0, y_0, \mu_0)$.

•3 $\Lambda_1 < 0 \Rightarrow$ supercritical Hopf bifurcation

$\Lambda_1 = 0 \Rightarrow$ degenerate Hopf bifurcation

$\Lambda_1 > 0 \Rightarrow$ subcritical Hopf bifurcation

EXAMPLE

a) Show that

$\dot{x} = \mu x - y + xy^2$

$\dot{y} = x + \mu y + y^3$

undergoes a subcritical Hopf bifurcation at $(x,y) = (0,0)$ and $\mu_0 = 0$

Solution

Let $f(x,y) = \mu x - y + xy^2$ and $g(x,y) = x + \mu y + y^3$ and $F(x,y) = (f(x,y), g(x,y))$. It follows that

$$DF(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} = \begin{bmatrix} \mu + y^2 & -1 + 2xy \\ 1 & \mu + 3y^2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(0,0) - \lambda I) = \begin{vmatrix} \mu-\lambda & -1 \\ 1 & \mu-\lambda \end{vmatrix} =$$

$$= (\mu-\lambda)^2 - 1\cdot(-1) = (\mu-\lambda)^2 + 1 = (\mu-\lambda-i)(\mu-\lambda+i) \Rightarrow$$

$\Rightarrow \lambda(DF(0,0)) = \{\mu + i, \mu - i\} \Rightarrow$ Hopf bifurcation at $(0,0)$ at $\mu = 0$.

- Classification.

We note that

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \mu x + xy^2 \\ \mu y + y^3 \end{bmatrix}$$

with $f(x,y) = \mu x + xy^2$

$g(x,y) = \mu y + y^3$.

At $(x,y)=0$:

$f_x(x,y)=\mu+y^2 \Rightarrow f_{xy}(x,y)=0 \Rightarrow f_{xy}(0,0)=0$

$f_{xx}(x,y)=(\partial/\partial x)f_x(x,y)=(\partial/\partial x)(\mu+y^2)=0 \Rightarrow f_{xx}(0,0)=0$

$f_{xxx}(x,y)=(\partial/\partial x)f_{xx}(x,y)=0 \Rightarrow f_{xxx}(0,0)=0$

$f_y(x,y)=(\partial/\partial y)(\mu x+xy^2)=2xy \Rightarrow f.$

$\Rightarrow f_{yy}(x,y)=(\partial/\partial y)f_y(x,y)=(\partial/\partial y)(2xy)=2x \Rightarrow$

$\Rightarrow f_{yy}(0,0)=0$

$f_{xyy}(x,y)=(\partial/\partial x)f_{yy}(x,y)=(\partial/\partial x)2x=2 \Rightarrow f_{xyy}(0,0)=2.$

$g_x(x,y)=(\partial/\partial x)g(x,y)=(\partial/\partial x)(\mu y+y^3)=0$

$\Rightarrow g_{xx}(x,y)=0 \Rightarrow g_{xxx}(x,y)=0$

It follows that $g_{xx}(0,0)=0$ and $g_{xxx}(0,0)=0$

Also $g_{xy}(x,y)=(\partial/\partial y)g_x(x,y)=0 \Rightarrow$

$\Rightarrow g_{xxy}(x,y)=(\partial/\partial x)g_{xy}(x,y)=0$

and therefore: $g_{xy}(0,0)=0$ and $g_{xxy}(0,0)=0$

$g_y(x,y)=(\partial/\partial y)(\mu y+y^3)=\mu+3y^2 \Rightarrow g_{yy}(x,y)=6xy \Rightarrow$

$\Rightarrow g_{yyy}(x,y)=6$ and therefore:

$g_{yy}(0,0)=0$ and $g_{yyy}(0,0)=6.$

From the above, using $\omega=1$, we have

$$\Lambda_1=[f_{xxx}+f_{xyy}+g_{xxy}+g_{yyy}]+$$
$$+\frac{1}{\omega}[f_{xy}(f_{xx}+f_{yy})-g_{xy}(g_{xx}+g_{yy})-f_{xx}g_{xx}+f_{yy}g_{yy}]=$$

$$=[0+2+0+6]+(1/1)[0(0+0)-0(0+0)-0\cdot0+0\cdot0]=8>0 \Rightarrow$$

$\Rightarrow$ subcritical Hopf bifurcation.

EXERCISES

(13) Show that the following systems undergo a Hopf bifurcation at $\mu = 0$ and classify it as supercritical, subcritical, or degenerate.

a) $$\begin{cases} \dot{x} = \mu x + y \\ \dot{y} = -x + \mu y - x^2 y \end{cases}$$

b) $$\begin{cases} \dot{x} = \mu x + y - x^3 \\ \dot{y} = -x + \mu y + 2y^3 \end{cases}$$

c) $$\begin{cases} \dot{x} = \mu x + y - x^2 \\ \dot{y} = -x + \mu y + 2x^2 \end{cases}$$

MM6: Center manifold reduction

Center Manifold Reduction

This technique is based on the following theorem

<u>Theorem</u> : Consider the following system of $n+m$ ordinary differential equations:

$$\begin{cases} \dot{x} = Ax + f(x,y) \\ \dot{y} = By + g(x,y) \end{cases}, \text{ with } \begin{cases} f(\mathbf{0}) = \mathbf{0} \wedge Df(\mathbf{0}) = \mathbf{0} \\ g(\mathbf{0}) = \mathbf{0} \wedge Dg(\mathbf{0}) = \mathbf{0} \end{cases}$$

with $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$, $g : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$.
Here, A is an $n \times n$ matrix, B is an $m \times m$ matrix, with

$$\begin{cases} \forall \lambda \in \lambda(A) : \operatorname{Re}(\lambda) = 0 \\ \forall \lambda \in \lambda(B) : \operatorname{Re}(\lambda) < 0 \end{cases}$$

Then there exists a center-manifold W^c given by

$$W^c = \{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = h(x) \}$$

with $h : \mathbb{R}^n \to \mathbb{R}^m$, and $h(\mathbf{0}) = \mathbf{0}$, and $Dh(\mathbf{0}) = \mathbf{0}$ such that the solution of the nonlinear system converges to W^c as $t \to +\infty$ if initialized near enough the fixed point $\mathbf{0}$.

The theorem can be used to analyze non-hyperbolic fixed points where all the eigenvalues of the corresponding Jacobian matrix are either zero or negative. The method cannot be applied if at least one eigenvalue is positive (in the real part $\operatorname{Re}(\lambda)$).

Methodology

Let $\dot{x} = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$ be an autonomous dynamical system. Let $x_0 \in \mathbb{R}^n$ be a fixed point with $f(x_0) = 0$. We assume that x_0 is a non-hyperbolic fixed point such that some of the eigenvalues of $Df(x_0)$ have zero real part and none of the eigenvalues have a strictly positive real part. In other words, we assume that

$$\begin{cases} \exists \lambda \in \lambda(Df(x_0)) : Re(\lambda) = 0 \\ \forall \lambda \in \lambda(Df(x_0)) : Re(\lambda) \leq 0 \end{cases}$$

The center manifold reduction technique consists of the following 3 steps:

1) Reduce system to canonical form
2) Apply the center-manifold theorem.
3) Determine series expansion for mapping h.

Reduction to canonical form

•₁ We linearize the autonomous system around x_0 and write:

$$\dot{x} = Df(x_0)x + G(x)$$

Here $G(x)$ captures the nonlinear terms of the system.

•₂ Assume that $Df(x_0)$ has distinct eigenvalues

$$\lambda(Df(x_0)) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$$

with corresponding eigenvectors $v_1, v_2, \dots, v_n$.

We daagonalize $Df(x_0)$ by defining

$$P = [\,V_1\ V_2\ \cdots\ V_n\,]$$

and writing

$$Df(x_0) = P\,\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) P^{-1}$$

Note that $Df(x_0)$ can be diagonalized even when the eigenvalues are not distinct, into a block-diagonal matrix.

• 3 Define the change of variables $y = P^{-1}x$.

It follows that $x = Py$, and therefore

$$\begin{aligned}
\dot{y} &= P^{-1}\dot{x} = P^{-1}(Df(x_0)x + G(x)) = P^{-1}(Df(x_0)Py + G(Py)) \\
&= [P^{-1}Df(x_0)P]\,y + G(Py) = \\
&= [P^{-1}P\,\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)P^{-1}P]\,y + G(Py) = \\
&= \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)\,y + G(Py)
\end{aligned}$$

which reduces to the following system of ODEs:

$$\begin{cases}
\dot{y}_1 = \lambda_1 y_1 + g_1(y_1, y_2, \dots, y_n) \\
\dot{y}_2 = \lambda_2 y_2 + g_2(y_1, y_2, \dots, y_n) \\
\text{-----} \\
\dot{y}_n = \lambda_n y_n + g_n(y_1, y_2, \dots, y_n)
\end{cases}$$

• Determining the center manifold

Let us now assume that $\mathrm{Re}(\lambda_a) = 0, \forall a \in [k]$ and $\mathrm{Re}(\lambda_a) < 0, \forall a \in [n] - [k]$. Then, according to the center manifold theorem, the first k equations for $y_1, y_2, \dots, y_k$

drive the dynamics of the system and the other $n-k$ equations are driven by <u>slaving principles</u> given by

$$\begin{cases} y_{k+1} = h_1(y_1, y_2, \dots, y_k) \\ y_{k+2} = h_2(y_1, y_2, \dots, y_k) \\ \vdots \\ y_n = h_{n-k}(y_1, y_2, \dots, y_k) \end{cases}$$

Let us define $u = (y_1, y_2, \dots, y_k)$ and $v = (y_{k+1}, y_{k+2}, \dots, y_n)$ and rewrite the above system as $v = h(u)$, with $h : \mathbb{R}^k \to \mathbb{R}^{n-k}$

Also, let $(u_0, v_0) = y_0 = P^{-1} x_0$ be the fixed point.

Then according to the center manifold theorem, h has to satisfy:

$$\begin{cases} h(u_0) = 0 \\ Dh(u_0) = 0 \end{cases}$$

Now let us rewrite the original system of ODEs for $y = (u, v)$ as follows:

$$\dot{u} = Au + G_1(u, v)$$
$$\dot{v} = Bv + G_2(u, v)$$

with $A = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ and $B = \mathrm{diag}(\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n)$ and G_1, G_2 are given by

$$G_1 = (g_1, g_2, \dots, g_k)$$
$$G_2 = (g_{k+1}, g_{k+2}, \dots, g_n)$$

To write governing PDEs for h, we note that

$$v = h(u) \Rightarrow \dot{v} = Dh(u)\,\dot{u} = Dh(u)\left[Au + G_1(u, v)\right] =$$
$$= Dh(u)\left[Au + G_1(u, h(u))\right]$$

and

$\dot{v} = Bv + G_2(u,v) = Bh(u) + G_2(u,h(u))$

and it follows that

$$Dh(u)[Au + G_1(u,h(u))] = Bh(u) + G_2(u,h(u))$$

Now, let us define the operator

$$\boxed{(Nh)(u) = Dh(u)[Au + G_1(u,h(u))] - [Bh(u) + G_2(u,h(u))]}$$

Then, h is given by the solution of the following initial value problem:

$$\boxed{\begin{cases} (Nh)(u) = 0 \\ h(u_0) = 0 \\ Dh(u_0) = 0 \end{cases}}$$

Note that in terms of components, $(Nh)(u)$ is given by:

$$\boxed{(Nh)_a(u) = \sum_{b=1}^{k} \left[(\lambda_b y_b + g_b(y)) \frac{\partial h_a}{\partial y_b} \right] - (\lambda_{k+a} h_a + g_{k+a}(y))}$$

● <u>The spectral gap theorem</u>

The mapping h can be determined via a power-series technique based on the following spectral gap theorem.

<u>Theorem</u> : Let an arbitrary $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ be given with $\psi(u_0) = \mathbb{0}$ and $D\psi(u_0) = \mathbb{0}$. Then, under the limit $u \to u_0$, we can show that

$$\exists q > 1 : (N\psi)(u) = O(\|u - u_0\|^q) \Rightarrow \|h(u) - \psi(u)\| = O(\|u - u_0\|^q)$$

It follows that power-series techniques can be used to approximate the center manifold to any degree of approximation by solving the equation $(N\psi)(u) = \mathbb{0}$ to the same degree of approximation, as shown in the examples below.

EXAMPLES

a) $\begin{cases} \dot{x} = x^2 y - x^5 \\ \dot{y} = -y + x^2 \end{cases}$ $\longleftarrow$ Analyze fixed points.

Solution

- Fixed points and Jacobian.

Let $f(x,y) = x^2 y - x^5$ and $g(x,y) = -y + x^2$.

(x,y) fixed point $\Leftrightarrow \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 y - x^5 = 0 \\ -y + x^2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x^2(y - x^3) = 0 \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x^2 = 0 \\ y = x^2 \end{cases} \vee \begin{cases} y = x^3 \\ y = x^2 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^3 - x^2 = 0 \\ y = x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^2(x - 1) = 0 \\ y = x^2 \end{cases}$

$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = 1 \end{cases} \Leftrightarrow (x,y) \in \{(0,0), (1,1)\}.$

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} 2xy - 5x^4 & x^2 \\ 2x & -1 \end{bmatrix}$$

- For $(x,y) = (1,1)$:

$DF(1,1) = \begin{bmatrix} 2-5 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow$

$\Rightarrow p(\lambda) = \det(DF(1,1) - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} =$

$= (-3-\lambda)(-1-\lambda) - 2 = (\lambda+3)(\lambda+1) - 2 = \lambda^2 + 4\lambda + 3 - 2$

$= \lambda^2 + 4\lambda + 1$

$\Delta = b^2 - 4ac = 4^2 - 4\cdot 1\cdot 1 = 16 - 4 = 12 = 4\cdot 3 \Rightarrow$

$\Rightarrow \lambda_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$ (both negative)

thus $\lambda(DF(1,1)) = \{-2-\sqrt{3}, -2+\sqrt{3}\} \Rightarrow (1,1)$ is a stable sink.

- For $(x,y) = (0,0)$:

$DF(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(DF(0,0)) = \{0, -1\} \Rightarrow$

$\Rightarrow (0,0)$ is a non-hyperbolic fixed point.

- Center-Manifold analysis: We note that $\dot{x}$ is the master equation and $\dot{y}$ is the slave equation, the system being already written in canonical form. So, let us consider $y = h(x)$ with $h(0) = 0$ and $h'(0) = 0$. It follows that

$$\left.\begin{aligned} \dot{y} &= h'(x)\dot{x} = h'(x)[x^2y - x^5] = h'(x)[x^2h(x) - x^5] \\ \dot{y} &= -y + x^2 = -h(x) + x^2 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow (Nh)(x) = h'(x)[x^2h(x) - x^5] + h(x) - x^2$

Let $h(x) = ax^2 + bx^3 + O(x^4) \Rightarrow h'(x) = 2ax + 3bx^2 + O(x^3)$

and it follows that

$$\begin{aligned} (Nh)(x) &= [2ax + 3bx^2 + O(x^3)][x^2(ax^2 + bx^3 + O(x^4)) - x^5] \\ &\quad + ax^2 + bx^3 + O(x^4) - x^2 = \\ &= x^2(2ax + 3bx^2)(ax^2 + bx^3) + O(x^9) - x^5[2ax + 3bx^2 + O(x^3)] \\ &\quad + ax^2 + bx^3 + O(x^4) - x^2 = \end{aligned}$$

$$= x^2(2a^2x^3+2abx^4+3abx^4+3b^2x^5) - 2ax^6 - 3bx^7 \underline{+ax^2} + \underline{\underline{bx^3}} \underline{-x^2} + O(x^4)$$

$$= (a-1)x^2 + bx^3 + O(x^4).$$

and therefore:

$$(Nh)(x) = 0 + O(x^4) \Leftrightarrow (a-1)x^2 + bx^3 + O(x^4) = O(x^4) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a-1=0 \\ b=0 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=0 \end{cases}$$

It follows that $h(x) = x^2 + O(x^4)$.

Then $\dot{x} = x^2y - x^5 = x^2h(x) - x^5 = x^2[x^2 + O(x^4)] - x^5$

$= x^4 + O(x^5)$

and the center-manifold reduction is:

$$\begin{cases} \dot{x} = x^4 + O(x^5) \leftarrow \text{master equation} \\ y = x^2 + O(x^4) \leftarrow \text{slave equation} \end{cases}$$

From the master equation we see that the $(0,0)$ fixed point is unstable.

Note that $\lambda(DF(0,0)) = \{0,-1\}$, thus linear stability analysis might suggest that $(0,0)$ is Lyapunov stable, and in the absense of positive eigenvalues there is no hint of instability. On the other hand, because $(0,0)$ is not hyperbolic, so linear stability analysis is not guaranteed to be accurate, and center manifold analysis shows that the $(0,0)$ fixed point is in fact unstable.

↳ Note that the center manifold ensures local convergence: if the initial condition is close to the center manifold, it will converge to the center manifold. We ~~co~~ can also investigate <u>global convergence</u>, i.e. whether or not ALL initial conditions converge to the center manifold via the following argument:

Since the center manifold is $y = x^2 + O(x^4)$, we define $U(x,y) = (y - x^2)^2$.

It is sufficient to show that $\dot{U}(x,y) < 0$.

$$\begin{aligned}
\dot{U}(x,y) &= (d/dt)\left[(y-x^2)^2\right] = 2(y-x^2)(\dot{y} - 2x\dot{x}) = \\
&= 2(y-x^2)(-y+x^2-2x(x^2y-x^5)) = \\
&= -2(y-x^2)^2 - 4x(y-x^2)(x^2y-x^5) \\
&= -2(y-x^2)^2 - 4(y-x^2)(x^3y-x^6) \\
&= -2(y-x^2)^2 - 4(x^3y^2 - x^6y - x^5y + x^8) = \\
&= -2(y-x^2)^2 - 4x^8 + 4x^3y(-y+x^3+x^2)
\end{aligned}$$

First two terms are negative, third term is unclear (negative or positive). Let us assume that $y = x^2 + \varepsilon$ with ε small. Then, it follows that

$$\begin{aligned}
4x^3y(-y+x^2+x^3) &= 4x^3(x^2+\varepsilon)(-x^2-\varepsilon+x^2+x^3) = \\
&= 4x^3(x^2+\varepsilon)(x^3-\varepsilon) \\
&= 4x^3(x^5 - \varepsilon x^2 + \varepsilon x^3 - \varepsilon^2) \\
&= 4x^8 + 4\varepsilon x^3(-x^2+x^3-\varepsilon) \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \dot{U}(x,y) &= -2(y-x^2)^2 - 4x^8 + 4x^8 + 4\varepsilon x^3(x^3-x^2-\varepsilon) = \\
&= -2(y-x^2)^2 + 4\varepsilon x^3(x^3-x^2-\varepsilon)
\end{aligned}$$

$$= -2(x^2+\varepsilon-x^2)^2 + 4\varepsilon x^3(x^3-x^2-\varepsilon)$$
$$= -2\varepsilon^2 + 4\varepsilon x^6 - 4\varepsilon x^5 - 4\varepsilon^2 x^3$$
$$= -2\varepsilon^2 - 4\varepsilon^2 x^3 + O(x^4) = -2\varepsilon^2(1+2x^3) + O(x^4) < 0$$

in the limit $x \to 0$, since for small x, $1+2x^3 > 0$.
It follows that we do not have global convergence towards the center manifold.

b) $\begin{cases} \dot{x} = xy \\ \dot{y} = -y - x^2 \end{cases}$ ← Find all fixed points and classify with respect to stability.

Solution

▸ Fixed points.

Let $f(x,y) = xy \wedge g(x,y) = -y - x^2$.

(x,y) fixed point $\Leftrightarrow \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} xy = 0 \\ -y - x^2 = 0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x(-x^2) = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Leftrightarrow (x,y) = (0,0).$

▸ Jacobian

Let $F(x,y) = (f(x,y), g(x,y))$.

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} y & x \\ -2x & -1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda(DF(0,0)) = \{0, -1\} \Rightarrow$$

$\Rightarrow (0,0)$ non-hyperbolic fixed point.

▸ Center-Manifold reduction.

System is already in canonical form with $\dot{x} = xy$ the master equation and $\dot{y} = -y - x^2$ the slave equation. Thus, in the limit $x \to 0$, let us define

$$y = h(x) = ax^2 + bx^3 + cx^4 + dx^5 + O(x^6)$$

Then:

$$\dot{y} = h'(x)\dot{x} = h'(x)\,xy = h'(x)\,x\,h(x) = x\,h(x)\,h'(x)$$

$$\dot{y} = -y - x^2 = -h(x) - x^2$$

therefore, let us define

$$N(x) = xh(x)h'(x) - [-h(x) - x^2] = xh(x)h'(x) + h(x) + x^2 =$$

$$= x(ax^2 + bx^3 + cx^4 + dx^5)(2ax + 3bx^2 + 4cx^3 + 5dx^4) + O(x^{12})$$

$$+ (ax^2 + bx^3 + cx^4 + dx^5) + O(x^6) + x^2 =$$

$$= (ax^3 + bx^4 + cx^5 + dx^6)(2ax + 3bx^2 + 4cx^3 + 5dx^4) +$$

$$+ ax^2 + bx^3 + cx^4 + dx^5 + x^2 + O(x^6) =$$

$$= 2a^2x^4 + 3abx^5 + 2abx^5 + O(x^6) + ax^2 + bx^3 + cx^4 + dx^5 + x^2 + O(x^6)$$

$$= (a+1)x^2 + bx^3 + (c + 2a^2)x^4 + (5ab + d)x^5 + O(x^6)$$

It follows that

$$N(x) = O(x^6) \Leftrightarrow (a+1)x^2 + bx^3 + (c+2a^2)x^4 + (5ab+d)x^5 + O(x^6) = O(x^6)$$

$$\Leftrightarrow \begin{cases} a+1=0 \\ b=0 \\ c+2a^2=0 \\ 5ab+d=0 \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=0 \\ c=-2a^2 \\ d=-5ab \end{cases} \Leftrightarrow \begin{cases} a=-1 \\ b=0 \\ c=-2 \\ d=0 \end{cases}$$

and therefore $h(x) = -x^2 - 2x^4 + O(x^6)$.

Thus, $\dot{x} = xy = xh(x) = x(-x^2 - 2x^4 + O(x^6)) =$

$$= -x^3 - 2x^5 + O(x^7)$$

$$y = -x^2 - 2x^4 + O(x^6)$$

and the centermanifold reduction reads:

$$\begin{cases} \dot{x} = -x^3 - 2x^5 + O(x^7) \\ y = -x^2 - 2x^4 + O(x^6) \end{cases}$$

It follows that $(0,0)$ is stable since the fixed point $x=0$ of $\dot{x} = -x^3 - 2x^5 + O(x^7)$ is stable.

↳ Local vs. global convergence

Let us consider the 1st-order approximation

$$y = -x^2 + O(x^4)$$

of the center manifold and therefore define

$$U(x,y) = (y+x^2)^2$$

It follows that

$$\dot U(x,y) = 2(y+x^2)(\dot y + 2x\dot x) = 2(y+x^2)(-y-x^2+2x(xy)) =$$
$$= 2(y+x^2)(-y-x^2+2x^2y)$$
$$= -2(y+x^2)^2 + 4x^2y(y+x^2)$$

Note that the 1st term is negative but the 2nd term can be positive or negative. Choose $y=-x^2+\varepsilon$ with ε small.

Then

$$\dot U(x,y) = -2(-x^2+\varepsilon+x^2)^2 + 4x^2(-x^2+\varepsilon)(-x^2+\varepsilon+x^2)$$
$$= -2\varepsilon^2 + 4\varepsilon x^2(-x^2+\varepsilon) = -2\varepsilon^2 - 4\varepsilon x^{+4} + 4\varepsilon^2x^2$$
$$= -4\varepsilon x^4 + 2\varepsilon^2(2x^2-1) = 2\varepsilon^2(2x^2-1) + O(x^4)$$

For small enough ε, $\dot U(x,y) < 0$, thus we have local but not global convergence

Inclusion of Linearly Unstable Directions

- The center manifold analysis is still applicable even if in the canonical formulation of the original ODE system some eigenvalues have $\mathrm{Re}(\lambda) > 0$.
- Consider the system, in canonical form

$$\begin{cases} \dot{x} = Ax + f(x,y,z) \\ \dot{y} = By + g(x,y,z) \\ \dot{z} = Cz + h(x,y,z) \end{cases}$$

with $(x,y,z) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$ and

$$\begin{cases} \forall \lambda \in \lambda(A) : \mathrm{Re}(\lambda) = 0 \\ \forall \lambda \in \lambda(B) : \mathrm{Re}(\lambda) < 0 \\ \forall \lambda \in \lambda(C) : \mathrm{Re}(\lambda) > 0 \end{cases}$$

and

$$\begin{cases} f(\mathbf{0}) = \mathbf{0} \wedge g(\mathbf{0}) = \mathbf{0} \wedge h(\mathbf{0}) = \mathbf{0} \\ Df(\mathbf{0}) = \mathbf{0} \wedge Dg(\mathbf{0}) = \mathbf{0} \wedge Dh(\mathbf{0}) = \mathbf{0} \end{cases}$$

(i.e. $\mathbf{0}$ is a fixed point and f, g, h capture only the nonlinear terms).

Then the center-manifold is given by

$$W^c = \{(x,y,z) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c \mid y = h_1(x) \wedge z = h_2(x)\}$$

with $h_1(\mathbf{0}) = \mathbf{0} \wedge h_2(\mathbf{0}) = \mathbf{0} \wedge Dh_1(\mathbf{0}) = \mathbf{0} \wedge Dh_2(\mathbf{0}) = \mathbf{0}$

- To determine h_1 and h_2, we note that

$$\dot{y} = Dh_1(x)\dot{x} = Dh_1(x)[Ax + f(x,y,z)] =$$
$$= Dh_1(x)[Ax + f(x, h_1(x), h_2(x))]$$

$$\dot{y} = By + g(x,y,z) = Bh_1(x) + g(x, h_1(x), h_2(x))$$
$$\dot{z} = Dh_2(x)\dot{x} = Dh_2(x)[Ax + f(x,y,z)] =$$
$$= Dh_2(x)[Ax + f(x, h_1(x), h_2(x))]$$
$$\dot{z} = Cz + h(x,y,z) = Ch_2(x) + h(x, h_1(x), h_2(x))$$

It follows that if we define

$$(N_1(h_1,h_2))(x) = Dh_1(x)[Ax + f(x,h_1(x),h_2(x))] - Bh_1(x) - g(x,h_1(x),h_2(x))$$
$$(N_2(h_1,h_2))(x) = Dh_2(x)[Ax + f(x,h_1(x),h_2(x))] - Ch_2(x) - h(x,h_1(x),h_2(x))$$

then $(N_1(h_1,h_2))(x) = \mathbf{0} \wedge (N_2(h_1,h_2))(x) = \mathbf{0}$.

Consequently, h_1 and h_2 are the solutions of the following initial value problem:

$$\begin{cases} (N_1(h_1,h_2))(x) = \mathbf{0} \\ (N_2(h_1,h_2))(x) = \mathbf{0} \\ h_1(x) = \mathbf{0} \wedge Dh_1(x) = \mathbf{0} \\ h_2(x) = \mathbf{0} \wedge Dh_2(x) = \mathbf{0} \end{cases}$$

which can still be solved via power-series methods.

EXAMPLES

a) Analyze the stability of the fixed point $(x,y,z) = (0,0,0)$ for the system

$$\begin{cases} \dot{x} = xz \\ \dot{y} = -y + x^2 \\ \dot{z} = z - xy \end{cases}$$

using center-manifold reduction.

Solution

Define $f(x,y,z) = xz \wedge g(x,y,z) = -y + x^2 \wedge h(x,y,z) = z - xy$. and $F = (f,g,h)$. It follows that

$$DF(x,y,z) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} z & 0 & x \\ 2x & -1 & 0 \\ -y & -x & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda(DF(0,0,0)) = \{0, -1, 1\}$$

$\Rightarrow (0,0,0)$ is a non-hyperbolic fixed point.

▸ Center Manifold analysis.

We note that $\dot{x} = xz$ is the master equation. Thus let $y = h_1(x)$ and $z = h_2(x)$. It follows that

$\dot{y} = h_1'(x)\dot{x} = h_1'(x)\,xz = x h_1'(x) h_2(x)$

$\dot{y} = -y + x^2 = -h_1(x) + x^2$

therefore, we define $N_1(x) = x h_1'(x) h_2(x) - [-h_1(x) + x^2]$

Likewise,

$$\dot{z} = h_2'(x)\dot{x} = h_2'(x)\, xz = x h_2'(x) h_2(x)$$

$$\dot{z} = z - xy = h_2(x) - x h_1(x)$$

therefore, we define

$$N_2(x) = x h_2'(x) h_2(x) - [h_2(x) - x h_1(x)].$$

For

$$h_1(x) = ax^2 + bx^3 + O(x^4)$$

$$h_2(x) = cx^2 + dx^3 + O(x^4)$$

we have

$$N_1(x) = x h_1'(x) h_2(x) + h_1(x) - x^2 =$$

$$= x(2ax + 3bx^2)(cx^2 + dx^3) + O(x^8) + ax^2 + bx^3 - x^2 + O(x^4)$$

$$= (a-1)x^2 + bx^3 + O(x^4)$$

$$N_2(x) = x h_2'(x) h_2(x) - h_2(x) + x h_1(x)$$

$$= x(2cx + 3dx^2)(cx^2 + dx^3) + O(x^8) - cx^2 - dx^3 + O(x^4)$$

$$+ x(ax^2 + bx^3) + O(x^5)$$

$$= -cx^2 - dx^3 + ax^3 + O(x^4) = -cx^2 + (a-d)x^3 + O(x^4)$$

It follows that

$$\begin{cases} N_1(x) = (a-1)x^2 + bx^3 + O(x^4) = O(x^4) \\ N_2(x) = -cx^2 + (a-d)x^3 + O(x^4) = O(x^4) \end{cases} \iff$$

$$\iff \begin{cases} a - 1 = 0 \\ b = 0 \\ -c = 0 \\ a - d = 0 \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = a \end{cases} \iff \begin{cases} a = 1 \\ b = 0 \\ c = 0 \\ d = 1 \end{cases}$$

and therefore $h_1(x) = x^2 + O(x^4)$ and $h_2(x) = x^3 + O(x^4)$

It follows that the master equation reads

$$\dot{x} = xz = x h_2(x) = x(x^3 + O(x^4)) = x^4 + O(x^5).$$

The center-manifold reduction near the (0,0,0) fixed point is given by:

$$\begin{cases} \dot{x} = x^4 + O(x^5) \\ y = x^2 + O(x^4) \\ z = x^3 + O(x^4) \end{cases}$$

$\longleftarrow$ thus (0,0,0) is unstable source.

EXERCISES

(1) Study the dynamics of the following systems near the origin $(x,y)=(0,0)$ via center-manifold analysis for the following autonomous dynamical systems.

a) $\begin{cases} \dot{x} = -x+y^2 \\ \dot{y} = -\sin x \end{cases}$

b) $\begin{cases} \dot{x} = x-2y \\ \dot{y} = x+y+x^4 \end{cases}$

c) $\begin{cases} \dot{x} = -2x+3y+y^3 \\ \dot{y} = 2x-3y+x^3 \end{cases}$

d) $\begin{cases} \dot{x} = y+x^2 \\ \dot{y} = -y-x^2 \end{cases}$

e) $\begin{cases} \dot{x} = -x+y \\ \dot{y} = -e^{x}+e^{-x}+2x \end{cases}$

f) $\begin{cases} \dot{x} = -x-y+z^2 \\ \dot{y} = 2x+y-z^2 \\ \dot{z} = x+2y-z \end{cases}$

Application of Center Manifold to Local Bifurcations

The center manifold technique can be used to investigate local bifurcations for multidimensional autonomous dynamical systems without an explicit determination of the local fixed point, as in the following example:

EXAMPLE

a) Investigate the possible bifurcation at $\mu=0$ for the following system, using center-manifold reduction.

$$\begin{cases} \dot{x} = \mu x - x^3 + xy \\ \dot{y} = -y + y^2 - x^2 \end{cases}$$

Solution

- Fixed point: There is an obvious fixed point at $(x,y)=(0,0)$.
- Linearization

Define: $f(x,y) = \mu x - x^3 + xy$ and $g(x,y) = -y + y^2 - x^2$ and $F(x,y) = (f(x,y), g(x,y))$. Then:

$$DF(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} = \begin{bmatrix} \mu - 3x^2 + y & x \\ -2x & -1+2y \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0) = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \cancel{(0,0)}\ \lambda(DF(0,0)) = \{\mu, -1\} \Rightarrow$$

$\Rightarrow (0,0)$ stable for $\mu<0$, unstable for $\mu>0$.
Note that stability is unknown for $\mu=0$.

▸ Center Manifold analysis: Note that center manifold reduction cannot be applied to the given dynamical system in the absense of zero-eigenvalues. However, we can "cheat" by turning μ into a variable and rewriting the system as:

$$\begin{cases} \dot{x} = \mu x - x^3 + xy \\ \dot{y} = -y + y^2 - x^2 \\ \dot{\mu} = 0 \end{cases}$$

Define $f(x,y,\mu) = \mu x - x^3 + xy$, $g(x,y,\mu) = -y + y^2 - x^2$, and $h(x,y,\mu) = 0$, and also define

$$F(x,y,\mu) = (f(x,y,\mu), g(x,y,\mu), h(x,y,\mu))$$

It follows that

$$DF(x,y,\mu) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial \mu \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial \mu \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial \mu \end{bmatrix} =$$

$$= \begin{bmatrix} \mu - 3x^2 + y & x & x \\ -2x & -1+2y & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \lambda(DF(0,0,0)) = \{0, -1\}$$

$\Rightarrow (0,0,0)$ is a non-hyperbolic fixed point.

Write the linearized equations around $(x,y,\mu) = (0,0,0)$ as follows:

$$\begin{cases} \dot{x} = 0x + (\mu x - x^3 + xy) \\ \dot{y} = -y + (y^2 - x^2) \\ \dot{\mu} = 0\mu \end{cases}$$

Note that $\dot{x}$ and $\dot{\mu}$ are the master equations and $\dot{y}$ is the slave equation. Therefore, let us write $y = H(x,\mu)$. It follows that

$$\dot{y} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \mu}\dot{\mu} = \frac{\partial H}{\partial x}\dot{x} = (\mu x - x^3 + xy)\frac{\partial H}{\partial x} =$$

$$= (\mu x - x^3 + x H(x,\mu))\frac{\partial H}{\partial x}$$

and

$$\dot{y} = -y + (y^2 - x^2) = -H(x,\mu) + H(x,\mu)^2 - x^2$$

therefore we define

$$N(x,\mu) = (\mu x - x^3 + xH(x,\mu))\frac{\partial H}{\partial x} + H(x,\mu) - H(x,\mu)^2 + x^2 .$$

Under the limit $x \to 0$, consider the expansion

$$H(x,\mu) = a(\mu)x^2 + b(\mu)x^3 + O(x^4) \Rightarrow \frac{\partial H(x,\mu)}{\partial x} = 2a(\mu)x + 3b(\mu)x^2 + O(x^3)$$

It follows that

$$N(x,\mu) = [\mu x - x^3 + x(a(\mu)x^2 + b(\mu)x^3 + O(x^4))][2a(\mu)x + 3b(\mu)x^2 + O(x^3)]$$
$$+ [a(\mu)x^2 + b(\mu)x^3 + O(x^4)] - [a(\mu)x^2 + b(\mu)x^3 + O(x^4)]^2 + x^2 =$$
$$= \underline{2\mu a(\mu)x^2} + \underline{\underline{3\mu b(\mu)x^3}} + O(x^4) + \underline{a(\mu)x^2} + \underline{\underline{b(\mu)x^3}} + O(x^4) - O(x^4) + \underline{x^2} =$$
$$= [2\mu a(\mu) + a(\mu) + 1]x^2 + [3\mu b(\mu) + b(\mu)]x^3 + O(x^4) =$$
$$= [(2\mu+1)a(\mu) + 1]x^2 + b(\mu)(3\mu+1)x^3 + O(x^4)$$

and therefore, if we restrict μ to $\mu \in (-1/3, 1/3)$, we have:

$$N(x,\mu) = O(x^4) \Leftrightarrow \begin{cases} (2\mu+1)\,a(\mu)+1=0 \\ b(\mu)(3\mu+1)=0 \end{cases} \Leftrightarrow \begin{cases} a(\mu) = \dfrac{-1}{2\mu+1} \\ b(\mu)=0 \end{cases}$$

and therefore

$$H(x,\mu) = \frac{-x^2}{2\mu+1} + O(x^4)$$

It follows that the $\dot{x}$ master equation reads:

$$\dot{x} = \mu x - x^3 + xy = \mu x - x^3 + xH(x,\mu) = \mu x - x^3 + x\left[\frac{-x^2}{2\mu+1} + O(x^4)\right]$$

$$= \mu x - x^3 - \frac{1}{2\mu+1}x^3 + O(x^4) = \mu x - \left(1 + \frac{1}{2\mu+1}\right)x^3 + O(x^4)$$

$$= \mu x - \frac{2\mu+2}{2\mu+1}x^3 + O(x^4)$$

and consequently, the center manifold reduction reads:

$$\begin{cases} \left.\begin{array}{l} \dot{x} = \mu x - \dfrac{2\mu+2}{2\mu+1}x^3 + O(x^4) \\ \\ \dot{\mu} = 0 \end{array}\right\} \text{master equations} \\ y = \dfrac{-x^2}{2\mu+1} + O(x^4). \quad \Big\} \text{slave equation} \end{cases}$$

► Local Bifurcation at $\mu=0$ and $(x,y)=(0,0)$

We can analyze the local bifurcation of $(x,y)=(0,0)$ at $\mu=0$, by studying the master equation $\dot{x}$ instead of the original two-dimensional system.

Define $G(x,\mu) = \mu x - \dfrac{2\mu+2}{2\mu+1}x^3 + O(x^4)$.

We note that

$G(0,0) = 0$

$$G_x(x,\mu) = \mu - \frac{2\mu+2}{2\mu+1}(3x^2) + O(x^3) \Rightarrow G_x(0,0) = 0$$

$$G_\mu(x,\mu) = x - x^3 \frac{\partial}{\partial\mu}\left(\frac{2\mu+2}{2\mu+1}\right) + O(x^4) \Rightarrow G_\mu(0,0) = 0$$

$$G_{xx}(x,\mu) = 0 - \frac{2\mu+2}{2\mu+1}(6x) + O(x^2) \Rightarrow G_{xx}(0,0) = 0$$

$$G_{x\mu}(x,\mu) = \frac{\partial}{\partial\mu}\left[\mu - \frac{2\mu+2}{2\mu+1}(3x^2) + O(x^3)\right] =$$

$$= 1 - (3x^2)\frac{\partial}{\partial\mu}\left(\frac{2\mu+2}{2\mu+1}\right) + O(x^3) \Rightarrow$$

$$\Rightarrow G_{x\mu}(0,0) = 1 - 0 = 1 \neq 0$$

$$G_{xxx}(x,\mu) = \frac{-(2\mu+2)}{2\mu+1} \cdot 6 + O(x) \Rightarrow$$

$$\Rightarrow G_{xxx}(0,0) = \frac{-(0+2)}{0+1} \cdot 6 + 0 = -12 \neq 0.$$

To summarize:

$$\begin{cases} G(0,0) = G_x(0,0) = G_\mu(0,0) = G_{xx}(0,0) \\ G_{x\mu}(0,0) = 1 \neq 0 \\ G_{xxx}(0,0) = -12 \neq 0 \end{cases} \Rightarrow$$

$\Rightarrow$ At $\mu = 0$, the $(x,y) = (0,0)$ fixed point undergoes a pitchfork bifurcation.

b) Analyze the bifurcation at the origin for the Lorenz equations, given below, using the center-manifold reduction method

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -bz + xy \end{cases} \quad \text{with } b>0,\ \sigma>0, \text{ and } \rho>0.$$

<u>Solution</u>

We note that $(x,y,z) = (0,0,0)$ is an obvious fixed point.

▶ Direct linearization

Define $f(x,y,z) = \sigma(y-x)$, $g(x,y,z) = \rho x - y - xz$, and $h(x,y,z) = -bz + xy$. Also define

$$F(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$$

It follows that

$$DF(x,y,z) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -b \end{bmatrix} \Rightarrow$$

$$\Rightarrow DF(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \Rightarrow$$

$$\Rightarrow p(\lambda) = \det(DF(0,0,0) - \lambda I) = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ \rho & -1-\lambda & 0 \\ 0 & 0 & -b-\lambda \end{vmatrix} =$$

$$= (-b-\lambda) \begin{vmatrix} -\sigma-\lambda & \sigma \\ \rho & -1-\lambda \end{vmatrix} =$$

$$= (-b-\lambda)\left[(-\sigma-\lambda)(-1-\lambda)-\rho\sigma\right] =$$
$$= (-b-\lambda)\left[(\lambda+\sigma)(\lambda+1)-\rho\sigma\right] =$$
$$= (-b-\lambda)(\lambda^2+(\sigma+1)\lambda+\sigma-\rho\sigma] =$$
$$= -(\lambda+b)(\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho))$$

$\hookrightarrow$ Note that for $\sigma(1-\rho)\neq 0$, the zeroes λ_1, λ_2 of the quadratic factor $\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho)$ will satisfy $\lambda_1\lambda_2\neq 0$ consequently none of the eigenvalues is zero and therefore we cannot apply the center manifold method. On the other hand for $\rho=1$, we have

$$p(\lambda) = -(\lambda+b)(\lambda^2+(\sigma+1)\lambda) = -\lambda(\lambda+b)(\lambda+\sigma+1) \Rightarrow$$
$$\Rightarrow \lambda(DF(0,0,0)) = \{-b, -\sigma-1, 0\} \Rightarrow$$
$\Rightarrow (0,0,0)$ non-hyperbolic fixed point.

▸ Center Manifold reduction

To make center manifold reduction applicable, we turn ρ into a variable governed by $\dot{\rho}=0$ with initial condition $\rho=1$ (at $t=0$). To center the 4D fixed point to the origin, we define $\mu=\rho-1$ and rewrite the Lorenz equations as:

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \mu x + x - y - xz \\ \dot{z} = -bz + xy \\ \dot{\mu} = 0 \end{cases}$$

This extended 4D system has an obvious fixed point at $(x,y,z,\mu) = (0,0,0,0)$.

We define $f(x,y,z,\mu) = \sigma(y-x)$,

$g(x,y,z,\mu) = \mu x + x - y - xz$,

$h(x,y,z,\mu) = -bz + xy$

and also we define

$$F(x,y,z,\mu) = (f(x,y,z,\mu), g(x,y,z,\mu), h(x,y,z,\mu))$$

$$\mathcal{F}(x,y,z,\mu) = (f(x,y,z,\mu), g(x,y,z,\mu), h(x,y,z,\mu), 0)$$

It follows that

$$DF(x,y,z,\mu) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \\ \partial h/\partial x & \partial h/\partial y & \partial h/\partial z \end{bmatrix} =$$

$$= \begin{bmatrix} -\sigma & \sigma & 0 \\ \mu+1-z & -1 & -x \\ y & x & -b \end{bmatrix} \Longrightarrow$$

$$\Rightarrow DF(0,0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

which is the same as the previous Jacobian matrix with $\rho = 1$, and therefore:

$$\lambda(DF(0,0,0,0)) = \{0, -\sigma-1, -b\}$$

Note that it is not necessary to write the full Jacobian for the 4×4 system explicitly since its Jacobian has a block diagonal structure

$$D\mathcal{F}(0,0,0,0) = \begin{bmatrix} DF(0,0,0,0) & 0 \\ 0 & 0 \end{bmatrix}$$

▸ To diagonalize the system we find the corresponding eigenvectors:

$\lambda_1 = -b$ has eigenvector $v_1 = (0,0,1)$

$\lambda_2 = 0$ has eigenvector $v_2 = (1,1,0)$

$\lambda_3 = -(\sigma+1)$ has eigenvector $v_3 = (-\sigma,1,0)$

Consequently, we define

$$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\sigma \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{\sigma+1} \begin{bmatrix} 0 & 0 & \sigma+1 \\ 1 & \sigma & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

and define

$$(u,v,w) = P^{-1}(x,y,z) \Leftrightarrow (x,y,z) = P(u,v,w)$$

Equivalently, we write:

$$\begin{cases} x = v - \sigma w \\ y = v + w \\ z = u \end{cases} \Leftrightarrow \begin{cases} u = z \\ v = (x+\sigma y)/(\sigma+1) \\ w = (-x+y)/(\sigma+1) \end{cases}$$

Now we rewrite the Lorenz equations in terms of the new variables u,v,w:

$$\dot{u} = \dot{z} = -bz + xy = -bu + (v-\sigma w)(v+w)$$

$$\dot{v} = \frac{\dot{x}+\sigma\dot{y}}{\sigma+1} = \frac{\sigma(y-x)+\sigma(\mu x + x - y - xz)}{\sigma+1} =$$

$$= \frac{\sigma(y-x+\mu x + x - y - xz)}{\sigma+1} = \frac{\sigma(\mu x - xz)}{\sigma+1} = \frac{\sigma x(\mu - z)}{\sigma+1} =$$

$$= \frac{\sigma(v-\sigma w)(\mu - u)}{\sigma+1}$$

$$\dot{w} = \frac{\dot{y} - \dot{x}}{\sigma+1} = \frac{(\mu x + x - y - xz) - \sigma(y-x)}{\sigma+1} =$$

$$= \frac{(\mu+\sigma+1)x - (\sigma+1)y - xz}{\sigma+1} = \frac{(\sigma+1)(x-y) + (\mu x - xz)}{\sigma+1} =$$

$$= (x-y) + \frac{x(\mu - z)}{\sigma+1} =$$

$$= (v - \sigma w) - (v + w) + \frac{(v-\sigma w)(\mu - u)}{\sigma+1} =$$

$$= -(\sigma+1)w + \frac{(\mu-u)(v-\sigma w)}{\sigma+1}$$

To summarize; the diagonalized equations read:

$$\begin{cases} \dot{u} = -bu + (v-\sigma w)(v+w) \\ \dot{v} = 0v + \sigma(v-\sigma w)(\mu-u)/(\sigma+1) \\ \dot{w} = -(\sigma+1)w + (\mu-u)(v-\sigma w)/(\sigma+1) \\ \dot{\mu} = 0\mu \end{cases}$$

We see that v, μ are the master variables and u, w are the slave variables. Let us write, therefore:

$$u = f(v) \quad \text{and} \quad w = g(v)$$

We note that

$$\dot{u} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial f}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} =$$

$$= \frac{\partial f}{\partial v}\,\frac{\sigma(v-\sigma w)(\mu-u)}{\sigma+1} = \frac{\partial f}{\partial v}\,\frac{\sigma(v-\sigma g(v))(\mu-f(v))}{\sigma+1}$$

$$\dot{u} = -bu + (v - \sigma w)(v + w) = -bf(v) + (v - \sigma g(v))(v + g(v))$$

$$\dot{w} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial g}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial g}{\partial v}\frac{\partial v}{\partial t} =$$

$$= \frac{\partial g}{\partial v}\frac{\sigma(v - \sigma w)(\mu - u)}{\sigma + 1} = \frac{\partial g}{\partial v}\frac{\sigma(v - \sigma g(v))(\mu - f(v))}{\sigma + 1}$$

$$\dot{w} = -(\sigma + 1)w + \frac{(\mu - u)(v - \sigma w)}{\sigma + 1} = -(\sigma + 1)g(v) + \frac{(\mu - f(v))(v - \sigma g(v))}{\sigma + 1}$$

consequently, we define:

$$N_u(v) = \frac{\partial f}{\partial v}\frac{\sigma(v - \sigma g(v))(\mu - f(v))}{\sigma + 1} + bf(v) - (v - \sigma g(v))(v + g(v))$$

$$= bf(v) + (v - \sigma g(v))\left[\frac{\partial f}{\partial v}\frac{\sigma(\mu - f(v))}{\sigma + 1} - (v + g(v))\right]$$

$$N_w(v) = \frac{\partial g}{\partial v}\frac{\sigma(v - \sigma g(v))(\mu - f(v))}{\sigma + 1} + (\sigma + 1)g(v) - \frac{(\mu - f(v))(v - \sigma g(v))}{\sigma + 1} =$$

$$= (\sigma + 1)g(v) + (v - \sigma g(v))\left[\frac{\partial g}{\partial v}\frac{\sigma(\mu - f(v))}{\sigma + 1} - \frac{\mu - f(v)}{\sigma + 1}\right]$$

$$= (\sigma + 1)g(v) + \frac{(v - \sigma g(v))(\mu - f(v))}{\sigma + 1}\left[\sigma\frac{\partial g}{\partial v} - 1\right]$$

- Use the expansions

$$f(v) = a_1(\mu)v^2 + a_2(\mu)v^3 + O(v^4) \Rightarrow \partial f/\partial v = 2a_1(\mu)v + 3a_2(\mu)v^2 + O(v^3)$$

$$g(v) = b_1(\mu)v^2 + b_2(\mu)v^3 + O(v^4) \Rightarrow \partial g/\partial v = 2b_1(\mu)v + 3b_2(\mu)v^2 + O(v^3)$$

and it follows that $N_u(v)$ and $N_w(v)$ are given by

$$N_u(v) = N_u^{(1)}(v) + N_u^{(2)}(v) + N_u^{(3)}(v)$$

with

$$N_u^{(1)}(v) = bf(v) = b(a_1 v^2 + a_2 v^3 + O(v^4)) =$$
$$= ba_1 v^2 + ba_2 v^3 + O(v^4)$$
$$N_u^{(2)}(v) = (v - \sigma g(v)) \frac{\partial f}{\partial v} \frac{\sigma(\mu - f(v))}{\sigma+1} =$$
$$= \frac{\sigma}{\sigma+1}(v - \sigma b_1 v^2 - \sigma b_2 v^3)(2a_1 v + 3a_2 v^2)(\mu - a_1 v^2 - a_2 v^3) + O(v^4)$$
$$= \frac{\sigma}{\sigma+1}(2a_1 v^2 + 3a_2 v^3 - 2\sigma a_1 b_1 v^3)(\mu - a_1 v^2 - a_2 v^3) + O(v^4)$$
$$= \frac{\sigma}{\sigma+1}(2a_1 \mu v^2) + O(v^4)$$ [Note: We drop μv^3 which is a 4th order]

$$N_u^{(3)}(v) = -(v - \sigma g(v))(v + g(v)) =$$
$$= -(v - \sigma b_1 v^2 - \sigma b_2 v^3)(v + b_1 v^2 + b_2 v^3) + O(v^4)$$
$$= -(v^2 + b_1 v^3 - \sigma b_1 v^3) + O(v^4) =$$
$$= -v^2 - b_1 v^3 + \sigma b_1 v^3 + O(v^4)$$

and therefore

$$N_u(v) = ba_1 v^2 + ba_2 v^3 + \frac{2\sigma a_1}{\sigma+1}\mu v^2 - v^2 - b_1 v^3 + \sigma b_1 v^3 + O(v^4)$$
$$= \left(ba_1 - 1 + \frac{2\mu\sigma}{\sigma+1} a_1\right) v^2 + (ba_2 - b_1 + \sigma b_1) v^3 + O(v^4)$$

Likewise $N_w(v) = N_w^{(1)}(v) + N_w^{(2)}(v) + N_w^{(3)}(v)$ with

$$N_w^{(1)}(v) = (\sigma+1) g(v) = (\sigma+1) b_1 v^2 + (\sigma+1) b_2 v^3 + O(v^4)$$
$$(v - \sigma g(v))(\mu - f(v)) = (v - \sigma b_1 v^2 - \sigma b_2 v^3)(\mu - a_1 v^2 - a_2 v^3) + O(v^4)$$
$$= \mu v - a_1 v^3 - \sigma b_1 \mu v^2 + O(v^4)$$
$$N_w^{(2)}(v) = \frac{\sigma}{\sigma+1}(v - \sigma g(v))(\mu - f(v)) \frac{\partial g}{\partial v} =$$

$$= \frac{\sigma}{\sigma+1}\left(\mu v - a_1 v^3 - \sigma b_1 \mu v^2\right)\left(2b_1 v + 3b_2 v^2\right) + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}\left(2b_1 \mu v^2\right) + O(v^4)$$ [Drop all μv^3 terms because they are 4th order]

$$N_w^{(3)}(v) = \frac{-1}{\sigma+1}\left(v - \sigma g(v)\right)\left(\mu - f(v)\right) =$$

$$= \frac{-1}{\sigma+1}\left(\mu v - a_1 v^3 - \sigma b_1 \mu v^2\right) + O(v^4)$$

and therefore

$$N_w(v) = (\sigma+1) b_1 v^2 + (\sigma+1) b_2 v^3 + \frac{2\sigma b_1}{\sigma+1}\mu v^2 +$$

$$+ \frac{-1}{\sigma+1}\left(\mu v - a_1 v^3 - \sigma b_1 \mu v^2\right) + O(v^4)$$

$$= \frac{-\mu}{\sigma+1} v + \left[b_1(\sigma+1) + \frac{2\sigma\mu}{\sigma+1} b_1 + \frac{\sigma\mu}{\sigma+1} b_1\right] v^2 + \left[b_2(\sigma+1) + \frac{a_1}{\sigma+1}\right] v^3 + O(v^4)$$

- We disregard the μv term on $N_w^{(3)}(v)$ since it can be paired up with other μv terms that we are not keeping track of. Now, we set the coefficients of v^2 and v^3 equal to zero:

$$\begin{cases} 6a_1 - 1 + \dfrac{2\mu\sigma}{\sigma+1} a_1 = 0 \\ 6a_2 - b_1 + \sigma b_1 = 0 \end{cases} \wedge \begin{cases} b_1(\sigma+1) + \dfrac{3\sigma\mu}{\sigma+1} = 0 \\ b_2(\sigma+1) + \dfrac{a_1}{\sigma+1} = 0 \end{cases} \Leftrightarrow$$

and it follows that:

$$a_1 = \frac{1}{b + \frac{2\mu\sigma}{\sigma+1}} = \frac{1}{b} - \frac{1}{b^2}\frac{2\mu\sigma}{\sigma+1} + O(\mu^2)$$

$$b_1 = \frac{3\sigma\mu}{(\sigma+1)^2}$$

$$a_2 = (1-\sigma)b_1 = \frac{3\sigma(1-\sigma)\mu}{b(\sigma+1)^2}$$

$$b_2 = \frac{-a_1}{(\sigma+1)^2} = \frac{-1}{b(\sigma+1)^2} + \frac{1}{b^2(\sigma+1)^2}\frac{2\mu\sigma}{\sigma+1} + O(\mu^2)$$

The master equation is given by:

$$\dot{v} = \frac{\sigma}{\sigma+1}(v-\sigma w)(\mu-u) = \frac{\sigma}{\sigma+1}(v-\sigma g(v))(\mu-f(v)) =$$

$$= \frac{\sigma}{\sigma+1}(\mu v - a_1 v^3 - \sigma b_1 \mu v^2) + O(v^4) =$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \left(\frac{1}{b} - \frac{2\mu\sigma}{\sigma+1}\frac{1}{b^2}\right)v^3 - \sigma \underbrace{\frac{3\sigma\mu}{(\sigma+1)^2}\mu v^2}_{\mu^2 v^2 \text{ term}}\right] + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \left(\frac{1}{b} - \underbrace{\frac{2\mu\sigma}{\sigma+1}\frac{1}{b^2}}_{\mu v^3 \text{ term can also be dropped}}\right)v^3\right] + O(v^4)$$

$$= \frac{\sigma}{\sigma+1}\left[\mu v - \frac{v^3}{b}\right]$$

which is the standard form of a $\boxed{\text{pitchfork bifurcation.}}$

For $\mu = 0$: $\dot{v} = -[b\sigma/(\sigma+1)]v^3$ which gives stable fixed point.

EXERCISE

(2) Use center manifold reduction to analyze the local bifurcation near the origin for the following autonomous dynamical systems

a) $\begin{cases} \dot{x} = -x + \mu y + y^2 \\ \dot{y} = -\sin x \end{cases}$ b) $\begin{cases} \dot{x} = -x + y + \mu x^2 \\ \dot{y} = -\sin x \end{cases}$

c) $\begin{cases} \dot{x} = 2x + 2y \\ \dot{y} = x + y + x^4 + \mu y^2 \end{cases}$ d) $\begin{cases} \dot{x} = -2x + 3y + y^3 + \mu x^2 \\ \dot{y} = 2x - 3y + x^3 \end{cases}$

e) $\begin{cases} \dot{x} = -x - y + z^2 \\ \dot{y} = 2x + y + \mu y - z^2 \\ \dot{z} = x + 2y - z \end{cases}$ f) $\begin{cases} \dot{x} = -2x + y + z + \mu x + y^2 z \\ \dot{y} = x - 2y + z + \mu x + x z^2 \\ \dot{z} = x + y - 2z + \mu x + x^2 y \end{cases}$

References

These lecture notes, and especially the exercises, follow the textbook by Strogatz, but from a more mathematically rigorous standpoint. Below is the list of references were consulted during the preparation of these lecture notes.

(1) S.H. Strogatz (1994): "Nonlinear dynamics and chaos", Addison-Wesley
(2) B. Deconinck (2009): "Dynamical systems", online lecture notes
(3) S. Wiggins (2003): "Introduction to Applied Nonlinear Dynamical Systems and Chaos", 2nd edition, Springer-Verlag