Lecture Notes on Linear Algebra

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Contents

LIN1: Brief introduction to Logic and Sets

BRIEF INTRODUCTION TO LOGIC AND SETS

Basic concepts

The basic concepts we wish to introduce informally are
a) Propositions
b) Sets
c) Predicates - Quantified statements.

Propositions

• A proposition p is any statement which is true or false.

• Given two propositions p, q we define the following composite propositions.

1) Conjunction $p \wedge q$: "p is true and q is true"
   ▸ True if both p and q are true, otherwise false.
2) Disjunction: $p \vee q$: "p is true or q is true (or both)"
   ▸ True if at least one of the two statements p or q is true, otherwise false.
3) Negation $\bar{p}$: "p is not true"
   ▸ True if p is false. False if p is true.
4) Exclusive Disjunction $p \veebar q$: "either p or q is true (not both)"
   ▸ True if either p or q but not both is true. Otherwise false.

5) Implication $p \Rightarrow q$: "If p is true then q is true"
True if the truth of p implies the truth of q. Note that if p is false, then we presume that $p \Rightarrow q$ is true regardless of whether q is true or false. If p is true and q is false then $p \Rightarrow q$ is false.

6) Equivalence $p \Leftrightarrow q$: "p is true if and only if q is true"
True if p and q always have the same truth value. False if p and q have opposite truth values.

Sets

- A <u>set</u> A is an <u>unordered</u> collection of <u>elements</u>. An element can be a number, a derived object (i.e. vectors, matrices, etc.) or another set.
- A set with a finite number of elements can be defined by listing the elements.
  e.g.: $A = \{2, 3, 6, 9, 12\}$.
- <u>Notation</u>: Let A, B be sets and let x be an element.
  1) $x \in A$: x belongs to A
     x is an element of A
  2) $x \notin A$: x does not belong to A
     x is not an element of A
  3) $A = B$: A and B have the same elements.
  4) $A \subseteq B$: All the elements of A belong to B

- We note that: $A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A)$
- <u>Special sets</u>

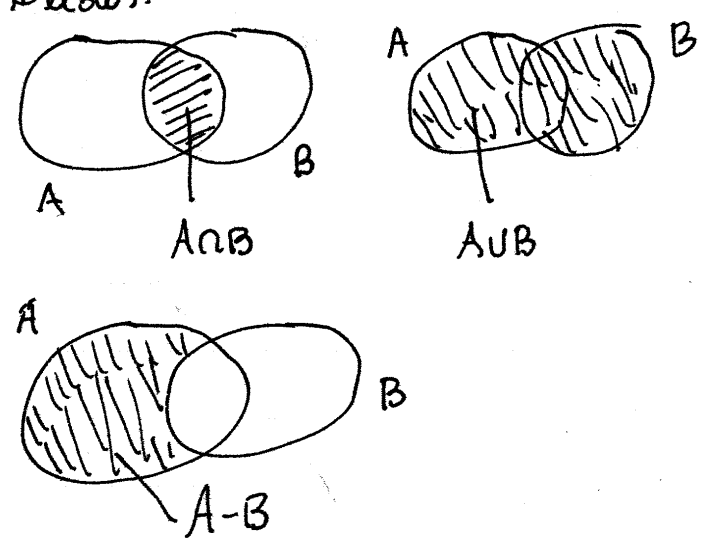
1) $\emptyset = \{\}$. The empty set.
   The empty set is the set that has no elements.
2) $\mathbb{C}$ = the set of all complex numbers
3) $\mathbb{R}$ = the set of all real numbers.
4) $\mathbb{Q}$ = the set of all rational numbers.
5) $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ = the set of all integers.
6) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ = the set of all natural numbers.
7) For $n \in \mathbb{N}$: $[n] = \{1, 2, 3, \dots, n\}$.

- We note that: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
- <u>Set operations</u>

Let A, B be two sets. We define the following set operations:

1) Intersection: $A \cap B$
   $x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$
2) Union: $A \cup B$
   $x \in A \cup B \Leftrightarrow x \in A \vee x \in B$
3) Difference: $A - B$
   $x \in A - B \Leftrightarrow x \in A \wedge x \notin B$

We represent these operations with Venn Diagrams as follows:



- Predicates and quantified statements

- A predicate $p(x)$ is a statement about x which is true or false depending on the value of x.
- Note that x can also be an ordered collection of elements $x = (x_1, x_2, \ldots, x_n)$. Then we write $p(x)$ as $p(x_1, x_2, \ldots, x_n)$.
- Given a predicate $p(x)$ and a set A, we define the following quantified statements:
  1) $\forall x \in A : p(x)$
     For all $x \in A$, $p(x)$ is satisfied.
  2) $\exists x \in A : p(x)$
     There is at least one $x \in A$ such that $p(x)$ is satisfied.
  3) $\exists! x \in A : p(x)$
     There is a unique $x \in A$ such that $p(x)$ is satisfied.
- If A is a finite set, then the above quantified statements are abbreviations for conjunction, disjunction, and exclusive disjunction: For example:

$$(\forall x \in \{a, b, c\} : p(x)) \Leftrightarrow (p(a) \land p(b) \land p(c))$$
$$(\exists x \in \{a, b, c\} : p(x)) \Leftrightarrow (p(a) \lor p(b) \lor p(c))$$

- Quantifiers can be nested to give compound quantified statements. For example:

1) $\forall x \in A : \exists y \in B : p(x, y)$
   For all $x \in A$, there is a $y \in B$, such that $p(x, y)$ is satisfied.

2) $\exists x \in A : \forall y \in B : p(x,y)$

There is an $x \in A$ such that for all $y \in B$, $p(x,y)$ is satisfied.

- Important quantified statements from algebra

$\forall a,b \in \mathbb{R} : (ab = 0 \Leftrightarrow a = 0 \lor b = 0)$

$\forall a,b \in \mathbb{R} : (a^2 + b^2 = 0 \Leftrightarrow a = 0 \land b = 0)$

$\forall a,b \in \mathbb{R} : (|a| + |b| = 0 \Leftrightarrow a = 0 \land b = 0)$

- <u>Definitions of sets</u>

There are 3 methods for defining sets:

1) <u>By listing</u>: For finite sets we can simply list the elements.

e.g.: $A = \{3, 7, 10, 12\}$

2) <u>By predicate</u>: $A = \{x \in U \mid p(x)\}$

with U a predefined set and $p(x)$ a predicate.

Belonging condition: $x \in A \Leftrightarrow (x \in U \land p(x))$

e.g.: We can use definition by predicate to define intervals:

$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

$[n] = \{x \in \mathbb{N} \mid 1 \leq x \leq n\} = \{1, 2, \dots, n\}$

3) <u>By mapping</u>: $A = \{\varphi(x) \mid x \in U \land p(x)\}$

with $\varphi(x)$ some <u>expression</u> of x, U a predefined set, and $p(x)$ a <u>predicate</u>.

Belonging condition: $y \in A \Leftrightarrow \exists x \in U : (\varphi(x) = y \land p(x))$

EXAMPLES

a) The set of complex numbers:

$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$.

$z \in \mathbb{C} \Leftrightarrow \exists a, b \in \mathbb{R} : z = a+bi$

b) The set of rational numbers:

$\mathbb{Q} = \{a/b \mid a \in \mathbb{Z} \land b \in \mathbb{N}-\{0\}\}$

$x \in \mathbb{Q} \Leftrightarrow \exists a \in \mathbb{Z} : \exists b \in \mathbb{N}-\{0\} : x = a/b$.

c) The set of even integers

$A = \{2k \mid k \in \mathbb{Z}\}$

$x \in A \Leftrightarrow \exists k \in \mathbb{Z} : x = 2k$

d) The set of odd integers

$A = \{2k+1 \mid k \in \mathbb{Z}\}$

$x \in A \Leftrightarrow \exists k \in \mathbb{Z} : x = 2k+1$.

e) $A = \{a^2+b^2 \mid a, b \in \mathbb{R} \land a+3b < 1\}$

$x \in A \Leftrightarrow \exists a, b \in \mathbb{R} : (x = a^2+b^2 \land a+3b < 1)$

- <u>Cartesian product</u>

We use definition by mapping to define the cartesian product between sets.

- An <u>ordered pair</u> (a,b) is an <u>ordered</u> collection of two elements a and b. We call a and b the <u>components</u> of (a,b).
- We note that: $(a,b) = (c,d) \Leftrightarrow (a = c \land b = d)$.

- Let A, B be two sets. We define the Cartesian product
  $A \times B = \{(a,b) \mid a \in A \land b \in B\}$.
  We also define:
  $A^2 = A \times A = \{(a,b) \mid a \in A \land b \in A\}$

EXAMPLE

For $A = \{1,2,3\}$ and $B = \{5,6\}$. Calculate $A \times B$, A^2, B^2.

Solution

$A \times B = \{1,2,3\} \times \{5,6\} =$
$= \{(1,5),(1,6),(2,5),(2,6),(3,5),(3,6)\}$

$A^2 = A \times A = \{1,2,3\} \times \{1,2,3\} =$
$= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

$B^2 = B \times B = \{5,6\} \times \{5,6\} =$
$= \{(5,5),(5,6),(6,5),(6,6)\}$

$\rightarrow$ The above can be generalized as follows

- An ordered n-tuplet $(x_1, x_2, \dots, x_n)$ is an ordered collection of n elements $x_1, x_2, \dots, x_n$.
- Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$.
  We note that:
  $x = y \Leftrightarrow \forall a \in [n] : x_a = y_a$
- Let $A_1, A_2, \dots, A_n$ be n sets. We define:
  $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \dots, x_n) \mid \forall a \in [n] : x_a \in A_a\}$
- Special case:
  $A_1 \times A_2 \times A_3 = \{(x_1, x_2, x_3) \mid x_1 \in A_1 \land x_2 \in A_2 \land x_3 \in A_3\}$.

EXERCISES

(1) Let $A = [7]$, $B = \{x \in A \mid x > 4\}$, and $C = \{x-1 \mid x \in B\}$.
List the elements of
a) B b) C c) $B \cap C$ d) $B \cup C$
e) $A - B$ f) $B - C$ g) $C - B$

(2) Write out the following statements in English
a) $\forall a \in A : \exists b \in B : (a, b) \in f$
b) $\exists a \in A : \forall b \in B : a + b > 3$
c) $\forall a \in A : \exists b \in B : (ab > 2 \wedge a + b > 1)$
d) $\forall a, b \in A : \exists c \in B : \forall d \in A : ab + bd < 3$
e) $\exists a \in A : \forall b \in B : (ab > 3 \Rightarrow b > 2)$
f) $\forall a \in A : \exists b \in B : (3a > b \veebar a + b < 0)$

(3) Write the following statements symbolically using quantifiers.
a) Every real number is equal to itself.
b) There is a real number x such that $3x - 1 = 2(x + 3)$
c) For every real number x, there is a natural number n such that $n > x$.
d) For every real number x, there is a complex number y such that $y^2 = x$.
e) There is a real number x such that for all real numbers y we have $x + y = 0$.

f) For all $\varepsilon > 0$, there is a $\delta > 0$ such that for all real numbers x, if $x_0 - \delta < x < x_0 + \delta$ then $|f(x) - a| < \varepsilon$.

g) There is a real number b such that for all natural numbers n we have $a_n < b$.

h) For all $\varepsilon > 0$, there is a natural number n_0 such that for any two natural numbers n_1 and n_2, if $n_1 > n_0$ and $n_2 > n_0$, then $|a_{n_1} - a_{n_2}| < \varepsilon$.

i) For any $M > 0$, there is a natural number n_0, such that for any other natural number n, if $n > n_0$ then $a_n > M$.

(4) Write the belonging condition $x \in A$ for the following sets, using quantifiers.

a) $A = \{x^2+1 \mid x \in \mathbb{Q} \wedge 2x < 1\}$

b) $A = \{3x+1 \mid x \in \mathbb{Z} \wedge x$ is a prime number$\}$

c) $A = \{x \in \mathbb{R} \mid x^2 + 3x > 0\}$

d) $A = \{a^3 + b^3 + c^3 \mid a, b \in \mathbb{R} \wedge c \in \mathbb{Q} \wedge a + b + c = 0\}$

e) $A = \{x \in \mathbb{R} \mid x^2 + 2x < 0 \vee 3x + 1 > -4 + x\}$

f) $A = \{a^2 - b^2 \mid a \in \mathbb{N} \wedge b \in \mathbb{R} \wedge a + b > 5\}$

g) $A = \{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z} : x = 3k\}$

h) $A = \{ab \mid a, b \in \mathbb{R} \wedge (a + b > 2 \vee a - b < -3)\}$

i) $A = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : y^2 + y = x\}$

j) $A = \{x \in \mathbb{R} \mid \forall y \in \mathbb{R} : x < y^2 + 1\}$

k) $A = \{a + b \mid a, b \in \mathbb{R} \wedge (ab > 1 \Rightarrow a^2 + b^2 > 2)\}$

l) $A = \{abc \mid a, b, c \in \mathbb{R} \wedge (a + b > 2 \underline{\vee} a - c < 3)\}$

m) $A = \{2a + 3b \mid a, b \in \mathbb{R} \wedge ab > 1 \wedge a - b < 0\}$

(5) List the elements for the following cartesian products

a) $A \times B$ with $A = \{2,3,4\}$ and $B = \{7,8\}$

b) $A \times B$ with $A = \{1\}$ and $B = \{3,9\}$

c) $A \times B$ with $A = \{3\}$ and $B = \{5\}$

d) $[2] \times [3]$

e) $A \times B$ with $A = [5] - [2]$ and $B = [2] \cap [4]$

f) $A \times B \times C$ with $A = [3] - \{1\}$, $B = [3] \cap [6]$, and $C = [2]$.

g) $A \times B \times C$ with $A = \{2\}$, $B = [2]$, $C = [4] - [2]$.

LIN2: Brief introduction to Proofs

BRIEF INTRODUCTION TO PROOF

▼ Negation and contrapositive of statements

- Let P, Q be compound statements. We say that $P \equiv Q$ (P and Q are equivalent) if and only if the compound statement $P \Leftrightarrow Q$ is always true, regardless of the truth value of the constituent statemends that compose P and Q.
- The following equivalences can be used to negate compound statements:

$\overline{p \wedge q} \equiv \bar{p} \vee \bar{q}$	$\overline{p \veebar q} \equiv p \Leftrightarrow q$
$\overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$	$\overline{p \Leftrightarrow q} \equiv p \veebar q$
$\overline{p \Rightarrow q} \equiv p \wedge \bar{q}$	

- Quantifred statements can be negated by the following rules

$$\overline{\forall x \in A : p(x)} \equiv \exists x \in A : \overline{p(x)}$$
$$\overline{\exists x \in A : p(x)} \equiv \forall x \in A : \overline{p(x)}$$

- Every statement of the form $P \Rightarrow Q$ is equivalent to the contrapositive statement $\bar{Q} \Rightarrow \bar{P}$. Consequently any proof of $P \Rightarrow Q$ also proves $\bar{Q} \Rightarrow \bar{P}$. The converse statement $Q \Rightarrow P$ is NOT equivalent to $P \Rightarrow Q$ and requires separate proof.

- We note that since

$$(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

the contrapositive statement of $P \Leftrightarrow Q$ is $\overline{P} \Leftrightarrow \overline{Q}$.

EXAMPLES

a) Write the negation of the definition of the limit from calculus

$$\lim_{x \to x_0} f(x) = \ell \Leftrightarrow \forall \varepsilon > 0 : \exists \delta > 0 : \forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)$$

Solution

$$\lim_{x \to x_0} f(x) \neq \ell \Leftrightarrow$$

$$\Leftrightarrow \exists \varepsilon > 0 : \overline{\exists \delta > 0 : \forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \overline{\forall x \in dom(f) : (0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : \overline{(0 < |x - x_0| < \delta \Rightarrow |f(x) - \ell| < \varepsilon)}$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : (0 < |x - x_0| < \delta \wedge \overline{|f(x) - \ell| < \varepsilon})$$

$$\Leftrightarrow \exists \varepsilon > 0 : \forall \delta > 0 : \exists x \in dom(f) : (0 < |x - x_0| < \delta \wedge |f(x) - \ell| \geq \varepsilon)$$

b) The contrapositive to the statement
$\forall a,b\in\mathbb{R}: (ab=0 \Rightarrow a=0 \vee b=0)$
is given by:

$$\forall a,b\in\mathbb{R}: (\overline{a=0 \vee b=0} \Rightarrow \overline{ab=0}) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (\overline{a=0} \wedge \overline{b=0} \Rightarrow ab\neq 0) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (a\neq 0 \wedge b\neq 0 \Rightarrow ab\neq 0).$$

c) The contrapositive to the statement
$\forall a,b\in\mathbb{R}: (a^2+b^2=0 \Rightarrow a=0 \wedge b=0)$
is given by:

$$\forall a,b\in\mathbb{R}: (\overline{a=0 \wedge b=0} \Rightarrow \overline{a^2+b^2=0}) \Leftrightarrow$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (\overline{a=0} \vee \overline{b=0} \Rightarrow a^2+b^2\neq 0)$$
$$\Leftrightarrow \forall a,b\in\mathbb{R}: (a\neq 0 \vee b\neq 0 \Rightarrow a^2+b^2\neq 0).$$

EXERCISES

① Write the negation of all the statements from Exercises 2 and 3 [Brief Introduction to Logic and Sets] both in terms of quantified statement notation and in English.

② Write the non-belonging condition $x \notin A$ for the sets given in Exercise 4 [Brief Introduction to Logic and Sets] both in terms of quantified statement notation and in English.

③ Write the contrapositive of the following statements, both in terms of quantified statement notation and in English.

a) $\forall a \in \mathbb{R}: a \geq 3 \Rightarrow a > 5$

b) $\forall a, b \in \mathbb{R}: |a| + |b| = 0 \Rightarrow (a = 0 \wedge b = 0)$

c) $\forall a, b \in \mathbb{R}: a^2 = b^2 \Leftrightarrow (a = b \vee a = -b)$

d) $\forall a, b, c, d \in \mathbb{R}: (a < b \wedge c < d) \Rightarrow a + c < b + d$

e) $\forall a, b, c \in \mathbb{R}: (a > 0 \wedge b > c > 0) \Rightarrow ab > ac$

(Hint: $b > c > 0$ is equivalent to $b > c \wedge c > 0$)

f) $\forall a, b, c \in \mathbb{R}: a^3 + b^3 + c^3 = 3abc \Rightarrow (a + b + c = 0 \vee a = b = c)$

(Hint: $a = b = c$ is equivalent to $a = b \wedge b = c$)

Methodology for writing proofs

Proving implications

1. To prove $\boxed{p \Rightarrow q}$

- **Direct Method**

  Assume p is true.
  [Prove q]

- **Contrapositive Method**

  We will show that $\bar{q} \Rightarrow \bar{p}$
  Assume $\bar{q}$ is true.
  [Prove $\bar{p}$]
  It follows that $p \Rightarrow q$

- **Contradiction Method**

  Assume p is true.
  To derive a contradiction, assume $\bar{q}$.
  [Prove r, using $p \wedge \bar{q}$]
  [Prove $\bar{r}$] ← Contradiction.
  It follows that q is true.

2. To prove $\boxed{p \Leftrightarrow q}$

($\Rightarrow$): Assume p is true
[Prove q]

($\Leftarrow$): Assume q is true
[Prove p]

Proofs involving sets

Let A, B be two sets.

(1) To prove $\boxed{A \subseteq B}$

[We prove $x \in A \Rightarrow x \in B$]

(2) To prove $\boxed{A = B}$

[We prove $x \in A \Rightarrow x \in B$]
It follows that $A \subseteq B$ (1)
[We prove $x \in B \Rightarrow x \in A$]
It follows that $B \subseteq A$ (2)
From (1) and (2): $A = B$.

- For proofs involving sets, we recall that

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$$
$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$$
$$x \in A - B \Leftrightarrow x \in A \wedge x \notin B$$
$$x \in \{x \in A \mid p(x)\} \Leftrightarrow x \in A \wedge p(x)$$
$$x \in \{\varphi(x) \mid x \in A \wedge p(x)\} \Leftrightarrow \exists y \in A : (\varphi(y) = x \wedge p(y))$$

Proofs involving identities

Let a, b be two expressions.
To prove $a = b$.

▶ Direct Method

$$a = \cdots = \cdots = \\ = \cdots = \cdots = b$$

▶ Indirect Method

$$a = \cdots = \cdots = c \quad (1)$$
$$b = \cdots = \cdots = c \quad (2)$$

From (1) and (2): $a = b$.

Proofs involving quantified statements

1 → To prove $\boxed{\forall x \in A : p(x)}$

Let $x \in A$ be given.
[Prove $p(x)$]
It follows that $\forall x \in A : p(x)$.

2 → To prove $\boxed{\exists x \in A : p(x)}$

▶ 1st method
[Define an $x \in A$]
[Prove that $p(x)$ is true]
It follows that $\exists x \in A : p(x)$

(Note that x can be indirectly defined by deducing a statement of the form $\exists x \in B : r(x)$ via a theorem or by constructing it from other variables that have been indirectly defined via existential statements)

▶ <u>2nd method</u>

$p(x) \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftarrow x \in S$

Choose an $x \in S$. Show that $x \in A \wedge p(x)$.

It follows that $\exists x \in A : p(x)$.

LIN3: Basic Linear Algebra

LINEAR ALGEBRA

▼ Matrices – Definitions

- An $n \times m$ matrix A is a collection of nm numbers $A_{ab} \in \mathbb{R}$ (with $a \in [n]$ and $b \in [m]$) arranged in n rows and m columns as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{bmatrix} \Bigg\downarrow \text{ rows } a = 1, 2, \ldots, n$$

$$\xrightarrow{\hspace{3em}} \text{ columns } b = 1, 2, \ldots, m$$

> Remember:
> A_{rc} : row, column
> A_{vh} : vertical, horizontal

We also write $A = [A_{ab}]$.
A_{ab} = the element of A at row a and column b.

- $M_{nm}(\mathbb{R})$ = the set of all $n \times m$ matrices with elements from $\mathbb{R}$.

- For $n = m$, an $n \times n$ matrix is called a square matrix and we write $M_n(\mathbb{R}) = M_{nn}(\mathbb{R})$.

- Let $A, B \in M_{nm}(\mathbb{R})$ be two matrices. Then
$A = B \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = B_{ab}$.

▶ Zero matrix:
Let $A \in M_{nm}(\mathbb{R})$ be a matrix. Then
$A = \mathbb{0} \iff \forall a \in [n] : \forall b \in [m] : A_{ab} = 0$

▶ Identity Matrix

We say that $I \in M_n(\mathbb{R})$ is an identity matrix if and only if
$$\forall a, b \in [n] : I_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$

▼ Basic operations with matrices

- Let $A, B, C \in M_{nm}(\mathbb{R})$ be given matrices, and let $\lambda \in \mathbb{R}$.
  Then, we define:
  $C = A + B \Leftrightarrow \forall a \in [n] : \forall b \in [m] : C_{ab} = A_{ab} + B_{ab}$ (addition)
  $C = \lambda A \Leftrightarrow \forall a \in [n] : \forall b \in [m] : C_{ab} = \lambda A_{ab}$. (scalar multiplication)
  We also define: $-A = (-1)A$ and $A - B = A + (-1)B$.
- Properties of matrix addition:
  $\forall A, B \in M_{nm}(\mathbb{R}) : A + B = B + A$
  $\forall A, B, C \in M_{nm}(\mathbb{R}) : (A+B) + C = A + (B+C)$
  $\forall A \in M_{nm}(\mathbb{R}) : A + \mathbf{0} = \mathbf{0} + A = A$
  $\forall A \in M_{nm}(\mathbb{R}) : \exists B \in M_{nm}(\mathbb{R}) : A + B = B + A = \mathbf{0}$
- Properties of scalar multiplication
  $\forall \lambda \in \mathbb{R} : \forall A, B \in M_{nm}(\mathbb{R}) : \lambda(A+B) = \lambda A + \lambda B$
  $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu) A$
  $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : (\lambda + \mu) A = \lambda A + \mu A$
  $\forall A \in M_{nm}(\mathbb{R}) : 1 \cdot A = A$
  $\forall \lambda \in \mathbb{R} : \lambda \mathbf{0} = \mathbf{0}$
  $\forall A \in M_{nm}(\mathbb{R}) : (-1)A = -A$

EXAMPLES

a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$.

Calculate $A+B$ and $2A-3B$.

Solution

$$A+B = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} =$$
$$= \begin{bmatrix} 1+2 & 3+3 & 2+1 \\ 3-1 & 1+0 & 4-2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ 2 & 1 & 2 \end{bmatrix}$$

$$2A-3B = 2\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} =$$
$$= \begin{bmatrix} 2 & 6 & 4 \\ 6 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 9 & 3 \\ -3 & 0 & -6 \end{bmatrix} =$$
$$= \begin{bmatrix} 2-6 & 6-9 & 4-3 \\ 6-(-3) & 2-0 & 8-(-6) \end{bmatrix} = \begin{bmatrix} -4 & -3 & 1 \\ 9 & 2 & 14 \end{bmatrix}$$

b) Prove: $\forall A,B,C \in M_{nm}(\mathbb{R}) : (A+B)+C = A+(B+C)$

Solution

Let $A,B,C \in M_{nm}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then:

$$[(A+B)+C]_{ab} = (A+B)_{ab} + C_{ab} = (A_{ab} + B_{ab}) + C_{ab} =$$
$$= A_{ab} + (B_{ab} + C_{ab}) = A_{ab} + (B+C)_{ab} =$$
$$= [A+(B+C)]_{ab}.$$

It follows that
$\forall a \in [n] : \forall b \in [m] : [(A+B)+C]_{ab} = [A+(B+C)]_{ab}$
$\Rightarrow (A+B)+C = A+(B+C)$
and therefore:
$\forall A, B, C \in M_{nm}(\mathbb{R}) : (A+B)+C = A+(B+C).$

c) Prove: $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu)A$

<u>Solution</u>

Let $\lambda, \mu \in \mathbb{R}$ and $A \in M_{nm}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then
$[\lambda(\mu A)]_{ab} = \lambda(\mu A)_{ab} = \lambda(\mu A_{ab}) = (\lambda\mu)A_{ab} = [(\lambda\mu)A]_{ab}.$
It follows that
$\forall a \in [n] : \forall b \in [m] : [\lambda(\mu A)]_{ab} = [(\lambda\mu)A]_{ab}$
$\Rightarrow \lambda(\mu A) = (\lambda\mu)A$
and therefore
$\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \lambda(\mu A) = (\lambda\mu)A.$

EXERCISES

① Let A, B be the matrices

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 1 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

a) Evaluate $C = 3A - 2B$

b) Solve with respect to X the equation
$2A + 3(X - B) = A + B$

② Let A, B be the matrices

$$A = \begin{bmatrix} a & -2a & c \\ 0 & -a & b \\ a+b & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2a & c \\ a & b-a & -b \\ a-b & 0 & -1 \end{bmatrix}$$

Evaluate and simplify $C = A + B$ and
$D = 2(A - B) - (A + 2B)$

③ Consider the matrix-valued functions

$$A(x) = \begin{bmatrix} 1 & x^2 \\ x & 3x \end{bmatrix}, \quad \forall x \in \mathbb{R}$$

$$B(x) = \begin{bmatrix} x-1 & 2x \\ x^2 & 1 \end{bmatrix}, \quad \forall x \in \mathbb{R}$$

a) Evaluate and simplify the function
$C(x) = 2A(2x+1) - B(x-2), \ \forall x \in \mathbb{R}$

b) Solve with respect to $Y(x)$ the matrix equation
$3A(x) + 2(Y(x) + A(x)) = A(x+1) - B(x)$

(4) Given the function

$$A(x) = \begin{bmatrix} 1 & 2x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}$$

Show that

$A(3x) + 3A(x) = A(0) + 3A(2x), \quad \forall x \in \mathbb{R}$

(5) Prove that

a) $\forall \lambda \in \mathbb{R} : \forall A, B \in M_{nm}(\mathbb{R}) : \lambda(A+B) = \lambda A + \lambda B$

b) $\forall \lambda, \mu \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : (\lambda + \mu) A = \lambda A + \mu A$

Matrix multiplication

The product AB of two matrices A, B can be defined only when $A \in M_{nl}(\mathbb{R})$ and $B \in M_{lm}(\mathbb{R})$. That is, the number of columns of A must be equal to the number of rows of B. Then we define the product as follows:

- For $A \in M_{nl}(\mathbb{R})$ and $B \in M_{lm}(\mathbb{R})$, we define $(AB) \in M_{nm}(\mathbb{R})$ such that

$$\boxed{\forall a \in [n] : \forall b \in [m] : \ (AB)_{ab} = \sum_{\gamma=1}^{l} A_{a\gamma} B_{\gamma b}}$$

$\mapsto$ To illustrate the definition, we consider the following special cases:

a) Row matrix X Column matrix: $A \in M_{1n}(\mathbb{R}) \wedge B \in M_{n1}(\mathbb{R})$.
Then $AB \in M_1(\mathbb{R})$ with

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} =$$

$$= \begin{bmatrix} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{bmatrix}$$

b) Product of 2×2 matrices: $A, B \in M_2(\mathbb{R})$.

$$AB = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_2 d_1 & a_1 c_2 + a_2 d_2 \\ b_1 c_1 + b_2 d_1 & b_1 c_2 + b_2 d_2 \end{bmatrix}$$

↳ From the above examples we see that the element $(AB)_{ab}$ is the product of row a of matrix A and column b of matrix B.

▸ Properties of Matrix Multiplication

$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{kl}(\mathbb{R}) : \forall C \in M_{lm}(\mathbb{R}) : (AB)C = A(BC)$

$\forall A \in M_{nk}(\mathbb{R}) : \forall B, C \in M_{km}(\mathbb{R}) : \quad A(B+C) = AB + AC$

$\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : \quad (B+C)A = BA + CA$

$\forall \lambda \in \mathbb{R} : \forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : \lambda(AB) = (\lambda A)B = A(\lambda B)$

$\forall A \in M_n(\mathbb{R}) : IA = AI = A \quad (I \in M_n(\mathbb{R})$ is the identity matrix$)$

$\forall A \in M_n(\mathbb{R}) : A\mathbb{0} = \mathbb{0}A = \mathbb{0}$

• It is not true for all matrices that $AB = BA$ (see homework for a counterexample). This creates some interesting complications.

▸ Manipulation Properties

$\forall A, B, C \in M_{nm}(\mathbb{R}) : A = B \Leftrightarrow A + C = B + C$

$\forall A, B \in M_{nk}(\mathbb{R}) : \forall C \in M_{km}(\mathbb{R}) : A = B \Rightarrow AC = BC$

$\forall C \in M_{nk}(\mathbb{R}) : \forall A, B \in M_{km}(\mathbb{R}) : A = B \Rightarrow CA = CB$

$\forall A, B, C \in M_{nm}(\mathbb{R}) : A + B = C \Leftrightarrow A = C - B$

• Note that the cancellation property $CA = CB \Rightarrow A = B$ is not true for all matrices

▸ Matrix powers

Let $A \in M_n(\mathbb{R})$ be a square matrix. We define

$$A^n = \underbrace{A \cdot A \cdot \ldots \cdot A}_{n \text{ times}}$$

$\forall a, b \in \mathbb{N} - \{0\} : \forall A \in M_n(\mathbb{R}) : A^a A^b = A^{a+b}$

$\forall a, b \in \mathbb{N} - \{0\} : \forall A \in M_n(\mathbb{R}) : (A^a)^b = A^{ab}$

EXAMPLES

a) Let $A=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$. Find all $x,y\in\mathbb{R}$ such that $A^2=xA-yI$.

Solution

We note that

$$A^2=AA=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}=\begin{bmatrix}2\cdot 2+1\cdot 3 & 2\cdot 1+1\cdot 2\\ 3\cdot 2+2\cdot 3 & 3\cdot 1+2\cdot 2\end{bmatrix}=$$

$$=\begin{bmatrix}4+3 & 2+2\\ 6+6 & 3+4\end{bmatrix}=\begin{bmatrix}7 & 4\\ 12 & 7\end{bmatrix}$$

and

$$xA-yI=x\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}-y\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2x & x\\ 3x & 2x\end{bmatrix}-\begin{bmatrix}y & 0\\ 0 & y\end{bmatrix}=$$

$$=\begin{bmatrix}2x-y & x\\ 3x & 2x-y\end{bmatrix}$$

It follows that

$$A^2=xA-yI\Leftrightarrow\begin{bmatrix}7 & 4\\ 12 & 7\end{bmatrix}=\begin{bmatrix}2x-y & x\\ 3x & 2x-y\end{bmatrix}\Leftrightarrow\begin{cases}2x-y=7\\ 3x=12\\ x=4\end{cases}$$

$$\Leftrightarrow\begin{cases}2\cdot 4-y=7\\ x=4\end{cases}\Leftrightarrow\begin{cases}8-y=7\\ x=4\end{cases}\Leftrightarrow\begin{cases}y=8-7=1\\ x=4\end{cases}\Leftrightarrow$$

$$\Leftrightarrow x=4\wedge y=1.$$

b) Let $A, B \in M_n(\mathbb{R})$ such that $A^2 = I$ and $B^2 = B$. Show that $(2B - I)^2 = I$ and $(A+I)^2 = 2(A+I)$.

<u>Solution</u>

Assume that $A, B \in M_n(\mathbb{R})$ with $A^2 = I$ and $B^2 = B$. Then

$$\begin{aligned}(2B-I)^2 &= (2B-I)(2B-I) = 2B(2B-I) - I(2B-I) = \\ &= (2B)(2B) - (2B)I - I(2B) + I^2 = \\ &= 4B^2 - 2B - 2B + I = 4B^2 - 4B + I \stackrel{*}{=} \\ &= 4B - 4B + I = 0B + I = \mathbf{0} + I = I.\end{aligned}$$

and

$$\begin{aligned}(A+I)^2 &= (A+I)(A+I) = A(A+I) + I(A+I) = \\ &= AA + AI + IA + I^2 = A^2 + A + A + I = \\ &= A^2 + 2A + I \stackrel{*}{=} I + 2A + I = 2A + 2I \\ &= 2(A+I).\end{aligned}$$

c) Prove: $\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = BA + CA$

<u>Solution</u>

Let $B, C \in M_{nk}(\mathbb{R})$ and $A \in M_{km}(\mathbb{R})$ be given. Let $a \in [n]$ and $b \in [m]$ be given. Then

$$\begin{aligned}[(B+C)A]_{ab} &= \sum_{\gamma \in [k]} (B+C)_{a\gamma} A_{\gamma b} = \sum_{\gamma \in [k]} (B_{a\gamma} + C_{a\gamma}) A_{\gamma b} = \\ &= \sum_{\gamma \in [k]} (B_{a\gamma} A_{\gamma b} + C_{a\gamma} A_{\gamma b}) = \\ &= \sum_{\gamma \in [k]} B_{a\gamma} A_{\gamma b} + \sum_{\gamma \in [k]} C_{a\gamma} A_{\gamma b} = \\ &= (BA)_{ab} + (CA)_{ab} = (BA + CA)_{ab}\end{aligned}$$

It follows that

$\forall a \in [n] : \forall b \in [m] : [(B+C)A]_{ab} = (BA+CA)_{ab} \Rightarrow$

$\Rightarrow (B+C)A = BA+CA$

and therefore

$\forall B, C \in M_{nk}(\mathbb{R}) : \forall A \in M_{km}(\mathbb{R}) : (B+C)A = BA + CA.$

d) Prove: $\forall A, B, C \in M_{nm}(\mathbb{R}) : (A = B \Rightarrow A + C = B + C)$

<u>Solution</u>

Let $A, B, C \in M_{nm}(\mathbb{R})$ be given and assume that $A = B$.

Then:

$A = B \Rightarrow \forall a \in [n] : \forall b \in [m] : A_{ab} = B_{ab}$ (1)

Let $a \in [n]$ and $b \in [m]$ be given. Then:

$(A+C)_{ab} = A_{ab} + C_{ab} = B_{ab} + C_{ab} = (B+C)_{ab}$

and it follows that

$\forall a \in [n] : \forall b \in [m] : (A+C)_{ab} = (B+C)_{ab}$

$\Rightarrow A + C = B + C$

and therefore we have shown that

$\forall A, B, C \in M_{nm}(\mathbb{R}) : (A = B \Rightarrow A + C = B + C)$

EXERCISES

(6) Consider the matrix

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

a) Find the unique $x, y \in \mathbb{R}$ such that $A^2 = xA + yI$

b) Use (a) to find $z, w \in \mathbb{R}$ such that $A^3 = zA + wI$.

(7) Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

evaluate and simplify

a) $C = AB$

b) $D = BA - 3B^2$

c) $E = 2A^3 - 3A + I$

(8) Given the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

evaluate and simplify $C = AB - BA$

(9) Given the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{R}$

show that: $A^2 - (a+d)A + (ad - bc)I = 0$

(10) Prove the following properties

a) $\forall A \in M_{nk}(\mathbb{R}): \forall B, C \in M_{km}(\mathbb{R}): A(B+C) = AB + AC$

b) $\forall A \in M_{nk}(\mathbb{R}): \forall B \in M_{k\lambda}(\mathbb{R}): \forall C \in M_{\lambda m}(\mathbb{R}): (AB)C = A(BC)$

(11) Rotation matrix

Let $R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$

Show that

a) $R(\vartheta_1)R(\vartheta_2) = R(\vartheta_1 + \vartheta_2)$, $\forall \vartheta_1, \vartheta_2 \in \mathbb{R}$

b) $R(\vartheta)R(-\vartheta) = I$, $\forall \vartheta \in \mathbb{R}$.

(12) For $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix}$

show that $AB \neq BA$

(13) For $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ with

$a, b \in \mathbb{R}$, show that $AB = BA$.

(14) Consider the function

$$\forall x \in \mathbb{R}: M(x) = \begin{bmatrix} 1 & 0 & x \\ -x & 1 & -x^2/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that

$\forall a, b \in \mathbb{R}: M(a)M(b) = M(a+b)$.

(15) Let $z = a + bi \in \mathbb{C}$ be a complex number, and define

$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Show that

a) $\forall z_1, z_2 \in \mathbb{C} : M(z_1 + z_2) = M(z_1) + M(z_2)$

b) $\forall z_1, z_2 \in \mathbb{C} : M(z_1 z_2) = M(z_1) M(z_2)$

→ This shows that $M(z)$ "imitates" the behaviour of complex number algebra.

(16) Let $A, B \in M_n(\mathbb{R})$. Show that

a) $AB = BA \Rightarrow (A - \lambda I)(B - \lambda I) = (B - \lambda I)(A - \lambda I), \forall \lambda \in \mathbb{R}$.

b) $(A+B)^2 = A^2 + 2AB + B^2 \Rightarrow AB = BA$

c) $A^2 = A \Rightarrow (A - I)^2 = I - A$

d) $AB = BA \Rightarrow A^2 B^2 = B^2 A^2$

e) $(B^2 = I \wedge AB = -AB) \Rightarrow AB = BA = 0$

(17) Let $A \in M_n(\mathbb{R})$ such that $A^2 = I$. Show that the matrices

$B = (1/2)(I + A)$

$C = (1/2)(I - A)$

satisfy $B^2 = B$ and $C^2 = C$.

Matrix Inverses

- Let $A \in M_n(\mathbb{R})$ be a square matrix. We say that
  B inverse of $A \Leftrightarrow AB = BA = I$.
- We define the set of all matrices $A \in M_n(\mathbb{R})$ that have an inverse as:
  $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \exists B \in M_n(\mathbb{R}) : AB = BA = I\}$
  We say that
  A non-singular $\Leftrightarrow A \in GL(n, \mathbb{R})$
  A singular $\Leftrightarrow A \notin GL(n, \mathbb{R})$
- Square matrices are not guaranteed to have an inverse. For example, for $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have:

$$\forall x, y, z, w \in \mathbb{R} : \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  However, we will argue that if a matrix does have an inverse, it is unique:
- $$\boxed{\forall A, B, C \in M_n(\mathbb{R}) : \begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C}$$

Proof

Let $A, B, C \in M_n(\mathbb{R})$ be given such that $AB = BA = I$ and $AC = CA = I$. Then

$$\begin{aligned} B &= BI && [\text{Identity matrix}] \\ &= B(AC) && [\text{Hypothesis } AC = I] \\ &= (BA)C && [\text{Associative property}] \end{aligned}$$

$$
\begin{aligned}
&= IC && [\text{Hypothesis } BA = I] \\
&= C && [\text{Identity matrix}]
\end{aligned}
$$

It follows that

$$\forall A, B, C \in M_n(\mathbb{R}) : \begin{cases} AB = BA = I \\ AC = CA = I \end{cases} \Rightarrow B = C \qquad \square$$

↳ The unique inverse of A is denoted as A^{-1}, as long as it exists.

▶ <u>Cancellation property.</u>

$$\boxed{\begin{array}{l} \forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (CA = CB \Leftrightarrow A = B) \\ \forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (AC = BC \Leftrightarrow A = B) \end{array}}$$

<u>Proof</u>

We show only the first statement. Let $A, B \in M_n(\mathbb{R})$ and $C \in GL(n, \mathbb{R})$ be given such that $CA = CB$. Then

$$
\begin{aligned}
A = IA &= && [\text{identity matrix}] \\
&= (C^{-1}C)A && [C^{-1} \text{ inverse of } C] \\
&= C^{-1}(CA) && [\text{associative property}] \\
&= C^{-1}(CB) && [\text{hypothesis: } CA = CB] \\
&= (C^{-1}C)B && [\text{associative property}] \\
&= IB && [C^{-1} \text{ inverse of } C] \\
&= B && [\text{identity matrix}]
\end{aligned}
$$

It follows that
$\forall A, B \in M_n(\mathbb{R}) : \forall C \in GL(n, \mathbb{R}) : (CA = CB \Rightarrow A = B)$

▸ Inverse of a 2x2 matrix

• Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ be a 2x2 square matrix

a) If $D = ad - bc \neq 0$, then A is non-singular with
$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

b) If $D = ad - bc = 0$, then A is singular.

▸ Application to 2x2 linear systems.

Any 2x2 linear system given by
$$\begin{cases} a_{11} x + a_{12} y = b_1 \\ a_{21} x + a_{22} y = b_2 \end{cases}$$
can be rewritten in terms of matrix algebra as:
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
and then solved using the following property:

• $\boxed{\forall A \in GL(n, \mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1} b)}$

Proof

Let $A \in GL(n,\mathbb{R})$ and $x,b \in M_{n1}(\mathbb{R})$ be given. Then:

$$
\begin{aligned}
Ax = b &\Leftrightarrow A^{-1}(Ax) = A^{-1}b && [\text{cancellation property}]\\
&\Leftrightarrow (A^{-1}A)x = A^{-1}b && [\text{associative property}]\\
&\Leftrightarrow Ix = A^{-1}b && [A^{-1} \text{ inverse of } A]\\
&\Leftrightarrow x = A^{-1}b && [\text{identity matrix}]
\end{aligned}
$$

It follows that

$\forall A \in GL(n,\mathbb{R}) : \forall x,b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b)$ □

Consequently, if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then:

$$
\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Leftrightarrow
$$

$$
\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} =
$$

$$
= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
$$

from which we may calculate the unique solutions for (x,y).

↳ <u>Notation : 2x2 determinant</u>

The expression $ad - bc$ is the 2x2 determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and we write:

$$
\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.
$$

EXAMPLES

a) Use the matrix inverse to solve the system

$$\begin{cases} 2x+5y=12 \\ 3x-y=1 \end{cases}$$

Solution

$$\begin{cases} 2x+5y=12 \\ 3x-y=1 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 12 \\ 1 \end{bmatrix} = \frac{1}{2(-1)-5\cdot 3}\begin{bmatrix} -1 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 12 \\ 1 \end{bmatrix} =$$

$$= \frac{-1}{17}\begin{bmatrix} (-1)\cdot 12 + (-5)\cdot 1 \\ (-3)\cdot 12 + 2\cdot 1 \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -12-5 \\ -36+2 \end{bmatrix} =$$

$$= \frac{-1}{17}\begin{bmatrix} -17 \\ -34 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ Thus solution set $S = \{(1,2)\}$

b) Similarly for the following parametric system:

$$\begin{cases} (a+1)x+(a-1)y = 4a+2 \\ 2ax+(a-1)y = 7a-1 \end{cases}$$

Solution

$$D = \begin{vmatrix} a+1 & a-1 \\ 2a & a-1 \end{vmatrix} = (a+1)(a-1) - 2a(a-1) = (a-1)(a+1-2a) =$$

$$= (a-1)(1-a) = -(a-1)^2.$$

Case 1 : For $a \neq 1 \Rightarrow D \neq 0$, and therefore:

$$\begin{cases} (a+1)x+(a-1)y = 4a+2 \\ 2ax+(a-1)y = 7a-1 \end{cases} \Leftrightarrow \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a+1 & a-1 \\ 2a & a-1 \end{bmatrix}^{-1} \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} a-1 & 1-a \\ -2a & a+1 \end{bmatrix} \begin{bmatrix} 4a+2 \\ 7a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(4a+2) + (1-a)(7a-1) \\ -2a(4a+2) + (a+1)(7a-1) \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(4a+2-7a+1) \\ -8a^2-4a+7a^2-a+7a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} (a-1)(-3a+3) \\ -a^2+2a-1 \end{bmatrix} =$$

$$= \frac{-1}{(a-1)^2} \begin{bmatrix} -3(a-1)^2 \\ -(a-1)^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\Leftrightarrow (x,y) = (3, 1)$. ; thus solution set $S = \{(3, 1)\}$.

<u>Case 2</u> : For $a = 1$; we have:

$$\begin{cases} (1+1)x + (1-1)y = 4\cdot 1+2 \\ 2\cdot 1x + (1-1)y = 7\cdot 1 - 1 \end{cases} \Leftrightarrow \begin{cases} 2x = 6 \\ 2x = 6 \end{cases} \Leftrightarrow x = 3$$

which gives as solution set:

$$S = \{(x,y) \in \mathbb{R}^2 \mid x = 3\} = \{(3,y) \mid y \in \mathbb{R}\}.$$

EXERCISES

(18) Use the matrix inverse to solve the following systems:

a) $\begin{cases} x+3y=4 \\ 2x+y=3 \end{cases}$ b) $\begin{cases} 2x-y=3 \\ x+2y=4 \end{cases}$ c) $\begin{cases} 3x+2y=7 \\ x+3y=10 \end{cases}$

d) $\begin{cases} 2ax+(a-3)y=a-1 \\ (a-3)x+2ay=a-a^2 \end{cases}$

e) $\begin{cases} x+(a+1)y=2 \\ (a+2)x+(1-a^2)y=5 \end{cases}$ f) $\begin{cases} ax-y=1-a \\ x-ay=a-a^2 \end{cases}$

↳ Distinguish between the values of the parameter $a\in\mathbb{R}$ where the corresponding matrix is non-singular vs. singular.

(19) Find all $a\in\mathbb{R}$ for which the matrix

$$A=\begin{bmatrix} a+3 & 2 \\ 1 & -a \end{bmatrix}$$

is non-singular.

(20) If $A,B\in M_n(\mathbb{R})$ are non-singular, show that AB is also non-singular with the inverse given by $(AB)^{-1}=B^{-1}A^{-1}$

(21) If $A, B, G \in M_n(\mathbb{R})$ and G is non-singular, then show that

a) $GA = GB \Rightarrow A = B$

b) $AG = BG \Rightarrow A = B$

(22) If $A \in M_n(\mathbb{R})$ with $A^3 = \mathbb{O}$, show that $I - A$ is non-singular with

$$(I - A)^{-1} = I + A + A^2$$

(23) If $A \in M_n(\mathbb{R})$ satisfies $A^2 + A + I = \mathbb{O}$, show that A is non-singular and $A^{-1} = A^2$.

(24) If $A, B \in M_n(\mathbb{R})$ with A being non-singular, show that

$$(A - B) A^{-1} (A + B) = (A + B) A^{-1} (A - B)$$

(25) Let $A, B \in M_n(\mathbb{R})$ with $A \neq \mathbb{O}$ and $B \neq \mathbb{O}$. Show that:

$$AB = \mathbb{O} \Rightarrow \begin{cases} A \text{ singular} \\ B \text{ singular} \end{cases}$$

▼ Matrix Transpose

- Let $A \in M_{nm}(\mathbb{R})$ be a matrix. We define the transpose matrix $A^T \in M_{mn}(\mathbb{R})$ as:
  $\forall a \in [m] : \forall b \in [n] : (A^T)_{ab} = A_{ba}$
- Let $A \in M_n(\mathbb{R})$ be a square matrix. We say that
  A symmetric $\Leftrightarrow A^T = A \Leftrightarrow \forall a, b \in [n] : \; A_{ab} = A_{ba}$

▶ Properties

$\forall A, B \in M_{nm}(\mathbb{R}) : \; (A+B)^T = A^T + B^T$
$\forall \lambda \in \mathbb{R} : \forall A \in M_{nm}(\mathbb{R}) : \; (\lambda A)^T = \lambda A^T$
$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T$
$\forall A \in GL(n, \mathbb{R}) : \; (A^T)^{-1} = (A^{-1})^T$
$\forall A \in M_{nm}(\mathbb{R}) : \; (A^T)^T = A$

EXAMPLES

a) Prove the property
$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T$.

Solution

Let $A \in M_{nk}(\mathbb{R})$ and $B \in M_{km}(\mathbb{R})$ be given. Let $a \in [m]$ and $b \in [n]$ be given. Then:

$$[(AB)^T]_{ab} = (AB)_{ba} = \sum_{\gamma \in [k]} A_{b\gamma} B_{\gamma a} = \sum_{\gamma \in [k]} A^T_{\gamma b} B^T_{a\gamma} =$$

$$= \sum_{\gamma \in [k]} B^T_{a\gamma} A^T_{\gamma b} = (B^T A^T)_{ab}$$

It follows that

$\forall a \in [n] : \forall b \in [m] : [(AB)^T]_{ab} = (B^T A^T)_{ab}$

$\Rightarrow (AB)^T = B^T A^T$

and therefore

$\forall A \in M_{nk}(\mathbb{R}) : \forall B \in M_{km}(\mathbb{R}) : (AB)^T = B^T A^T.$ $\square$

b) Show that

$$\forall A,B \in M_n(\mathbb{R}) : \begin{cases} A,B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB \text{ symmetric.}$$

<u>Solution</u>

Let $A,B \in M_n(\mathbb{R})$ be given such that A,B symmetric and $AB = BA$. Then

$$\begin{aligned} (AB)^T &= B^T A^T && [\text{transpose of matrix product}] \\ &= BA && [\text{hypothesis: } A,B \text{ symmetric}] \\ &= AB && [\text{hypothesis: } AB = BA] \end{aligned}$$

$\Rightarrow AB$ symmetric.

It follows that

$$\forall A,B \in M_n(\mathbb{R}) : \begin{cases} A,B \text{ symmetric} \\ AB = BA \end{cases} \Rightarrow AB \text{ symmetric.}$$

EXERCISES

(26) Give proofs for all properties of the matrix transpose.

(27) Show that if $A \in M_n(\mathbb{R})$ is symmetric and non-singular, then A^{-1} is also symmetric.

(28) Let $A, B \in M_n(\mathbb{R})$. Show that
A, B, AB symmetric $\Rightarrow AB = BA$.

(29) Given $A, P \in M_n(\mathbb{R})$, show that
A symmetric $\Rightarrow B = P^T A P$ symmetric

(30) Consider the rotation matrix

$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

a) Show that $R(\vartheta)$ is non-singular with
$[R(\vartheta)]^{-1} = R(-\vartheta)$

b) Show that
$[R(\vartheta)]^T = R(-\vartheta)$

c) For what angles $\vartheta \in \mathbb{R}$ is $R(\vartheta)$ symmetric?

(31) Let $A \in M_n(\mathbb{R})$ be a square matrix.
Show that

a) $A + A^T$ symmetric
b) $A^T A$ symmetric

(32) Let $z = a + bi \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and i the imaginary unit and define

$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Show that

a) $\forall z \in \mathbb{C} - \{0\} : M(1/z) = M(z)^{-1}$
b) $\forall z_1 \in \mathbb{C} : \forall z_2 \in \mathbb{C} - \{0\} : M(z_1/z_2) = M(z_1) M(z_2)^{-1}$

(33) Let $A, B \in M_n(\mathbb{R})$ be two square matrices. We say that

$$B \text{ skew-symmetric} \Leftrightarrow \forall a, b \in [n] : B_{ab} = -B_{ba}$$
$$\Leftrightarrow B^T = -B$$

Show that

$$\forall A, B \in M_n(\mathbb{R}) : \begin{cases} A \text{ symmetric} \\ B \text{ skew-symmetric} \end{cases} \Rightarrow A^2 B A^2 \text{ skew-symmetric}$$

LIN4: Determinants and Linear Systems

DETERMINANTS AND LINEAR SYSTEMS

Determinants

Determinants are used to find the inverse of $n\times n$ matrices, and solve $n\times n$ linear systems.

Leibnitz definition of determinants

1) Permutations

Let $[n]=\{1,2,3,\dots,n\}$. A permutation σ is a reshuffling of the order of the elements of n. Formally, σ is a bijection $\sigma:[n]\to[n]$ whereby each element of $[n]$ is mapped into a distinct element of $[n]$.

S_n = set of all permutations on $[n]$

EXAMPLE

For $n=3$:

$S_3=\{(1,2,3),(2,3,1),(3,1,2),(3,2,1),(1,3,2),(2,1,3)\}$

are the six permutations of $[3]$.

For $\sigma=(2,3,1)$: $\sigma(1)=2$, $\sigma(2)=3$, $\sigma(3)=1$.

2) Permutation parity

Let $\sigma\in S_n$ be a permutation of $[n]$. We define the parity $s(\sigma)$ of σ as:

$$\boxed{s(\sigma)=\operatorname{sign}\left[\prod_{b=1}^{n-1}\prod_{a=b+1}^{n}(\sigma(a)-\sigma(b))\right]}$$

with sign(x) defined as

$$\operatorname{sign}(x) = \begin{cases} 1 & , \text{ if } x > 0 \\ 0 & , \text{ if } x = 0 \\ -1 & , \text{ if } x < 0 \end{cases}$$

For $\sigma \in S_n$, $s(\sigma) = 1$ or $s(\sigma) = -1$. We say that

σ even permutation $\Leftrightarrow s(\sigma) = 1$

σ odd permutation $\Leftrightarrow s(\sigma) = -1$

EXAMPLE

For $\sigma = (3,1,4,2)$, the parity of σ is:

$$s(\sigma) = \operatorname{sign}\left[\prod_{b=1}^{3} \prod_{a=b+1}^{4} (\sigma(a) - \sigma(b))\right] =$$

$$= \operatorname{sign}[(\sigma(2)-\sigma(1))(\sigma(3)-\sigma(1))(\sigma(4)-\sigma(1))(\sigma(3)-\sigma(2))$$
$$\times (\sigma(4)-\sigma(2))(\sigma(4)-\sigma(3))]$$
$$= \operatorname{sign}[(1-3)(4-3)(2-3)(4-1)(2-1)(2-4)]$$
$$= \operatorname{sign}[(-2)(1)(-1)(3)(1)(-2)] = -1.$$

- A transposition is a permutation that switches only two elements of $[n]$. Every permutation can be constructed as a sequence of transpositions. An even permutation can be constructed by an even number of transpositions. An odd permutation requires an odd number of transpositions.

EXAMPLE

a) For $\sigma = (3,1,4,2)$, we can construct σ with 3 transpositions:

$$(1,2,3,4) \longrightarrow (3,2,1,4) \longrightarrow (3,2,4,1)$$

$$\longrightarrow (3,1,4,2)$$

and therefore σ is odd.

b) For $n=3$, S_3 has 3 even permutations and 3 odd permutations:

$A = \{\sigma \in S_3 \mid \sigma \text{ even}\}$
$= \{(1,2,3), (2,3,1), (3,1,2)\} \leftarrow$ even permutations

$B = \{\sigma \in S_3 \mid \sigma \text{ odd}\}$
$= \{(3,2,1), (1,3,2), (2,1,3)\} \leftarrow$ odd permutations

3) <u>Determinants</u>

We now use permutations to define determinants as follows:

$$\boxed{\forall A \in M_n(\mathbb{R}) : \det(A) = \sum_{\sigma \in S_n} \left[s(\sigma) \prod_{a=1}^{n} A_{a,\sigma(a)} \right]}$$

$\vdash$ 1x1, 2x2, 3x3 determinants

For $n=1$: $\quad |A_{11}| = A_{11}$

For $n=2$: $\quad \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$

For $n=3$: we use the Sarrus scheme:

$$\begin{array}{ccc|cc} & & - & - & - \\ A_{11} & A_{12} & A_{13} & A_{11} & A_{12} \\ A_{21} & A_{22} & A_{23} & A_{21} & A_{22} \\ A_{31} & A_{32} & A_{33} & A_{31} & A_{32} \\ & & + & + & + \end{array} =$$

$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}$$
$$- A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33}$$

▶ Note that there are 3 positive terms corresponding to the 3 even permutations of S_3 and 3 negative terms corresponding to the 3 odd permutations.

$\vdash$ Fundamental properties of determinants

1) $I \in M_n(\mathbb{R})$ identity matrix $\Rightarrow \det(I) = 1$
2) $\forall A \in M_n(\mathbb{R})$: $\det(A^T) = \det(A)$

3) $\forall A, B \in M_n(\mathbb{R})$: $\det(AB) = \det(A)\det(B)$

4) $\forall A \in M_n(\mathbb{R})$: $(A \text{ non-singular} \Leftrightarrow \det(A) \neq 0)$

$\forall A \in M_n(\mathbb{R})$: $(A \text{ singular} \Leftrightarrow \det(A) = 0)$

↳ It follows that the set $GL(n, \mathbb{R})$ of non-singular matrices satisfies

$$GL(n, \mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det A \neq 0 \}$$

5) $\forall A \in GL(n, \mathbb{R})$: $\det(A^{-1}) = \dfrac{1}{\det(A)}$

↳ <u>Determinant of lower/upper triangular matrices</u>

- Let $A \in M_n(\mathbb{R})$ be a matrix. We say that

  A lower-triangular $\Leftrightarrow \forall a, b \in [n]: (a < b \Rightarrow A_{ab} = 0)$

  A upper-triangular $\Leftrightarrow \forall a, b \in [n]: (a > b \Rightarrow A_{ab} = 0)$

- It can be shown that if A is upper-triangular or lower-triangular, its determinant $\det(A)$ is given by the product of all diagonal components:

$$\boxed{\begin{array}{l} \forall A \in M_n(\mathbb{R}): \left(A \text{ lower-triangular} \Rightarrow \det A = \prod_{a=1}^{n} A_{aa}\right) \\[1em] \forall A \in M_n(\mathbb{R}): \left(A \text{ upper-triangular} \Rightarrow \det A = \prod_{a=1}^{n} A_{aa}\right) \end{array}}$$

EXAMPLES

a) Evaluate the determinant of

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 3\cdot 1 - 5\cdot 2 = 3 - 10 = -7.$$

b) Evaluate the determinant of

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \text{ using the Sarrus rule.}$$

Solution

$$\det(A) = \left|\begin{array}{ccc|cc} 1 & 2 & -2 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 0 & 3 \end{array}\right| =$$

$$= 1\cdot 1\cdot 1 + 2\cdot 0\cdot 0 + (-2)\cdot 1\cdot 3 - 0\cdot 1\cdot(-2) - 3\cdot 0\cdot 1 - 1\cdot 1\cdot 2$$

$$= 1 + 0 - 6 - 0 - 0 - 2 = 1 - 6 - 2 = 1 - 8 = -7.$$

c) Evaluate the determinant of

$$A = \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Solution:

A upper triangular $\Rightarrow$

$\Rightarrow \det A = A_{11} A_{22} A_{33} A_{44}$

$= 1(-1)\cdot 2\cdot 7 = -14.$

d) Let $A, B \in M_n(\mathbb{R})$. Show that $AB = I \Rightarrow BA = I$.

<u>Solution</u>

Assume that $AB = I$. It follows that

$\det(A)\det(B) = \det(AB) = \det(I) = 1 \Rightarrow$

$\Rightarrow \det(A)\det(B) \neq 0 \overset{*}{\Rightarrow}$

$\Rightarrow \det(A) \neq 0 \wedge \det(B) \neq 0 \Rightarrow$

$\Rightarrow A, B$ are non-singular

Let A^{-1}, B^{-1} be the corresponding inverse matrices. Then

$BA = I(BA)$	[identity matrix]
$= (A^{-1}A)(BA)$	[A^{-1} inverse of A]
$= A^{-1}[A(BA)]$	[associative property]
$= A^{-1}[(AB)A]$	[associative property]
$= A^{-1}IA$	[hypothesis $AB = I$]
$= A^{-1}A$	[identity matrix]
$= I$	[A^{-1} inverse of A]

* Note that we use the contrapositive of the statement

$\forall a, b \in \mathbb{R} : (ab = 0 \Leftrightarrow (a = 0 \vee b = 0))$

which is given by

$\forall a, b \in \mathbb{R} : (ab \neq 0 \Leftrightarrow (a \neq 0 \wedge b \neq 0))$.

EXERCISES

① Which of the following permutations are odd and which are even? Show using both the product definition and enumeration of transpositions.

a) $\sigma = (1,3,2,4)$ c) $\sigma = (2,3,4,1)$

b) $\sigma = (3,1,4,2)$ d) $\sigma = (1,4,3,2)$

② Calculate the following determinants:

a) $\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix}$ b) $\begin{vmatrix} 2a & a+1 \\ a-1 & a \end{vmatrix}$

c) $\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix}$ d) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

③ Given the matrix

$$A = \begin{bmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{bmatrix}$$

show that

$$A \text{ singular} \Leftrightarrow x = 1$$

④ Solve with respect to x the following equations:

a) $\begin{vmatrix} 1 & 3x-4 \\ -1 & 4x+1 \end{vmatrix} = 0$ b) $\begin{vmatrix} x-1 & x^2-1 \\ 1-x^2 & x^3-1 \end{vmatrix} = 0$

(5) Let $A, B \in M_n(\mathbb{R})$. Show that
AB singular $\Rightarrow$ (A singular $\vee$ B singular)

(6) Rotation matrix.
Consider the rotation matrix

$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

Show that $\det R(\vartheta) = 1$.

(7) Complex number matrix
Let $z = a + bi \in \mathbb{C}$ be a complex number with $a, b \in \mathbb{R}$, and define

$$M(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Show that $\det M(z) = |z|$, $\forall z \in \mathbb{C}$.

Co-factor expansion of determinants

Let $A \in M_n(\mathbb{R})$ be a square matrix. Let $a, b \in [n]$.
The minor matrix $M_{ab}(A)$ is defined as the $(n-1)\times(n-1)$ square matrix obtained from A by deleting: (a) The a^{th} row of A
(b) The b^{th} column of A.
The formal definition of $M_{ab}(A)$ is given by:

$$\boxed{\forall c,d \in [n-1] : (M_{ab}(A))_{cd} = \begin{cases} A_{cd} & , \text{ if } c<a \wedge d<b \\ A_{c,d+1} & , \text{ if } c<a \wedge d \geq b \\ A_{c+1,d} & , \text{ if } c \geq a \wedge d<b \\ A_{c+1,d+1} & , \text{ if } c \geq a \wedge d \geq b \end{cases}}$$

EXAMPLE

Given $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 5 & 7 & 2 \\ 3 & 1 & 5 & 2 \\ 1 & 4 & 7 & 3 \end{bmatrix} \Rightarrow$

Note that $A_{23} = 7$

$$\Rightarrow M_{23}(A) = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

- We may use minor matrices to calculate determinants <u>recursively</u> as follows:

1) Row Expansion

$$\boxed{\forall a \in [n]: \det A = \sum_{b=1}^{n} (-1)^{a+b} A_{ab} \det(M_{ab}(A))}$$

2) Column expansion

$$\boxed{\forall b \in [n]: \det A = \sum_{a=1}^{n} (-1)^{a+b} A_{ab} \det(M_{ab}(A))}$$

EXAMPLE

$$\begin{vmatrix} 3 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \quad \text{sign of } (-1)^{a+b} \leftrightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= (-1)\cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + (+1)\cdot 5 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + (-1)\cdot 1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} =$$

$$= -(1\cdot 1 - 2\cdot 3) + 5(3\cdot 1 - 2\cdot 2) - (3\cdot 3 - 1\cdot 2) =$$

$$= -(1-6) + 5(3-4) - (9-2) =$$

$$= -(-5) + 5(-1) - 7 = 5 - 5 - 7 = -7.$$

▸ Method: Zero is your FRIEND.

example

$$\begin{vmatrix} 4 & 1 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 1 & 3 & 1 \end{vmatrix} \rightarrow =$$

$$= 4 \cdot (-2) \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 4 \cdot (-2) \cdot (2 \cdot 1 - 3 \cdot 3)$$

$$= (-8)(2-9) = (-8)(-7) = 56$$

EXERCISE

(34) Evaluate the determinants

(a) $\begin{vmatrix} 3 & 2 & 1 & 3 \\ 0 & 1 & 9 & 2 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 1 & 5 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 3 & 7 \\ 2 & 0 & 2 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 3 & 2 & 7 & 5 \\ 5 & 0 & 7 & 0 & 0 \\ 2 & 0 & 2 & 3 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 8 & 0 & 1 & 4 & 2 \end{vmatrix}$

Simplification of determinants

The calculation of determinants can be simplified considerably by using the following properties:

1) If we transpose 2 rows or 2 columns, the determinant changes sign.

e.g. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

2) If 2 rows or 2 columns are identical, then the determinant is equal to 0.

e.g. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

3) If we multiply a row or column by $\lambda \in \mathbb{R}$, then the determinant itself is multiplied by λ.

e.g. $$\lambda \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & \lambda a_2 & a_3 \\ b_1 & \lambda b_2 & b_3 \\ c_1 & \lambda c_2 & c_3 \end{vmatrix} = \begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It follows that

- 1 We can pull out a common factor from any row or column.

e.g. $$\begin{vmatrix} 3 & 2 & 7 \\ 5 & 4 & -1 \\ 8 & -8 & 12 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 & 7 \\ 5 & 2 & -1 \\ 8 & -4 & 12 \end{vmatrix} = 2\cdot 4 \cdot \begin{vmatrix} 3 & 1 & 7 \\ 5 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

- 2 If all the elements of a row or column are 0, then the determinant is 0.

e.g. $$\begin{vmatrix} 2 & 0 & 3 \\ 7 & 0 & 2 \\ 1 & 0 & 4 \end{vmatrix} = 0$$

- 3 If $A \in M_{nn}(\mathbb{R}) \Rightarrow \det(\lambda A) = \lambda^n \det(A), \forall \lambda \in \mathbb{R}$.

4) When every element of a row or column is written as a sum of two numbers, then the determinant can be rewritten as the sum of two determinants.

e.g. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

5) If we add to the elements of a row (or column) the elements of another row (or column) multiplied by a common factor λ, then the value of the determinant does not change.

e.g.
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+\lambda a_1 & b_2+\lambda a_2 & b_3+\lambda a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6) When the elements above OR bellow the diagonal are all 0, then the determinant is equal to the product of the diagonal elements.

e.g.
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

So, to simplify a determinant

- 1 Check if common factors can be pulled out via (3)
- 2 Use (1), (5) to diagonalize the determinant (i.e. create ZEROES!) so you can then use (6)
- 3 If you run into identical rows or columns then use (2).

EXAMPLES

a) Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & -4 & -3 & 3 \\ 0 & 4 & 0 & 1 \\ 1 & 6 & 0 & 1 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & -4 & -3 & 3 \\ 0 & 4 & 0 & 1 \\ 1 & 6 & 0 & 1 \end{vmatrix} \begin{matrix} \cdot(-2) \ \cdot(-1) \\ \\ \\ \\ \end{matrix} =$$

$$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 2+(-2)\cdot 1 & -4+(-2)\cdot 2 & -3+(-2)(-1) & 3+(-2)\cdot 2 \\ 0 & 4 & 0 & 1 \\ 1+(-1)\cdot 1 & 6+(-1)\cdot 2 & 0+(-1)(-1) & 1+(-1)2 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -8 & -1 & -1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 1 & -1 \end{vmatrix} = (+1)\cdot 1 \cdot \begin{vmatrix} -8 & -1 & -1 \\ 4 & 0 & 1 \\ 4 & 1 & -1 \end{vmatrix} =$$

$$= 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} \cdot 1 = 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 1-2 & 1-1 & -1-1 \end{vmatrix} =$$

$$= 4\begin{vmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{vmatrix} = 4 \cdot (-1)(-1)\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} =$$

$$= 4\left(1(-2) - 1(-1)\right) = 4(-2+1) = -4$$

We use determinant properties to zero out a column or a row. Then we perform a co-factor expansion. This reduces the size of the determinant. We repeat, all the way down to 2x2 size.

8) Show that

$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} = 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

Solution

$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} (a+b)+(b+c)+(c+a) & b+c & c+a \\ (c+a)+(a+b)+(b+c) & a+b & b+c \\ (b+c)+(c+a)+(a+b) & c+a & a+b \end{vmatrix} =$$

$\cdot 1$

$\cdot 1$

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & c+a & a+b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 1 & a+b & b+c \\ 1 & c+a & a+b \end{vmatrix} (-1)$$

$$= 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 0 & (a+b)-(b+c) & (b+c)-(c+a) \\ 0 & (c+a)-(b+c) & (a+b)-(c+a) \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & b+c & c+a \\ 0 & a-c & b-a \\ 0 & a-b & b-c \end{vmatrix} = 2(a+b+c)\cdot(+1)(1)\begin{vmatrix} a-c & b-a \\ a-b & b-c \end{vmatrix}$$

$$= 2(a+b+c)\left[(a-c)(b-c)-(b-a)(a-b)\right]$$

$$= 2(a+b+c)\left[(a-c)(b-c)+(a-b)^2\right] =$$
$$= 2(a+b+c)(\underline{ab} - ac - bc + c^2 + a^2 - \underline{2ab} + b^2)$$
$$= 2(a+b+c)(a^2+b^2+c^2 - ab - bc - ca).$$

EXERCISES

(9) Evaluate the following determinants:

a) $\begin{vmatrix} 3 & 5 & 8 \\ 3 & 6 & 9 \\ 3 & 7 & 4 \end{vmatrix}$ b) $\begin{vmatrix} x & x+1 & x+3 \\ y & y+1 & y+3 \\ 1 & 1 & 1 \end{vmatrix}$

c) $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$ d) $\begin{vmatrix} 1 & -2 & 0 & 3 \\ 1 & 1 & 2 & 1 \\ 3 & 1 & -1 & 4 \\ 5 & 1 & 2 & -1 \end{vmatrix}$

e) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$ f) $\begin{vmatrix} -1 & 0 & 2 & 1 & -3 \\ 1 & 2 & 3 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 4 & 5 & 1 & -1 \\ 0 & 1 & 2 & 0 & -2 \end{vmatrix}$

(10) Solve the following equations

a) $\begin{vmatrix} x-3 & 4 & x \\ 3x-2 & -6 & 2x-1 \\ 4x-3 & 2 & x^2-3 \end{vmatrix} = 0$ b) $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & x \\ 1 & 2 & 1 & x^2 \\ 1 & 3 & 3 & x^3 \end{vmatrix} = 0$

(11) Show that

a) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

c) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

d) $\begin{vmatrix} x+y & z & z \\ y & z+x & y \\ x & x & z+y \end{vmatrix} = 4xyz$

e) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

f) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} = ab$

g) $$\begin{vmatrix} 1 & -c & b \\ c & 1 & -a \\ -b & a & 1 \end{vmatrix} = a^2+b^2+c^2+1$$

h) $$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = a(b-a)(c-b)(d-c)$$

i) $$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = 0$$

j) $$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0$$

k) $$\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 \\ a^4 & 1 & a & a^2 & a^3 \\ a^3 & a^4 & 1 & a & a^2 \\ a^2 & a^3 & a^4 & 1 & a \\ a & a^2 & a^3 & a^4 & 1 \end{vmatrix} = (1-a^5)^4$$

l) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

m) $\begin{vmatrix} a_1 & b_1 & a_1x^2+b_1x+c_1 \\ a_2 & b_2 & a_2x^2+b_2x+c_2 \\ a_3 & b_3 & a_3x^2+b_3x+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

n) $\begin{vmatrix} 1 & a & b & 1 \\ 1 & a & a & a \\ a & 1 & ab & b \\ a & a & ab & 1 \end{vmatrix} = (a-b)(a-1)(1-ab)$

o) $\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = 4(a^2+b^2)(c^2+d^2)$

p) $\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ b & b & a & b \\ a & a & a & b \end{vmatrix} = -(a-b)^4$

Matrix inverse, in general

Recall that for a 2×2 matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
we have
$$ad-bc \neq 0 \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
For larger matrices, we use the following theory:

Def: Let $A \in M_n(\mathbb{R})$ be a square matrix. We define the adjugate matrix $\mathrm{adj}(A)$ such that

$$\forall a,b \in [n]: [\mathrm{adj}(A)]_{ab} = (-1)^{a+b} \det(M_{ba}(A))$$

Thm: Let $A \in GL(n,\mathbb{R})$ be a non-singular square matrix. Then

$$A^{-1} = \left(\frac{1}{\det(A)}\right) \mathrm{adj}(A)$$

EXAMPLE

Find the matrix inverse of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

Solution

Since,

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} \begin{matrix} (-2) & (-1) \\ & \\ & \end{matrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1+(-2)\cdot 0 & -1+(-2)(-1) \\ 0 & 2+(-1)\cdot 0 & 5+(-1)(-1) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 6 \end{vmatrix} = (+1)\cdot 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} = 1\cdot 6 - 1\cdot 2 = 4$$

and

$$[\operatorname{adj}(A)]_{11} = (-1)^{1+1} \det(M_{11}(A)) = \begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = 1\cdot 5 - (-1)\cdot 2$$

$$= 5+2 = 7$$

$$[\operatorname{adj}(A)]_{12} = (-1)^{1+2} \det(M_{21}(A)) = - \begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix} =$$

$$= -(0\cdot 5 - (-1)\cdot 2) = -(0+2) = -2$$

$$[\operatorname{adj}(A)]_{13} = (-1)^{1+3} \det(M_{31}(A)) = \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} =$$

$$= 0(-1) - (-1)\cdot 1 = 1$$

$$[\operatorname{adj}(A)]_{21} = (-1)^{2+1} \det(M_{12}(A)) = - \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} =$$

$$= -(2 \cdot 5 - (-1)1) = -(10+1) = -11$$

$$[\operatorname{adj}(A)]_{22} = (-1)^{2+2} \det(M_{22}(A)) = \begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix} =$$

$$= 1 \cdot 5 - (-1) \cdot 1 = 5+1 = 6$$

$$[\operatorname{adj}(A)]_{23} = (-1)^{2+3} \det(M_{32}(A)) = - \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} =$$

$$= -[1(-1) - (-1)2] = -(-1+2) = -1$$

$$[\operatorname{adj}(A)]_{31} = (-1)^{3+1} \det(M_{13}(A)) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 3$$

$$[\operatorname{adj}(A)]_{32} = (-1)^{3+2} \det(M_{23}(A)) = - \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} =$$

$$= -(1 \cdot 2 - 0 \cdot 1) = -2$$

$$[\operatorname{adj}(A)]_{33} = (-1)^{3+3} \det(M_{33}(A)) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \cdot 1 - 2 \cdot 0 = 1$$

it follows that

$$\operatorname{adj}(A) = \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow A^{-1} = \frac{1}{\det A} \operatorname{adj}(A) = \frac{1}{4} \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

EXERCISES

(12) Find the inverse matrix A^{-1} for the following matrices

a) $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

(13) If $(2I - A)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, evaluate the matrix A.

(14) If $A^{-1}B^{-1} = \begin{bmatrix} 5 & 0 \\ 2 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

then evaluate the matrix B.

(15) If $A^{-1}B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$, then evaluate the matrix BA.

(16) Solve for the matrix $X \in M_3(\mathbb{R})$:

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 5 \\ 5 & 0 & 8 \end{bmatrix} X = 7 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

▼ $n \times n$ linear system of equations

- An $n \times n$ linear system of equations is a system of the form:

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \\ A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n = b_n \end{cases} \qquad (1)$$

- This system can be rewritten as a matrix equation:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or equivalently as

$$Ax = b \qquad (2)$$

with $A \in M_n(\mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$.

- We say that

Eq.(2) is homogeneous $\Leftrightarrow$ $b = \mathbf{0}$

Eq.(2) is inhomogeneous $\Leftrightarrow$ $b \neq \mathbf{0}$

- Solution techniques for solving linear systems include:
  a) Matrix inverse method
  b) Cramer's rule
  c) Gaussian elimination

Matrix inverse method

We have already explained that if $\det A \neq 0$, then any linear system can be solved via the property

$$\forall A \in GL(n,\mathbb{R}) : \forall x, b \in M_{n1}(\mathbb{R}) : (Ax = b \Leftrightarrow x = A^{-1}b)$$

However, due to the difficulty of calculating A^{-1}, this method is recommended only for 2×2 systems, as was explained earlier.

Cramer's rule

We write

$$A = [A_1 \ A_2 \ A_3 \cdots A_n]$$

with $A_1, A_2, A_3, \ldots, A_n \in M_{n1}(\mathbb{R})$ the columns of the matrix A, and define the determinants

$$D = \det A = \det([A_1 \ A_2 \ A_3 \cdots A_n])$$
$$D_1 = \det([b \ A_2 \ A_3 \cdots A_n])$$
$$D_2 = \det([A_1 \ b \ A_3 \cdots A_n])$$

$$D_3 = \det([A_1 \ A_2 \ b \cdots A_n])$$
$$\vdots$$
$$D_n = \det([A_1 \ A_2 \ A_3 \cdots \ b])$$

Note that for any $k \in [n]$, in the determinant D_k, we replace the k^{th} column of A with the column matrix b.

- Cramer's method for solving linear systems is based on the following theorem:

<u>Thm</u>: Given the linear system $Ax=b$ with $A \in M_n(\mathbb{R})$ and $x, b \in M_{n1}(\mathbb{R})$.

a) If $D \neq 0$, then $Ax=b$ has a unique solution given by $\forall k \in [n]: x_k = D_k / D$.

b) $\begin{cases} D = 0 \\ \exists k \in [n] : D_k \neq 0 \end{cases} \Rightarrow Ax=b$ has no solutions

---

<u>Remark</u>: Note that the theorem is inconclusive when
$$\begin{cases} D=0 \\ \forall k \in [n] : D_k = 0 \end{cases}$$
Then, the system needs to be investigated via Gaussian Elimination method.

a) $\begin{cases} \lambda x + (\lambda-2)y = \lambda+1 & (1) \\ (\lambda+1)x - (\lambda-2)y = \lambda \end{cases}$

Solution

$$D = \begin{vmatrix} \lambda & \lambda-2 \\ \lambda+1 & -(\lambda-2) \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda & 1 \\ \lambda+1 & -1 \end{vmatrix} = (\lambda-2)[\lambda(-1) - (\lambda+1)\cdot 1]$$

$$= (\lambda-2)(-\lambda-\lambda-1) = -(\lambda-2)(2\lambda+1)$$

$$D_x = \begin{vmatrix} \lambda+1 & \lambda-2 \\ \lambda & -(\lambda-2) \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda+1 & 1 \\ \lambda & -1 \end{vmatrix} = (\lambda-2)[(\lambda+1)(-1) - 1\cdot\lambda].$$

$$= (\lambda-2)(-\lambda-1-\lambda) = -(\lambda-2)(2\lambda+1)$$

$$D_y = \begin{vmatrix} \lambda & \lambda+1 \\ \lambda+1 & \lambda \end{vmatrix} = \lambda^2 - (\lambda+1)^2 = (\lambda-(\lambda+1))(\lambda+(\lambda+1)) =$$

$$= (\lambda-\lambda-1)(\lambda+\lambda+1) = -(2\lambda+1)$$

Note that $D = 0 \Leftrightarrow -(\lambda-2)(2\lambda+1) = 0 \Leftrightarrow$

$\Leftrightarrow \lambda - 2 = 0 \lor 2\lambda+1 = 0 \Leftrightarrow \lambda = 2 \lor \lambda = -1/2 \Leftrightarrow$

$\Leftrightarrow \lambda \in \{2, -1/2\}$.

Case 1: If $\lambda \in \mathbb{R} - \{-1/2, 2\}$, then the system has a unique solution given by:

$$x = \frac{D_x}{D} = \frac{-(\lambda-2)(2\lambda+1)}{-(\lambda-2)(2\lambda+1)} = 1$$

$$y = \frac{D_y}{D} = \frac{-(2\lambda+1)}{-(\lambda-2)(2\lambda+1)} = \frac{1}{\lambda-2}$$

Case 2: If $\lambda = 2$, then

$D = 0 \land D_x = 0 \land D_y = -5 \neq 0 \Rightarrow$

$\Rightarrow$ the system is inconsistent (i.e. no solutions).

Case 3: If $\lambda = -1/2$, then

$$D = 0 \wedge D_x = 0 \wedge D_y = 0$$

so we have to solve the system explicitly:

$$(1) \Leftrightarrow \begin{cases} (-1/2)x + (-1/2 - 2)y = -1/2 + 1 \\ (-1/2+1)x - (-1/2-2)y = -1/2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} -x + (-1-4)y = -1+2 \\ (-1+2)x - (-1-4)y = -1 \end{cases} \Leftrightarrow \begin{cases} -x - 5y = 1 \\ x + 5y = -1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x + 5y = 1 \Leftrightarrow x = 1 - 5y \Leftrightarrow (x, y) = (1-5y, y)$$

$$\Leftrightarrow (x, y) \in \{(1-5y, y) \mid y \in \mathbb{R}\}.$$

To summarize, the solution set is:

$$S = \begin{cases} \{(1, 1/(\lambda - 2))\} &, \text{ if } \lambda \in \mathbb{R} - \{-1/2, 2\} \\ \varnothing &, \text{ if } \lambda = 2 \\ \{(1-5y, y) \mid y \in \mathbb{R}\} &, \text{ if } \lambda = -1/2 \end{cases}$$

EXERCISES

(17) Use Cramer's rule to solve the following linear systems:

a) $\begin{cases} x-y=0 \\ 3x+2y=5 \end{cases}$

$(1,1)$

b) $\begin{cases} x+2y-z=0 \\ 2x-y+3z=0 \\ x+y+z=2 \end{cases}$

$(-2,2,2)$

c) $\begin{cases} x-y+z=3 \\ 2x+y-3z=10 \\ x+5y-9z=8 \end{cases}$

(no solutions)

d) $\begin{cases} 2x-y-z-w=-1 \\ x-2y+z+w=-2 \\ x+y-2z+w=4 \\ x+y+z-2w=-8 \end{cases}$

$(-2,-1,-3,1)$

e) $\begin{cases} x+y+z+w=2 \\ 2x-w+3z=9 \\ -x+2y-z+2w=-5 \\ 3x+y-w=4 \end{cases}$

$(x,y,z,w)=(1,0,2,-1)$

(18) Solve the following linear systems in terms of the parameter $a \in \mathbb{R}$.

a) $\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases}$

b) $\begin{cases} x-ay+z=a \\ x+y+z=-1 \\ ax+y-a^2z=1. \end{cases}$

c) $\begin{cases} x+y+z=1 \\ ax+by+cz=d \\ a^2x+b^2y+c^2z=d^2 \end{cases}$

d) $\begin{cases} x+y+z=1+c \\ x+(1+a)y+z=1 \\ x+y+(1+b)z=1 \end{cases}$

Method of Gaussian elimination

- We represent the linear system $Ax=b$ in terms of an augmented matrix M:

$$\left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1n} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} & b_m \end{array}\right] \longleftarrow \text{each row represents an equation}$$

- We say that two augmented matrices M_1 and M_2 are equivalent (notation: $M_1 \sim M_2$) if and only if the corresponding linear systems have the same solution set.

Properties

- The following transformations map an augmented matrix M_1 to an equivalent augmented matrix M_2.

1) Transposition: We can swap any two rows (but not columns).

e.g. $$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \end{array}\right]$$

2) Scalar Multiplication: We can multiply any row (but not a column) with a non-zero scalar $\lambda \in \mathbb{R}-\{0\}$.

e.g. $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \cdot \lambda \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ \lambda a_2 & \lambda b_2 & \lambda c_2 & \lambda d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

3) Linear combination : We can add to any row (but not a column) any other row multiplied by a non-zero scalar $\lambda \in \mathbb{R} - \{0\}$

e.g. $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right] \begin{array}{l} \cdot\lambda \\ \hookleftarrow \\ \end{array} \sim \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2+\lambda a_1 & b_2+\lambda b_1 & c_2+\lambda c_1 & d_2+\lambda d_1 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

- Note that these properties are somewhat different from the corresponding properties of determinants.

▸ Method

- Using these properties, we try to diagonalize the augmented matrix defering fractional arithmetic as much as possible to the very last step. We work on the augmented matrix one column at a time.
- If during this process we get a row of the form

$$0 \ 0 \ 0 \ \dots \ 0 \mid a$$

then: a) If $a \neq 0$, the system is inconsistent and we stop work.

b) If $a=0$, then the row corresponds to an identity and may then be deleted from the augmented matrix.

EXAMPLES

a) $\begin{cases} 2x-y=1 \\ x+y=3 \\ 3x+y=0 \end{cases}$ ⟵ An overdetermined system: more equations than unknowns

Solution

$$M=\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 3 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{array}\right]\begin{array}{l} (-2)\ (-3) \\ \\ \\ \end{array} \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1+(-2)\cdot 1 & 1+(-2)3 \\ 0 & 1+(-3)\cdot 1 & 0+(-3)3 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -5 \\ 0 & -2 & -9 \end{array}\right]\begin{array}{l} \\ \cdot 2 \\ \cdot(-3) \end{array} \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & 6 & 27 \end{array}\right]\begin{array}{l} \\ \cdot 1 \\ \\ \end{array} \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & 0 & 27-10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -6 & -10 \\ 0 & 0 & 17 \end{array}\right]$$

therefore the system has no solutions.

b) $\begin{cases} x+z+4w+2v=3 \\ y+2w-v=-1 \\ -x+3y+2z=-2 \end{cases}$ ⟵ An underdetermined system: less equations than unknowns.

Solution

Since,

$$\begin{cases} x+z+4w+2v=3 \\ y+2w-v=-1 \\ -x+3y+2z=-2 \end{cases} \Longleftrightarrow \begin{cases} 1x+0y+1z+4w+2v=3 \\ 0x+1y+0z+2w-v=-1 \\ -1x+3y+2z+0w+0v=-2 \end{cases} \qquad (1)$$

the corresponding augmented matrix is:

$$M=\begin{array}{c}\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ -1 & 3 & 2 & 0 & 0 & -2 \end{array}\right]\end{array}\begin{array}{c}\cdot 1 \\ \\ \\ \end{array}$$

$$\sim\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 3+0 & 2+1 & 0+4 & 0+2 & -2+3 \end{array}\right]$$

$$\sim\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 3 & 3 & 4 & 2 & 1 \end{array}\right]\begin{array}{c} \\ (-3) \\ \\ \end{array}$$

$$\sim\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3+(-3)\cdot 0 & 4+(-3)2 & 2+(-3)(-1) & 1+(-1)(-3) \end{array}\right]$$

$$\sim\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 4 & 2 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right]\begin{array}{c}\cdot 3 \\ \\ \\ \end{array} \sim$$

$$\sim\left[\begin{array}{ccccc|c} 3 & 0 & 3 & 12 & 6 & 9 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right]\begin{array}{c} \\ \\ (-1) \end{array} \sim$$

$$\sim \left[\begin{array}{ccccc|c} 3 & 0 & 0 & 12+(-1)(-2) & 6+(-1)5 & 9+(-1)4 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 3 & 0 & 0 & 14 & 1 & 5 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -2 & 5 & 4 \end{array}\right]$$

and it follows that

$$\text{Eq.}(1) \Leftrightarrow \begin{cases} 3x+14w+v=5 \\ y+2w-v=-1 \\ 3z-2w+5v=4 \end{cases} \Leftrightarrow \begin{cases} 3x=5-14w-v \\ y=-1-2w+v \\ 3z=4+2w-5v \end{cases}$$

$$\Leftrightarrow \begin{cases} x=(5/3)-(14/3)w-(1/3)v \\ y=-1-2w+v \\ z=(4/3)+(2/3)w-(5/3)v \end{cases}$$

$$\Leftrightarrow (x,y,z) = ((5/3)-(14/3)w-(1/3)v, \\ -1-2w+v, (4/3)+(2/3)w-(5/3)v)$$

$$= (5/3, -1, 4/3) + w(-14/3, -2, 2/3) + \\ + v(-1/3, 1, -5/3)$$

$$\Leftrightarrow (x,y,z,w,v) = (5/3,-1,4/3,0,0) + w(-14/3,-2,2/3,1,0) \\ + v(-1/3,1,-5/3,0,1)$$

We conclude that the solution set is given by

$$S = \{a + wb + vc \mid w, v \in \mathbb{R}\}$$

with $a, b, c \in \mathbb{R}^5$ given by

$$a = (5/3, -1, 4/3, 0, 0)$$
$$b = (-14/3, -2, 2/3, 1, 0)$$
$$c = (-1/3, 1, -5/3, 0, 1)$$

EXERCISES

(19) Solve the following systems using Gaussian Elimination

a) $\begin{cases} x-2y=-4 \\ 3x+y=9 \\ x+5y=17 \end{cases}$ $(2,3)$

b) $\begin{cases} x+z=4 \\ 2x-y+3z=9 \\ 2y-z=1 \\ 3x+y-2z=-1 \end{cases}$ $(1,2,3)$

c) $\begin{cases} x-y-2z=6 \\ 3x-3y-6z=1 \end{cases}$ (inconsistent)

d) $\begin{cases} x+y+w=4 \\ y+w+z=-2 \\ x+w+z=1 \end{cases}$ $(z+6,\ z+3,\ -2z-5,\ z)$

e) $\begin{cases} x+2y+4z=0 \\ y-2z=0 \\ x+8z=0 \end{cases}$ $(-8z,\ 2z,\ z)$

LIN5: Eigenvalues and Eigenvectors

EIGENVALUES AND EIGENVECTORS

Eigenvalues and Eigenvectors

- Let $A \in M_{nn}(\mathbb{R})$ be a square matrix. We say that $\lambda \in \mathbb{C}$ is an eigenvalue of A with eigenvector $x \in M_{n1}(\mathbb{C})$ if and only if

$$Ax = \lambda x .$$

with $x \neq 0$

- With every eigenvalue λ we associate an eigenvector space $E_\lambda(A)$ which consists of all vectors x that are eigenvectors to the eigenvalue λ. Thus:

$$E_\lambda(A) = \{ x \in M_{n1}(\mathbb{C}) \mid Ax = \lambda x \}$$

How to find the eigenvalues

- $\boxed{\lambda \text{ eigenvalue of } A \Leftrightarrow \det(A - \lambda I) = 0}$

Proof

Note that

$$Ax = \lambda x \Leftrightarrow Ax - \lambda x = 0 \Leftrightarrow Ax - \lambda I x = 0$$
$$\Leftrightarrow (A - \lambda I)x = 0 \qquad (1)$$

Eq. (1) has an obvious solution $x = \mathbf{0}$ and if $\det(A - \lambda I) \neq 0$, then this solution is unique. It follows that

λ eigenvalue of $A \Leftrightarrow$
$\Leftrightarrow$ Eq. (1) has a solution $x \neq 0$
$\Leftrightarrow$ $x = \mathbf{0}$ is NOT a unique solution
$\Leftrightarrow$ $\det(A - \lambda I) = 0$.

example

Find the eigenvalues of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) =$$

$$= \begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 3-\lambda & -1 \\ 2 & 1 & 3-\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} 2-\lambda & 2-\lambda & 1 \\ 0 & 0 & -1 \\ 2 & (3-\lambda)^2+1 & 3-\lambda \end{vmatrix} \rightarrow =$$

$$= -(-1) \begin{vmatrix} 2-\lambda & 2-\lambda \\ 2 & (3-\lambda)^2+1 \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1 & 1 \\ 2 & (3-\lambda)^2+1 \end{vmatrix}$$

$$= (2-\lambda)\left[(3-\lambda)^2+1-2\right] = (2-\lambda)\left[(3-\lambda)^2-1\right]$$

$$= (2-\lambda)(3-\lambda-1)(3-\lambda+1) = (2-\lambda)(2-\lambda)(4-\lambda).$$

λ eigenvalue of $A \Leftrightarrow \det(A-\lambda I)=0 \Leftrightarrow$

$\Leftrightarrow (2-\lambda)(2-\lambda)(4-\lambda)=0$

$\Leftrightarrow \underline{\lambda = 2 \text{ or } \lambda = 4}$

$\lambda = 2$: <u>double</u> eigenvalue

How to find the eigenvectors

For each eigenvalue λ we use Gaussian Elimination to solve the equation

$$(A-\lambda I)\,x=0$$

The solution space of this equation coincides with the eigenvector space $E_\lambda(A)$.

example

In the previous example:
For $\lambda=2$

$$M \sim \left[\begin{array}{ccc|c} 2-\lambda & -1 & 1 & 0 \\ 0 & 3-\lambda & -1 & 0 \\ 2 & 1 & 3-\lambda & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & +1 & 0 \end{array}\right] (+1)$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] (-1) \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \cdot (1/2) \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x + z = 0 \\ y - z = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = -z \\ y = z \end{cases} \Leftrightarrow (x, y, z) = (-z, z, z) = z(-1, 1, 1)$$

thus, the corresponding eigenvector space is given by $E_2(A) = \{z(-1,1,1) \mid z \in \mathbb{R}\}$.

EXAMPLES – THEORETICAL

a) Let $A \in M_n(\mathbb{R})$ be a matrix with $A^2 = 5A - 6I$. Show that

λ eigenvalue of $A \Rightarrow \lambda = 2 \vee \lambda = 3$.

Solution

Let $\lambda \in \lambda(A)$ be an eigenvalue of A with x a corresponding eigenvector. It follows that

$Ax = \lambda x$ and

$A^2 x = (AA)x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$.

and therefore:

$$A^2 = 5A - 6I \Rightarrow A^2 - 5A + 6I = \mathbb{0} \Rightarrow$$
$$\Rightarrow (A^2 - 5A + 6I)x = \mathbb{0}x = \mathbb{0} \qquad (1)$$

and $(A^2 - 5A + 6I)x = A^2x - 5Ax + 6Ix = \lambda^2 x - 5\lambda x + 6x =$
$$= (\lambda^2 - 5\lambda + 6)x \qquad (2)$$

From (1) and (2):

$$(\lambda^2 - 5\lambda + 6)x = \mathbb{0} \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0$$
$$\Rightarrow \lambda = 2 \vee \lambda = 3$$

EXERCISES

① Find the eigenvalues and corresponding eigenvector spaces for the following matrices.

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$ $(\lambda = 1 \longrightarrow t(0,0,1))$

b) $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}$ $(\lambda = 1 \longrightarrow t(1,-1,0)$
$\lambda = 2 \longrightarrow t(3,3,-1)$
$\lambda = 21 \longrightarrow t(1,1,6))$

c) $C = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ $(\lambda = 1 \longrightarrow t(3,-1,3)$
$\lambda = 2 \longrightarrow t(2,2,-1))$

[answers can be confirmed by matlab or octave]

(2) Let $A = \begin{bmatrix} 1 & 2a+1 \\ 2a-1 & 1 \end{bmatrix}$
For what values of a does A have only one eigenvalue?

(3) Rotation matrix
Let $R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$
Show that $R(\vartheta)$ has real eigenvalues if and only if $\sin\vartheta = 0$.

(4) Find the eigenvalues of the following matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$ (ans: $\lambda = +1, -1$)

(5) Let $A \in M_{nn}(\mathbb{R})$ be a matrix such that $A^2 = I$. Show that if λ is an eigenvalue of A, then $\lambda = 1$ or $\lambda = -1$.

(6) Let $A \in M_{nn}(\mathbb{R})$ be a non-singular matrix. Show that if $\lambda \neq 0$ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

(7) Let $A \in M_n(\mathbb{R})$ with $A^2 + 3A = -2I$. Show that if:
λ eigenvalue of $A \Rightarrow \lambda = -1 \vee \lambda = -2$.

(8) Let $A \in M_n(\mathbb{R})$ with $A^{-1} = 2I - A$. Show that:
λ eigenvalue of $A \Rightarrow \lambda = 1$.

Characteristic polynomial

Thm: Let $A \in M_n(\mathbb{R})$ be a square matrix. Then the determinant $\det(A-\lambda I)$ simplifies to a polynomial of the form
$$\det(A-\lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- We call the polynomial obtained by expanding $\det(A-\lambda I)$ the <u>characteristic polynomial</u> of A.

<u>Proof</u>

Let $\sigma_0 \in S_n$ be the do-nothing permutation such that $\forall a \in [n]: \sigma_0(a) = a$

Then, we have $S(\sigma) = 1$, and therefore

$$\begin{aligned}
\det(A-\lambda I) &= \sum_{\sigma \in S_n} \left[S(\sigma) \prod_{a=1}^{n} (A-\lambda I)_{a,\sigma(a)} \right] \\
&= \prod_{a=1}^{n} (A-\lambda I)_{a,\sigma_0(a)} + g(\lambda) \\
&= \prod_{a=1}^{n} (A-\lambda I)_{aa} + g(\lambda) \\
&= \prod_{a=1}^{n} (A_{aa}-\lambda) + g(\lambda) \qquad (1)
\end{aligned}$$

with

$$g(\lambda) = \sum_{\sigma \in S_n - \{\sigma_0\}} \left[S(\sigma) \prod_{a=1}^{n} (A-\lambda I)_{a,\sigma(a)} \right] \qquad (2)$$

From Eq.(1), the highest-order term from the first contribution is $(-\lambda)^n = (-1)^n \lambda^n$. We also note that for $\sigma \neq \sigma_0$, the products that appear in $g(\lambda)$ involve at least two non-diagonal elements, since σ is at least one transposition away from σ_0, and therefore $\deg(g(\lambda)) \leq n-2$. It follows that $g(\lambda)$ does not contribute additional λ^n terms. The conclusion follows $\square$

- Recall that according to the fundamental theorem of algebra, a polynomial

$\forall x \in \mathbb{R}: p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

has zeroes $x_1, x_2, \ldots, x_n \in \mathbb{C}$, and it can be factored as

$$p(x) = a_n (x-x_1)(x-x_2)\cdots(x-x_n) = a_n \prod_{b=1}^{n} (x-x_b)$$

- Combining the previous result with the fundamental theorem of algebra gives; the following theorem:

Thm: Let $A \in M_n(\mathbb{R})$ be a square matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of A. Then,

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n = \prod_{a \in [n]} \lambda_a$$

Proof

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ are the n zeroes of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

Using the fundamental theorem of algebra, it follows that

$$\det(A-\lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$$
$$= (-1)^n \prod_{a\in[n]} (\lambda-\lambda_a) = \prod_{a\in[n]} (\lambda_a-\lambda) \Rightarrow$$
$$\Rightarrow \det A = \det(A-0I) = \prod_{a\in[n]} (\lambda_a - 0) = \prod_{a\in[n]} \lambda_a \quad \square$$

▶ Trace of a matrix

Def: Let $A \in M_n(\mathbb{R})$ be a square matrix. We define the trace of A as:
$$\operatorname{tr}(A) = \sum_{a=1}^{n} A_{aa} = A_{11} + A_{22} + \dots + A_{nn}$$

We can now show that:

Prop: Let $A \in M_n(\mathbb{R})$ be a matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$. Then, we have:
$$\operatorname{tr}(A) = \sum_{a=1}^{n} \lambda_a = \lambda_1 + \lambda_2 + \dots + \lambda_n.$$

Proof

We note that
$$\det(A-\lambda I) = \prod_{a=1}^{n} (\lambda_a - \lambda) =$$
$$= (-1)^n \lambda^n + \left(\sum_{a=1}^{n} \lambda_a\right)(-\lambda)^{n-1} + \dots + c_1\lambda + c_0$$

$$= (-1)^n \lambda^n + (-1)^{n-1}\left(\sum_{a=1}^{n} \lambda_a\right)\lambda^{n-1} + \dots + c_1\lambda + c_0$$

thus the coefficient c_{n-1} of the λ^{n-1} term is

$$c_{n-1} = (-1)^{n-1}\left(\sum_{a=1}^{n} \lambda_a\right)$$

We also note, from Eq.(1) in the proof of the first theorem of this section that we have:

$$\det(A-\lambda I) = \prod_{a=1}^{n} (A_{aa}-\lambda) + g(\lambda) =$$

$$= (-1)^n\lambda^n + (-1)^{n-1}\left(\sum_{a=1}^{n} A_{aa}\right)\lambda^{n-1} + \dots + d_1\lambda + d_0 + g(\lambda)$$

$$= (-1)^n\lambda^n + (-1)^{n-1}\,\mathrm{tr}(A)\,\lambda^{n-1} + \dots + d_1\lambda + d_0 + g(\lambda)$$

with $\deg(g(\lambda)) \leq n-2$. It follows that $g(\lambda)$ does not contribute to the coefficient of λ^{n-1} and thus

$$c_{n-1} = (-1)^{n-1}\,\mathrm{tr}(A)$$

We conclude that

$$(-1)^{n-1}\,\mathrm{tr}(A) = (-1)^{n-1}\left(\sum_{a=1}^{n} \lambda_a\right) \Rightarrow \mathrm{tr}(A) = \sum_{a=1}^{n} \lambda_a \qquad \square$$

EXAMPLES

a) Let $A = \begin{bmatrix} a+3 & 1 \\ 2a & 2 \end{bmatrix}$ and let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the eigenvalues of A. Find all $a \in \mathbb{R}$ such that $\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1$

Solution

$$\lambda_1 \lambda_2 = \det A = \begin{vmatrix} a+3 & 1 \\ 2a & 2 \end{vmatrix} = 2(a+3) - 1 \cdot 2a = 2a+6-2a = 6$$

and

$$\lambda_1 + \lambda_2 = \operatorname{tr} A = (a+3) + 2 = a+5$$

so it follows that

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} = \frac{(\lambda_1+\lambda_2)^2 - 2(\lambda_1\lambda_2)}{(\lambda_1\lambda_2)^2} =$$

$$= \frac{(\operatorname{tr} A)^2 - 2\det A}{(\det A)^2} = \frac{(a+5)^2 - 2 \cdot 6}{6^2} =$$

$$= \frac{(a+5)^2 - 12}{36}.$$

and therefore:

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = 1 \Leftrightarrow \frac{(a+5)^2 - 12}{36} = 1 \Leftrightarrow (a+5)^2 - 12 = 36$$

$$\Leftrightarrow (a+5)^2 = 12 + 36 \Leftrightarrow (a+5)^2 = 48 = 16 \cdot 3 = 4^2 \cdot 3$$

$$\Leftrightarrow a+5 = 4\sqrt{3} \vee a+5 = -4\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow a = -5 + 4\sqrt{3} \vee a = -5 - 4\sqrt{3}$$

EXERCISES

(9) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2x+3 & 1 \\ 1 & 1 & 2x-1 \end{bmatrix}$$

If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A, find $x \in \mathbb{C}$ such that

a) $\lambda_1 \lambda_2 \lambda_3 = 1$

b) $\lambda_1 + \lambda_2 + \lambda_3 = 4$

(10) Let $A = \begin{bmatrix} x+1 & 2x \\ 3 & x-1 \end{bmatrix}$

with eigenvalues λ_1, λ_2. Find $x \in \mathbb{C}$ such that

a) $\lambda_1 + \lambda_2 = 3$

b) $\dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} = 1$

c) $\lambda_1^2 + \lambda_2^2 = 1$ (Hint: $a^2 + b^2 = (a+b)^2 - 2ab$)

(11) Let $A = \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix}$

with eigenvalues λ_1, λ_2. Find $a \in \mathbb{C}$ such that $\lambda_1^3 + \lambda_2^3 = 0$.

(Hint: $(a+b)^3 = (a^3 + b^3) + 3ab(a+b)$)

The following problems use
a) $\det(AB) = \det(A)\det(B)$
b) $\det(I) = 1$.

(12) Let $A \in M_{nn}(\mathbb{R})$ be a matrix and let $B = P^{-1}AP$ with $P \in M_{nn}(\mathbb{R})$ a non-singular matrix. Show that A, B have the same eigenvalues.

(13) Let $A, B \in M_{nn}(\mathbb{R})$ with A non-singular. Show that AB and BA have the same eigenvalues.

(14) Let $A \in M_{22}(\mathbb{R})$ be a 2×2 matrix. If A is non-singular, show that

$$\operatorname{tr}(A^{-1}) = \frac{\operatorname{tr}(A)}{\det(A)}$$

(see exercise (6))

• (15) Let $A \in M_{33}(\mathbb{R})$ be a 3×3 non-singular matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ that satisfy

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

Show that

$$\operatorname{tr}(A^{-1}) = \frac{(\operatorname{tr}(A)+1)(\operatorname{tr}(A)-1)}{2\det A}$$

Cayley-Hamilton theorem

> **Thm:** Let $A \in M_n(\mathbb{R})$ be a square matrix ~~we~~ with characteristic polynomial
> $$\det(A-\lambda I) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$
> Then A satisfies
> $$(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = \mathbf{0}$$

▶ **Method:** The Cayley-Hamilton theorem provides a second method for calculating the matrix inverse A^{-1} as shown in the following example

EXAMPLE

Given the matrix

$$A = \begin{bmatrix} 5 & 4 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

use the Cayley-Hamilton theorem to write A^{-1} in terms of A.

Solution

Since,

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & 4 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda) \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} =$$

$$= (2-\lambda)\left[(5-\lambda)(2-\lambda) - 4\cdot 1\right] =$$

$$= (2-\lambda)(10 - 5\lambda - 2\lambda + \lambda^2 - 4) =$$

$$= (2-\lambda)(\lambda^2 - 7\lambda + 6) =$$

$$= 2\lambda^2 - 14\lambda + 12 - \lambda^3 + 7\lambda^2 - 6\lambda =$$

$$= -\lambda^3 + (2+7)\lambda^2 + (-14-6)\lambda + 12$$

$$= -\lambda^3 + 9\lambda^2 - 20\lambda + 12 \Rightarrow$$

$$\Rightarrow -A^3 + 9A^2 - 20A + 12I = 0 \Rightarrow$$

$$\Rightarrow A^3 - 9A^2 + 20A = 12I \Rightarrow$$

$$\Rightarrow A(A^2 - 9A + 20I) = 12I \Rightarrow$$

$$\Rightarrow A\left[\frac{1}{12}(A^2 - 9A + 20I)\right] = I \Rightarrow$$

$$\Rightarrow A^{-1} = \frac{1}{12}(A^2 - 9A + 20I).$$

► <u>Method</u>: We can also use the Cayley-Hamilton theorem to write higher powers A^n in terms of a few powers of A.

EXAMPLE

Given the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Show that $\forall n \in \mathbb{N} - \{0,1\}: A^n = nA - (n-1)I$.

Solution

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1 \Rightarrow$$

$\Rightarrow A^2 - 2A + I = 0 \Rightarrow A^2 = 2A - I$.

Assume that for $n=k$: $A^k = kA - (k-1)I$

For $n=k+1$, we will show that : $A^{k+1} = (k+1)A - kI$

We have:

$$A^{k+1} = A^k A = [kA - (k-1)I]A =$$
$$= kA^2 - (k-1)A = k(2A-I) - (k-1)A =$$
$$= 2kA - kI - kA + A = (2k-k+1)A - kI =$$
$$= (k+1)A - kI.$$

EXERCISES

(16) For the following matrices, write A^{-1} and A^3 in terms of A and I.

a) $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ d) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(17) For the following matrices, write A^{-1} and A^4 in terms of I, A, A^2.

a) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ d) $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Linear systems of differential equations

- Recall that the solution to the differential equation

$$\frac{dy(t)}{dt} = a\,y(t) + f(t)$$

with initial condition

$$y(0) = y_0$$

is given by:

$$y(t) = e^{at} y_0 + e^{at} \int_0^t e^{-a\tau} f(\tau)\, d\tau$$

- We want the solution to the more general system of linear differential equations:

$$\begin{cases} \dfrac{dy_1(t)}{dt} = A_{11} y_1(t) + A_{12} y_2(t) + \dots + A_{1n} y_n(t) + f_1(t) \\ \dfrac{dy_2(t)}{dt} = A_{21} y_1(t) + A_{22} y_2(t) + \dots + A_{2n} y_n(t) + f_2(t) \\ \quad\vdots \\ \dfrac{dy_n(t)}{dt} = A_{n1} y_1(t) + A_{n2} y_2(t) + \dots + A_{nn} y_n(t) + f_n(t) \end{cases}$$

with initial condition:

$$y_1(0) = b_1 \ , \ y_2(0) = b_2 \ , \ \dots , \ y_n(0) = b_n$$

If we define

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} , \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} , \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

then the system can be rewritten as:

$$\boxed{\begin{cases} \dfrac{dy(t)}{dt} = A\,y(t) + f(t) \\ y(0) = b \end{cases}}$$

The solution to this system is based on the matrix exponential.

The matrix exponential

- Let $A \in M_{nn}(\mathbb{R})$ be a square matrix. The exponential of A is defined as

$$\boxed{\exp(A) = \sum_{n=0}^{+\infty} \frac{1}{n!} A^n}$$

with $0! = 1$

$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

This generalizes the identity

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

- The matrix exponential $\exp(A)$ converges for all matrices A.

▸ <u>Properties</u>

a) $$\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$$

b) $$[\exp(A)]^{-1} = \exp(-A)$$

c) $$AB = BA \Rightarrow \exp(A+B) = \exp(A)\exp(B)$$

d) $$\frac{dy(t)}{dt} = Ay(t) \rightarrow y(t) = \exp(tA)y(0)$$

e) $$\exp((t_1+t_2)A) = \exp(t_1A)\exp(t_2A)$$

Solution to linear system of ODEs

The solution to the ODE system

$$\frac{dy(t)}{dt} = Ay(t) + f(t)$$

is given by

$$\boxed{y(t) = \exp(tA)\,y(0) + \exp(tA)\int_0^t \exp(-\tau A) f(\tau)\,d\tau}$$

How to calculate the matrix exponential

To calculate $\exp(tA)$ we work as follows:

- 1 Let $A \in M_{nn}(\mathbb{R})$. From the Cayley-Hamilton theorem we conclude that there are coefficients $c_0, c_1, \dots, c_{n-1}$ such that

$$\exp(tA) = c_{n-1} A^{n-1} t^{n-1} + \dots + a_1 A t + a_0 I$$

  Before we find $c_0, \dots, c_{n-1}$ we simplify the right-hand side of the expression above.
- 2 We find the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the matrix A.

• 3 Let
$f(x) = c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$
To find the coefficients $c_0, c_1, \ldots, c_{n-1}$:
a) If λ_k is an eigenvalue of $\underline{\underline{tA}}$, then

$$e^{\lambda_k} = f(\lambda_k)$$

b) If λ_k is an eigenvalue of A with multiplicity m, then we also have:

$$\begin{cases} e^{\lambda_k} = f'(\lambda_k) \\ e^{\lambda_k} = f''(\lambda_k) \\ \vdots \\ e^{\lambda_k} = f^{(m-1)}(\lambda_k) \end{cases}$$

Thus we get a system of n equations from which we find $c_0, \ldots, c_{n-1}$.

• 4 Knowing the coefficients $c_0, \ldots, c_{n-1}$ we now calculate the exponential $\exp(tA)$.

example

For $A = \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix}$

we have

$$\exp(tA) = c_1 tA + c_0 I = c_1 t \begin{bmatrix} 1 & 1 \\ 9 & 1 \end{bmatrix} + c_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 t + c_0 & c_1 t \\ 9c_1 t & c_1 t + c_0 \end{bmatrix}$$

- Eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 9 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 =$$

$$= (1-\lambda-3)(1-\lambda+3) =$$

$$= (-\lambda-2)(-\lambda+4) =$$

$$= (\lambda+2)(\lambda-4) \Rightarrow$$

$\Rightarrow$ The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 4$

$\Rightarrow$ The eigenvalues of tA are $\lambda_1 = -2t$ and $\lambda_2 = 4t \Rightarrow$

$$\Rightarrow \begin{cases} e^{4t} = c_1(4t) + c_0 \\ e^{-2t} = c_1(-2t) + c_0 \end{cases} \Leftrightarrow \dots \Leftrightarrow \begin{cases} c_1 = \dfrac{1}{6t}\left(e^{4t} - e^{-2t}\right) \\ c_0 = \dfrac{1}{3}\left(e^{4t} + 2e^{-2t}\right) \end{cases}$$

It follows that

$$\exp(tA) = \cdots = \frac{1}{6}\begin{bmatrix} 3e^{4t}+3e^{-2t} & e^{4t}-e^{-2t} \\ 9e^{4t}-9e^{-2t} & 3e^{4t}+3e^{-2t} \end{bmatrix}$$

<u>example</u>

For $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$

we have

$$\exp(tA) = c_1 t A + c_0 I = \cdots$$

$$= \begin{bmatrix} c_0 & c_1 t \\ -9c_1 t & 6c_1 t + c_0 \end{bmatrix}$$

Eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ -9 & 6-\lambda \end{vmatrix} = -\lambda(6-\lambda) - (-9) =$$

$$= -6\lambda + \lambda^2 + 9 = (\lambda-3)^2 \Rightarrow$$

$\Rightarrow \lambda = 3$ double eigenvalue of A

$\Rightarrow \lambda = 3t$ double eigenvalue of tA

For $f(x) = c_1 x + c_0 \Rightarrow f'(x) = c_1$ thus

$$\begin{cases} e^{3t} = c_1(3t) + c_0 \\ e^{3t} = c_1 \end{cases} \Leftrightarrow \cdots \Leftrightarrow \begin{cases} c_0 = e^{3t}(1-3t) \\ c_1 = e^{3t} \end{cases}$$

thus

$$\exp(tA) = \cdots = \begin{bmatrix} (1-3t)e^{3t} & te^{3t} \\ -9te^{3t} & (1+3t)e^{3t} \end{bmatrix}$$

Now let us consider

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -9y_1 + 6y_2 \end{cases} \Leftrightarrow \frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

It follows that

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \exp(tA)\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} =$$

$$= \begin{bmatrix} (1-3t)e^{3t} & te^{3t} \\ -9te^{3t} & (1+3t)e^{3t} \end{bmatrix}\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} y_1(t) = (1-3t)e^{3t}y_1(0) + te^{3t}y_2(0) \\ y_2(t) = -9te^{3t}y_1(0) + (1+3t)e^{3t}y_2(0) \end{cases}$$

2nd method

For a 2x2 matrix A with eigenvalues λ_1, λ_2, the matrix exponential is given by:

a) If $\lambda_1 \neq \lambda_2$ then

$$\boxed{\exp(tA) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} A}$$

b) If $\lambda_1 = \lambda_2 = \lambda$ then

$$\boxed{\exp(tA) = e^{\lambda t}(1 - \lambda t) I + t e^{\lambda t} A}$$

EXERCISES

(18) Use the matrix exponential to solve the following systems in terms of $y_1(0)$ and $y_2(0)$:

a) $$\begin{cases} dy_1/dt = 4y_1 + y_2 \\ dy_2/dt = -2y_1 + y_2 \end{cases}$$

b) $$\begin{cases} dy_1/dt = -5y_1 - y_2 \\ dy_2/dt = y_1 - 3y_2 \end{cases}$$

c) $$\begin{cases} dy_1/dt = y_1 \\ dy_2/dt \quad y_1 + y_2 \end{cases}$$

(19) Rotation matrix.
Show that the rotation matrix

$$R(\vartheta) = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$$

satisfies:

$$R(\vartheta) = \exp\left[\vartheta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right], \quad \vartheta \in \mathbb{R}.$$

↳ Use $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.

LIN6: Vector Spaces

VECTOR SPACES

▼ Internal Operations

Def : Let A, B, C be sets with $A \times B \neq \emptyset$ and $C \neq \emptyset$.

An operation is a mapping $f: A \times B \to C$ such that every $(a, b) \in A \times B$ is mapped to $afb \in C$.

- afb = the result of the operation.
- notation : We typically represent operations with notations such as:

  $a+b$, $a \cdot b$, $a * b$, $a \circ b$

  with $+, \cdot, *, \circ$ being the operations.
- remark : Let $*: A \times B \to C$ be an operation. An immediate consequence of the definition of the operation as a mapping is the following statements:

$$\boxed{\begin{array}{l} \forall a, b \in A : \forall c \in B : (a = b \Rightarrow a * c = b * c) \\ \forall a, b \in B : \forall c \in A : (a = b \Rightarrow c * a = c * b) \end{array}}$$

Def : Let $A \neq \emptyset$ be a set. An internal operation on A is any mapping $f: A \times A \to A$ such that all $(a, b) \in A \times A$ are mapped to $afb \in A$.

<u>Def</u> : Let $*$ be an internal operation on A. Let $A_1 \subseteq A$ with $A_1 \neq \emptyset$. We say that

$$\boxed{* \text{ closed on } A_1 \Leftrightarrow \forall a, b \in A_1 : a * b \in A_1}$$

↳ <u>Properties of operations</u>

<u>Def</u> : Let $*$ be an internal operation on A. We say that

$*$ commutative $\Leftrightarrow \forall a, b \in A : a * b = b * a$
$*$ associative $\Leftrightarrow \forall a, b, c \in A : (a * b) * c = a * (b * c)$
e unit element of $(A, *) \Leftrightarrow \forall a \in A : a * e = e * a = a$

<u>Def</u> : Let $*$ be an internal operation on A with unit element $e \in A$. We say that

a, a' symmetric with respect to $*$ $\Leftrightarrow a * a' = a' * a = e$

- We now show that if $(A, *)$ has a unit element, then it is unique. Likewise, given a unique unit element $e \in A$, every $a \in A$ can have no more than than one symmetric element $a' \in A$.

Thm: Let $*$ be an internal operation on A.

a) $\boxed{e_1, e_2 \text{ unit elements of } (A,*) \Rightarrow e_1 = e_2}$

b)
$$\left.\begin{array}{l} * \text{ associative} \\ (A,*) \text{ has unit element } e \in A \\ a, a_1 \text{ symmetric} \\ a, a_2 \text{ symmetric} \end{array}\right\} \Rightarrow a_1 = a_2$$

Proof

a) Assume that e_1, e_2 unit elements of $(A,*)$. Then

$e_1 * e_2 = e_2$ [e_1 unit element]

$e_1 * e_2 = e_1$ [e_2 unit element]

Then $e_1 = e_2$ □

b)
$$\begin{aligned} a_1 &= a_1 * e && [e \text{ unit element}] \\ &= a_1 * (a * a_2) && [a, a_2 \text{ symmetric}] \\ &= (a_1 * a) * a_2 && [\text{associative}] \\ &= e * a_2 && [a, a_1 \text{ symmetric}] \\ &= a_2 && [e \text{ unit element}] \end{aligned}$$
□

EXAMPLES

a) Addition in $\mathbb{R}$.

1) "+" in $\mathbb{R}$ is associative and commutative
2) $0 \in \mathbb{R}$ is a unit element of $(\mathbb{R}, +)$
3) If $a \in \mathbb{R}$, then $-a$ is symmetric to a with respect to "+".

b) Multiplication in $\mathbb{R}$

1) "$\cdot$" in $\mathbb{R}$ is associative and commutative
2) $1 \in \mathbb{R}$ is a unit element of $(\mathbb{R}, \cdot)$
3) If $a \in \mathbb{R} - \{0\}$, then $1/a$ is symmetric to a with respect to "$\cdot$".

c) Multiplication in $M_n(\mathbb{R})$

1) "$\cdot$" in $M_n(\mathbb{R})$ is associative but NOT commutative
2) The identity matrix $I = [\delta_{ab}]$ with
$$\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$
is the unique unit element of $(M_n(\mathbb{R}), \cdot)$ since $\forall A \in M_n(\mathbb{R}): AI = IA$.
3) If $\det(A) \neq 0$, then A^{-1} is the symmetric element of A because $AA^{-1} = A^{-1}A = I$.

EXAMPLES

a) Let $A = \mathbb{R} - \{\lambda\}$ and define

$x * y = xy - \lambda(x+y) + \lambda(\lambda+1)$

i) Show that "$*$" is closed on A.

ii) Show that "$*$" is commutative

iii) Show that "$*$" has a unit element on A.

Solution

i) It is sufficient to show that $\forall x,y \in A : x*y \in A$.
Let $x,y \in A$ be given. To derive a contradiction, let us assume that $x*y \notin A$. Then:

$x*y \notin A \Leftrightarrow x*y \notin \mathbb{R} - \{\lambda\} \Leftrightarrow x*y = \lambda \Leftrightarrow xy - \lambda(x+y) + \lambda(\lambda+1) = \lambda$

$\Leftrightarrow xy - \lambda x - \lambda y + \lambda^2 + \lambda = \lambda \Leftrightarrow xy - \lambda x - \lambda y + \lambda^2 = 0 \Leftrightarrow$

$\Leftrightarrow x(y-\lambda) - \lambda(y-\lambda) = 0 \Leftrightarrow (x-\lambda)(y-\lambda) = 0 \Leftrightarrow$

$\Leftrightarrow x - \lambda = 0 \vee y - \lambda = 0 \Leftrightarrow x = \lambda \vee y = \lambda \Leftrightarrow x \notin \mathbb{R} - \{\lambda\} \vee y \notin \mathbb{R} - \{\lambda\}$

$\Leftrightarrow x \notin A \vee y \notin A \longleftarrow$ contradiction, since $x \in A \wedge y \in A$.

It follows that $x*y \in A$. We have thus shown that
$(\forall x,y \in A : x*y \in A) \Rightarrow$ "$*$" closed under A.

ii) Sufficient to show that $\forall x,y \in A : x*y = y*x$
Let $x,y \in A$ be given. Then:

$x*y = xy - \lambda(x+y) + \lambda(\lambda+1) = yx - \lambda(y+x) + \lambda(\lambda+1) = y*x$

It follows that
$(\forall x,y \in A : x*y = y*x) \Rightarrow$ "$*$" commutative on A.

iii) It is sufficient to find an $e \in A$ such that

$\forall x \in A : x * e = e * x = x$

We note that

$$
\begin{aligned}
x * e - x &= ex - \lambda(e+x) + \lambda(\lambda+1) - x = \\
&= \underline{ex} - \lambda e - \underline{\lambda x} + \lambda^2 + \lambda - \underline{x} = \\
&= x(e-\lambda-1) - \lambda(e-\lambda-1) = \\
&= (x-\lambda)(e-\lambda-1)
\end{aligned}
$$

and therefore

$x * e = x \Leftrightarrow x * e - x = 0 \Leftrightarrow (x-\lambda)(e-\lambda-1) = 0 \Leftrightarrow$

$\Leftrightarrow x - \lambda = 0 \vee e - \lambda - 1 = 0 \Leftrightarrow e - \lambda - 1 = 0$ [since $x \in A \Rightarrow x \neq \lambda$]

$\Leftrightarrow e = \lambda + 1$.

It follows that for $e = \lambda + 1$, $\left.\begin{array}{r}\forall x \in A : x * e = x \\ \text{"}*\text{" commutative on } A\end{array}\right\} \Rightarrow$

$\Rightarrow \forall x \in A : (x * e = e * x = x) \Rightarrow e = \lambda + 1$ is a unit element of "$*$" on A.

↳ We note that in the argument above:

a) We use proof by contradiction in part (i) to show that $x * y \in A$.

b) We have also used the following theorem: $\forall a, b \in \mathbb{R} : (ab = 0 \Leftrightarrow a = 0 \vee b = 0)$.

b) We define $x*y = xy + 2ax + by$, $\forall x, y \in \mathbb{R}$. Find all $a, b \in \mathbb{R}$ such that "$*$" is associative on $\mathbb{R}$.

Solution

Let $x, y, z \in \mathbb{R}$. We note that

$$x*(y*z) = x*(yz+2ay+bz) =$$
$$= x(yz+2ay+bz) + 2ax + b(yz+2ay+bz) =$$
$$= \underline{xyz} + \underline{2axy} + bxz + 2ax + \underline{byz} + \underline{2aby} + b^2 z =$$
$$(x*y)*z = (xy+2ax+by)*z =$$
$$= (xy+2ax+by)z + 2a(xy+2ax+by) + bz =$$
$$= xyz + 2axz + byz + 2axy + 2a^2x + 2aby + bz =$$
$$= \underline{xyz} + \underline{2axy} + 2axz + 4a^2x + \underline{byz} + \underline{2aby} + bz$$

and it follows that

$$x*(y*z) - (x*y)*z = (bxz + 2ax + b^2 z) - (2axz + 4a^2x + bz) =$$
$$= (b-2a)xz + (2a - 4a^2)x + (b^2 - b)z =$$
$$= (b-2a)xz + 2a(1-2a)x + b(b-1)z.$$

It follows that:

"$*$" is associative on $\mathbb{R} \Leftrightarrow \forall x, y, z \in \mathbb{R}: x*(y*z) = (x*y)*z$

$\Leftrightarrow \forall x, y, z \in \mathbb{R}: (x*(y*z) - (x*y)*z = 0) \Leftrightarrow$

$\Leftrightarrow \forall x, z \in \mathbb{R}: (b-2a)xz + 2a(1-2a)x + b(b-1)z = 0 \overset{*}{\Leftrightarrow}$

$$\Leftrightarrow \begin{cases} b-2a=0 \\ 2a(1-2a)=0 \\ b(b-1)=0 \end{cases} \Leftrightarrow \begin{cases} b=2a \\ a=0 \\ b=0 \end{cases} \vee \underbrace{\begin{cases} b=2a \\ a=1/2 \\ b=0 \end{cases} \vee \begin{cases} b=2a \\ a=0 \\ b=1 \end{cases}}_{\text{contradictions}} \vee \begin{cases} b=2a \\ a=1/2 \\ b=1 \end{cases}$$

$$\Leftrightarrow \begin{cases} a=0 \\ b=0 \end{cases} \vee \begin{cases} a=1/2 \\ b=1 \end{cases} \Leftrightarrow (a,b)\in\{(0,0),(1/2,1)\}$$

↳ In the above solution, the main argument is:
"$*$" associative on $\mathbb{R} \Leftrightarrow \cdots \Leftrightarrow$
$\Leftrightarrow (a,b)\in\{(0,0),(1/2,1)\}$
The preceding calculations are the <u>preamble</u> of the solution. The purpose ~~of~~ the pream

EXERCISES

(1) Show that $a*b=a+b+5$ defined on $\mathbb{R}$ is both commutetive and associative.

(2) We define on $*\mathbb{R}$ the operation $a*b=ab+a+b$.
Show that

a) $*$ is commutative and associative

b) Find the unit element of $*$

c) Find which elements of $\mathbb{R}$ have an inverse with respect to the operation $*$.

(3) We define on $\mathbb{R}$ the following operations:
$x*y=x^2y^2$ and $x\circ y=y(x+y)$
Explore whether

a) the operations are commutative

b) the operations are associative

c) the operations have a unit element

d) every element of $\mathbb{R}$ has an inverse.

(4) We define on the set $(0,+\infty)$ the operation

$$x*y=\frac{xy}{x+y}, \quad \forall x,y\in(0,+\infty)$$

Show that

a) $*$ is commutative and associative, using the definition

b) $\forall x,y\in(0,+\infty)$: $\frac{1}{x}*\frac{1}{y}=\frac{1}{x+y}$

c) Use (b) to provide an alternate proof of (a).

(5) We define on $\mathbb{R}$ the operation $a*b=(a-1)b^2-(a-1)+ab$. Find (if it exists) the unit element of this operation.

(6) Given the set

$$A=\left\{\begin{bmatrix} a & 0 \\ 2a & 0 \end{bmatrix} \middle| a\in\mathbb{R}\right\}$$

show that the matrix $E=\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ is a unit element with respect to regular matrix multiplication restricted to the set A. Also show that A is closed with respect to matrix multiplication.

(7) Let A be a set with an operation $*$ and unit element e such that

$\forall a,b,c,d\in A : (a*b)*(c*d)=(a*c)*(b*d)$

Show that $*$ is associative and commutative.

(8) Let A be a set with an operation $*$ and unit element e such that

$\forall x,y,z\in A : (x*y)*z = x*(z*y)$

Show that $*$ is associative and commutative.

↳ From exercise 6 we see that the same operation may have a different unit element, if it is restricted into a smaller set.

V Groups

Def: Let G be a set with "$*$" an internal operation on U with $G \subseteq U$. We say that:

a) $(G,*)$ is a group if and only if:

1) "$*$" is closed on G
2) $\forall a,b,c \in G: a*(b*c) = (a*b)*c$
3) $\exists e \in G: \forall a \in G: e*a = a*e = a$
4) $\forall a \in G: \exists a' \in G: a'*a = a*a' = e$

b) $(G,*)$ is an abelian group if and only if:

1) $(G,*)$ is a group
2) $\forall a,b \in G: a*b = b*a$.

- Therefore, $(G,*)$ is a group if and only if "$*$" is closed on G, "$*$" is associative, has a unit element, and every element of G has a symmetric element. $(G,*)$ is an abelian group if and only if it is already a group and furthermore "$*$" is commutative.

EXAMPLES

a) $(\mathbb{R},+)$ and $(\mathbb{R}-\{0\},\cdot)$ are abelian groups.

b) $(M_n(\mathbb{R}),+)$ is an abelian group.

c) We define the general linear group

$$GL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

Then $(GL(n,\mathbb{R}),\cdot)$ is a group but NOT an abelian group. Note that

1) Matrix multiplication is NOT commutative
2) We need the restriction $\det(A)\neq 0$ to ensure that every A has a symmetric element.

Sufficient condition

- To show that $(G,*)$ is a group, we may in fact weaken conditions (c) and (d) according to the following theorem:

Thm: Let $(G,*)$ with G a set and "$*$" an internal operation on U with $G\subseteq U$. Assume that:

a) $\forall a,b\in G: a*b\in G$
b) $\forall a,b,c\in G: a*(b*c)=(a*b)*c$
c) $\exists e\in G: \forall a\in G: e*a=a$
d) $\forall a\in G: \exists a'\in G: a'*a=e$

Then $(G,*)$ is a group.

Proof

Let $a\in G$ be given. Let a' be the symmetric element of a such that $a'*a=e$. (exists by hypothesis (d)).

Let $a'' \in G$ be the symmetric element of a' such that $a'' * a' = e$ (exists by hypothesis (d)).
It is sufficient to show that $a * a' = e \wedge a * e = a$.
We note that:

$$\begin{aligned}
a * a' &= e * (a * a') = && [\text{hypothesis (c)}]\\
&= (a'' * a') * (a * a') = && [\text{definition}]\\
&= a'' * [a' * (a * a')] = && [\text{associative}]\\
&= a'' * [(a' * a) * a'] = && [\text{associative}]\\
&= a'' * (e * a') = && [\text{definition}]\\
&= a'' * a' = && [\text{hypothesis (c)}]\\
&= e && [\text{hypothesis (d)}]
\end{aligned}$$

and

$$\begin{aligned}
a * e &= a * (a' * a) = && [\text{definition}]\\
&= (a * a') * a = && [\text{associative}]\\
&= e * a = && [\text{definition}]\\
&= a && [\text{hypothesis (c)}]
\end{aligned}$$

From (a), (b), (c), (d) and the above results it follows that $(G, *)$ is a group □

Consequences of group definition

Thm: Let $(G, *)$ be a group. Then:

$$\boxed{\begin{array}{l} \forall a, b \in G : (a * b)' = b' * a' \\ \forall a \in G : a'' = a \end{array}}$$

Proof

a) To show $\forall a,b \in G: (a*b)' = b'*a'$

Let $a,b \in G$ be given. Then

$$
\begin{aligned}
(a*b)*(b'*a') &= a*[b*(b'*a')] && \text{[associative]}\\
&= a*[(b*b')*a'] && \text{[associative]}\\
&= a*(e*a') && \text{[}b,b'\text{ symmetric]}\\
&= a*a' && \text{[unit element]}\\
&= e && \text{[}a,a'\text{ symmetric]}
\end{aligned}
$$

and

$$
\begin{aligned}
(b'*a')*(a*b) &= b'*[a'*(a*b)] && \text{[associative]}\\
&= b'*[(a'*a)*b] && \text{[associative]}\\
&= b'*(e*b) && \text{[}a',a\text{ symmetric]}\\
&= b'*b && \text{[unit element]}\\
&= e && \text{[}b',b\text{ symmetric]}
\end{aligned}
$$

It follows, by uniqueness of the symmetric element, that

$$
\left\{\begin{array}{l} (a*b)*(b'*a') = e \\ (b'*a')*(a*b) = e \end{array}\right. \Rightarrow (a*b)' = b'*a'.
$$

b) To show $\forall a \in G: a'' = a$

Let $a \in G$ be given. Then

$$
\begin{aligned}
a'' &= a''*e && \text{[unit element]}\\
&= a''*(a'*a) && \text{[}a',a\text{ symmetric]}\\
&= (a''*a')*a && \text{[associative]}\\
&= e*a && \text{[}a'',a'\text{ symmetric]}\\
&= a && \text{[unit element].}
\end{aligned}
$$

$\square$

↳ For the multiplication group $(M_n(\mathbb{R}), \cdot)$ of matrices, this theorem gives:
$\forall A, B \in M_n(\mathbb{R}) : (AB)^{-1} = B^{-1}A^{-1}$
$\forall A \in M_n(\mathbb{R}) : (A^{-1})^{-1} = A$.

EXAMPLE

Show that $(\mathbb{R} - \{1/3\}, *)$ with $a*b = a+b-3ab$ is an abelian group.

Proof

- Closure : Let $a, b \in \mathbb{R} - \{1/3\}$ with $a \neq 1/3$ and $b \neq 1/3$ be given.

To show that $a*b \neq 1/3$, assume that $a*b = 1/3$.
It follows that:

$$(a*b) - (1/3) = a+b-3ab-(1/3) = (b-3ab) - (1/3-a) =$$
$$= b(1-3a) - (1/3)(1-3a) =$$
$$= (1-3a)(b-(1/3)) = 3(1/3-a)(b-1/3);$$

and therefore:

$$a*b = 1/3 \Rightarrow (a*b) - 1/3 = 0 \Rightarrow 3(1/3-a)(b-1/3) = 0$$
$$\Rightarrow 1/3 - a = 0 \lor b - 1/3 = 0 \Rightarrow$$
$$\Rightarrow a = 1/3 \lor b = 1/3 \leftarrow \text{Contradiction.}$$

Therefore: $a*b \neq 1/3 \Rightarrow a*b \in \mathbb{R} - \{1/3\}$.
Thus: $\forall a, b \in \mathbb{R} - \{1/3\} : a*b \in \mathbb{R} - \{1/3\} \Rightarrow$
$\Rightarrow$ "$*$" closed on $\mathbb{R} - \{1/3\}$.

- <u>Commutative</u> : Let $a, b \in \mathbb{R}-\{1/3\}$ be given. Then:

$$a*b = a+b-3ab = b+a-3ba = b*a, \ \forall a, b \in G \Rightarrow$$
$\Rightarrow$ "$*$" commutative.

- <u>Associative</u> : Let $a, b, c \in \mathbb{R}-\{1/3\}$ be given. Then:

$$\begin{aligned} a*(b*c) &= a*(b+c-3bc) = \\ &= a+(b+c-3bc)-3a(b+c-3bc) = \\ &= a+b+c-3bc-3ab-3ac+9abc = \\ &= (a+b+c)-3(ab+bc+ca)+9abc \qquad (1) \end{aligned}$$

and

$$\begin{aligned} (a*b)*c &= (a+b-3ab)*c = \\ &= (a+b-3ab)+c-3(a+b-3ab)c \\ &= a+b-3ab+c-3ac-3bc+9abc \\ &= (a+b+c)-3(ab+bc+ca)+9abc \qquad (2) \end{aligned}$$

From (1) and (2):
$\forall a, b, c \in \mathbb{R}-\{1/3\}: \ a*(b*c) = (a*b)*c \Rightarrow$
$\Rightarrow$ "$*$" associative.

- <u>Unit element</u> : Let $a \in \mathbb{R}-\{1/3\}$ be given.

We solve the equation:

$$\begin{aligned} e*a = a &\Leftrightarrow e+a-3ea = a \Leftrightarrow e-3ea = 0 \Leftrightarrow \\ &\Leftrightarrow e(1-3a) = 0 \Leftrightarrow e = 0 \vee 1-3a = 0. \qquad (3) \end{aligned}$$

Note that $a \in \mathbb{R}-\{1/3\} \Rightarrow a \neq 1/3 \Rightarrow 1-3a \neq 0$ and therefore $(3) \Leftrightarrow e = 0$.

Thus $\forall a \in \mathbb{R}-\{1/3\}: \ 0*a = a$.

- Symmetric elements:

Let $a \in \mathbb{R}-\{1/3\}$ be given. We solve the equation

$$b * a = 0 \Leftrightarrow b+a-3ba=0 \Leftrightarrow b(1-3a)+a=0 \Leftrightarrow$$
$$\Leftrightarrow b(1-3a)=-a \Leftrightarrow b(3a-1)=a.$$

Since $a \in \mathbb{R}-\{1/3\} \Rightarrow a \neq 1/3 \Rightarrow 3a-1 \neq 0$, and therefore:

$$b * a = 0 \Leftrightarrow b = \frac{a}{3a-1}$$

To show that $\dfrac{a}{3a-1} \neq \dfrac{1}{3}$, assume that $\dfrac{a}{3a-1} = \dfrac{1}{3}$

Then:

$$\frac{a}{3a-1} = \frac{1}{3} \Leftrightarrow 3a = 3a-1 \Leftrightarrow \underline{0a=-1} \longleftarrow \text{inconsistent}$$

It follows that $\dfrac{a}{3a-1} \neq \dfrac{1}{3} \Rightarrow b = \dfrac{a}{3a-1} \in \mathbb{R}-\left\{\dfrac{1}{3}\right\}$

- It follows that $(\mathbb{R}-\{1/3\}, *)$ is an abelian group.

EXERCISES

(9) Given the set $A = \{x \in \mathbb{R} \mid -1 < x < 1\}$ we define the operation $*$ with $a * b = (a+b)/2$.
Explore whether $(A, *)$ is a group.

(10) Given the set $G = \{x \in \mathbb{R} \mid -1 < x < 1\}$ we define the operation $*$ with $a * b = \dfrac{a+b}{1+ab}$.
Show that $(G, *)$ is an abelian group.

(11) We define on $\mathbb{R}$ the operation $*$ with $x * y = x + y + 1$.
Show that $(\mathbb{R}, *)$ is a group.

(12) We define on $G = \mathbb{R} - \{2\}$ the operation $*$ with $x * y = 2(x+y-1) - xy$. Show that $(G, *)$ is an abelian group.

(13) We define on $G = \mathbb{R} - \{1\}$ the operation $*$ with $x * y = xy - x - y + 2$. Show that $(G, *)$ is an abelian group.

(14) We define on $G = (-\sqrt{2}, \sqrt{2})$ the operation $*$ with $x * y = \dfrac{2x+2y}{xy+2}$. Show that $(G, *)$ is an abelian group.

(15) Let $(G, *)$ be a group, and let $x, y \in G$ such that $x * y = y$. Show that x is the unit element of $(G, *)$.

(16) Let $(G, *)$ be a group such that
$\forall a, b \in G: (a * b) * (a * b) = (a * a) * (b * b)$
Show that $(G, *)$ is an abelian group.

▼ Vector spaces

Def : An external operation on A with coefficients from G is any mapping $f: G \times A \to A$ such that every $(\lambda, a) \in G \times A$ is mapped into $\lambda a \in A$.

▶ notation : For external operations we prefer to use multiplicative notation. In the expression $\lambda a \in A$ we say that λ is the coefficient of λa.

Def : Let $(V, +, \cdot)$ be endowed with an internal operation "$+$" : $V \times V \to V$ and an external operation "$\cdot$" : $\mathbb{R} \times V \to V$. We say that $(V, +, \cdot)$ is a real vector space if and only if the following conditions are satisfied:

a) $(V, +)$ is a group

b) $\forall \lambda \in \mathbb{R} : \forall x, y \in V : \lambda(x+y) = \lambda x + \lambda y$

c) $\forall \lambda, \mu \in \mathbb{R} : \forall x \in V : (\lambda + \mu) x = \lambda x + \mu x$

d) $\forall \lambda, \mu \in \mathbb{R} : \forall x \in V : \lambda(\mu x) = (\lambda \mu) x$

e) $\forall x \in V : 1x = x$.

▶ In the above definition, $\lambda + \mu$ and $\lambda \mu$ represent regular addition and multiplication in $\mathbb{R}$.

$\rightarrow$ $(V,+)$ is an abelian group

We will now show that if $(V,+,\cdot)$ is a real vector space then, although not demanded by the above definition, $(V,+)$ will be an abelian group. The proof is dependent on the following general property of groups:

Lemma : Let $(G,*)$ be a group. Then:

$$\boxed{\forall a,b,c \in G : (c*a = c*b \lor a*c = b*c \Rightarrow a = b)}$$

Proof

Let $a,b,c \in G$ be given. Let $e \in G$ be the unit element of G.

Case 1 : Assume that $c*a = c*b$. Then:

$$a = e*a = (c'*c)*a = c'*(c*a) = c'*(c*b)$$
$$= (c'*c)*b = e*b = b.$$

Case 2 : Assume that $a*c = b*c$. Then

$$a = a*e = a*(c*c') = (a*c)*c' = (b*c)*c' =$$
$$= b*(c*c') = b*e = b \qquad \square$$

Thm : $\boxed{(V,+,\cdot) \text{ real vector space} \Rightarrow (V,+) \text{ abelian group}}$

Proof

By definition:

$(V,+,\cdot)$ real vector space $\Rightarrow (V,+)$ group (1)

Let $x,y \in V$ be given. Then

$(1+1)(x+y) = (1+1)x + (1+1)y = x+x+y+y$ (2)

$(1+1)(x+y) = 1(x+y) + 1(x+y) = x+y+x+y$ (3)

From (2) and (3), using the above lemma we have:

$$x+x+y+y = x+y+x+y \Rightarrow x+x+y = x+y+x \Rightarrow$$
$$\Rightarrow x+y = y+x.$$

It follows that

$$\left.\begin{array}{l}\forall x,y \in V : x+y = y+x \Rightarrow \text{"+" commutative} \\ (V,+) \text{ group}\end{array}\right\} \Rightarrow$$

$\Rightarrow (V,+)$ abelian group. $\square$

Properties of vector spaces.

- Let $\mathbb{0} \in V$ be the unit element of the abelian group $(V,+)$.
- Denote as $-x$ the symmetric element of $x \in V$.
- By definition, we know that for all $\lambda, \mu \in \mathbb{R}$ and for all $x,y,z \in V$, we have:

$(x+y)+z = x+(y+z)$	$\lambda(x+y) = \lambda x + \lambda y$
$x+y = y+x$	$(\lambda+\mu)x = \lambda x + \mu x$
$x+\mathbb{0} = x$	$\lambda(\mu x) = (\lambda\mu)x$
$x+(-x) = \mathbb{0}$	$1x = x$

- We will now show that:

① $\boxed{\forall \lambda \in \mathbb{R} : \lambda \mathbf{0} = \mathbf{0}}$

Proof

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given. Then:
$\lambda x + \lambda \mathbf{0} = \lambda(x + \mathbf{0}) = \lambda x = \lambda x + \mathbf{0} \Rightarrow$
$\Rightarrow \lambda \mathbf{0} = \mathbf{0}$. □

② $\boxed{\forall x \in V : 0x = \mathbf{0}}$

Proof

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given. Then
$\lambda x + 0x = (\lambda + 0)x = \lambda x = \lambda x + \mathbf{0} \Rightarrow 0x = \mathbf{0}$. □

③ $\boxed{\forall \lambda \in \mathbb{R} : \forall x \in V : (\lambda x = \mathbf{0} \Rightarrow \lambda = 0 \lor x = \mathbf{0})}$

Proof

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given with $\lambda x = \mathbf{0}$.
Case 1 : If $\lambda = 0 \Rightarrow \lambda = 0 \lor x = \mathbf{0}$
Case 2 : If $\lambda \neq 0 \Rightarrow \lambda^{-1}\lambda = 1$. It follows that

$$x = 1x = (\lambda^{-1}\lambda)x = \lambda^{-1}(\lambda x) = \lambda^{-1}\mathbf{0} = \mathbf{0} \Rightarrow$$
$$\Rightarrow \lambda = 0 \lor x = \mathbf{0} \qquad \square$$

④ $\boxed{\forall \lambda \in \mathbb{R} : \forall x \in V : (-\lambda)x = \lambda(-x) = -\lambda x}$

Proof

Let $\lambda \in \mathbb{R}$ and $x \in V$ be given. We note that
$(-\lambda)x + \lambda x = [(-\lambda) + \lambda]x = 0x = \mathbf{0} \Rightarrow \lambda x$ symmetric of $(-\lambda)x$
$\Rightarrow (-\lambda)x = -\lambda x$.
Similarly:
$\lambda(-x) + \lambda x = \lambda[(-x) + x] = \lambda \mathbf{0} = \mathbf{0} \Rightarrow$
$\Rightarrow \lambda x$ symmetric of $\lambda(-x) \Rightarrow \lambda(-x) = -\lambda x$.
It follows that $(-\lambda)x = \lambda(-x) = -\lambda x$ $\square$

- From the above properties we can also show that:

$$\boxed{\begin{array}{l}
\forall \lambda \in \mathbb{R} - \{0\} : \forall x, y \in V : (\lambda x = \lambda y \Rightarrow x = y) \\
\forall \lambda, \mu \in \mathbb{R} : \forall x \in V - \{\mathbf{0}\} : (\lambda x = \mu x \Rightarrow \lambda = \mu) \\
\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V : \begin{cases} \lambda(x-y) = \lambda x - \lambda y \\ (\lambda - \mu)x = \lambda x - \mu x \end{cases} \\
\forall x \in V : (-1)x = -x
\end{array}}$$

Basic Vector Spaces

1 The space $\mathbb{R}^2$

For $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$ we define:

$$(x_1,y_1)=(x_2,y_2) \Leftrightarrow x_1=x_2 \wedge y_1=y_2$$
$$(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$$
$$\forall \lambda\in\mathbb{R}: \lambda(x_1,y_1)=(\lambda x_1,\lambda y_1)$$

Then $(\mathbb{R}^2,+,\cdot)$ is a vector space.

2 The space $\mathbb{R}^n$

The previous vector space can be generalized for n dimensions as follows:
Let $[n]=\{1,2,3,\dots,n\}$, let $\lambda\in\mathbb{R}$, and let $x,y\in\mathbb{R}^n$ with $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_n)$.
We define:

$$x=y \Leftrightarrow \forall a\in[n]: x_a=y_a$$

Let $z\in\mathbb{R}^n$ with $z=(z_1,z_2,\dots,z_n)$. Then define:

$$z=x+y \Leftrightarrow \forall a\in[n]: z_a=x_a+y_a$$
$$z=\lambda x \Leftrightarrow \forall a\in[n]: z_a=\lambda x_a$$

Then $(\mathbb{R}^n,+,\cdot)$ is a vector space.

③ → The space $F(A)$

We define $F(A)$ as the set of all functions $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$. Let $\lambda \in \mathbb{R}$ and let $f, g, h \in F(A)$.
We define:
$f = g \iff \forall x \in A: f(x) = g(x)$
$h = f + g \iff \forall x \in A: h(x) = f(x) + g(x)$
$h = \lambda f \iff \forall x \in A: h(x) = \lambda f(x)$.
Then $(F(A), +, \cdot)$ is a vector space.

④ → The space $M_{nm}(A)$

Recall that we have defined $M_{nm}(\mathbb{R})$ as the set of all $n \times m$ matrices. Combined with matrix addition "$+$" and scalar multiplication "$\cdot$", $(M_{nm}(\mathbb{R}), +, \cdot)$ is a vector space.

EXAMPLES

a) Show vector addition, defined on $\mathbb{R}^2$ is associative.

Solution

Sufficient to show that

$$\forall x,y,z \in \mathbb{R}^2 : x+(y+z) = (x+y)+z$$

Let $x,y,z \in \mathbb{R}^2$ be given with $x=(x_1,x_2)$, $y=(y_1,y_2)$ and $z=(z_1,z_2)$. Then

$$\begin{aligned} x+(y+z) &= (x_1,x_2)+[(y_1,y_2)+(z_1,z_2)] = \\ &= (x_1,x_2)+(y_1+z_1, y_2+z_2) = \\ &= (x_1+y_1+z_1, x_2+y_2+z_2) = \\ &= (x_1+y_1, x_2+y_2)+(z_1,z_2) = \\ &= [(x_1,x_2)+(y_1,y_2)]+(z_1,z_2) = \\ &= (x+y)+z \end{aligned}$$

It follows that

$$\forall x,y,z \in \mathbb{R}^2 : x+(y+z) = (x+y)+z \Rightarrow$$
$$\Rightarrow \text{"+" associative on } \mathbb{R}^2.$$

b) Show that function addition, defined on $F(A)$ with $A \subseteq \mathbb{R}$ is associative.

Solution

Sufficient to show that
$\forall f,g,h \in F(A) : \forall x \in A : (f+(g+h))(x) = ((f+g)+h)(x)$
Let $f,g,h \in F(A)$ and $x \in A$ be given. Then
$$\begin{aligned}(f+(g+h))(x) &= f(x) + (g+h)(x) = \\ &= f(x) + g(x) + h(x) = \\ &= (f+g)(x) + h(x) = \\ &= ((f+g)+h)(x)\end{aligned}$$
It follows that
$\forall f,g,h \in F(A) : \forall x \in A : (f+(g+h))(x) = ((f+g)+h)(x) \Rightarrow$
$\Rightarrow \forall f,g,h \in F(A) : f+(g+h) = (f+g)+h$
$\Rightarrow$ "+" associative on $F(A)$.

EXERCISES

(17) Give the detailed proof that $\mathbb{R}^2$ is a vector space with respect to vector addition and scalar multiplication, defined as:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$\lambda(x, y) = (\lambda x, \lambda y), \quad \forall \lambda \in \mathbb{R}$$

(18) Give the detailed proof that $F(A)$ with $A \subseteq \mathbb{R}$ is a vector space with respect to function addition and scalar multiplication, defined as

$$h = f + g \iff \forall x \in A : h(x) = f(x) + g(x)$$
$$h = \lambda f \iff \forall x \in A : h(x) = \lambda f(x).$$

Vector subspaces

Def: Let $(V,+,\cdot)$ be a vector space. We say that

$$\boxed{V_0 \text{ subspace of } V \Leftrightarrow \begin{cases} V_0 \subseteq V \land V_0 \neq \varnothing \\ (V_0,+,\cdot) \text{ is a vector space} \end{cases}}$$

Subspace criteria

① Main subspace criterion

Thm: Let $(V,+,\cdot)$ be a vector space and let $V_0 \subseteq V$ and $V_0 \neq \varnothing$. Then:

$$\boxed{V_0 \text{ subspace of } V \Leftrightarrow \forall \lambda \in \mathbb{R} : \forall x,y \in V_0 : (x+y \in V_0 \land \lambda x \in V_0)}$$

② Condensed subspace criterion

Thm: Let $(V,+,\cdot)$ be a vector space and let $V_0 \subseteq V$ and $V_0 \neq \varnothing$. Then:

$$\boxed{V_0 \text{ subspace of } V \Leftrightarrow \forall \lambda,\mu \in \mathbb{R} : \forall x,y \in V_0 : \lambda x + \mu y \in V_0}$$

Proof

($\Rightarrow$): Assume that V_0 subspace of $V \Rightarrow$
$\Rightarrow \forall \lambda \in \mathbb{R} : \forall x,y \in V_0 : (x+y \in V_0 \wedge \lambda x \in V_0)$.
Let $\lambda,\mu \in \mathbb{R}$ and $x,y \in V_0$ be given. Then:

$$\left.\begin{array}{l}\lambda \in \mathbb{R} \wedge x \in V_0 \Rightarrow \lambda x \in V_0 \\ \mu \in \mathbb{R} \wedge y \in V_0 \Rightarrow \mu y \in V_0\end{array}\right\} \Rightarrow \lambda x + \mu y \in V_0$$

It follows that
$\forall \lambda,\mu \in \mathbb{R} : \forall x,y \in V_0 : \lambda x + \mu y \in V_0$.

($\Leftarrow$): Assume that:
$$\forall \lambda,\mu \in \mathbb{R} : \forall x,y \in V_0 : (\lambda x + \mu y \in V_0).$$
Let $\lambda \in \mathbb{R}$ and $x,y \in V_0$ be given. Then:
$1x + 1y \in V_0 \Rightarrow x+y \in V_0$
$\lambda x + 0y \in V_0 \Rightarrow \lambda x \in V_0$
It follows that
$\forall \lambda \in \mathbb{R} : \forall x,y \in V_0 : (\lambda x \in V_0 \wedge x+y \in V_0) \Rightarrow$
$\Rightarrow V_0$ subspace of V.

3 → Unit element belongs to every subspace

Thm: Let $(V,+,\cdot)$ be a vector space with $\mathbf{0} \in V$ the unit element of the group $(V,+)$ and let $V_0 \subseteq V$ and $V_0 \neq \varnothing$. Then:

$$\boxed{V_0 \text{ subspace of } V \Rightarrow \mathbf{0} \in V_0}$$

Proof

Assume that V_0 is a subspace of V. Since $V_0 \neq \emptyset$, choose an $x \in V_0$.

Then: $x \in V_0 \Rightarrow 0x \in V_0 \Rightarrow \mathbf{0} \in V_0$ $\square$

↳ The contrapositive statement is:

$$\boxed{\mathbf{0} \notin V_0 \Rightarrow V_0 \text{ NOT a subspace of } V}$$

Thus showing $\mathbf{0} \notin V_0$ is sufficient to show that V_0 is not a subspace of V.

4 → Intersection of subspaces

Thm: Let $(V, +, \cdot)$ be a vector space. Then:

$$\boxed{\begin{cases} V_1 \text{ subspace of } V \\ V_2 \text{ subspace of } V \end{cases} \Rightarrow V_1 \cap V_2 \text{ subspace of } V}$$

Proof

Assume that V_1, V_2 are subspaces of V.

Then: $\begin{cases} \forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_1 : \lambda x + \mu y \in V_1 \\ \forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_2 : \lambda x + \mu y \in V_2 \end{cases}$

Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in V_1 \cap V_2$ be given. Then:

$$\begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_1 \cap V_2 \end{cases} \Rightarrow \begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_1 \end{cases} \land \begin{cases} \lambda, \mu \in \mathbb{R} \\ x, y \in V_2 \end{cases} \Rightarrow$$

$$\Rightarrow \lambda x + \mu y \in V_1 \land \lambda x + \mu y \in V_2 \Rightarrow$$

$$\Rightarrow \lambda x + \mu y \in V_1 \cap V_2.$$

It follows that

$$\forall \lambda, \mu \in \mathbb{R} : \forall x, y \in V_1 \cap V_2 : \lambda x + \mu y \in V_1 \cap V_2 \Rightarrow$$

$\Rightarrow V_1 \cap V_2$ subspace of V. □

EXAMPLES

a) Let $V=\{(a,b)\in\mathbb{R}^2 \mid 2a+3b=0\}$. Show that V is a subspace of $\mathbb{R}^2$.

Solution

Let $\lambda,\mu\in\mathbb{R}$ and $x,y\in V$ be given.

$x\in V \Rightarrow \exists a_1,b_1\in\mathbb{R}: (x=(a_1,b_1)\wedge 2a_1+3b_1=0)$

$y\in V \Rightarrow \exists a_2,b_2\in\mathbb{R}: (y=(a_2,b_2)\wedge 2a_2+3b_2=0)$

It follows that

$$\begin{aligned}\lambda x+\mu y &= \lambda(a_1,b_1)+\mu(a_2,b_2)=\\ &=(\lambda a_1,\lambda b_1)+(\mu a_2,\mu b_2)=\\ &=(\lambda a_1+\mu a_2,\lambda b_1+\mu b_2)\equiv(c_1,c_2)\Rightarrow\end{aligned}$$

$$\Rightarrow\begin{cases} c_1=\lambda a_1+\mu a_2 \\ c_2=\lambda b_1+\mu b_2\end{cases}\Rightarrow$$

$$\begin{aligned}\Rightarrow 2c_1+3c_2 &= 2(\lambda a_1+\mu a_2)+3(\lambda b_1+\mu b_2)=\\ &=\lambda(2a_1+3b_1)+\mu(2a_2+3b_2)=\\ &=\lambda 0+\mu 0=0\Rightarrow \lambda x+\mu y=(c_1,c_2)\in V\end{aligned}$$

It follows that

$$\left.\begin{array}{l}\forall\lambda,\mu\in\mathbb{R}:\forall x,y\in V:\lambda x+\mu y\in V\\ \emptyset\neq V\subseteq\mathbb{R}^2\end{array}\right\}\Rightarrow V \text{ subspace of } \mathbb{R}^2 \ \square$$

$\hookrightarrow$ Note that the belonging condition for V is:

$x\in V\Leftrightarrow \exists a,b\in\mathbb{R}: (x=(a,b)\wedge 2a+3b=0)$.

b) Let $V = \{f \in F(\mathbb{R}) \mid f \text{ continuous in } \mathbb{R}\}$. Show that V is a subspace of $F(\mathbb{R})$.

Solution

Let $\lambda, \mu \in \mathbb{R}$ and $f, g \in V$ be given. Then

$f \in V \Rightarrow f$ continuous in $\mathbb{R} \Rightarrow \forall x_0 \in \mathbb{R}: \lim_{x \to x_0} f(x) = f(x_0)$

$g \in V \Rightarrow g$ continuous in $\mathbb{R} \Rightarrow \forall x_0 \in \mathbb{R}: \lim_{x \to x_0} g(x) = g(x_0)$.

It follows that:

$$\lim_{x \to x_0} [(\lambda f + \mu g)(x)] = \lim_{x \to x_0} [(\lambda f)(x) + (\mu g)(x)] =$$
$$= \lim_{x \to x_0} [\lambda f(x) + \mu g(x)] = \lambda \lim_{x \to x_0} f(x) + \mu \lim_{x \to x_0} g(x) =$$
$$= \lambda f(x_0) + \mu g(x_0) = (\lambda f)(x_0) + (\mu g)(x_0) =$$
$$= (\lambda f + \mu g)(x_0), \ \forall x_0 \in \mathbb{R} \Rightarrow$$

$\Rightarrow \lambda f + \mu g$ continuous in $\mathbb{R} \Rightarrow \lambda f + \mu g \in V$.

It follows that:

$\forall \lambda, \mu \in \mathbb{R}: \forall f, g \in V: \lambda f + \mu g \in V \Rightarrow$

$\Rightarrow V$ subspace of $\mathbb{R}^2$ $\square$.

c) Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix and let $V = \{X \in M_n(\mathbb{R}) \mid AX = XA\}$. Show that V is a subspace of $M_n(\mathbb{R})$.

<u>Solution</u>

Let $\lambda, \mu \in \mathbb{R}$ and $X, Y \in V$ be given. Then:

$X \in V \Rightarrow AX = XA$

$Y \in V \Rightarrow AY = YA$.

It follows that

$$\begin{aligned} A(\lambda X + \mu Y) &= A(\lambda X) + A(\mu Y) = \lambda(AX) + \mu(AY) = \\ &= \lambda(XA) + \mu(YA) = (\lambda X)A + (\mu Y)A = \\ &= (\lambda X + \mu Y)A \Rightarrow \lambda X + \mu Y \in V \end{aligned}$$

and therefore:

$\forall \lambda, \mu \in \mathbb{R} : \forall X, Y \in V : \lambda X + \mu Y \in V \Rightarrow$

$\Rightarrow V$ subspace of $M_n(\mathbb{R})$.

d) Show that $V = \{f \in F(\mathbb{R}) \mid f \text{ even}\}$ is a subspace of $F(\mathbb{R})$. Recall that we define on $F(\mathbb{R})$:

$$f \text{ even} \Leftrightarrow \forall x \in \mathbb{R} : f(-x) = f(x)$$

Solution

Let $\lambda, \mu \in \mathbb{R}$ and $f, g \in V$ be given.

$$f, g \in V \Rightarrow \begin{cases} f \text{ even} \\ g \text{ even} \end{cases} \Rightarrow \begin{cases} \forall x \in \mathbb{R} : f(-x) = f(x) \\ \forall x \in \mathbb{R} : g(-x) = g(x) \end{cases} \quad (1)$$

Let $x \in \mathbb{R}$ be given. Then

$$\begin{aligned}(\lambda f + \mu g)(-x) &= (\lambda f)(-x) + (\mu g)(-x) = \\ &= \lambda f(-x) + \mu g(-x) = \lambda f(x) + \mu g(x) = \\ &= (\lambda f)(x) + (\mu g)(x) = (\lambda f + \mu g)(x)\end{aligned}$$

It follows that

$\forall x \in \mathbb{R} : (\lambda f + \mu g)(-x) = (\lambda f + \mu g)(x) \Rightarrow$

$\Rightarrow \lambda f + \mu g$ even $\Rightarrow \lambda f + \mu g \in V$

Consequently:

$\forall \lambda, \mu \in \mathbb{R} : \forall f, g \in V : \lambda f + \mu g \in V \Rightarrow$

$\Rightarrow V$ subspace of $F(\mathbb{R})$.

EXERCISES

(19) Show that $V=\{(x,y)\in\mathbb{R}^2 \mid 3x+7y=0\}$ is a subspace of $\mathbb{R}^2$.

(20) Show that
$$V=\{(x,y,z)\in\mathbb{R}^3 \mid x+2y+2z=0 \wedge x-y-3z=0\}$$
is a subspace of $\mathbb{R}^3$.

(21) Show that $V=\{(x,y)\in\mathbb{R}^2 \mid 4x+y=2\}$ is NOT a subspace of $\mathbb{R}^2$.

(22) Show that $V=\{f\in F(\mathbb{R}) \mid f \text{ odd}\}$ is a subspace of $F(\mathbb{R})$. Recall that: f odd $\Leftrightarrow \forall x\in\mathbb{R}: f(-x)=-f(x)$.

(23) Show that $V=\{f\in F(\mathbb{R}) \mid f \text{ periodic}\}$ is a subspace of $F(\mathbb{R})$. Recall that
$$f \text{ periodic} \Leftrightarrow \exists T>0 : \forall x\in\mathbb{R}: f(x+T)=f(x)$$

(24) Show that
$$V=\{f\in F(\mathbb{R}) \mid \forall a,b\in\mathbb{R}: |f(a)-f(b)|\leq k|a-b|\}$$
is a subspace of $F(\mathbb{R})$, with $k\in(0,+\infty)$.
[Hint: Use the following properties of absolute values:
$$\forall a,b\in\mathbb{R}: |a+b|\leq|a|+|b|$$
$$\forall a,b\in\mathbb{R}: |ab|=|a||b|$$]

(25) Show that
$$V=\{f\in F(\mathbb{R}) \mid f \text{ differentiable in } \mathbb{R} \wedge f'+3f=0\}.$$
is a subspace of $F(\mathbb{R})$.

(26) Let $A, B \in M_n(\mathbb{R})$ be two $n \times n$ matrices and let
$V = \{X \in M_n(\mathbb{R}) \mid AX + XB = 0\}$.
Show that V is a subspace of $M_n(\mathbb{R})$.

(27) Let $A \in M_n(\mathbb{R})$ be a non-singular $n \times n$ matrix and let $V = \{X \in M_n(\mathbb{R}) \mid AXA^{-1} = I\}$
Show that V is NOT a subspace of $M_n(\mathbb{R})$.

(28) Let V be a vector space and let A, B be subspaces of V. We define
$A + B = \{x + y \mid x \in A \wedge y \in B\}$
Show that $A+B$ is a subspace of V.

Subspaces spanned by vectors

Let $(V, +, \cdot)$ be a vector space and let $x_1, x_2, \ldots, x_n \in V$ be n vectors of V.

<u>Def</u>: The set V_0 spanned by $\{x_1, x_2, \ldots, x_n\}$ is defined as:

$$\boxed{V_0 = \text{span}\{x_1, x_2, \ldots, x_n\} = \left\{ \sum_{a=1}^{n} \lambda_a x_a \,\middle|\, \forall a \in [n] : \lambda_a \in \mathbb{R} \right\}}$$

- We note that the belonging condition for V_0 reads:

$$\boxed{x \in \text{span}\{x_1, x_2, \ldots, x_n\} \Leftrightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R} : x = \sum_{a=1}^{n} \lambda_a x_a}$$

- We will now show that V_0 is a subspace of V.

<u>Thm</u>: $$\boxed{\left.\begin{array}{l} A = \{x_1, x_2, \ldots, x_n\} \subseteq V \\ (V, +, \cdot) \text{ vector space} \end{array}\right\} \Rightarrow \begin{array}{l} \text{span}(A) \text{ subspace} \\ \text{of } V. \end{array}}$$

<u>Proof</u>

Let $\lambda, \mu \in \mathbb{R}$ and $x, y \in \text{span}(A)$ be given.
It follows that:

$$x \in \mathrm{span}(A) \Rightarrow \exists p_1, p_2, \dots, p_n \in \mathbb{R}: \ x = \sum_{a=1}^{n} p_a x_a$$

$$y \in \mathrm{span}(A) \Rightarrow \exists q_1, q_2, \dots, q_n \in \mathbb{R}: \ y = \sum_{a=1}^{n} q_a x_a$$

and therefore:

$$\lambda x + \mu y = \lambda \sum_{a=1}^{n} p_a x_a + \mu \sum_{a=1}^{n} q_a x_a =$$

$$= \sum_{a=1}^{n} (\lambda p_a + \mu q_a) x_a \Rightarrow \lambda x + \mu y \in \mathrm{span}(A).$$

It follows that
$\forall \lambda, \mu \in \mathbb{R}: \forall x, y \in \mathrm{span}(A): \lambda x + \mu y \in \mathrm{span}(A) \Rightarrow$
$\Rightarrow \mathrm{span}(A)$ subspace of V. □

Basic properties of spanned spaces

Let $A \subseteq V$ and $B \subseteq V$ with A, B finite sets. Then

$$\boxed{\begin{array}{l} A \subseteq \mathrm{span}(A) \\ A \subseteq B \Rightarrow \mathrm{span}(A) \subseteq \mathrm{span}(B) \end{array}}$$

Proof

a) To show $A \subseteq \mathrm{span}(A)$.
Let $A = \{x_1, x_2, \dots, x_n\}$. Let $u \in A$ be given.
Then $u \in A \Rightarrow \exists a \in [n]: u = x_a$

Define $\lambda_b = \begin{cases} 1 & , \text{if } a = b \\ 0 & , \text{if } a \neq b \end{cases}$

Then $u = x_a = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \Rightarrow u \in \mathrm{span}(A)$.

It follows that $\forall u \in A : u \in \mathrm{span}(A) \Rightarrow A \subseteq \mathrm{span}(A)$.

b) To show $A \subseteq B \Rightarrow \mathrm{span}(A) \subseteq \mathrm{span}(B)$

For $A = B$, the statement is trivial, so we assume with no loss of generality that $A \neq B$ and write

$A = \{x_1, x_2, \ldots, x_p\}$ and $B = \{x_1, x_2, \ldots, x_n\}$ with $p < n$.

Let $u \in \mathrm{span}(A)$ be given. Since

$u \in \mathrm{span}(A) \Rightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{R} : u = \lambda_1 x_1 + \cdots + \lambda_p x_p$

Therefore,

$$u = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p =$$
$$= \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_p x_p + 0\, x_{p+1} + \cdots + 0\, x_n \Rightarrow$$
$$\Rightarrow u \in \mathrm{span}(B).$$

It follows that

$\forall u \in \mathrm{span}(A) : u \in \mathrm{span}(B) \Rightarrow \mathrm{span}(A) \subseteq \mathrm{span}(B)$

EXAMPLES

a) Define the vector space spanned by the vectors $x_1 = (1,3,0)$ and $x_2 = (0,2,-1)$.

Solution

We note that

$$a x_1 + b x_2 = a(1,3,0) + b(0,2,-1) =$$
$$= (a,3a,0) + (0,2b,-b)$$
$$= (a, 3a+2b, -b)$$

It follows that

$$V = \text{span}\{x_1, x_2\} = \{a x_1 + b x_2 \mid a,b \in \mathbb{R}\} =$$
$$= \{(a, 3a+2b, -b) \mid a,b \in \mathbb{R}\}.$$

b) Show that $V = \{(a+b, 2b, b-3a) \mid a,b \in \mathbb{R}\}$ is a vector space.

Solution

We note that

$$(a+b, 2b, b-3a) = (a,0,-3a) + (b,2b,b) =$$
$$= a(1,0,-3) + b(1,2,1)$$
$$= ax + by$$

with $x = (1,0,-3)$ and $y = (1,2,1)$. It follows that

$$V = \{(a+b, 2b, b-3a) \mid a,b \in \mathbb{R}\} = \{ax + by \mid a,b \in \mathbb{R}\}$$
$$= \text{span}\{x,y\} \Rightarrow V \text{ subspace of } \mathbb{R}^2 \Rightarrow$$
$$\Rightarrow (V,+,\cdot) \text{ is a vector space.}$$

c) Define by description the vector subspace of $F(\mathbb{R})$ spanned by the functions:

$f(x) = \sin x \ , \ \forall x \in \mathbb{R}$

$g(x) = \cos x \ , \ \forall x \in \mathbb{R}$.

Solution

Let $a, b \in \mathbb{R}$ and note that

$$(af+bg)(x) = (af)(x) + (bg)(x) = af(x) + bg(x) = $$
$$= a\sin x + b\cos x \ , \ \forall x \in \mathbb{R}$$

It follows that

$$V = \text{span}\{f, g\} = \{af+bg \mid a, b \in \mathbb{R}\} =$$
$$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : h = af+bg\}$$
$$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : h(x) = (af+bg)(x)\}$$
$$= \{h \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : h(x) = a\sin x + b\cos x\}.$$

d) Show that the space defined as

$$V = \{f \in F(\mathbb{R}) \mid \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = (ax+b)\sin x + (ax^2+bx+b)\cos x\}$$

is a subspace of $F(\mathbb{R})$

Solution

$$f \in V \Leftrightarrow \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} :$$
$$: f(x) = (ax+b)\sin x + (ax^2+bx+b)\cos x =$$
$$= ax\sin x + b\sin x + ax^2\cos x + b(x+1)\cos x$$
$$= a(x\sin x + x^2\cos x) + b(\sin x + (x+1)\cos x)$$
$$= ag_1(x) + bg_2(x) = (ag_1)(x) + (bg_2)(x) =$$
$$= (ag_1+bg_2)(x)$$

with $g_1, g_2 \in V$ defined as

$\forall x \in \mathbb{R}: \; g_1(x) = x\sin x + x^2 \cos x$

$\forall x \in \mathbb{R}: \; g_2(x) = \sin x + (x+1)\cos x$

It follows that:

$$f \in V \Leftrightarrow \exists a, b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = (ag_1 + bg_2)(x) \Leftrightarrow$$
$$\Leftrightarrow \exists a, b \in \mathbb{R} : f = ag_1 + bg_2$$
$$\Leftrightarrow f \in \operatorname{span}\{g_1, g_2\}$$

and therefore

$$V = \operatorname{span}\{g_1, g_2\} \Rightarrow V \text{ subspace of } \mathcal{F}(\mathbb{R}).$$

e) Consider the space

$$V = \left\{ \begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

Show that V is a subspace of $M_3(\mathbb{R})$.

<u>Solution</u>

We note that

$$\begin{bmatrix} a & 2c & c \\ 2b+c & b & -2b \\ a+3c & 2a+b & 3a \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ a & 2a & 3a \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2b & b & -2b \\ 0 & b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2c & c \\ c & 0 & 0 \\ 3c & 0 & 0 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -2 \\ 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} =$$

$$= aA_1 + bA_2 + cA_3$$

and it follows that

$$V = \{aA_1 + bA_2 + cA_3 \mid a,b,c \in \mathbb{R}\} =$$

$$= \operatorname{span}\{A_1, A_2, A_3\} \Rightarrow V \text{ subspace of } M_3(\mathbb{R}).$$

EXERCISES

(29) Define the vector space spanned by:

a) $x_1 = (2,1,3)$ and $x_2 = (1,3,5)$

b) $x_1 = (1,-1,3,2)$ and $x_2 = (2,-2,5,3)$

c) $x_1 = (0,3,2,5)$, $x_2 = (1,-3,-4,2)$, and $x_3 = (-2,1,3,-1)$

(30) Show that the following sets are vector spaces that are subspaces of $\mathbb{R}^n$ for some n.

a) $V = \{(a+3b, b, 2a) \mid a,b \in \mathbb{R}\}$

b) $V = \{(a-2b+c, 3b, c+2a, b-c) \mid a,b,c \in \mathbb{R}\}$

c) $V = \{(2a-b, b+a, 4b) \mid a,b \in \mathbb{R}\}$.

(31) Define by description the vector spaces of $F(\mathbb{R})$ spanned by

a) $\begin{cases} \forall x \in \mathbb{R} : f(x) = e^x \\ \forall x \in \mathbb{R} : g(x) = e^{-x} \end{cases}$ b) $\begin{cases} \forall x \in \mathbb{R} : f(x) = 2x \\ \forall x \in \mathbb{R} : g(x) = x^2 - 1 \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R} : f(x) = xe^x \\ \forall x \in \mathbb{R} : g(x) = (x+1)^2 e^x \\ \forall x \in \mathbb{R} : h(x) = (x-1)^2 e^x \end{cases}$

(32) Show that the following sets are vector spaces that are subspaces of $F(\mathbb{R})$.

a) $V = \{f \in F(\mathbb{R}) \mid \exists a,b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = x^2(ax+b)\}$

b) $V = \{f \in F(\mathbb{R}) \mid \exists a,b \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = e^{-x}(a\sin x + b\cos x)\}$

c) $V = \{f \in F(\mathbb{R}) \mid \exists a,b,c \in \mathbb{R} : \forall x \in \mathbb{R} : f(x) = xe^x(ax^2+bx+c)\}$

(33) Show that the following sets are vector spaces that are subspaces of $M_n(\mathbb{R})$ for some n.

a) $V = \left\{ \begin{bmatrix} a+b & 3b \\ 2b & a-b \end{bmatrix} \middle| \, a,b \in \mathbb{R} \right\}$

b) $V = \left\{ \begin{bmatrix} a+c & 2a+b \\ 2a-b & a-c \end{bmatrix} \middle| \, a,b,c \in \mathbb{R} \right\}$

c) $V = \left\{ \begin{bmatrix} a+2c & 2a-b & 3c \\ b+c & a+c & 2a+c \\ 3c & 2b-c & a+b \end{bmatrix} \middle| \, a,b,c \in \mathbb{R} \right\}$

Linear Independence

Let $(V,+,\cdot)$ be a vector space and let $A=\{x_1,x_2,\dots,x_n\}\subseteq V$ be a set of vectors in V.

Def:

$$\boxed{\begin{array}{l} A \text{ linearly dependent} \Leftrightarrow \exists x\in A : x\in \operatorname{span}(A-\{x\}) \\ A \text{ linearly independent} \Leftrightarrow A \text{ NOT linearly dependent} \end{array}}$$

- It follows that A is <u>linearly dependent</u> if at least one vector $x\in A$ belongs to the subspace $\operatorname{span}(A-\{x\})$ generated by all vectors in A except for x.
- By negating the definition of linear dependence, we can rewrite the definition of linear independence as follows:

$$\boxed{A \text{ linearly independent} \Leftrightarrow \forall x\in A : x\notin \operatorname{span}(A-\{x\})}$$

Characterization of linear independence/dependence

Thm:

$$\boxed{\begin{array}{l} A \text{ linearly dependent} \Leftrightarrow \\ \Leftrightarrow \exists (\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathbb{R}^n : \begin{cases} (\lambda_1,\lambda_2,\dots,\lambda_n)\neq \mathbf{0} \\ \lambda_1 x_1+\lambda_2 x_2+\dots+\lambda_n x_n = \mathbf{0} \end{cases} \end{array}}$$

Proof

$(\Rightarrow)$: Assume that A is linearly dependent.
Since A linearly dependent $\Rightarrow$
$$\Rightarrow \exists x \in A : x \in \operatorname{span}(A - \{x\}).$$
Without loss of generality assume a reordering of the elements of A such that:
$$x_n \in \operatorname{span}(A - \{x_n\}) \Rightarrow x_n \in \operatorname{span}\{x_1, x_2, \ldots, x_{n-1}\}$$
$$\Rightarrow \exists \mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R} : x_n = \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_{n-1} x_{n-1}$$
It follows that:
$$x_n = \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_{n-1} x_{n-1} \Rightarrow$$
$$\Rightarrow \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_{n-1} x_{n-1} - x_n = 0$$
For $(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\mu_1, \mu_2, \ldots, \mu_{n-1}, -1)$ we have
$$\begin{cases} (\lambda_1, \lambda_2, \ldots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \end{cases}$$
This concludes the argument.

$(\Leftarrow)$: Assume that:
$$\exists (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n : \begin{cases} (\lambda_1, \lambda_2, \ldots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \end{cases}$$
Note that
$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \neq \mathbf{0} \Rightarrow \lambda_1 \neq 0 \lor \lambda_2 \neq 0 \lor \cdots \lor \lambda_n \neq 0.$$
Assume without loss of generality that $\lambda_1 \neq 0$.
It follows that:

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \Rightarrow$$
$$\Rightarrow \lambda_1 x_1 = -\lambda_2 x_2 - \cdots - \lambda_n x_n \Rightarrow$$
$$\Rightarrow x_1 = \frac{-\lambda_2}{\lambda_1} x_2 + \frac{-\lambda_3}{\lambda_1} x_3 + \cdots + \frac{-\lambda_n}{\lambda_1} x_n \Rightarrow$$
$$\Rightarrow x_1 \in \operatorname{span}\{x_2, x_3, \ldots, x_n\} \Rightarrow$$
$$\Rightarrow x_1 \in \operatorname{span}(A - \{x_1\}) \Rightarrow A \text{ linearly dependent.} \quad \square$$

The negation of the previous theorem gives the following equivalent statement.

$$\boxed{\begin{array}{l} A \text{ linearly independent} \Leftrightarrow \\ \Leftrightarrow \forall (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : (\lambda_1 x_1 + \cdots + \lambda_n x_n = \mathbf{0} \Rightarrow \lambda_1 = \cdots = \lambda_n = 0) \end{array}}$$

- Note that for $A = \{u\}$ with $u \in \mathbb{R}^n$ and $u \neq \mathbf{0}$, A is linearly independent because
$$\left.\begin{array}{l} \lambda u = \mathbf{0} \Rightarrow \lambda = 0 \vee u = \mathbf{0} \\ u \neq \mathbf{0} \end{array}\right\} \Rightarrow \lambda = 0 .$$
- For $A = \{\mathbf{0}\}$, A is linearly dependent because $1\mathbf{0} = 0$ and $1 \neq 0$.

Properties of linear dependence/independence

1. $\boxed{B \subseteq A \wedge B \text{ linearly dependent} \Rightarrow A \text{ linearly dependent}}$

<u>Proof</u>

Let $A = \{x_1, x_2, \dots, x_n\}$ and $B = \{x_1, x_2, \dots, x_p\}$ with $p < n$ (since $B \subset A$).

Assume that B is linearly dependent.

Since: B linearly dependent $\Rightarrow$

$$\Rightarrow \exists (\mu_1, \mu_2, \dots, \mu_p) \in \mathbb{R}^p : \begin{cases} (\mu_1, \mu_2, \dots, \mu_p) \neq \mathbf{0} & (1) \\ \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_p x_p = \mathbf{0} & (2) \end{cases}$$

Define $(\lambda_1, \lambda_2, \dots, \lambda_n) = (\mu_1, \mu_2, \dots, \mu_p, 0, \dots, 0)$

From (1): $(\lambda_1, \lambda_2, \dots, \lambda_n) \neq \mathbf{0}$. (3)

Furthermore:

$$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = \mu_1 x_1 + \dots + \mu_p x_p + 0 x_{p+1} + \dots + 0 x_n$$
$$= \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_p x_p = \mathbf{0} \quad (4)$$

From (3) and (4):

$A = \{x_1, \dots, x_n\}$ is linearly dependent. $\square$.

2 $\rightarrow$ $\boxed{\begin{cases} B \subset A \\ A \text{ linearly independent} \end{cases} \Rightarrow B \text{ linearly independent}}$

Proof

Assume that A linearly independent and $B \subset A$.

To show that B linearly independent, assume that B is NOT linearly independent.

Since: B NOT linearly independent $\Rightarrow$

$$\left. \begin{array}{l} \Rightarrow B \text{ linearly dependent} \\ B \subset A \end{array} \right\} \Rightarrow$$

$\Rightarrow A$ linearly dependent $\leftarrow$ Contradiction with hypothesis.

It follows that B linearly independent. $\square$

(3) $\rightarrow$ $\boxed{\left.\begin{array}{l} A \text{ linearly independent} \\ A \cup \{u\} \text{ linearly dependent} \end{array}\right\} \Rightarrow u \in \operatorname{span}(A).}$

Proof

Let $A = \{x_1, x_2, \ldots, x_n\}$.

Assume that A is linearly independent and $A \cup \{u\}$ linearly dependent.

Since: $A \cup \{u\}$ linearly dependent $\Rightarrow$

$$\Rightarrow \exists (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n+1} : \begin{cases} (\lambda_0, \lambda_1, \ldots, \lambda_n) \neq \mathbf{0} & (1) \\ \lambda_0 u + \lambda_1 x_1 + \cdots + \lambda_n x_n = \mathbf{0} & (2) \end{cases}$$

We claim that $\lambda_0 \neq 0$.

To show that $\lambda_0 \neq 0$, assume that $\lambda_0 = 0$.

From (2):

$$\left.\begin{array}{l} \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \\ A = \{x_1, x_2, \ldots, x_n\} \text{ linearly independent} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \Rightarrow$$

$$\Rightarrow (\lambda_1, \ldots, \lambda_n) = \mathbf{0} \Rightarrow (\lambda_0, \lambda_1, \ldots, \lambda_n) = \mathbf{0} \leftarrow \text{Contradiction.}$$

It follows that $\lambda_0 \neq 0$, and therefore from (2):

$$\lambda_0 u = -\lambda_1 x_1 - \lambda_2 x_2 - \cdots - \lambda_n x_n \Rightarrow$$

$$\Rightarrow u = -(\lambda_1/\lambda_0)x_1 - (\lambda_2/\lambda_0)x_2 - \cdots - (\lambda_n/\lambda_0)x_n$$
$$\Rightarrow u \in \operatorname{span}\{x_1, x_2, \ldots, x_n\} \Rightarrow$$
$$\Rightarrow u \in \operatorname{span}(A). \quad \square$$

EXAMPLES

a) Let $x, y \in \mathbb{R}^3$ with $x = (3,1,2)$ and $y = (1,0,3)$. Show that x, y are linearly independent.

Solution

Let $a, b \in \mathbb{R}$ be given and assume that $ax + by = \mathbf{0}$.
We note that:

$$ax + by = \mathbf{0} \Rightarrow a(3,1,2) + b(1,0,3) = (0,0,0) \Rightarrow$$

$$\Rightarrow \begin{cases} 3a + b = 0 \\ a + 2b = 0 \\ 2a + 3b = 0 \end{cases} \Rightarrow \begin{cases} 3(-2b) + b = 0 \\ a = -2b \\ 2(-2b) + 3b = 0 \end{cases} \Rightarrow \begin{cases} -6b + b = 0 \\ a = -2b \\ -4b + 3b = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} -5b = 0 \\ a = -2b \\ -b = 0 \end{cases} \Rightarrow \begin{cases} a = -2b \\ b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \end{cases} \Rightarrow a = b = 0.$$

It follows that

$$\forall a, b \in \mathbb{R} : (ax + by = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$$

$\Rightarrow x, y$ linearly independent.

↳ Note that the steps taken to solve $ax + by = \mathbf{0}$ are valid in both directions:

$$ax + by = \mathbf{0} \Leftrightarrow \cdots \Leftrightarrow \cdots \Leftrightarrow a = b = 0$$

however the definition of linear independence only requires the "$\Rightarrow$" direction.

b) Let $x, y, z \in \mathbb{R}^3$ with $x = (1,2,2)$, $y = (3,1,4)$, and $z = (-1,3,0)$. Show that x, y, z is linearly dependent.

Solution

Let $a, b, c \in \mathbb{R}$. Note that

$$ax + by + cz = \mathbf{0} \Leftrightarrow a(1,2,2) + b(3,1,4) + c(-1,3,0) = (0,0,0)$$

$$\Leftrightarrow \begin{cases} a + 3b - c = 0 \\ 2a + b + 3c = 0 \\ 2a + 4b = 0 \end{cases} \Leftrightarrow \begin{cases} (-2b) + 3b - c = 0 \\ 2(-2b) + b + 3c = 0 \\ a = -2b \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} +b - c = 0 \\ -4b + b + 3c = 0 \\ a = -2b \end{cases} \Leftrightarrow \begin{cases} b - c = 0 \\ -3b + 3c = 0 \\ a = -2b \end{cases} \Leftrightarrow \begin{cases} b - c = 0 \\ a = -2b \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c = b \\ a = -2b. \end{cases} \Leftrightarrow (a,b,c) = (-2b, b, b) = b(-2,1,1)$$

Thus: For $(a,b,c) = (-2,1,1) \Rightarrow$

$$\Rightarrow \quad -2x + y + z = \mathbf{0} \Rightarrow x, y, z \text{ linearly dependent.}$$

↳ Note that in solving $ax + by + cz = \mathbf{0}$ we only need the "$\Leftarrow$" direction so we can claim that: $(a,b,c) = (-2,1,1) \Rightarrow ax + by + cz = \mathbf{0}$. Contrast this remark with the previous example.

c) Let $f, g \in F(\mathbb{R})$ with $f(x) = \sin x, \forall x \in \mathbb{R}$ and $g(x) = \cos x, \forall x \in \mathbb{R}$. Show that f, g are linearly independent.

Solution

Let $a, b \in \mathbb{R}$ be given such that $af + bg = 0$.
We note that

$$af + bg = 0 \Rightarrow \forall x \in \mathbb{R}: af(x) + bg(x) = 0 \Rightarrow$$
$$\Rightarrow \forall x \in \mathbb{R}: a \sin x + b \cos x = 0. \quad (1)$$

From (1), for $x = 0$:

$$a \sin 0 + b \cos 0 = 0 \Rightarrow a \cdot 0 + b \cdot 1 = 0 \Rightarrow b = 0$$

From (1), for $x = \pi/2$:

$$a \sin(\pi/2) + b \cos(\pi/2) = 0 \Rightarrow a \cdot 1 + b \cdot 0 = 0 \Rightarrow a = 0$$

It follows that

$$\forall a, b \in \mathbb{R}: (af + bg = 0 \Rightarrow a = b = 0) \Rightarrow$$
$$\Rightarrow f, g \text{ linearly independent.} \quad \square$$

d) Let $f, g \in F(\mathbb{R})$ with $f(x) = 2x, \forall x \in \mathbb{R}$ and $g(x) = x^2, \forall x \in \mathbb{R}$. Show that f, g linearly independent.

Solution

Let $a, b \in \mathbb{R}$ be given such that $af + bg = 0$.
We note that:

$\underline{af+bg=\mathbf{0}} \Rightarrow \forall x\in\mathbb{R}: af(x)+bg(x)=0 \Rightarrow$

$\Rightarrow \forall x\in\mathbb{R}: a(2x)+bx^2=0.$ (1).

From (1), for $x=1$: $2a+b=0$ (2)

From (1), for $x=2$: $4a+4b=0$ (3)

From (2) and (3):

$$\begin{cases}2a+b=0\\4a+4b=0\end{cases} \Rightarrow \begin{cases}2a+b=0\\a+b=0\end{cases} \Rightarrow \begin{cases}a+(a+b)=0\\a+b=0\end{cases} \Rightarrow \begin{cases}a=0\\a+b=0\end{cases}$$

$$\Rightarrow \begin{cases}a=0\\0+b=0\end{cases} \Rightarrow \begin{cases}a=0\\b=0\end{cases} \Rightarrow \underline{a=b=0}.$$

It follows that

$\forall a,b\in\mathbb{R}: (af+bg=\mathbf{0} \Rightarrow a=b=0) \Rightarrow$

$\Rightarrow f,g$ linearly independent.

e) Let $f,g,h\in F(\mathbb{R})$ with $f(x)=\cos x$, $\forall x\in\mathbb{R}$, $g(x)=\cos x\cos 2x$, $\forall x\in\mathbb{R}$, and $h(x)=\sin x\sin 2x$, $\forall x\in\mathbb{R}$. Show that f,g,h are linearly dependent.

<u>Solution</u>

We note that

$f(x)=\cos x=\cos(2x-x)=\cos 2x\cos x+\sin 2x\sin x=$

$=g(x)+h(x)$, $\forall x\in\mathbb{R} \Rightarrow f=g+h \Rightarrow f\in\operatorname{span}\{g,h\}$

$\Rightarrow f,g,h$ linearly dependent.

EXERCISES

(34) Let $x, y \in \mathbb{R}^3$ with $x = (1,2,1)$ and $y = (3,-1,1)$. Show that x, y are linearly independent.

(35) Let $x, y, z \in \mathbb{R}^4$ with $x = (2,1,1,3)$, $y = (-1,2,1,-1)$, and $z = (0,5,3,1)$. Show that x, y, z are linearly dependent.

(36) Let $f, g, h \in F(\mathbb{R})$ be 3 functions that belong to ~~F(R)~~ the vector space $F(\mathbb{R})$. Show that given the following definitions, f, g, h are linearly independent.

a) $\begin{cases} \forall x \in \mathbb{R} : f(x) = 3x \\ \forall x \in \mathbb{R} : g(x) = x+2 \\ \forall x \in \mathbb{R} : h(x) = (x-1)^2 \end{cases}$ b) $\begin{cases} \forall x \in \mathbb{R} : f(x) = \sin x \\ \forall x \in \mathbb{R} : g(x) = \cos x \\ \forall x \in \mathbb{R} : h(x) = x \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R} : f(x) = 1-x \\ \forall x \in \mathbb{R} : g(x) = 1+x \\ \forall x \in \mathbb{R} : h(x) = 1-x^2 \end{cases}$

(37) Let $f, g, h \in F(\mathbb{R})$ be 3 functions that belong to the vector space $F(\mathbb{R})$. Show that given the following definitions, f, g, h are linearly dependent.

a) $\begin{cases} \forall x \in \mathbb{R} : f(x) = x-1 \\ \forall x \in \mathbb{R} : g(x) = x^3-1 \\ \forall x \in \mathbb{R} : h(x) = x-x^3 \end{cases}$ b) $\begin{cases} \forall x \in \mathbb{R} : f(x) = \sin^2 x \\ \forall x \in \mathbb{R} : g(x) = \cos 2x \\ \forall x \in \mathbb{R} : h(x) = 2-\cos 2x \end{cases}$

c) $\begin{cases} \forall x \in \mathbb{R} : f(x) = \cos 2x \\ \forall x \in \mathbb{R} : g(x) = \cos^2 x \\ \forall x \in \mathbb{R} : h(x) = \sin^2 x \end{cases}$ (Hint: Use your trigonometric identities from precalculus)

(38) Let $f, g, h \in F(\mathbb{R})$ with

$$\begin{cases} \forall x \in \mathbb{R} : f(x) = 1 \\ \forall x \in \mathbb{R} : g(x) = e^x \\ \forall x \in \mathbb{R} : h(x) = e^{2x} \end{cases}$$

Show that f, g, h are linearly independent.
(Hint: Starting with $\forall x \in \mathbb{R} : a + be^x + ce^{2x} = 0$ we can obtain additional equations by differentiating twice with respect to x. Then set $x = 0$ to obtain a 3×3 system of equations for a, b, c).

(39) Let $f, g, h \in F(\mathbb{R})$ with

$$\begin{cases} \forall x \in \mathbb{R} : f(x) = 1 \\ \forall x \in \mathbb{R} : g(x) = e^x \\ \forall x \in \mathbb{R} : h(x) = xe^x \end{cases}$$

Show that f, g, h are linearly independent.

(40) Let $f_1, f_2, \ldots, f_n \in F(\mathbb{R})$ with
$\forall k \in [n] : \forall x \in \mathbb{R} : f_k(x) = \sin(kx)$
with $[n] = \{1, 2, 3, \ldots, n\}$.

a) For any $k, m \in [n]$, evaluate the integral

$$I_{km} = \int_{-\pi}^{\pi} f_k(x) f_m(x)\, dx.$$

(Hint: Distinguish between the cases $k = m$ and $k \neq m$ and use the identity

$$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$$

to do the integral).

b) Use (a) to show that $f_1, f_2, \ldots, f_n$ are linearly independent.

(41) Let $a, b, c \in (0, 2\pi)$ with $a \neq b \neq c \neq a$. Show that

a) $\sin(x+a)\sin(c-b) + \sin(x+b)\sin(a-c) + \sin(x+c)\sin(b-a) = 0$

b) Let $f, g, h \in F(\mathbb{R})$ with

$$\begin{cases} \forall x \in \mathbb{R} : f(x) = \sin(x+a) \\ \forall x \in \mathbb{R} : g(x) = \sin(x+b) \\ \forall x \in \mathbb{R} : h(x) = \sin(x+c) \end{cases}$$

Show that f, g, h are linearly dependent.

(42) Let $x, y, z \in V$ with V a vector space. Show that

a) x, y, z linearly independent $\Rightarrow$ $x+y, y+z, z+x$ linearly independent.

b) x, y, z linearly independent
$\Rightarrow$ $x+y, y-x, y+z-2x$ linearly independant.

Linear Independence in $\mathbb{R}^n$

In the previous examples we have used the following characterizations directly to establish linear independence and linear dependence:

- For $A = \{x_1, x_2, \dots, x_n\} \subset V$

  a) A linearly dependent $\Leftrightarrow$
  $$\exists(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \begin{cases} (\lambda_1, \lambda_2, \dots, \lambda_n) \neq \mathbf{0} \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \end{cases}$$

  b) A linearly independent $\Leftrightarrow$
  $$\forall (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \mathbf{0} \Rightarrow$$
  $$\Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = \mathbf{0})$$

For the special case of the vector space $V = \mathbb{R}^n$, linear dependence and independence can be determined via the following specialized theory:

Def: Let $A = \{x_1, x_2, \dots, x_k\} \subseteq \mathbb{R}^n$ be a set of k n-dimensional vectors. We define a corresponding matrix
$$M = \mathrm{Mat}(A) = [x_1 \; x_2 \cdots x_k] \in M_{nk}(\mathbb{R})$$
as an $n \times k$ matrix where for $a \in \mathbb{N}$ with $1 \leq a \leq k$, the a^{th} column of M consists of the components of the vector x_a. In other words: $M_{ab} = (x_b)_a$

<u>Def</u> : Let $M \in M_{nk}(\mathbb{R})$ be an $n \times k$ matrix with $k \leq n$ (i.e more rows than columns). We define the set $Sub(M)$ of submatrices of M as the set of all matrices $S \in M_k(\mathbb{R})$ obtained from M by deleting any arbitrary selection of $n-k$ rows.

↳ For a square matrix $M \in M_n(\mathbb{R})$, no rows can be deleted therefore $Sub(M) = \{M\}$.

EXAMPLE

For $x_1 = (2,5,3,1)$ and $x_2 = (3,1,4,7)$ it follows that

$$M = Mat(\{x_1, x_2\}) = [x_1 \; x_2] = \begin{bmatrix} 2 & 3 \\ 5 & 1 \\ 3 & 4 \\ 1 & 7 \end{bmatrix}, \text{ and}$$

$$Sub(M) = \left\{ \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 1 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix} \right\}$$

□

<u>Thm</u> : Let $A = \{x_1, x_2, \dots, x_k\} \subseteq \mathbb{R}^n$ with $k \leq n$. Then

$$\boxed{\begin{aligned} &A \text{ linearly independent} \iff \exists M \in Sub(Mat(A)) : \det(M) \neq 0 \\ &A \text{ linearly dependent} \iff \forall M \in Sub(Mat(A)) : \det(M) = 0 \end{aligned}}$$

For the case $k=n$, the above theorem reduces to the following simpler statement:

Corollary: Let $A=\{x_1,x_2,\dots,x_n\}\subseteq\mathbb{R}^n$. Then

$$\boxed{\begin{array}{l}\{x_1,\dots,x_n\}\text{ linearly independent} \Leftrightarrow \det([x_1\cdots x_n])\neq 0\\ \{x_1,\dots,x_n\}\text{ linearly dependent} \Leftrightarrow \det([x_1\cdots x_n])=0\end{array}}$$

EXAMPLES

a) Show that the vectors $x_1 = (1,0,2)$, $x_2 = (1,1,2)$, and $x_3 = (2,2,2)$ are linearly dependent.

Solution

$$\det([x_1\ x_2\ x_3]) = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \end{vmatrix} \begin{matrix} (-1) \\ \leftarrow \\ \leftarrow \end{matrix} = \begin{vmatrix} 1 & 1 & 2 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} =$$

$$= (+1)\cdot 2 \cdot \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = (+1)\cdot 2\cdot((-1)\cdot 1 - 0\cdot 1) =$$

$$= 2(-1-0) = -2 \neq 0 \Rightarrow \{x_1, x_2, x_3\} \text{ linearly dependent.}$$

b) Show that $x_1 = (1,2,1)$ and $x_2 = (2,-1,1)$ are linearly independent.

Solution

$$\text{Let } B = [x_1\ x_2] = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \mathrm{Sub}(B) = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\text{Since } \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 1(-1) - 2\cdot 2 = -1-4 = -5 \neq 0 \Rightarrow$$

$$\Rightarrow \exists M \in \mathrm{Sub}(B) : \det(M) \neq 0 \Rightarrow \{x_1, x_2\} \text{ linearly independent.}$$

c) Show that $x_1 = (1,2,0,1)$, $x_2 = (2,1,3,2)$, and $x_3 = (2,2,2,2)$ are linearly dependent

<u>Solution</u>

Let $B = [x_1\ x_2\ x_3] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \Rightarrow$

$$\Rightarrow \mathrm{Sub}(B) = \left\{ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \right\} = \{B_1, B_2, B_3, B_4\}.$$

We note that

$$\det(B_1) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} (-1) = \begin{vmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{vmatrix} =$$

$$= (+1)\cdot 2 \cdot \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = (+1)\cdot 2\cdot [1\cdot 1 - (-1)(-1)] =$$

$$= 2(1-1) = 0$$

$$\det(B_2) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 0, \quad \det(B_3) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 0,$$

$$\det(B_4) = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{vmatrix} = - \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 3 & 2 \end{vmatrix} = + \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= \det(B_1) = 0$$

and therefore

$\forall M \in Sub(B) : \det(M) = 0 \Rightarrow$

$\Rightarrow \{x_1, x_2, x_3\}$ linearly dependent.

d) Let $x=(1,2,1)$, $y=(1,1,0)$, $z=(a,2a+3,2)$. Find all $a\in\mathbb{R}$ such that x,y,z are linearly dependent.

Solution

$$\det([x\ y\ z]) = \begin{vmatrix} 1 & 1 & a \\ 2 & 1 & 2a+3 \\ 1 & 0 & 2 \end{vmatrix} \overset{(-1)}{} = \begin{vmatrix} 1 & 1 & a \\ 1 & 0 & a+3 \\ 1 & 0 & 2 \end{vmatrix} =$$

$$= (-1)\cdot 1\cdot \begin{vmatrix} 1 & a+3 \\ 1 & 2 \end{vmatrix} = -(1\cdot 2 - 1\cdot(a+3)) = -2+(a+3) =$$

$$= a+1.$$

It follows that

x,y,z linearly dependent $\Leftrightarrow \det([x\ y\ z])=0$

$\Leftrightarrow a+1=0 \Leftrightarrow a=-1$

Thus:

x,y,z linearly dependent $\Leftrightarrow a=-1$.

e) Let $x=(3,9,1)$ and $y=(a,2a-1,1-3a)$. Find all $a\in\mathbb{R}$ such that x,y linearly independent.

<u>Solution</u>

Let $M=[x\ y]=\begin{bmatrix} 3 & a \\ 9 & 2a-1 \\ 1 & 1-3a \end{bmatrix} \Rightarrow$

$\Rightarrow \text{Sub}(M)=\left\{\begin{bmatrix} 3 & 0 \\ 9 & 2a-1 \end{bmatrix}, \begin{bmatrix} 3 & a \\ 1 & 1-3a \end{bmatrix}, \begin{bmatrix} 9 & 2a-1 \\ 1 & 1-3a \end{bmatrix}\right\}$

$=\{M_1, M_2, M_3\}$

and note that

$\det M_1=\begin{vmatrix} 3 & 0 \\ 9 & 2a-1 \end{vmatrix}=3(2a-1)=6a-3$

$\det M_2=\begin{vmatrix} 3 & a \\ 1 & 1-3a \end{vmatrix}=3(1-3a)-1\cdot a=3-9a-a=3-10a$

$\det M_3=\begin{vmatrix} 9 & 2a-1 \\ 1 & 1-3a \end{vmatrix}=9(1-3a)-(2a-1)=9-27a-2a+1=$

$=-29a+10$

It follows that:

x,y linearly independent $\Leftrightarrow \forall A\in\text{Sub}(M): \det A\neq 0 \Leftrightarrow$

$\Leftrightarrow \det M_1\neq 0 \wedge \det M_2\neq 0 \wedge \det M_3\neq 0 \Leftrightarrow$

$\Leftrightarrow 6a-3\neq 0 \wedge 3-10a\neq 0 \wedge -29a+10\neq 0$

$\Leftrightarrow a\neq 1/2 \wedge a\neq 3/10 \wedge a\neq 10/29$

$\Leftrightarrow a\in\mathbb{R}-\{1/2, 3/10, 10/29\}$.

EXERCISES

(43) Show that the following vectors are linearly independent

a) $x=(1,2)$ and $y=(-1,1)$

b) $x=(3,1,1)$ and $y=(0,4,5)$

c) $x=(2,1,0,3)$, $y=(1,3,3,1)$, and $z=(3,4,3,2)$

(44) Show that the following vectors are linearly dependent

a) $x=(3,2)$, $y=(4,-1)$, and $z=(5,-2)$

b) $x=(9,-3,7)$, $y=(1,8,8)$, and $z=(5,-5,1)$

c) $x=(2,-1,5,7)$, $y=(3,1,5,-2)$, and $z=(1,1,1,-4)$

(45) Let $x=(1,3,-1)$, $y=(1,a,4)$, and $z=(3,-2,b)$. Find the set of all $a,b\in\mathbb{R}$ such that x,y,z are linearly independent on $\mathbb{R}^3$.

(46) Find all $a\in\mathbb{R}$ such that $x=(1,1,1)$, $y=(1,a,-1)$, and $z=(a,1,1)$ are linearly independent on $\mathbb{R}^3$.

(47) Find all $a\in\mathbb{R}$ such that $x=(3,1,-4,6)$, $y=(1,1,4,4)$, and $z=(1,0,-4,a)$ are linearly dependent on $\mathbb{R}^4$.

(48) Find the set of all $(a,b)\in\mathbb{R}^2$ such that $x=(3,-2,-1,3)$, $y=(1,0,2,4)$, and $z=(1,-3,a,b)$ are linearly dependent on $\mathbb{R}^4$.

(49) Show that the vectors $x=(1,3,5,p)$, $y=(a,3a,5a,p)$, and $z=(-b,-3b,-5b,r)$ are always linearly dependent on $\mathbb{R}^4$

(50) Let $x=(1,a,a^2)$, $y=(1,b,b^2)$, and $z=(1,c,c^2)$. Show that x,y,z linearly dependent $\Leftrightarrow x=y \vee y=z \vee z=x$.

Basis and dimension of vector spaces

Let $(V,+,\cdot)$ be a vector space and let $B=\{x_1,\ldots,x_n\}\subseteq V$. We use the notation $|B|=n$ to denote the cardinality of B (i.e. the number of elements in the set B).

Def : $$\boxed{B \text{ basis of } V \Leftrightarrow \begin{cases} B \text{ linearly independent} \\ V=\operatorname{span}(B) \end{cases}}$$

Thm : Assume that B is a basis of the vector space V. Then:

$$\boxed{\forall u\in V : \exists!\,(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbb{R}^n : u=\lambda_1 x_1+\cdots+\lambda_n x_n}$$

(For all $u\in V$, there is a unique $(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbb{R}^n$ such that $u=\lambda_1x_1+\ldots+\lambda_nx_n$).

Proof

Assume that B is a basis of V. Let $u\in V$ be given.

$$\left.\begin{array}{l} B \text{ basis of } V \Rightarrow V=\operatorname{span}(B) \\ u\in V \end{array}\right\} \Rightarrow u\in\operatorname{span}(B)\Rightarrow$$

$\Rightarrow \exists\,(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbb{R}^n : u=\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n$.

To show that $(\lambda_1,\ldots,\lambda_n)$ is unique, assume that it is not unique and therefore:

$$\exists (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n : \begin{cases} (\mu_1, \dots, \mu_n) \neq (\lambda_1, \dots, \lambda_n) \\ \mu_1 x_1 + \mu_2 x_2 + \dots + \mu_n x_n = u \end{cases}$$

Then :

$$\sum_{a=1}^{n} (\lambda_a - \mu_a) x_a = \sum_{a=1}^{n} \lambda_a x_a - \sum_{a=1}^{n} \mu_a x_a = u - u = \mathbb{0} \quad (1)$$

and

B basis of $V \Rightarrow x_1, x_2, \dots, x_n$ linearly independent. (2)

From (1) and (2) :

$\forall a \in [n] : \lambda_a - \mu_a = 0 \Rightarrow \forall a \in [n] : \lambda_a = \mu_a \Rightarrow$

$\Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = (\mu_1, \mu_2, \dots, \mu_n) \leftarrow$ Contradiction

It follows that $(\lambda_1, \lambda_2, \dots, \lambda_n)$ is unique.

and therefore

$\forall u \in V : \exists! (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : u = \lambda_1 x_1 + \dots + \lambda_n x_n$. $\square$

↳ This result shows that the basis B functions as a <u>coordinate system</u> for the vector space V which allows every vector $u \in V$ to be written as

$$u = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$$

in a unique way. The numbers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ are the <u>coordinates</u> of the vector u with respect to the coordinate system defined by the basis B.

Dimension of vector space V

Let $(V,+,\cdot)$ be a vector space and let $A=\{x_1,\ldots,x_n\} \subseteq V$ and $B=\{y_1,y_2,\ldots,y_m\}\subseteq V$. We show that:

(1) $\boxed{\begin{cases} B \text{ basis of } V \\ |A|>|B| \end{cases} \Rightarrow A \text{ linearly dependent}}$

Proof

$$\left.\begin{array}{l} B \text{ basis of } V \Rightarrow V=\mathrm{span}(B) \\ \forall a\in[n]: x_a\in V \end{array}\right\} \Rightarrow \forall a\in[n]: x_a\in \mathrm{span}(B)$$

$$\Rightarrow \forall a\in[n]: \exists (M_{a1},\ldots,M_{am})\in\mathbb{R}^m : x_a=\sum_{b=1}^{m} M_{ab}y_b \quad (1)$$

Let $(\lambda_1,\lambda_2,\ldots,\lambda_m)\in\mathbb{R}^m$ and solve:

$$\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n=\mathbb{0} \Leftrightarrow \sum_{a=1}^{n}\lambda_a x_a=\mathbb{0}$$

$$\overset{(1)}{\Leftrightarrow} \sum_{a=1}^{n}\lambda_a\left[\sum_{b=1}^{m}M_{ab}y_b\right]=0 \Leftrightarrow \sum_{b=1}^{m}\left[\sum_{a=1}^{n}\lambda_aM_{ab}\right]y_b=0 \quad (2)$$

Since B basis of $V \Rightarrow y_1,y_2,\ldots,y_m$ linearly independent (3)
From (2) and (3), it follows that

$$\sum_{a=1}^{n}\lambda_aM_{ab}=0,\ \forall b\in[m]. \quad (4).$$

Since (4) is a system of m equations with n unknowns and since $|A|>|B| \Rightarrow n>m$ it follows that (4) is either inconsistent or has non-zero solutions. Since $\forall a\in[n]: \lambda_a=0$ satisfies (4), it follows that (4) is not inconsistent and therefore it has a non-zero solution $(\lambda_1,\lambda_2,\ldots,\lambda_n)\neq \mathbf{0}$.

Therefore:

$$\left\{\begin{array}{l}(\lambda_1,\lambda_2,\ldots,\lambda_n)\neq \mathbf{0} \\ \lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_n x_n=\mathbf{0}\end{array}\right. \Rightarrow$$

$\Rightarrow x_1,x_2,\ldots,x_n$ linearly dependent

$\Rightarrow A$ linearly dependent.

(2) $\rightarrow$ $\boxed{\left.\begin{array}{l}B_1 \text{ basis of } V \\ B_2 \text{ basis of } V\end{array}\right\} \Rightarrow |B_1|=|B_2|}$

Proof

Assume that B_1 and B_2 are basis of V.

To show that $|B_1|=|B_2|$, assume with no loss of generality that $|B_1|>|B_2|$. Then:

$$\left\{\begin{array}{l}|B_1|>|B_2| \\ B_2 \text{ basis of } V\end{array}\right. \Rightarrow B_1 \text{ linearly dependent} \Rightarrow$$

$\Rightarrow B_1$ NOT linearly independent $\Rightarrow$

$\Rightarrow B_1$ NOT basis of $V \leftarrow$ Contradiction.

Similar argument if $|B_1|<|B_2|$. It follows that $|B_1|=|B_2|$.

From this statement we conclude that any basis B of a vector space V has the same number n of elements. We call this number, the <u>dimension</u> of V and write $\dim V = n$.

Let V be a vector space with $\dim V = n \in \mathbb{N}$ and let $\{x_1, x_2, \dots, x_p\} \subseteq V$. From property ① it immediately follows that

$$\boxed{p > \dim V \Rightarrow \{x_1, x_2, \dots, x_p\} \text{ linearly dependent}}$$

The contrapositive statement gives:

$$\boxed{\{x_1, x_2, \dots, x_p\} \text{ linearly independent} \Rightarrow p \leq \dim V}$$

Let V be a vector space and let $\mathbf{0}$ be the unit element of the group $(V, +)$. Then:

a) $\{\mathbf{0}\}$ is a subspace of V

b) $\{\mathbf{0}\} = \operatorname{span}\{\mathbf{0}\}$

c) However, $\{\mathbf{0}\}$ does not have a basis since $\{\mathbf{0}\}$ is linearly dependent.

d) Consequently, the dimension of $\{\mathbf{0}\}$ is <u>defined</u> to be $\dim\{\mathbf{0}\} = 0$

It is possible to have vector spaces with no finite set basis B. For example $F(A)$, the set of all functions $f: A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$.

Dimension and canonical basis of $\mathbb{R}^n$

- We define the n-dimensional vectors

$e_1 = (1, 0, 0, \dots, 0)$
$e_2 = (0, 1, 0, \dots, 0)$
$e_3 = (0, 0, 1, \dots, 0)$
$\vdots$
$e_n = (0, 0, 0, \dots, 1)$

Then it follows that

a) $B = \{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$

b) $\dim(\mathbb{R}^n) = n$

Proof

a) $$\det([e_1\, e_2 \cdots e_n]) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} =$$

$$= \det(I) = 1 \neq 0 \Rightarrow$$

$\Rightarrow B = \{e_1, e_2, \dots, e_n\}$ is linearly independent (1)

Since $B \subseteq \mathbb{R}^n \Rightarrow \operatorname{span}(B) \subseteq \mathbb{R}^n$. (2)

It is sufficient to show that $\mathbb{R}^n \subseteq \operatorname{span}(B)$.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ be given. Then

$$x = (x_1, x_2, \dots, x_n) =$$
$$= (x_1, 0, \dots, 0) + (0, x_2, \dots, 0) + \cdots + (0, 0, \dots, x_n)$$
$$= x_1(1, 0, \dots, 0) + x_2(0, 1, \dots, 0) + \cdots + x_n(0, 0, \dots, 1)$$
$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \Rightarrow$$

$\Rightarrow x \in \text{span}\{e_1, e_2, \dots, e_n\} \Rightarrow x \in \text{span}(B)$.

It follows that $\forall x \in \mathbb{R}^n : x \in \text{span}(B) \Rightarrow \mathbb{R}^n \subseteq \text{span}(B)$. (3)

From (1),(2),(3):

$$\begin{cases} B \text{ linearly independent} \\ \text{span}(B) \subseteq \mathbb{R}^n \\ \mathbb{R}^n \subseteq \text{span}(B) \end{cases} \Rightarrow \begin{cases} B \text{ linearly independent} \\ \text{span}(B) = \mathbb{R}^n \end{cases} \Rightarrow$$

$\Rightarrow B$ basis of $\mathbb{R}^n$.

b) B basis of $\mathbb{R}^n \Rightarrow$

$\Rightarrow \dim V = |B| = |\{e_1, e_2, \dots, e_n\}| = n$. $\square$

$\hookrightarrow$ Using similar arguments, it can be shown that

$\dim M_{mn}(\mathbb{R}) = m \cdot n$

$\dim M_n(\mathbb{R}) = n^2$.

Basis of a vector space with known dimension

- Let V be a vector space with $\dim V = n$ and let $A = \{x_1, x_2, \ldots, x_n\} \subseteq V$. The problem is to explore whether A is a basis of V.
- We note that by definition:

$$\boxed{A \text{ linearly dependent} \Rightarrow A \text{ NOT basis of } V}$$

What happens if A is linearly independent?

$$\boxed{\left.\begin{array}{l} A \subseteq V \text{ linearly independent} \\ \dim V = |A| \end{array}\right\} \Rightarrow A \text{ is basis of } V}$$

Proof

Assume that $\dim V = n$ and $A = \{x_1, x_2, \ldots, x_n\} \subseteq V$ be linearly independent.
It is sufficient to show that $\operatorname{span}(A) \subseteq V \wedge V \subseteq \operatorname{span}(A)$.

(a) To show $\operatorname{span}(A) \subseteq V$:

Since $\left\{\begin{array}{l} A \subseteq V \\ V \text{ vector space} \end{array}\right. \Rightarrow \underline{\operatorname{span}(A) \subseteq V.}$

(b) To show $V \subseteq \operatorname{span}(A)$.

Let $u \in V$ be given.

Case 1 : If $\exists a \in [n] : u = x_a$

Then since $\left.\begin{array}{l} x_a \in A \\ A \subseteq \mathrm{span}(A) \end{array}\right\} \Rightarrow u = x_a \in \mathrm{span}(A) \Rightarrow$

$\Rightarrow u \in \mathrm{span}(A)$.

Case 2 : If $\forall a \in [n] : u \neq x_a$

Then, it follows that

$|\{x_1, x_2, \ldots, x_n, u\}| = n+1 > n = \dim V \Rightarrow$

$\Rightarrow \left.\begin{array}{l} \{x_1, x_2, \ldots, x_n, u\} \text{ linearly dependent} \\ \{x_1, x_2, \ldots, x_n\} \text{ linearly independent} \end{array}\right\} \Rightarrow$

$\Rightarrow u \in \mathrm{span}\{x_1, x_2, \ldots, x_n\} = \mathrm{span}(A)$

It follows that

$\forall u \in V : u \in \mathrm{span}(A) \Rightarrow \underline{V \subseteq \mathrm{span}(A)}$

In both cases above we find that $V \subseteq \mathrm{span}(A)$.

It follows that

$$\left\{\begin{array}{l} V \subseteq \mathrm{span}(A) \\ \mathrm{span}(A) \subseteq V \\ A \text{ linearly independent} \end{array}\right. \Rightarrow \left\{\begin{array}{l} V = \mathrm{span}(A) \\ A \text{ linearly independent} \end{array}\right. \Rightarrow$$

$\Rightarrow A$ basis of V $\square$

$\hookrightarrow$ Recall that we have shown previously that

$\left.\begin{array}{l} B \text{ basis of } V \\ |A| > |B| \end{array}\right\} \Rightarrow A$ linearly dependent

It follows that

$$\boxed{\begin{array}{l} p > \dim V \Rightarrow \{x_1, x_2, \ldots, x_p\} \subseteq V \text{ linearly dependent} \\ \{x_1, x_2, \ldots, x_p\} \text{ linearly independent} \Rightarrow p \leq \dim V. \end{array}}$$

EXAMPLES

a) Let $x = (2,1,3)$, $y = (1,3,0)$, and $z = (1,2,3)$.
Show that $B = \{x, y, z\}$ is a basis of $\mathbb{R}^3$.

<u>Solution</u>

$$\det([x\ y\ z]) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ 3 & 0 & 0 \end{vmatrix} =$$

(-1)

$$= (+1)\cdot 3 \cdot \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 3\left(1\cdot 1 - (-1)\cdot 3\right) = 3\,(1+3) = 12 \neq 0$$

$$\left.\begin{array}{l} \Rightarrow \{x,y,z\} \text{ linearly independent} \\ \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow \{x,y,z\} \text{ basis of } \mathbb{R}^3.$$

b) Let $x = (1,1,0)$, $y = (2,0,1)$, and $z = (6,2,2)$. Show that $B = \{x, y, z\}$ is NOT basis of $\mathbb{R}^3$.

<u>Solution</u>

$$\det([x\ y\ z]) = \begin{vmatrix} 1 & 2 & 6 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} (-1) \\ \hookleftarrow \\ \end{matrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{vmatrix} =$$

$$= (+1)\cdot 1 \cdot \begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} = (-2)\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = (-2)\cdot 0 = 0$$

$\Rightarrow \{x,y,z\}$ linearly dependent $\Rightarrow \{x,y,z\}$ NOT basis of $\mathbb{R}^3$.

c) Show that $B = \{(a, a+1), (a+1, a+2)\}$ is a basis of $\mathbb{R}^2$ for all $a \in \mathbb{R}$.

Solution

Define $x = (a, a+1)$ and $y = (a+1, a+2)$. It follows that

$$\det([x\ y]) = \begin{vmatrix} a & a+1 \\ a+1 & a+2 \end{vmatrix} = a(a+2) - (a+1)^2 =$$

$$= (a^2 + 2a) - (a^2 + 2a + 1) =$$

$$= a^2 + 2a - a^2 - 2a - 1 = -1 \neq 0 \Rightarrow$$

$\Rightarrow x, y$ linearly independent (1).

Also: $|B| = |\{x, y\}| = 2 = \dim \mathbb{R}^2$ (2)

From (1) and (2): B basis of $\mathbb{R}^2$.

d) Let $B = \{(3a-1, a), (3a, a+1)\}$. Find all $a \in \mathbb{R}$ such that B is a basis of $\mathbb{R}^2$.

Solution

Define $x = (3a-1, a)$ and $y = (3a, a+1)$. Then

$$\det([x\ y]) = \begin{vmatrix} 3a-1 & 3a \\ a & a+1 \end{vmatrix} = (3a-1)(a+1) - 3a^2 =$$

$$= 3a^2 + 3a - a - 1 - 3a^2 = 2a - 1.$$

Since $|B| = |\{x, y\}| = 2 = \dim \mathbb{R}^2$, it follows that

B basis of $\mathbb{R}^2 \Leftrightarrow x, y$ linearly independent $\Leftrightarrow$

$$\Leftrightarrow \det([x\ y]) \neq 0 \Leftrightarrow$$

$$\Leftrightarrow 2a - 1 \neq 0 \Leftrightarrow 2a \neq 1 \Leftrightarrow a \neq 1/2$$

$$\Leftrightarrow a \in \mathbb{R} - \{1/2\}.$$

e) Let $B=\{x,y\}$ be a basis of $\mathbb{R}^2$. Let $u=x+3y$ and $v=2x-y$. Show that $B=\{u,v\}$ is also a basis of $\mathbb{R}^2$.

Solution

$B=\{x,y\}$ basis of $\mathbb{R}^2 \Rightarrow x,y$ linearly independent $\Rightarrow$

$$\Rightarrow \forall a,b\in\mathbb{R}: (ax+by=\mathbf{0} \Rightarrow (a,b)=(0,0)) \qquad (1)$$

Let $a,b\in\mathbb{R}$ be given and assume that $au+bv=\mathbf{0}$.

We note that

$$au+bv = a(x+3y)+b(2x-y) = ax+3ay+2bx-by =$$
$$=(a+2b)x+(3a-b)y$$

and therefore

$$au+bv=\mathbf{0} \Rightarrow (a+2b)x+(3a-b)y=\mathbf{0} \overset{(1)}{\Rightarrow} \begin{cases} a+2b=0 \\ 3a-b=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1\cdot(-1)-2\cdot 3 = -1-6 = -7 \neq 0$

it follows that $a=b=0 \Rightarrow (a,b)=(0,0)$.

We have thus shown that

$$\forall a,b\in\mathbb{R}: (au+bv=\mathbf{0} \Rightarrow (a,b)=(0,0)) \Rightarrow$$
$$\Rightarrow u,v \text{ linearly independent} \Rightarrow$$
$$\Rightarrow B=\{u,v\} \text{ basis of } \mathbb{R}^2 \quad (\text{since } |B|=2=\dim\mathbb{R}^2). \quad \square$$

EXERCISES

(51) Show that the following sets are a basis for $\mathbb{R}^2$.

a) $B = \{(1,1),(0,1)\}$

b) $B = \{(a,0),(a,b)\}$ with $ab \neq 0$

c) $B = \{(\cos\vartheta, \sin\vartheta), (-\sin\vartheta, \cos\vartheta)\}$ with $\vartheta \in \mathbb{R}$.

d) $B = \{(\cos\vartheta - \sin\vartheta, -\cos\vartheta - \sin\vartheta), (\cos\vartheta + \sin\vartheta, \cos\vartheta - \sin\vartheta)\}$

(52) Find all $a \in \mathbb{R}$ such that the following sets are a basis of $\mathbb{R}^2$

a) $x = (a-1, 3)$ and $y = (-a+1, a+1)$

b) $x = (a-1, a^2-2a+1)$ and $y = (0, a+1)$

(53) Let $B = \{x, y\}$ be a basis of $\mathbb{R}^2$. Show that $B' = \{u, v\}$ with $u = 3x - y$ and $v = x + 2y$ is also a basis of $\mathbb{R}^2$.

(54) Let $x = (2,1,0)$, $y = (2,1,1)$, $z = (2,2,1)$. Show that $B = \{x, y, z\}$ is a basis of $\mathbb{R}^3$.

(55) Let $x = (-1,1,1)$, $y = (1, a^2, 2)$, and $z = (-2, 2a, 1)$. Find all $a \in \mathbb{R}$ such that $B = \{x, y, z\}$ is a basis of $\mathbb{R}^3$.

(56) Let $B = \{x, y, z\}$ be a basis of $\mathbb{R}^3$, and let $u = 2x + y$, $v = z$, $w = u + 2v$. Show that $B' = \{u, v, w\}$ is also a basis of $\mathbb{R}^3$.

(57) Show that $B = \{x, y, z, w\}$ is a basis of $\mathbb{R}^4$ with

a) $x = (0,1,1,1)$, $y = (1,0,1,1)$, $z = (1,1,0,1)$, and $w = (1,1,1,0)$

b) $x = (2,-1,0,1)$, $y = (1,3,2,0)$, $z = (0,-1,-1,0)$, and $w = (-2,1,2,1)$

c) $x = (1,-1,2,0)$, $y = (1,1,2,0)$, $z = (3,0,0,1)$, and $w = (2,1,-1,0)$

(58) Let $B=\{x,y,z,w\}$ be a basis of $\mathbb{R}^4$. Let $u=x+y$, $v=z+w$, $p=-x+z+w$, and $q=w-y$. Show that $B'=\{u,v,p,q\}$ is also a basis of $\mathbb{R}^4$.

(59) Let V be a vector space with $\dim V=4$. Let $x_1,x_2,x_3,x_4\in V$ with x_1,x_2,x_3 linearly independent. Define $u=\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+\lambda_4x_4$ with $\lambda_1+\lambda_2+\lambda_3+\lambda_4\neq 1$. Show that $B=\{u-x_1,u-x_2,u-x_3,u-x_4\}$ is a basis of V.

Dimension of span(B)

- Let V be a vector space, let $B = \{x_1, \dots, x_n\} \subseteq V$ be a set of n vectors and consider the subspace of V given by $V_0 = \mathrm{span}(B)$. The problem is to determine the dimension $\dim V_0$.
- By the basis definition, it follows immediately that

$$\boxed{B \text{ linearly independent} \Rightarrow \dim(\mathrm{span}(B)) = |B| = n}$$

So the question becomes, what if B is linearly dependent?

Thm : Let V be a vector space, and $x_1, x_2, \dots, x_n \in V$, and let $p < n$. Then

$$\boxed{\left.\begin{array}{l} \{x_1, \dots, x_p\} \text{ linearly independent} \\ \forall u \in \{x_{p+1}, \dots, x_n\} : \{x_1, \dots, x_p, u\} \text{ linearly dependent} \end{array}\right\} \Rightarrow \\ \Rightarrow \{x_1, \dots, x_p\} \text{ basis of } \mathrm{span}\{x_1, \dots, x_n\}}$$

Proof

Since $\{x_1, \dots, x_p\} \subseteq \{x_1, \dots, x_n\} \Rightarrow$

$$\Rightarrow \mathrm{span}(\{x_1, \dots, x_p\}) \subseteq \mathrm{span}(\{x_1, \dots, x_n\}) \quad (1)$$

It is sufficient to show that

$$\mathrm{span}(\{x_1, \dots, x_n\}) \subseteq \mathrm{span}(\{x_1, \dots, x_p\})$$

Preliminary argument:
Let $a \in \mathbb{N}$ with $1 \leq a \leq n-p$. Then

$$x_{p+a} \in \{x_{p+1}, \dots, x_n\} \Rightarrow$$

$$\left.\begin{array}{l}\Rightarrow \{x_1, x_2, \dots, x_p, x_{p+a}\} \text{ linearly dependent} \\ \quad \{x_1, x_2, \dots, x_p\} \text{ linearly independent}\end{array}\right\} \Rightarrow$$

$$\Rightarrow x_{p+a} \in \mathrm{span}\{x_1, \dots, x_p\} \Rightarrow$$

$$\Rightarrow \exists \mu_{a1}, \mu_{a2}, \dots, \mu_{ap} \in \mathbb{R} : x_{p+a} = \mu_{a1} x_1 + \dots + \mu_{ap} x_p.$$

Main argument:
Let $u \in \mathrm{span}(\{x_1, \dots, x_n\})$ be given. Then

$$\exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R} : u = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n.$$

It follows that

$$u = \sum_{a \in [n]} \lambda_a x_a = \sum_{b \in [p]} \lambda_b x_b + \sum_{a \in [n-p]} \lambda_{p+a} x_{p+a} =$$

$$= \sum_{b \in [p]} \lambda_b x_b + \sum_{a \in [n-p]} \lambda_{p+a} \left[\sum_{b \in [p]} \mu_{ab} x_b \right]$$

$$= \sum_{b \in [p]} \lambda_b x_b + \sum_{b \in [p]} \left[\sum_{a \in [n-p]} \lambda_{p+a} \mu_{ab} \right] x_b =$$

$$= \sum_{b \in [p]} \left[\lambda_b + \sum_{a \in [n-p]} (\lambda_{p+a} \mu_{ab}) \right] x_b \Rightarrow$$

$$\Rightarrow u \in \mathrm{span}\{x_1, x_2, \dots, x_p\}.$$

It follows that span$\{x_1,$

$$\forall u \in \mathrm{span}(\{x_1, \dots, x_n\}) : u \in \mathrm{span}(\{x_1, \dots, x_p\}) \Rightarrow$$

$$\Rightarrow \mathrm{span}(\{x_1, \dots, x_n\}) \subseteq \mathrm{span}(\{x_1, \dots, x_p\}). \quad (2)$$

From (1) and (2):

$$\left.\begin{array}{l}\operatorname{span}(\{x_1,\dots,x_p\}) = \operatorname{span}(\{x_1,\dots,x_n\}) \\ \{x_1,\dots,x_p\} \text{ linearly independent}\end{array}\right\} \Rightarrow$$

$\Rightarrow \{x_1,\dots,x_p\}$ basis of $\operatorname{span}(\{x_1,\dots,x_n\})$. □

Belonging condition to span(B)

Let $V = \operatorname{span}(B)$. If B is shown to be a basis of B, then the following proposition gives a belonging condition to V. We stress that that if B is linearly dependent, then the theorem below will not work.

Prop: If $V = \operatorname{span}(B)$ and $B = \{x_1, x_2, \dots, x_n\}$ be a basis of V. Then

$$\boxed{x \in V \Leftrightarrow x, x_1, x_2, \dots, x_n \text{ linearly dependent}}$$

Proof

$(\Rightarrow)$: Assume $x \in V$. Then

$x \in V \Rightarrow x \in \operatorname{span}\{x_1, x_2, \dots, x_n\} \Rightarrow$

$\Rightarrow x, x_1, x_2, \dots, x_n$ linearly dependent.

$(\Leftarrow)$: Assume that $x, x_1, x_2, \dots, x_n$ linearly dependent. Then

$$\left\{\begin{array}{l} x_1, x_2, \dots, x_n \text{ basis of } V \\ x, x_1, x_2, \dots, x_n \text{ linearly dependent}\end{array}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} x_1, x_2, \dots, x_n \text{ linearly independent} \\ x, x_1, x_2, \dots, x_n \text{ linearly dependent}\end{array}\right. \Rightarrow x \in \operatorname{span}\{x_1, x_2, \dots, x_n\}$$

$\Rightarrow x \in V$.

EXAMPLES

a) Let $V = \text{span}\{x_1, x_2, x_3, x_4\}$ with $x_1 = (1,2,0,3)$, $x_2 = (2,0,3,1)$, $x_3 = (-1,2,-3,2)$, $x_4 = (3,-2,6,-1)$.
Find $\dim V$ and a belonging condition for $(a,b,c,d) \in V$.

Solution

Sufficient to find a basis B of V.

- Check x_1, x_2, x_3, x_4:

$$\det([x_1\ x_2\ x_3\ x_4]) = \begin{array}{|cccc|l} 1 & 2 & -1 & 3 & (-2)\ (-3) \\ 2 & 0 & 2 & -2 & \\ 0 & 3 & -3 & 6 & \\ 3 & 1 & 2 & -1 & \end{array} =$$

$$= \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & -4 & 4 & -8 \\ 0 & 3 & -3 & 6 \\ 0 & -5 & 5 & -10 \end{vmatrix} = 0 \Rightarrow x_1, x_2, x_3, x_4 \text{ linearly dependent.}$$

- Check x_1, x_2, x_3.

$$\text{Let } A_{123} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \text{Sub}(A_{123}) = \left\{ \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{bmatrix} \right\}$$

$$= \{ A_{123}^{(1)}, A_{123}^{(2)}, A_{123}^{(3)}, A_{123}^{(4)} \}.$$

Since:

(+1)

$$\det A_{123}^{(1)} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & 0 \end{vmatrix} = 0$$

$$\det A_{123}^{(2)} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -4 & 4 \\ 3 & -5 & 5 \end{vmatrix} = 0$$

(-2)

(+1)

$$\det A_{123}^{(3)} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

(+1)

$$\det A_{123}^{(4)} = \begin{vmatrix} 2 & 0 & 2 \\ 0 & 3 & -3 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

(+1)

it follows that

$\forall B \in Sub(A_{123}) : (\det B = 0) \Rightarrow x_1, x_2, x_3$ linearly dependent.

- Check x_1, x_2, x_4

Let $A_{124} = [x_1 \; x_2 \; x_4] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{bmatrix} \Rightarrow$

$$\mathrm{Sub}(A_{124}) = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 0 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{bmatrix} \right\} =$$

$$= \left\{ A_{124}^{(1)}, A_{124}^{(2)}, A_{124}^{(3)}, A_{124}^{(4)} \right\}$$

Since

$$\det A_{124}^{(1)} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 0 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 0 \\ 0 & 3 & 6 \end{vmatrix} \rightarrow = (-1) \cdot 2 \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} =$$

$(+1)$

$$= (-1) \cdot 2 \cdot 2 \cdot 3 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = (-1) \cdot 2 \cdot 2 \cdot 3 \cdot 0 = 0$$

$$\det A_{124}^{(2)} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix} \rightarrow = (-1) \cdot 2 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} =$$

$(+1)$

$$= (-1) \cdot 2 \cdot 2 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$\det A_{124}^{(3)} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{vmatrix} \begin{matrix} (-3) \\ \\ \\ \end{matrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & -5 & -10 \end{vmatrix} = (+1) \cdot 1 \cdot \begin{vmatrix} 3 & 6 \\ -5 & -10 \end{vmatrix} =$$

$$= (+1) \cdot 1 \cdot 3 \cdot (-5) \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

$$\det A_{124}^{(4)} = \begin{vmatrix} 2 & 0 & -2 \\ 0 & 3 & 6 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 6 \\ 3 & 1 & 2 \end{vmatrix} = (+1)\cdot 2 \cdot \begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} =$$

$(+1)$

$$= (1)\cdot 2 \cdot 3 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

and therefore:

$\forall B \in Sub(A_{124}) : (\det B = 0) \Rightarrow x_1, x_2, x_4$ linearly dependent;

- Check x_1, x_2.

Let $A_{12} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 3 \\ 3 & 1 \end{bmatrix} \Rightarrow$

$$\Rightarrow Sub(A_{12}) = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix} \right\} = \{A_{12}^{(1)}, \ldots, A_{12}^{(6)}\}$$

$$\det A_{12}^{(1)} = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = 1\cdot 0 - 2\cdot 2 = -4 \neq 0 \Rightarrow$$

$\Rightarrow \exists B \in Sub(A_{12}) : (\det B \neq 0) \Rightarrow x_1, x_2$ linearly independent.

- Basis of V.

It follows from the above results that:

$$\left\{\begin{array}{l} x_1, x_2 \text{ linearly independent} \\ x_1, x_2, x_3 \text{ linearly dependent} \\ x_1, x_2, x_4 \text{ linearly dependent} \end{array}\right. \Rightarrow$$

$\Rightarrow \quad V = \operatorname{span}\{x_1, x_2, x_3, x_4\} = \operatorname{span}\{x_1, x_2\}$

and therefore:

$$\left\{\begin{array}{l} V = \operatorname{span}\{x_1, x_2\} \\ x_1, x_2 \text{ linearly independent} \end{array}\right. \Rightarrow \{x_1, x_2\} \text{ basis of } V \Rightarrow$$

$\Rightarrow \dim V = |\{x_1, x_2\}| = 2.$

• Belonging condition for V.

Let $x = (a, b, c, d) \in \mathbb{R}^4$ and define

$$A = [x_1 \; x_2 \; x] = \begin{bmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 0 & 3 & c \\ 3 & 1 & d \end{bmatrix} \Rightarrow$$

$$\Rightarrow \operatorname{Sub}(A) = \left\{ \begin{bmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 0 & 3 & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 3 & 1 & d \end{bmatrix}, \begin{bmatrix} 1 & 2 & a \\ 0 & 3 & c \\ 3 & 1 & d \end{bmatrix}, \begin{bmatrix} 2 & 0 & b \\ 0 & 3 & c \\ 3 & 1 & d \end{bmatrix} \right\}$$

$$= \{A_1, A_2, A_3, A_4\}.$$

We calculate the determinants of A_1, A_2, A_3, A_4:

$$\det A_1 = \begin{vmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 0 & 3 & c \end{vmatrix} \begin{array}{l} (-2) \\ \hookleftarrow \\ \end{array} = \begin{vmatrix} 1 & 2 & a \\ 0 & -4 & -2a+b \\ 0 & 3 & c \end{vmatrix} =$$

$$= \begin{vmatrix} -4 & -2a+b \\ 3 & c \end{vmatrix} = (-4c)-3(-2a+b) = 6a-3b-4c$$

$$\det A_2 = \begin{vmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 3 & 1 & d \end{vmatrix} \begin{matrix} \leftarrow \\ \\ (-2) \end{matrix} = \begin{vmatrix} -5 & 0 & a-2d \\ 2 & 0 & b \\ 3 & 1 & d \end{vmatrix} =$$

$$= (-1)\cdot 1 \cdot \begin{vmatrix} -5 & a-2d \\ 2 & b \end{vmatrix} = -(-5b-2(a-2d)) =$$

$$= 2a+5b-4d$$

$$\det A_3 = \begin{vmatrix} 1 & 2 & a \\ 0 & 3 & c \\ 3 & 1 & d \end{vmatrix} \begin{matrix} (-3) \\ \\ \leftarrow \end{matrix} = \begin{vmatrix} 1 & 2 & a \\ 0 & 3 & c \\ 0 & -5 & -3a+d \end{vmatrix} = \begin{vmatrix} 3 & c \\ -5 & -3a+d \end{vmatrix} =$$

$$= 3(-3a+d)-(-5)c = -9a+5c+3d$$

$$\det A_4 = \begin{vmatrix} 2 & 0 & b \\ 0 & 3 & c \\ 3 & 1 & d \end{vmatrix} \begin{matrix} \\ \leftarrow \\ (-3) \end{matrix} = \begin{vmatrix} 2 & 0 & b \\ -9 & 0 & c-3d \\ 3 & 1 & d \end{vmatrix} = (-1)\cdot 1 \cdot \begin{vmatrix} 2 & b \\ -9 & c-3d \end{vmatrix}$$

$$= -[2(c-3d) - (-9)b] = -9b-2c+6d$$

Main argument:

$x \in V \Leftrightarrow x \in \operatorname{span}\{x_1, x_2\} \Leftrightarrow$

$\Leftrightarrow x_1, x_2, x$ linearly dependent $\Leftrightarrow$

$\Leftrightarrow \forall A \in \operatorname{Sub}([x_1\ x_2\ x]) : \det A = 0 \Leftrightarrow$

$\Leftrightarrow \det A_1 = 0 \wedge \det A_2 = 0 \wedge \det A_3 = 0 \wedge \det A_4 = 0$

$\Leftrightarrow 6a-3b-4c = 0 \wedge 2a+5b-4d = 0 \wedge -9a+5c+3d = 0$

$\wedge -9b-2c+6d = 0.$

b) Let $f, g, h \in F(\mathbb{R})$ with:

$\forall x \in \mathbb{R} : f(x) = 1$

$\forall x \in \mathbb{R} : g(x) = \sin^2 x$

$\forall x \in \mathbb{R} : h(x) = \cos^2 x$

Find the dimension of $V = \text{span}\{f, g, h\}$.

Solution

• Check f, g, h.

We note that

$$\forall x \in \mathbb{R} : (g+h)(x) = g(x) + h(x) = \sin^2 x + \cos^2 = $$
$$= 1 = f(x) \Rightarrow$$

$\Rightarrow f = g + h \Rightarrow f \in \text{span}\{g, h\} \Rightarrow f, g, h$ linearly dependent. (1)

• Check g, h

We will show that $\forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 g + \lambda_2 h = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = 0)$.

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be given. Assume that $\lambda_1 g + \lambda_2 h = \mathbf{0}$ (2)

Then

$$\forall x \in \mathbb{R} : (\lambda_1 g + \lambda_2 h)(x) = (\lambda_1 g)(x) + (\lambda_2 h)(x) = \lambda_1 g(x) + \lambda_2 h(x)$$
$$= \lambda_1 \sin^2 x + \lambda_2 \cos^2 x \quad (3)$$

From (2) and (3): $\forall x \in \mathbb{R} : \lambda_1 \sin^2 x + \lambda_2 \cos^2 x = 0$

For $x = 0$: $\lambda_1 \sin^2 0 + \lambda_2 \cos^2 0 = 0 \Rightarrow 0\lambda_1 + 1\lambda_2 = 0 \Rightarrow \lambda_2 = 0$

For $x = \pi/2$: $\lambda_1 \sin^2(\pi/2) + \lambda_2 \cos^2(\pi/2) = 0 \Rightarrow 1\lambda_1 + 0\lambda_2 = 0 \Rightarrow$
$\Rightarrow \lambda_1 = 0$.

It follows that

$\forall \lambda_1, \lambda_2 \in \mathbb{R} : (\lambda_1 g + \lambda_2 h = \mathbf{0} \Rightarrow \lambda_1 = \lambda_2 = 0) \Rightarrow$

$\Rightarrow g, h$ linearly independent (4).

From (1) and (4):

$$\left\{\begin{array}{l} f, g, h \text{ linearly dependent} \\ g, h \text{ linearly independent} \end{array}\right. \Rightarrow$$

$\Rightarrow V = \operatorname{span}\{f, g, h\} = \operatorname{span}\{g, h\}$. (5)

From (4) and (5):

$$\left\{\begin{array}{l} V = \operatorname{span}\{g, h\} \\ g, h \text{ linearly independent} \end{array}\right. \Rightarrow \{g, h\} \text{ basis of } V \Rightarrow$$

$\Rightarrow \dim V = |\{g, h\}| = 2.$

c) Let us define

$$M(a,b,c) = \begin{bmatrix} a & b & c \\ 3c & a-3c & b \\ 3b & -3b+3c & a-3c \end{bmatrix}$$

and consider the set

$V = \{ M(a,b,c) \mid (a,b,c) \in \mathbb{R}^3 \}$

Show that V is a subspace of $M_3(\mathbb{R})$ and determine the dimension $\dim V$.

<u>Solution</u>

We note that

$$M(a,b,c) = \begin{bmatrix} a & b & c \\ 3c & a-3c & b \\ 3b & -3b+3c & a-3c \end{bmatrix} =$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & b \\ 3b & -3b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ 3c & -3c & 0 \\ 0 & 3c & -3c \end{bmatrix} =$$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -3 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 3 & -3 & 0 \\ 0 & 3 & -3 \end{bmatrix}$$

$= aA_1 + bA_2 + cA_3$

with

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -3 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 3 & -3 & 0 \\ 0 & 3 & -3 \end{bmatrix}$$

It follows that

$$V = \{M(a,b,c) \mid a,b,c \in \mathbb{R}\} =$$
$$= \{aA_1 + bA_2 + cA_3 \mid a,b,c \in \mathbb{R}\} =$$
$$= \operatorname{span}\{A_1, A_2, A_3\} \qquad (1)$$

From (1): V subspace of $M_2(\mathbb{R})$

• Check A_1, A_2, A_3 dependence

Let $a,b,c \in \mathbb{R}$ be given. Assume that $aA_1 + bA_2 + cA_3 = \mathbf{0}$.

It follows that

$$\begin{bmatrix} a & b & c \\ 3c & a-3c & b \\ 3b & -3b+3c & a-3c \end{bmatrix} = M(a,b,c) = aA_1 + bA_2 + cA_3 = \mathbf{0} \Rightarrow$$

$$\Rightarrow a = 0 \wedge b = 0 \wedge c = 0 \Rightarrow (a,b,c) = (0,0,0).$$

We have thus shown

$$\forall a,b,c \in \mathbb{R} : (aA_1 + bA_2 + cA_3 = \mathbf{0} \Rightarrow (a,b,c) = (0,0,0)) \Rightarrow$$

$\Rightarrow A_1, A_2, A_3$ linearly independent. (2)

From (1) and (2):

$$\begin{cases} V = \operatorname{span}\{A_1, A_2, A_3\} \\ A_1, A_2, A_3 \text{ linearly independent} \end{cases} \Rightarrow$$

$\Rightarrow \{A_1, A_2, A_3\}$ basis of $V \Rightarrow$

$$\Rightarrow \dim V = |\{A_1, A_2, A_3\}| = 3.$$

d) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and consider the space V given by

$V = \{X \in M_2(\mathbb{R}) : AX = XA\}$.

Show that V is a subspace of $M_2(\mathbb{R})$ and evaluate $\dim V$.

Solution

• Determine V.

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We note that

$$X \in V \Leftrightarrow X \in \{X \in M_2(\mathbb{R}) : AX = XA\} \Leftrightarrow AX = XA \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} a+2c = a \\ c = c \\ b+2d = 2a+b \\ d = 2c+d \end{cases} \Leftrightarrow \begin{cases} 2c = 0 \\ 2d = 2a \\ 2c = 0 \end{cases} \Leftrightarrow \begin{cases} a = d \\ c = 0 \end{cases}$$

$$\Leftrightarrow X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} =$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = aA_1 + bA_2$$

with $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

It follows that

$x \in V \Leftrightarrow \exists a, b \in \mathbb{R}: X = aA_1 + bA_2 \Leftrightarrow X \in \text{span}\{A_1, A_2\}$

and therefore $V = \text{span}\{A_1, A_2\}$ (1)

From (1): V subspace of $M_2(\mathbb{R})$.

• Check dependence of A_1, A_2.

Let $a, b \in \mathbb{R}$ be given. Assume that $aA_1 + bA_2 = \mathbf{0}$.

It follows that

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = aA_1 + bA_2 = \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = 0.$$

Thus:

$\forall a, b \in \mathbb{R}: (aA_1 + bA_2 = \mathbf{0} \Rightarrow a = b = 0) \Rightarrow$

$\Rightarrow A_1, A_2$ linearly independent. (2)

From (1) and (2):

$$\left.\begin{cases} V = \text{span}\{A_1, A_2\} \\ A_1, A_2 \text{ linearly independent} \end{cases}\right. \Rightarrow \{A_1, A_2\} \text{ basis of } V$$

$\Rightarrow \dim V = |\{A_1, A_2\}| = 2.$

e) Given $x=(1,2,3)$, $y=(-1,4,5)$, $z=(-5,2,1)$, and $w=(9,12,19)$. Show that $\text{span}\{x,y\}=\text{span}\{z,w\}$.

<u>Solution</u>

Let $A_1=\begin{bmatrix}1 & -1\\ 2 & 4\end{bmatrix}\in \text{Sub}([x\ y]) \Rightarrow$

$\Rightarrow \det A_1 = 1\cdot 4-(-1)\cdot 2 = 4+2 = 6\neq 0 \Rightarrow$

$\Rightarrow$ x,y linearly independent. (1)

Let $A_2=\begin{bmatrix}-1 & -5\\ 4 & 2\end{bmatrix}\in \text{Sub}([z\ w]) \Rightarrow$

$\Rightarrow \det A_2 = (-1)\cdot 2-(-5)\cdot 4 = -2+20 = 18\neq 0 \Rightarrow$

$\Rightarrow$ z,w linearly independent (2)

We also note that

$$\det([x\ y\ z]) = \begin{vmatrix}1 & -1 & -5\\ 2 & 4 & 2\\ 3 & 5 & 1\end{vmatrix}\ (-2)\ (-3) = \begin{vmatrix}1 & -1 & -5\\ 0 & 6 & 12\\ 0 & 8 & 16\end{vmatrix} =$$

$$= 6\cdot 8\cdot \begin{vmatrix}1 & -1 & -5\\ 0 & 1 & 2\\ 0 & 1 & 2\end{vmatrix} = 0 \Rightarrow$$

$$\left.\begin{matrix}\Rightarrow x,y,z \text{ linearly dependent}\\ x,y \text{ linearly independent}\end{matrix}\right\}\Rightarrow \underline{z\in \text{span}\{x,y\}}\quad (3)$$

$$\det([x\ y\ w]) = \begin{vmatrix}1 & -1 & 9\\ 2 & 4 & 12\\ 3 & 5 & 19\end{vmatrix}\ (-2)\ (-3) = \begin{vmatrix}1 & -1 & 9\\ 0 & 6 & -6\\ 0 & 8 & -8\end{vmatrix}$$

$$= 6\cdot 8 \cdot \begin{vmatrix} 1 & -1 & 9 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} x, y, w \text{ linearly dependent} \\ x, y \text{ linearly independent} \end{array}\right\} \Rightarrow \underline{w \in \operatorname{span}\{x, y\}}. \quad (4)$$

From (3), (4):

$$w, z \in \operatorname{span}\{x, y\} \Rightarrow \forall a, b \in \mathbb{R}: (az + bw) \in \operatorname{span}\{x, y\} \Rightarrow$$

$$\Rightarrow \operatorname{span}\{z, w\} = \{az + bw \mid a, b \in \mathbb{R}\} \subseteq \operatorname{span}\{x, y\} \Rightarrow$$

$$\Rightarrow \underline{\operatorname{span}\{z, w\} \subseteq \operatorname{span}\{x, y\}} \quad (5)$$

Furthermore:

$$\det([x\ z\ w]) = \begin{vmatrix} 1 & -5 & 9 \\ 2 & 2 & 12 \\ 3 & 1 & 19 \end{vmatrix} \begin{matrix} (-2) & (-3) \\ & \\ & \end{matrix} = \begin{vmatrix} 1 & -5 & 9 \\ 0 & 12 & -6 \\ 0 & 16 & -8 \end{vmatrix} =$$

$$= 6\cdot 8 \cdot \begin{vmatrix} 1 & -5 & 9 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} x, z, w \text{ linearly dependent} \\ z, w \text{ linearly independent} \end{array}\right\} \Rightarrow \underline{x \in \operatorname{span}\{z, w\}} \quad (6)$$

$$\det([y\ z\ w]) = \begin{vmatrix} -1 & -5 & 9 \\ 4 & 2 & 12 \\ 5 & 1 & 19 \end{vmatrix} \begin{matrix} 4 & 5 \\ & \\ & \end{matrix} = \begin{vmatrix} -1 & -5 & 9 \\ 0 & -18 & 48 \\ 0 & -24 & 64 \end{vmatrix} =$$

$$= 6 \cdot 8 \cdot \begin{vmatrix} -1 & -5 & 9 \\ 0 & -3 & 8 \\ 0 & -3 & 8 \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} y, z, w \text{ linearly dependent} \\ z, w \text{ linearly independent} \end{matrix}\right\} \Rightarrow \underline{y \in \operatorname{span}\{z, w\}.} \quad (7)$$

From (6) to (7):

$$x, y \in \operatorname{span}\{z, w\} \Rightarrow \forall a, b \in \mathbb{R}: (ax + by) \in \operatorname{span}\{z, w\} \Rightarrow$$

$$\Rightarrow \operatorname{span}\{x, y\} = \{ax + by \mid \alpha, b \in \mathbb{R}\} \subseteq \operatorname{span}\{z, w\} \Rightarrow$$

$$\Rightarrow \underline{\operatorname{span}\{x, y\} \subseteq \operatorname{span}\{z, w\}.} \quad (8)$$

From (5) and (8):

$$\left\{\begin{matrix} \operatorname{span}\{z, w\} \subseteq \operatorname{span}\{x, y\} \\ \operatorname{span}\{x, y\} \subseteq \operatorname{span}\{z, w\} \end{matrix}\right. \Rightarrow \underline{\underline{\operatorname{span}\{x, y\} = \operatorname{span}\{z, w\}.}}$$

EXERCISES

(60) Let $x=(1,-1,2,1)$, $y=(1,2,1,0)$, and $z=(-1,1,-2,-1)$. Find a basis and the dimension of $V=\text{span}\{x,y,z\}$.

(61) Let $x=(1,4,-5,2)$ and $y=(1,2,3,1)$, and define $V=\text{span}\{x,y\}$. Explore whether $u=(2,14,-34,7)$ belongs to V.

(62) Let $x=(2,1,0)$, $y=(1,-1,2)$, and $z=(0,3,4)$, and define $V=\text{span}\{x,y,z\}$. Show that

$$(a,b,c)\in V \Leftrightarrow 2a-4b-3c=0$$

(63) Let $x=(1,1,1)$, $y=(1,-1,0)$, $z=(0,2,1)$, and $w=(3,1,2)$. Show that $\text{span}\{x,y\}=\text{span}\{z,w\}$.
(Hint: First we use linear dependance and independence to show that $z,w\in\text{span}\{x,y\}$ and $x,y\in\text{span}\{z,w\}$.)

(64) Let $x=(1,-1,2)$, $y=(2,1,3)$, and $z=(3,3,4)$. Show that $z\in\text{span}\{x,y\}$.

(65) Find the dimension and a basis for the subspace $F(\mathbb{R})$ spanned by:

a) $\begin{cases} \forall x\in\mathbb{R}: f(x)=\sin x\cos x \\ \forall x\in\mathbb{R}: g(x)=\sin 2x \\ \forall x\in\mathbb{R}: h(x)=\cos 2x \end{cases}$

b) $\begin{cases} \forall x\in\mathbb{R}: f(x)=\sin^2 x \\ \forall x\in\mathbb{R}: g(x)=\cos 2x \\ \forall x\in\mathbb{R}: h(x)=1+\cos 2x \end{cases}$

c) $\begin{cases} \forall x\in\mathbb{R}: f(x)=\sin^2 x \\ \forall x\in\mathbb{R}: g(x)=\cos^2 x \\ \forall x\in\mathbb{R}: h(x)=\cos 2x \end{cases}$

d) $\begin{cases} \forall x\in\mathbb{R}: f(x)=xe^x \\ \forall x\in\mathbb{R}: g(x)=x^2e^x \\ \forall x\in\mathbb{R}: h(x)=x^3e^x \end{cases}$

(66) Let $M(a,b) = \begin{bmatrix} 3a+b & 2a \\ 2b & a+b \end{bmatrix}$ and define

$V = \{M(a,b) \mid a,b \in \mathbb{R}\}$.

Show that V is a subspace of $M_2(\mathbb{R})$ and find a basis and the dimension of V.

(67) Let

$$M(a,b,c) = \begin{bmatrix} a+b+c & b+c & a+b \\ a-b+c & c+a & b+c \\ a+b-c & a+b & c+a \end{bmatrix}$$

and define $V = \{M(a,b,c) \mid a,b,c \in \mathbb{R}\}$. Show that V is a subspace of $M_2(\mathbb{R})$ and find a basis and the dimension of V.

LIN7: Vector Spaces – Theory Questions

THEORY QUESTIONS ON VECTOR SPACES

Internal operations

(1) What is the definition of an operation?

(2) What is the definition of an internal operation?

(3) Let $*$ be an internal operation on a set $A \neq \emptyset$. State the necessary and sufficient condition for the following statements

a) $*$ is associative

b) $*$ is commutative

c) e is a unit element of $(A, *)$

d) $*$ is NOT associative

e) $*$ is NOT commutative

f) e is NOT a unit element of $(A, *)$

(4) Let $*$ be an internal operation on A with unit element $e \in A$. If $a, a' \in A$, write the necessary and sufficient condition for the statement:

a, a' are symmetric

(5) Show that if $*$ is an internal operation in A with e a unit element, then that unit element is unique. State and prove the corresponding mathematical statement.

(6) Show that if $*$ is an associative internal operation on A with a unit element $e \in A$, then any element $a \in A$ cannot have more than one symmetric element $a' \in A$. State and prove the corresponding mathematical statement.

(7) Let $*$ be an internal operation on A and let $A_1 \subseteq A$ be a subset of A. When do we say that "$*$" is closed on the set A_1?

Groups

1. Let $*$ be an internal operation on U and let $G \subseteq U$ be a subset of U. Give the definitions for the following statements:
   a) $(G, *)$ is a group
   b) $(G, *)$ is an abelian group.

2. Let $*$ be an internal operation on U and let $G \subseteq U$ be a subset of U. Give the theorem stating the <u>sufficient conditions</u> for showing that $(G, *)$ is a group.

3. Let $(G, *)$ be a group and let $a' \in G$ be the symmetric element of $a \in G$. Prove that:
   a) $\forall a, b \in G : (a * b)' = b' * a'$
   b) $\forall a \in G : a'' = a$ (note: $a'' = (a')'$)

Vector spaces

① What is the definition of an external operation?

② What is the definition of a real vector space?

③ Show that if $(V,+,\cdot)$ is a real vector space then $(V,+)$ is an abelian group.

④ Let $(V,+,\cdot)$ be a vector space and let $\mathbf{0}$ be the unit element of the group $(V,+)$. Show that:

a) $\forall \lambda \in \mathbb{R}: \lambda \mathbf{0} = \mathbf{0}$

b) $\forall x \in V: 0x = \mathbf{0}$

References

The following references were consulted during the preparation of these lecture notes.

(1) A. Pistofides (1992), "Algebra IV", unpublished lecture notes.
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(3) T.M. Apostol (1969), "Calculus, Vol. 2", Wiley.

Lecture notes by Pistofides are available for download at

http://www.math.utpa.edu/lf/OGS/pistofides.html